Laplace's method and high temperature generalized Hopfield models

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Abstract

We consider a class of disordered mean-field spin systems that generalize the Hopfield model with many patterns in two ways: (i) General multi-spin interactions are permitted and (ii) the disorder variables have arbitrary distributions with finite exponential moments. We prove that for all models in this class the high temperature normalized partition function fluctuates according to (essentially) the same log-normal distribution. We also give an analogue statement concerning the fluctuations of the joint distribution of the overlaps of any number of replicas. The key ingredient in the proof of these results is an asymptotic expansion of the Laplace's integral that we perform up to the $1/N$-term.

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1 Introduction and results

The mathematical study of spin glass models is by now more than 20 years old and still a flourishing subject. These models were first introduced in the field of theoretical physics in an attempt to understand the singular low temperature behavior of disordered magnetic materials. Historically, the first model developed to this end was a random interactions version of the classical nearest neighbor Ising model called Edwards-Anderson model [12]. Soon after a mean-field version of the EA model was introduced by Sherrington and Kirkpatrick [23]: It played and continues to play a central rôle in the development of the subject. Although it was announced as being "solvable" the SK model happened to be the source of many very difficult problems and it is only recently that its thermodynamic limit was proved to exist for all values of the temperature and the external field [14] and its free energy computed [27].

In parallel to the activity around the SK model and with the same objective in mind, physicists and mathematicians started considering what can be regarded as a disordered version of the Curie-Weiss model, namely the Hopfield model (see [19, 20, 21] for seminal works). In this paper we will be concerned with this model and some of its generalizations. It can be introduced as follows: Let $\Sigma_N = \{-1, 1\}^N$ be the state space of spin configurations $\sigma = (\sigma_1, \dots, \sigma_N)$. The set Σ_N is furnished with the a-priori measure $\mathsf{P}_\sigma = [1/2(\delta_1 + \delta_{-1})]^{\otimes N}$ and expectation w.r.t. P_σ is denoted by E_σ. The configurations $\sigma \in \Sigma_N$ are weighted according to a *random* Gibbs measure at inverse temperature $\beta > 0$

$$\mathcal{G}_{N,M,\beta}[\omega](\sigma) = \frac{1}{\mathcal{Z}_{N,M,\beta}[\omega]} e^{\beta H^{Hopfield}_{N,M}[\omega](\sigma)} \, \mathsf{P}_\sigma(\sigma) \tag{1}$$

with

$$\mathcal{Z}_{N,M,\beta}[\omega] = \mathsf{E}_\sigma \, e^{\beta H^{Hopfield}_{N,M}[\omega](\sigma)}, \tag{2}$$

where the randomness is brought to the Hamiltonian

$$H^{Hopfield}_{N,M}[\omega](\sigma) = \frac{N}{2} \sum_{k=1}^{M} \Big(\frac{\sigma \cdot \xi^k[\omega]}{N} \Big)^2 \tag{3}$$

by a family of independent and identically distributed fair Bernoulli random variables $(\xi_i^k)_{1 \le i \le N}^{1 \le k \le M}$ called *disorder* variables and defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $\mathbb{E}$ expectation w.r.t. $\mathbb{P}$, by $\cdot$ the usual scalar product on $\mathbb{R}^N$ and for every k, $1 \le k \le M$, we denote by ξ^k the vector $(\xi_1^k, \dots, \xi_N^k) \in \Sigma_N$. The way the disorder enters into play is easily seen by keeping track of the ω's in the formulas (1,2,3). A distinctive feature of this model is that its definition involves two size parameters N and M. However, as it is customary, we will take $M = M(N)$. The ξ^k's are traditionally called *patterns* while the random maps $m^k \; : \; \sigma \in \Sigma_N \mapsto m^k(\sigma) = N^{-1} \sum_{i=1}^N \xi_i^k \sigma_i$ are called *overlaps*. This vocabulary is reminiscent of the field of neural networks where the Hamiltonian (3) became popular as a model of associative memories [15]. From the point of view of statistical physics, the partition function $\mathcal{Z}_{N,M,\beta}$, that turns to be a *random variable*, conveys important information on the system. One of the main issues is the computation of the free energy

$$f_{N,M,\beta} = \frac{1}{N} \log \mathcal{Z}_{N,M,\beta} \tag{4}$$

in the thermodynamic limit $N \to \infty$. It is known (see e.g. [3]) that this quantity is actually self-averaging

$$\lim_{N \to \infty} |f_{N,M,\beta} - \mathbb{E} f_{N,M,\beta}| = 0 \quad a.s. \tag{5}$$

a fact that makes the existence and computation of $\lim_{N\to\infty} \mathbb{E} f_{N,M,\beta}$ an important question. It crucially depends on the value of the ratio $\lim_{N\to\infty} M/N = \alpha$ and it is known [2] that

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \mathcal{Z}_{N,M,\beta} = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \mathcal{Z}_{N,M,\beta} \tag{6}$$

in the closure of the set $\beta(1 + \sqrt{\alpha}) < 1$ (Note that the relation $\le$ in (6) holds true for any $\beta \ge 0$ and $\alpha \ge 0$ due to Jensen's inequality). The system is then said to be in its *high temperature phase* and behaves in a simple way. For instance its free energy is easy to compute as a direct consequence of (5) and (6). The situation is completely different in the *low temperature phase*, i.e. outside the closure of $\beta(1+\sqrt{\alpha}) < 1$, where two different phenomenon which are refereed to as *ferromagnetic* and *spin glass* behaviors can be observed. For instance a rigorous proof for the expression of the free energy of the Hopfield model on large parts of the low temperature phase is still missing.

In view of the difficulties presented by the study of the SK and Hopfield models it appeared more rewarding to consider their variants. Indeed, some are at the same time amenable to a rigorous analysis and already exhibit part of the rich behavior expected from the original models. In particular the study of $p-$bodies interaction versions have played an important role in the progresses made during the last years in this field (see, among many others, [5] for the $p-$spins SK model and its formal limit as $p \to \infty$ the so-called Random Energy Model, [3, 6] for the $p-$spins Hopfield model and finally [26] for a broad survey of the topic).

Advances in the understanding of variants of the original models naturally led to the study of *families* of models. The interest in considering families rather than particular cases obviously lies on the fact that only an upper level of abstraction can reveal what are the mechanisms that really enter into play. In the case of generalized Hopfield models, Bovier and Gayrard [4] gave a construction of the extremal Gibbs states of systems which Hamiltonians write

$$H^{BG}_{N,M}(\sigma) = NE_M\Big((m^k(\sigma))_{1\le k\le M}\Big) \tag{7}$$

with E_M smooth convex functions defined on $\mathbb{R}^M$. Actually their framework is more abstract and far more general than what (7) means. But, when restricted to the family of Hopfield-type models, it reduces to Hamiltonians like (7). In particular, the assumption that E_M is convex is a crucial ingredient of their study.

Comets and Dembo [7], through a Large Deviations analysis of conditional probabilities, reached a rather complete understanding of the thermodynamics of a family of models which Hamiltonians defined on $[-1,1]^N$ (continuous spins) have the form

$$H^{CD}_{N,M}(\sigma) = Ng\Big(\sum_{k=1}^{M}\Big\{f\Big(m^k(\sigma)\Big) - f(0) - m^k(\sigma)f'(0)\Big\}\Big). \tag{8}$$

They assume that the disorder variables are bounded, that the maps f and g are continuous and that f is C^2 in a neighborhood of 0. However, their LD analysis is restricted to the $M = o(N)$ regime.

This paper's aim is to investigate the high temperature properties of a class of models related to those already analyzed in [4, 7]. The class is defined as follows:

1) The disorder variables $(\xi_i^k)_{1\le i\le N}^{1\le k\le M}$ are independent, identically distributed and defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. They satisfy

$$\gamma(y) = \log \mathbb{E}\exp(\xi_i^k y) < \infty \tag{9}$$

for every $y \in \mathbb{R}$. Thus, unlike [7], we do not assume the disorder variables to be bounded. We even allow their tail distribution to be significantly larger than e.g. Gaussian tails. We further assume that the distribution of ξ_i^k is symmetric and that $\mathbb{E}(\xi_i^k)^2 = 1$. We denote by I the Cramér's transform of the distribution of the ξ_i^k's

$$I(x) = \sup_{y \in \mathbb{R}} \{yx - \gamma(y)\}. \tag{10}$$

2) Let f be an even, smooth function defined on $\mathbb{R}$ such that $f(0) = 0$ and $f''(0) = 1$. By smooth we mean that f is continuous on $\mathbb{R}$ and at least C^6 in a neighborhood of 0. In the class of models we consider the Hamiltonians write

$$H_{N,M}(\sigma) = N \sum_{k=1}^{M} f(m^k(\sigma)). \tag{11}$$

Thus, unlike [4], we do not assume the Hamiltonians to be convex functions of the overlaps $(m^k(\sigma))_{1 \le k \le M}$. Moreover, unlike [3], we do not restrict the choice of f to power functions. This is not only for the sake of generality but also (and mainly) because, as it is showed in [13], Curie-Weiss models with general f already exhibit a rich behavior of multiple phase transitions.

3) Finally we consider models such that the critical inverse temperature

$$\boldsymbol{\beta}_c = \sup\{\beta > 0 : x \mapsto \beta f(x) - I(x) \text{ has its unique global maximum at } x = 0\}$$

is well-defined. Due to the properties of f and I around 0 we have $\boldsymbol{\beta}_c \le 1$.

With all these elements in hand we can define the Gibbs measure, the partition function and the free energy as in (1,2,4). Let $\mathbf{D}$ be the following domain of parameters (β, α):

$$\mathbf{D} = \left\{ \begin{array}{l} \beta < \boldsymbol{\beta}_c \\ \beta(1 + \sqrt{\alpha}) < 1. \end{array} \right. \tag{12}$$

For any fixed $(\beta, \alpha) \in \mathbf{D}$ we prove a weak Law of Large Numbers for $f_{N,M,\beta}$ and the corresponding Central Limit Theorem in the scale N. These results are obtained as a consequence of the convergence in distribution of the *normalized* partition function $\mathbf{Z}_{N,M,\beta} = \mathcal{Z}_{N,M,\beta}/\mathbb{E}\mathcal{Z}_{N,M,\beta}$. In proving the convergence of $\mathbf{Z}_{N,M,\beta}$ we follow the martingale approach first introduced by Comets and Neveu in the study of the SK model [9]. It showed to be powerful in many related situations (see [29, 5, 8, 28]). More precisely, following [8], we take advantage of the linear relation between M and N when $N \to \infty$ and "replace" α by a dynamical parameter $t \ge 0$:

$$H_N(t, \sigma) = N \sum_{k=1}^{[Nt]} f(m^k(\sigma)). \tag{13}$$

For every integer N and every $\sigma \in \Sigma_N$, $H_N(\cdot, \sigma)$ can be viewed as a random walk. Let us denote by $\varphi_N(\beta, \sigma) = \log \mathbb{E} \exp\left\{\beta N f(\frac{\sigma \cdot \xi^1}{N})\right\}$. Due to the symmetry of the

ξ_i^k' s it is clear that $\varphi_N(\beta, \sigma) = \varphi_N(\beta)$ does not depend on σ. For every fixed β, σ and N the process

$$b_N^\beta(t, \sigma) = \exp\{\beta H_N(t, \sigma) - [Nt]\varphi_N(\beta)\}$$

is a positive mean one martingale w.r.t. the filtration generated by the sequence of patterns:

$$\mathcal{F}_t^N = \sigma(\xi_i^k; 1 \le i, 1 \le k \le [tN]) \ . \tag{14}$$

We notice that $\mathbf{Z}_{N,M,\beta}$ can then be represented by

$$\mathbf{Z}_N^\beta(t) = \mathsf{E}_\sigma \, b_N^\beta(t, \sigma)$$

which is a positive mean one martingale w.r.t. $(\mathcal{F}_t^N)_{t \ge 0}$ as well. Hence, investigating the convergence in distribution of $\mathbf{Z}_{N,M,\beta}$ is reduced to the study of the convergence in distribution of the sequence of martingales $\mathbf{Z}_N^\beta(t)$ for which we prove

Theorem 1 *Let $\beta_0 \in [0, \boldsymbol{\beta}_c)$ and $t_0 > 0$ be such that (12) holds. Then the process $\mathbf{Z}_N^{\beta_0}(t)$ converges in distribution as $N \to \infty$ on the Skorohod space $D([0, t_0), \mathbf{R}^+)$ to the process*

$$Z_\infty^{\beta_0}(t) = \exp\left\{M_\infty^{\beta_0}(t) - \frac{1}{2}\Gamma(\beta_0, t)\right\} \tag{15}$$

where $M_\infty^{\beta_0}(t)$ is an independent increments Gaussian process on $[0, t_0)$ with continuous paths, mean zero and variance $\Gamma(\beta_0, t)$ given by

$$\Gamma(\beta_0, t) = -\frac{1}{2}\ln\left(1 - t\Big(\frac{\beta_0}{1-\beta_0}\Big)^2\right) + 2\lambda t\Big(\frac{\beta_0}{1-\beta_0}\Big)^2 \tag{16}$$

with $\lambda = \frac{1}{8}[\mathbb{E}(\xi_i^k)^4 - 3]$.

Remarks
1) The variance of the limiting process Z_∞^β is defined only when $\beta(1 + \sqrt{t}) < 1$.
2) If $\beta > \boldsymbol{\beta}_c$ and $\alpha > 0$ the annealed free energy is not even defined: $N^{-1}\log \mathbb{E}\mathcal{Z}_{N,M,\beta}$ is divergent.
3) Theorem 1 in [8] is a particular case of Theorem 1 here: The domain of validity of the result we give is optimal for the particular choice of the Hopfield model.

Actually, Theorem 1 is not proved for $\mathbf{Z}_N^\beta$ directly but for an approximating process Z_N^β. Beyond the fact that we deal with a family of models, the main difficulty we have to face in the present paper is due to the distribution of the disorder variables. Indeed, as we shall use the logarithm martingale of Z_N^β, we need to have it controlled from below. This is achieved in [8] for the Hopfield model through an appeal to a concentration result established by Talagrand in [25]. This is no longer the case here, as it seems impossible to adapt the proof of Talagrand's result to heavily tailed disorder variables like those considered in the present paper.

Let us denote by S_N the sum of N independent and identically distributed random variables ξ_i^k. The following asymptotic expansion of the Laplace's integral is a key result in our paper.

Theorem 2 *1. For every $\beta < 1$ we have*

$$\log \mathbb{E} e^{\beta N f(\frac{S_N}{N})} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} = \varphi(\beta) + \frac{\lambda}{N}\left(\frac{\beta}{1-\beta}\right)^2 + \frac{\mu}{N}\frac{\beta}{(1-\beta)^2} + o(\frac{1}{N}) \quad (17)$$

with

$$\varphi(\beta) = -\frac{1}{2}\log(1-\beta) \qquad \text{and} \quad \mu = \frac{f^{(4)}(0)}{8}. \quad (18)$$

2. For every $\beta < 1$ there exists a $d_0 > 0$ such that

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \log \mathbb{E} e^{\beta N f(\frac{S_N}{N})} \mathbf{1}_{\frac{1}{N^{1/4}} \leq \left|\frac{S_N}{N}\right| < d_0} < 0. \quad (19)$$

3. For every $\beta < \boldsymbol{\beta}_c$ and every $d > 0$ we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} e^{\beta N f(\frac{S_N}{N})} \mathbf{1}_{d \leq |\frac{S_N}{N}|} < 0. \quad (20)$$

4. As a consequence of (17), (19) and (20), for every $\beta < \boldsymbol{\beta}_c$ we have

$$\varphi_N(\beta) = \varphi(\beta) + \frac{\lambda}{N}\left(\frac{\beta}{1-\beta}\right)^2 + \frac{\mu}{N}\frac{\beta}{(1-\beta)^2} + o(\frac{1}{N}). \quad (21)$$

Remark
Combining Theorem 1 and Theorem 2 leads to

$$f_{N,M,\beta} \xrightarrow{\mathbb{P}} -\frac{\alpha}{2}\log(1-\beta)$$

and the corresponding Central Limit Theorem for every $(\beta, \alpha) \in \mathbf{D}$.

Theorem 2 is related to Theorem 3 in [18]. It is, in some sense, an improvement on the latter reference. Indeed, we perform the asymptotic expansion up to the $1/N$-term. To our knowledge such statement has not been published before. Some computations in this direction are already present in [30], but under the restrictive assumption that $f(x) \leq c_0 x^2/2$ and $\gamma(x) \leq c_1 x^2/2$, for some positive constants c_0 and c_1. While these conditions include the classical Hopfield model as a particular case, they are not satisfactory for applications in statistical mechanics, since they do not give insights on the competition between the energy H_N (defined on the ground of f) and the entropy I.

Finally, Theorem 2 is also a key ingredient in the study of the fluctuations of the joint distribution of any number of replicas:

Theorem 3 *Let $\beta_0 \in [0, \boldsymbol{\beta}_c)$ and $\alpha > 0$ be such that (12) holds. For any fixed $n \geq 2$ and any real bounded continuous functions $F_{i,j}(x)$ with $1 \leq i < j \leq n$ the following convergence holds in probability:*

$$\sum_{\sigma^1,\dots,\sigma^n} \prod_{1\leq i<j\leq n} F_{i,j}\Big(\frac{\sigma^i \cdot \sigma^j}{\sqrt{N}}\Big)\mathcal{G}_{N,M,\beta_0}(\sigma^1)\cdots\mathcal{G}_{N,M,\beta_0}(\sigma^n)$$

$$\longrightarrow \prod_{1\leq i<j\leq n} \sqrt{\frac{1-\frac{\alpha\beta_0^2}{(1-\beta_0)^2}}{2\pi}} \int_{\mathbb{R}} F_{i,j}(x) e^{-\frac{1}{2}\left(1-\frac{\alpha\beta_0^2}{(1-\beta_0)^2}\right)x^2} dx. \tag{22}$$

In order to keep the paper's size within reasonable limits, we will not give a proof of Theorem 3. It is already established for the Hopfield model in [17]. Actually, the proof of Theorem 3 is to the proof given in [17] what the proof of Theorem 1 here is to the proof of Theorem 1 in [8].

The paper is organized as follows: In section 2 we introduce the process Z_N^β and prove that it approximates $\mathbf{Z}_N^\beta$ provided Theorem 2 holds. We also detail our strategy in proving the convergence of Z_N^β. In Section 3 we prove Theorem 2. Section 4 is (thus) devoted to the proof of Theorem 1 (with Z_N^β instead of $\mathbf{Z}_N^\beta$).

2 An auxiliary process

Theorem 2 is assumed to hold all through this Section. Moreover $0 < \beta_0 < \boldsymbol{\beta}_c$ and $t_0 > 0$ will be fixed values satisfying (12) and we shall drop the corresponding superscripts when no confusion can occur. It is known from the theory of Large Deviations that for every fixed $M > 0$ the distribution of $(m^k(\sigma))_{1\leq k\leq M}$ is concentrated around $(0,\dots,0)$ up to exponentially in N unlikely events. This suggests to consider the process

$$Z_N(t) = \mathsf{E}_\sigma\, e_N(t,\sigma) \tag{23}$$

where

$$e_N(t,\sigma) = \prod_{k=1}^{[Nt]} \left\{ \left(e^{\beta_0 N f(\frac{\sigma\cdot\xi^k}{N})} \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right| < \frac{1}{N^{1/4}}} + \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right| \geq \frac{1}{N^{1/4}}} \right) e^{-\phi_N(\beta_0)} \right\} \tag{24}$$

and

$$\phi_N(\beta) = \log \mathbb{E}\left\{ e^{\beta N f(\frac{S_N}{N})} \mathbf{1}_{\left|\frac{S_N}{N}\right| < \frac{1}{N^{1/4}}} + \mathbf{1}_{\left|\frac{S_N}{N}\right| \geq \frac{1}{N^{1/4}}} \right\}. \tag{25}$$

It is a consequence of Theorem 2 that for every $\beta < \boldsymbol{\beta}_c$ we have $\phi_N(\beta) = \varphi_N(\beta) + o(1/N)$. For every fixed $N \in \mathbb{N}$, the process $Z_N(t)$ is a positive, mean one martingale w.r.t. the filtration $(\mathcal{F}_t^N)_{t\geq 0}$. Its relevance comes from the

Lemma 1 *Let $\beta_0 \in [0, \boldsymbol{\beta}_c)$ and $t_0 > 0$ be such that (12) holds. If $Z_N(t)$ converges in distribution as $N \to \infty$ on the Skorohod space $D([0,t_0), \mathbf{R}^+)$ then $\mathbf{Z}_N(t)$ converges as well and admits the same limit distribution.*

Proof The process $\widehat{Z_N(t)} := \mathsf{E}_\sigma\, e^{\beta_0 H_N(t,\sigma)-[Nt]\phi_N(\beta_0)} = \mathbf{Z}_N(t)e^{o(1)}$ with $o(1)$ uniform on $0 \le t \le t_0$ converges on the Skorohod space $D([0,t_0),\mathbf{R}^+)$ if and only if $\mathbf{Z}_N(t)$ converges too. It is therefore sufficient to prove that there exist $N_0, K, h > 0$ such that for all $N \ge N_0$ we have

$$\mathbb{P}\left(\sup_{0\le t\le t_0}\left|Z_N(t)-\widehat{Z_N(t)}\right| > e^{\frac{-h\sqrt{N}}{2}}\right) \le [Nt_0]^2 K e^{\frac{-h\sqrt{N}}{2}}.$$

To this end we first notice that for every $t \ge 0$

$$\begin{aligned}
\left|\widehat{Z_N(t)} - Z_N(t)\right| \;&\le\; \mathsf{E}_\sigma \exp\left\{\beta_0 H_N(t,\sigma) - [Nt]\phi_N(\beta_0)\right\}\mathbf{1}_{\cup_{l=1}^{[Nt]}\{\left|\frac{\sigma\cdot\xi^l}{N}\right|\ge\frac{1}{N^{1/4}}\}} + \\
&+ \mathsf{E}_\sigma \prod_{k=1}^{[Nt]}\left\{\left(e^{\beta_0 N f(\frac{\sigma\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}} + \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|\ge\frac{1}{N^{1/4}}}\right)e^{-\phi_N(\beta_0)}\right\}\mathbf{1}_{\cup_{l=1}^{[Nt]}\{\left|\frac{\sigma\cdot\xi^l}{N}\right|\ge\frac{1}{N^{1/4}}\}} \\
&\le \sum_{l=1}^{[Nt]} \mathsf{E}_\sigma \exp\left\{\beta_0 H_N(t,\sigma) - [Nt]\phi_N(\beta_0)\right\}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^l}{N}\right|\ge\frac{1}{N^{1/4}}} + \\
&+ \sum_{l=1}^{[Nt]} \mathsf{E}_\sigma \prod_{k=1}^{[Nt]}\left\{\left(e^{\beta_0 N f(\frac{\sigma\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}} + \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|\ge\frac{1}{N^{1/4}}}\right)e^{-\phi_N(\beta_0)}\right\}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^l}{N}\right|\ge\frac{1}{N^{1/4}}}.
\end{aligned}$$

Combining Chernoff's bound (see e.g. Section IV.5.2 in [24]) and Theorem 2, we obtain that for every $\beta_0 < \boldsymbol{\beta}_c$ there exist $N_0, K, h > 0$ such that for every $N \ge N_0$

$$\begin{aligned}
\mathbb{E}\left|Z_N(t) - \widehat{Z_N(t)}\right| \;&\le [Nt]\mathbb{E}\,\mathsf{E}_\sigma \exp\left\{\beta_0 H_N(([Nt]-1)/N,\sigma) - ([Nt]-1)\phi_N(\beta_0)\right\} \times \\
&\times \mathbb{E}\exp\left\{\beta_0 N f(\frac{S_N}{N}) - \phi_N(\beta_0)\right\}\mathbf{1}_{\frac{1}{N^{1/4}}\le\left|\frac{S_N}{N}\right|} + \\
&+[Nt]\mathbb{E}\,\mathsf{E}_\sigma \prod_{k=1}^{[Nt]-1}\left\{\left(e^{\beta_0 N f(\frac{\sigma\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}} + \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|\ge\frac{1}{N^{1/4}}}\right)e^{-\phi_N(\beta_0)}\right\} \times \\
&\times \mathbb{P}(\frac{1}{N^{1/4}} \le \left|\frac{S_N}{N}\right|) \\
&\le [Nt]K e^{-h\sqrt{N}}.
\end{aligned}$$

Hence for every $N \ge N_0$ and $t \le t_0$

$$\mathbb{P}\left(\left|Z_N(t) - \widehat{Z_N(t)}\right| > e^{-h\sqrt{N}/2}\right) \le [Nt_0]K e^{-h\sqrt{N}/2}$$

due to Chebichev's inequality. Now, let us recall that both $Z_N(t)$ and $\widehat{Z_N(t)}$ take at most $[Nt_0]$ different values on $[0,t_0]$, whence

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\le t\le t_0}\left|Z_N(t) - \widehat{Z_N(t)}\right| > e^{-h\sqrt{N}/2}\right) \;&\le \sum_{k=1}^{[Nt_0]}\mathbb{P}\left(\left|Z_N(\frac{k}{N}) - \widehat{Z_N(\frac{k}{N})}\right| > e^{-h\sqrt{N}/2}\right) \\
&\le [Nt_0]^2 K e^{-h\sqrt{N}/2}.
\end{aligned}$$

■

Our strategy in the proof of Theorem 1 is to prove the convergence in distribution of Z_N on the Skorohod space $D([0,t_0),\mathbf{R}^+)$ rather than working out a direct proof on $\mathbf{Z}_N$. In order to prove the convergence of Z_N we introduce its logarithm martingale defined by

$$M_N(t)=\sum_{k=0}^{[Nt]-1}\Delta M_N\Big(\frac{k}{N}\Big),\qquad \Delta M_N(t)=\frac{\Delta Z_N(t)}{Z_N(t)},\tag{26}$$

where we denote by Δ the difference operator

$$\Delta f(t)=f(t+1/N)-f(t).\tag{27}$$

Notice that for every $N,k\in\mathbb{N}$

$$\begin{aligned}&\Delta M_N(\frac{k}{N})\\ =\;&\mathsf{E}_\sigma\,\frac{e_N(\frac{k+1}{N},\sigma)-e_N(\frac{k}{N},\sigma)}{Z_N(\frac{k}{N})}\\ =\;&\mathsf{E}_\sigma\left(\left(e^{\beta_0 Nf(\frac{\sigma\cdot\xi^{k+1}}{N})}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^{k+1}}{N}\right|<\frac{1}{N^{1/4}}}+\mathbf{1}_{\left|\frac{\sigma\cdot\xi^{k+1}}{N}\right|\geq\frac{1}{N^{1/4}}}\right)e^{-\phi_N(\beta_0)}-1\right)\frac{e_N(\frac{k}{N},\sigma)}{Z_N(\frac{k}{N})}\end{aligned}$$

which is the reason why we introduce the notation

$$Re_N(\frac{k}{N},\sigma)=\left(e^{\beta_0 Nf(\frac{\sigma\cdot\xi^{k+1}}{N})}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^{k+1}}{N}\right|<\frac{1}{N^{1/4}}}+\mathbf{1}_{\left|\frac{\sigma\cdot\xi^{k+1}}{N}\right|\geq\frac{1}{N^{1/4}}}\right)e^{-\phi_N(\beta_0)}-1,$$

that allows us to write

$$\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})=\mathsf{E}_\sigma\,Re_N(\frac{k}{N},\sigma)e_N(\frac{k}{N},\sigma).$$

Due to the presence of truncations in the "modified" Boltzmann weight (24) the relative increment $Re_N(\frac{k}{N},\sigma)$ has the following property: There exists a constant $K>0$ such that for every k,N and every $\sigma\in\Sigma_N$ we have

$$\left|Re_N(\frac{k}{N},\sigma)\right|\leq Ke^{\sqrt{N}}.\tag{28}$$

Our proof of the convergence of Z_N is organized as follows. First we shall prove that

Lemma A For every $\alpha>0$

$$\mathrm{I}^N=\sum_{k=0}^{[Nt_0]-1}\mathbb{P}\Big(|\Delta M_N(\frac{k}{N})|Z_N(\frac{k}{N})>\alpha\;\Big|\;\mathcal{F}^N_{\frac{k}{N}}\Big)\stackrel{\mathbb{P}}{\longrightarrow}0.\tag{29}$$

Lemma B For every $0 \le t < t_0$ and every $\eta > 0$

$$\mathrm{I\!I}^N = \sum_{k=0}^{[Nt]-1} \Big| \mathbb{E}\Big(\Big[(\Delta M_N(\frac{k}{N}))^2 - \Delta\Gamma(\beta_0, \frac{k}{N}) \Big] \times \\ \times (Z_N(\frac{k}{N}))^2 \mathbf{1}_{\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N}) < \eta} \Bigm| \mathcal{F}^N_{\frac{k}{N}} \Big) \Big| \xrightarrow{\mathbb{P}} 0. \quad (30)$$

Lemma C For every $0 \le t < t_0$ and every $\eta > 0$

$$\mathrm{I\!I\!I}^N = \sum_{k=0}^{[Nt]-1} \Big| \mathbb{E}\Big(\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N}) \mathbf{1}_{Z_N(\frac{k}{N}) \Delta M_N(\frac{k}{N}) > \eta} \Bigm| \mathcal{F}^N_{\frac{k}{N}} \Big) \Big| \xrightarrow{\mathbb{P}} 0. \quad (31)$$

Lemma D For every $u > 0$ there exists a $\delta > 0$ such that for every $N \ge N(u)$ we have

$$\mathbb{P}(\min_{0 \le t \le t_0} Z_N(t) \ge \delta) \ge 1 - e^{-u}. \quad (32)$$

The proof of these four lemmas is carried out in Sections 4.2-4.5. From (29, 30, 31, 32) we deduce that

(i) For every $0 \le t < t_0$

$$\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Big[\Delta M_N(\frac{k}{N}) \Big]^2 \mathbf{1}_{|\Delta M_N(\frac{k}{N})| < 1} \Bigm| \mathcal{F}^N_{\frac{k}{N}} \Big) \xrightarrow{\mathbb{P}} \Gamma(\beta_0, t). \quad (33)$$

(ii) For every $\alpha > 0$

$$\sum_{k=0}^{[Nt_0]-1} \mathbb{P}\Big(|\Delta M_N(\frac{k}{N})| > \alpha \Bigm| \mathcal{F}^N_{\frac{k}{N}} \Big) \xrightarrow{\mathbb{P}} 0. \quad (34)$$

(iii) For every $0 \le t < t_0$

$$\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Delta M_N(\frac{k}{N}) \mathbf{1}_{|\Delta M_N(\frac{k}{N})| > 1} \Bigm| \mathcal{F}^N_{\frac{k}{N}} \Big) \xrightarrow{\mathbb{P}} 0. \quad (35)$$

This is achieved in Section 4.6. Altogether (i,ii,iii) are sufficient conditions for the convergence of $Z_N(t)$ to $Z_\infty(t)$, see [8]. Combined with Lemma 1 it leads to Theorem 1.

Concluding this section, we recall that we are left to prove Theorem 2 and formulas (29)-(35). This is worked out respectively in Section 3 and Section 4.

3 Proof of Theorem 2

We start this section with a brief account on the basic properties of γ and I that will be needed in the sequel. Then we prove that for every $\beta < \boldsymbol{\beta}_c$ the sequence $(e^{\beta N f(\frac{S_N}{N})})_{N\in\mathbb{N}}$ is uniformly bounded in $L^\kappa(\mathbb{P})$ for some $\kappa > 1$. This allows us to use Large Deviations results, in particular Varadhan's Lemma. Finally, we start the proper analysis of $\varphi_N(\beta)$ in Section 3.3.

3.1 Basic properties of γ and I

Since (9) holds $\gamma''(y) > 0$ for every $y \in \mathbb{R}$ hence γ is strictly convex. Thus γ' is strictly increasing and defines a 1-1 mapping from $\mathbb{R}$ onto an open interval D_I: Every $y \in \mathbb{R}$ is associated to $x \in D_I$ by $x = \gamma'(y)$. One can prove that I is infinitely differentiable on D_I and that for every $x \in D_I$ and $y \in \mathbb{R}$ such that $x = \gamma'(y)$ we have

$$I(x) = yx - \gamma(y), \quad I'(x) = y, \quad \text{and} \quad I''(x) = 1/\gamma''(y).$$

Thus I is strictly convex on D_I. Furthermore, I is even, non-negative, $I(0) = 0$ and $\lim_{|x|\to\infty} I(x)/|x| = \infty$ since (9) holds, see Lemma 2.2.20 in [11].

3.2 A uniform bound

Lemma 2 *For every $\beta < \boldsymbol{\beta}_c$ there exists a $\kappa > 1$ such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E} e^{\kappa\beta N f(\frac{S_N}{N})} < \infty. \tag{36}$$

Proof We consider two different cases

<u>*First case:*</u> There exists a $d \in \mathbb{R}$ such that $I(d) = \infty$.

Then, due to the symmetry of I, we have $I(x) = \infty$ for every $|x| \geq d$. Thus, according to Cramér's Theorem (see both Theorem 2.2.3 and Remark c page 27 in [11]), we get $\mathbb{P}\left(\left|\frac{S_N}{N}\right| \geq d\right) \leq 2e^{-N \inf_{|x|\geq d} I(x)} = 0$ for every $N \in \mathbb{N}$. Hence it suffices to prove that for every $\beta < \boldsymbol{\beta}_c$ there exists a $\kappa > 1$ such that $\sup_{N\in\mathbb{N}} \mathbb{E} e^{\kappa\beta N f(\frac{S_N}{N})} \mathbf{1}_{|\frac{S_N}{N}|<d} < \infty$, but this is a consequence of Theorem 3 in [18].

<u>*Second case:*</u> For every $x \in \mathbb{R}$ we have $I(x) < \infty$.

Then I is strictly increasing on $\mathbb{R}_+$, with $I(0) = 0$ and $\lim_{|x|\to\infty} I(x) = \infty$. Hence we can define a sequence $(a_l)_{l\in\mathbb{N}}$ such that $a_0 = 0$ and $I(a_{l+1}) = I(a_l)+1$ for every $l \in \mathbb{N}$. This sequence is strictly increasing and unbounded, thus divergent. Moreover, since $\beta < \boldsymbol{\beta}_c$, there exists a $\rho < 1$ such that $\beta f(x) \leq \rho I(x)$ for every $x \in \mathbb{R}$. Let κ be such that $\kappa > 1$ but $\rho\kappa < 1$ and $\kappa\beta < \boldsymbol{\beta}_c$, and let $\alpha < 1$ be such that $\rho\kappa < \alpha$. There exists an $l_0 \in \mathbb{N}$ such that for every $l \geq l_0$

$$I(a_l) > \frac{\rho\kappa}{\alpha - \rho\kappa}.$$

As a consequence, we have $\rho\kappa I(a_{l+1}) < \alpha I(a_l)$ whenever $l \geq l_0$. Now we write

$$\mathbb{E}e^{\kappa\beta N f(\frac{S_N}{N})} = \mathbb{E}e^{\kappa\beta N f(\frac{S_N}{N})}\mathbf{1}_{\left|\frac{S_N}{N}\right| \leq a_{l_0+1}} + \mathbb{E}e^{\kappa\beta N f(\frac{S_N}{N})}\mathbf{1}_{\left|\frac{S_N}{N}\right| > a_{l_0+1}}.$$

Once again, the first term in the latter sum is uniformly bounded in $N \in \mathbb{N}$ as a consequence of Theorem 3 in [18]. For the second term, let us proceed as follows: First notice that

$$\begin{aligned}
e^{\beta\kappa N f(\frac{S_N}{N})}\mathbf{1}_{|\frac{S_N}{N}|>a_{l_0+1}} &= \lim_{M\to\infty}\sum_{l=l_0+1}^{M} e^{\beta\kappa N f(\frac{S_N}{N})}\mathbf{1}_{\frac{S_N}{N}\in(a_l,a_{l+1}]} \text{ a.s.} \\
&\leq \lim_{M\to\infty}\sum_{l=l_0+1}^{M} e^{\rho\kappa N I(\frac{S_N}{N})}\mathbf{1}_{\frac{S_N}{N}\in(a_l,a_{l+1}]} \text{ a.s.}
\end{aligned}$$

Thus, combining Beppo-Lévi's Theorem (see Propriété 3.2.15 in [10]) and Cramér's Theorem we obtain

$$\begin{aligned}
\mathbb{E}e^{\beta\kappa N f(\frac{S_N}{N})}\mathbf{1}_{|\frac{S_N}{N}|>a_{l_0+1}} &\leq \mathbb{E}\lim_{M\to\infty}\sum_{l=l_0+1}^{M} e^{\rho\kappa N\sup_{x\in(a_l,a_{l+1}]}I(x)}\mathbf{1}_{\frac{S_N}{N}\in[a_l,a_{l+1}]} \\
&\leq \lim_{M\to\infty}\sum_{l=l_0+1}^{M} e^{\rho\kappa N\sup_{x\in(a_l,a_{l+1}]}I(x)}e^{-N\inf_{x\in[a_l,a_{l+1}]}I(x)} \\
&\leq \lim_{M\to\infty}\sum_{l=l_0+1}^{M} e^{-N(1-\alpha)I(a_l)} < \infty.
\end{aligned}$$

The announced result follows. ∎

3.3 Asymptotic expansion of ϕ_N

First of all let us notice that since $f(x) = \frac{x^2}{2} + o(x^2)$ as $x \to 0$, for any $\delta > 0$ there exists a $d_0 > 0$ small enough such that for every $x \in [-d_0, d_0]$ we have

$$(1-\delta)\frac{x^2}{2} \leq f(x) \leq (1+\delta)\frac{x^2}{2}. \tag{37}$$

Let $\beta < 1$ and $\delta > 0$ be such that

$$\beta(1+\delta) < 1 \tag{38}$$

and denote by d_0 the quantity associated to δ by (37).

We are interested in $A^N = \mathbb{E}\exp\left\{\beta N f(\frac{S_N}{N})\right\}$ that we split in the sum of three terms

$$
\begin{aligned}
A^N = \quad & \mathbb{E}e^{\beta N f(\frac{S_N}{N})}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} && (:= A_1^N)\\
& + \mathbb{E}e^{\beta N f(\frac{S_N}{N})}\mathbf{1}_{\frac{1}{N^{1/4}}\le|\frac{S_N}{N}|<d_0} && (:= A_2^N)\\
& + \mathbb{E}e^{\beta N f(\frac{S_N}{N})}\mathbf{1}_{d_0\le|\frac{S_N}{N}|} && (:= A_3^N).
\end{aligned}
$$

We will obtain estimates on each of these quantities separately. As we already noticed in the introduction, $\beta < \boldsymbol{\beta}_c$ implies $\beta < 1$. Hence it is sufficient, in order to prove both (17), (19) and (21), to find a control of A_1^N and A_2^N under this latter condition. We will prove that A_3^N decays exponentially fast to 0 whatever the value of $d_0 > 0$ is. This proves (20) and is obtained under the stronger condition that $\beta < \boldsymbol{\beta}_c$.

3.3.1 Asymptotic expansion of A_1^N

We have

$$
\begin{aligned}
A_1^N \quad &= \quad \mathbb{E}e^{\frac{\beta N}{2}(\frac{S_N}{N})^2}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}\\
&\quad + \mathbb{E}e^{\frac{\beta N}{2}(\frac{S_N}{N})^2}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \frac{\mathbb{E}e^{\frac{\beta N}{2}(\frac{S_N}{N})^2}\left(e^{\beta N\tilde{f}(\frac{S_N}{N})}-1\right)\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}}{\mathbb{E}e^{\frac{\beta N}{2}(\frac{S_N}{N})^2}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}}\\
&= \quad A_{1,1}^N + A_{1,2}^N
\end{aligned}
$$

where $\tilde{f}(x) = f(x) - \frac{x^2}{2}$.

Asymptotic expansion of $A_{1,1}^N$

To deal with $A_{1,1}^N$ we use the well known Gaussian trick

$$
e^{x^2/2} = \int_{\mathbb{R}} e^{ux} g_1(du) \tag{39}
$$

where g_{σ^2} stands for the density w.r.t. the Lebesgue measure of the centered Gaussian probability measure on $\mathbb{R}$ with variance σ^2, and write

$$
A_{1,1}^N = \mathbb{E}\int_{\mathbb{R}} e^{\sqrt{\frac{\beta}{N}}(S_N)u} g_1(du)\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}.
$$

Next we follow Martin-Löf [18], and associate to the interval $[-\frac{1}{N^{1/4}}, \frac{1}{N^{1/4}}]$ an interval $\mathcal{J} = [-J, J]$ with accurate properties. Indeed, γ' maps every $y \in \mathbb{R}$ into x such that $I'(x) = y$. Thus, let us fix J such that $\gamma'(\pm\sqrt{\frac{\beta}{N}}J) = \pm\frac{1}{N^{1/4}}$. Note that J is well-defined provided N is large enough. Then for every $u \in \mathcal{J}$ there exists an $x \in [-\frac{1}{N^{1/4}}, \frac{1}{N^{1/4}}]$ such that

$$
\sqrt{\frac{\beta}{N}}u - I'(x) = 0. \tag{40}
$$

We remark that since $\gamma'(x) = x + o(x)$ when $x \to 0$ we get $J = \frac{N^{1/4}}{\sqrt{\beta}} + o(N^{1/4})$. Now we write

$$\begin{aligned} A^N_{1,1} = \; & \mathbb{E}\int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u} g_1(du) && (:= A^N_{1,1,1}) \\ & - \mathbb{E}\int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u} g_1(du) \mathbf{1}_{|\frac{S_N}{N}| \geq \frac{1}{N^{1/4}}} && (:= A^N_{1,1,2}) \\ & + \mathbb{E}\int_{\overline{\mathcal{J}}} e^{\sqrt{\frac{\beta}{N}}(S_N)u} g_1(du) \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}} && (:= A^N_{1,1,3}). \end{aligned}$$

It is a well known result that

$$\mathbb{E} e^{\sqrt{\frac{\beta}{N}}(S_N)u} \mathbf{1}_{|\frac{S_N}{N}| \geq \frac{1}{N^{1/4}}} \leq e^{N \sup_{x \geq \frac{1}{N^{1/4}}} (\sqrt{\frac{\beta}{N}} ux - I(x))} + e^{N \sup_{x \leq -\frac{1}{N^{1/4}}} (\sqrt{\frac{\beta}{N}} ux - I(x))}$$

see e.g., Lemma 1 in [18]. Because of (40) and the convexity of I the sup's in the last display are attained for $x = \pm\frac{1}{N^{1/4}}$. Since we have $I(x) = x^2/2 + o(x^2)$ when $x \to 0$ we obtain

$$\begin{aligned} A^N_{1,1,2} & \leq \int_{\mathcal{J}} e^{(\sqrt{\beta} u N^{1/4} - \frac{\sqrt{N}}{2}) + O(1)} g_1(du) du \\ & \quad + \int_{\mathcal{J}} e^{(-\sqrt{\beta} u N^{1/4} - \frac{\sqrt{N}}{2}) + O(1)} g_1(du) du \\ & \leq 2 e^{-\sqrt{N}\frac{(1-\beta)}{2} + O(1)}. \end{aligned}$$

By the same kind of arguments we can prove that $A^N_{1,1,3}$ decays exponentially fast to 0 as well. Next we consider $A^N_{1,1,1}$. We have

$$\begin{aligned} A^N_{1,1,1} & = \mathbb{E}\int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u} g_1(u) du \\ & = \int_{\mathcal{J}} e^{N\gamma(\sqrt{\frac{\beta}{N}}u)} g_1(u) du \\ & = \frac{1}{\sqrt{1-\beta}} \int_{\mathcal{J}} \frac{\sqrt{1-\beta}}{\sqrt{2\pi}} e^{-\frac{u^2(1-\beta)}{2}} e^{\frac{u^2(1-\beta)}{2}} e^{N\gamma(\sqrt{\frac{\beta}{N}}u)} e^{-\frac{u^2}{2}} du \\ & = e^{\phi(\beta)} \left[1 + \int_{\mathcal{J}} \left(e^{-\frac{\beta u^2}{2} + N\gamma(\sqrt{\frac{\beta}{N}}u)} - 1 \right) g_{\frac{1}{1-\beta}}(u) du \right]. \end{aligned}$$

Due to the definition of $\mathcal{J}$ we get

$$\int_{\mathcal{J}} \left(e^{-\frac{\beta u^2}{2} + N\gamma(\sqrt{\frac{\beta}{N}}u)} - 1 \right) g_{\frac{1}{1-\beta}}(u) du = \frac{1}{N} \int_{\mathbb{R}} \left(\frac{\gamma^{(4)}(0)\beta^2 u^4}{4!} \right) g_{\frac{1}{1-\beta}}(u) du + o(\frac{1}{N}),$$

whence

$$A^N_{1,1,1} = e^{\phi(\beta)} \left[1 + \frac{\lambda}{N} \left(\frac{\beta}{1-\beta} \right)^2 + o(\frac{1}{N}) \right].$$

Since both $A_{1,1,2}^N$ and $A_{1,1,3}^N$ decay exponentially fast to 0 we obtain

$$A_{1,1}^N = e^{\phi(\beta)} \left[1 + \frac{\lambda}{N} \left(\frac{\beta}{1-\beta} \right)^2 + o(\frac{1}{N}) \right]. \tag{41}$$

Asymptotic control of $A_{1,2}^N$

To deal with $A_{1,2}^N$ we need to introduce the sequence $(h_N)_{N\in\mathbb{N}}$ of real valued functions defined on $\mathbb{R}$ by

$$h_N(u) = N \left(e^{\beta N \tilde{f}(\frac{u}{\sqrt{N}})} - 1 \right)$$

and the sequence $(\nu_N)_{N\in\mathbb{N}}$ of probability measures on $\mathbb{R}$ which are the distribution of $\frac{S_N}{\sqrt{N}}$ under the probability $\mathbb{Q}_N$ defined by

$$d\mathbb{Q}_N = \frac{e^{\frac{\beta N}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}}{\mathbb{E} e^{\frac{\beta N}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}} . d\mathbb{P}.$$

Notice that for every $N \in \mathbb{N}$ we have $NA_{1,2}^N = A_{1,1}^N \cdot \int_{\mathbb{R}} h_N d\nu_N$. We prove that $(\int_{\mathbb{R}} h_N d\nu_N)_{N\in\mathbb{N}}$ is convergent:

Lemma 3 *1. The sequence $(h_N)_{N\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$ to*

$$h(u) = \beta \frac{f^{(4)}(0)}{4!} u^4.$$

2. *The sequence $(\nu_N)_{N\in\mathbb{N}}$ converges weakly to the Gaussian probability measure on $\mathbb{R}$ with mean 0 and variance $\frac{1}{1-\beta}$.*

3. *There exists an $\alpha > 1$ such that*

$$\sup_{N\in\mathbb{N}} \int_{\mathbb{R}} |h_N|^\alpha d\nu_N < \infty. \tag{42}$$

4. *As a consequence of 1), 2) and 3) we have $\int_{\mathbb{R}} h_N d\nu_N \to \int_{\mathbb{R}} h(u) g_{\frac{1}{1-\beta}}(u) du$.*

Proof

1. There is no loss of generality in considering compact sets of the form $[-a, a]$ for some $a > 0$. Then for N large enough

$$N\beta \widetilde{f}(\frac{u}{\sqrt{N}}) = \frac{\beta}{N} \frac{f^{(4)}(0)}{4!} u^4 + o(\frac{1}{N}).$$

where $o(\frac{1}{N})$ is uniform in $u \in [-a, a]$. Hence we get

$$|h_N(u) - h(u)| = \left| N\left[e^{N\beta\tilde{f}(\frac{u}{\sqrt{N}})} - 1\right] - \beta u^4 \frac{f^{(4)}(0)}{4!}\right| = o(1),$$

where $o(1)$ is uniform in $u \in [-a, a]$. The announced result follows.

2. We compute the limit as $N \to \infty$ of the characteristic function of ν_N given by

$$B^N(v) = \frac{\mathbb{E} e^{iv\frac{S_N}{\sqrt{N}}} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}}}{\mathbb{E} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}}}.$$

We already have an estimate of the denominator, so we can concentrate on the numerator. Using the Gaussian trick (39) once again we obtain

$$\begin{aligned}
\mathbb{E} e^{iv\frac{S_N}{\sqrt{N}}} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}} &= \mathbb{E} \int_{\mathbb{R}} e^{\sqrt{\frac{\beta}{N}}(S_N)u + iv\frac{S_N}{\sqrt{N}}} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}} g_1(u) du \\
&= \mathbb{E} \int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u + iv\frac{S_N}{\sqrt{N}}} g_1(u) du \\
&\quad - \mathbb{E} \int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u + iv\frac{S_N}{\sqrt{N}}} \mathbf{1}_{|\frac{S_N}{N}| \geq \frac{1}{N^{1/4}}} g_1(u) du \\
&\quad + \mathbb{E} \int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u + iv\frac{S_N}{\sqrt{N}}} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}} g_1(u) du \\
&= B_1^N(v) + B_2^N(v) + B_3^N(v).
\end{aligned}$$

For every $v \in \mathbb{R}$ we have $|B_2^N(v)| \leq A_{1,1,2}^N$ and $|B_3^N(v)| \leq A_{1,1,3}^N$. We can thus concentrate on $B_1^N(v)$. We have

$$\begin{aligned}
B_1^N(v) &= \mathbb{E} \int_{\mathcal{J}} e^{\sqrt{\frac{\beta}{N}}(S_N)u + iv\frac{S_N}{\sqrt{N}}} g_1(u) du \\
&= \int_{\mathcal{J}} e^{N\gamma(\sqrt{\frac{\beta}{N}}u + \frac{iv}{\sqrt{N}})} g_1(u) du \\
&= \int_{\mathcal{J}} e^{\frac{1}{2}(\sqrt{\beta}u + iv)^2} g_1(u) du + o(1) \\
&= \frac{1}{\sqrt{1-\beta}} e^{-\frac{v^2}{2(1-\beta)}} + o(1).
\end{aligned}$$

Thus $B^N(v) \to e^{-\frac{v^2}{2(1-\beta)}}$ which is the characteristic function of the centered Gaussian probability measure on $\mathbb{R}$ with variance $\frac{1}{1-\beta}$.

3. We will see that any $\alpha > 1$ such that actually $\alpha\beta < 1$ works. Let us recall that

$$\int_{\mathbb{R}} |h_N|^\alpha d\nu_N = \frac{\mathbb{E} N^\alpha \left| e^{N\beta \widetilde{f}(\frac{S_N}{N})} - 1 \right|^\alpha e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}}{\mathbb{E} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}}.$$

Since the denominator is nothing but $A_{1,1}^N$, we are only concerned with the numerator. In order to keep notations as light as possible, we will denote by K a finite quantity that only depends on α and β and that may vary from line to line. We start with

$$\begin{aligned} C^N &= \mathbb{E} N^\alpha \left| e^{N\beta \widetilde{f}(\frac{S_N}{N})} - 1 \right|^\alpha e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \\ &\leq \mathbb{E} N^\alpha \left| e^{N\beta f(\frac{S_N}{N})} - e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \right|^\alpha \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \\ &\leq K\mathbb{E}\left(N^\alpha \left| e^{N\beta f(\frac{S_N}{N})} - e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} - \beta N \widetilde{f}(\frac{S_N}{N}) e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \right|^\alpha \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \right) \\ &\quad + K\mathbb{E}\left(N^\alpha \left| \beta N \widetilde{f}(\frac{S_N}{N}) e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \right|^\alpha \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \right) \\ &= C_1^N + C_2^N. \end{aligned}$$

It is a consequence of the Taylor-Lagrange formula that for every $x, y \in \mathbb{R}$ we have

$$|e^x - e^y - (x-y)e^y| \leq 2|x-y|(e^y + e^x). \tag{43}$$

Thus

$$\begin{aligned} &\left| e^{N\beta f(\frac{S_N}{N})} - e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} - \beta N \widetilde{f}(\frac{S_N}{N}) e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \right| \leq \\ &\qquad \leq 2 \left| \beta N \widetilde{f}(\frac{S_N}{N}) \right| \left(e^{N\beta f(\frac{S_N}{N})} + e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \right) \end{aligned}$$

whence

$$\begin{aligned} C_1^N \leq \quad & K\mathbb{E} \left| N^2 \widetilde{f}(\tfrac{S_N}{N}) \right|^\alpha e^{\frac{N\beta\alpha}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \quad (:= C_{1,1}^N) \\ & + K\mathbb{E} \left| N^2 \widetilde{f}(\tfrac{S_N}{N}) \right|^\alpha e^{N\beta\alpha f(\frac{S_N}{N})} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \quad (:= C_{1,2}^N). \end{aligned}$$

According to Hölder's inequality, for conjugates p and q with p small enough to ensure that $\alpha\beta p < 1$ we have

$$C_{1,1}^N \leq \left(\mathbb{E} \left| N^2 \widetilde{f}(\frac{S_N}{N}) \right|^{q\alpha} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \right)^{1/q} \left(\mathbb{E} e^{\frac{N\beta\alpha p}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \right)^{1/p}$$

and

$$C_{1,2}^N \;\leq\; \left(\mathbb{E}\left|N^2\widetilde{f}(\frac{S_N}{N})\right|^{q\alpha}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}\right)^{1/q}\left(\mathbb{E}e^{N\beta\alpha p f(\frac{S_N}{N})}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}}\right)^{1/p}.$$

First notice that it results from the asymptotic expansion of $A_{1,1}^N$ that for p and α satisfying $\alpha\beta p < 1$ we have

$$\sup_{N\in\mathbb{N}}\mathbb{E}e^{\frac{N\beta\alpha p}{2}(\frac{S_N}{N})^2}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} < \infty. \tag{44}$$

On the other hand we know that if β, α and p are such that $\alpha\beta p < 1$ then

$$\beta\alpha p f(x) - I(x) = (\beta\alpha p - 1)\frac{x^2}{2} + o(x^2)$$

as $x \to 0$. Hence, for N large enough, $\beta\alpha p f(x) - I(x)$ achieves its unique global maximum on $[-\frac{1}{N^{1/4}}, \frac{1}{N^{1/4}}]$ at 0. Thus, according to Theorem 3 in [18] we obtain

$$\sup_{N\in\mathbb{N}}\mathbb{E}e^{N\beta\alpha p f(\frac{S_N}{N})}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} < \infty.$$

Next we need to show that

$$\mathbb{E}\left|N^2\widetilde{f}(\frac{S_N}{N})\right|^{q\alpha}\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} < \infty. \tag{45}$$

Due to the presence of the truncation we have

$$\begin{aligned}\left|\tilde{f}(\frac{S_N}{N})\right|\mathbf{1}_{|\frac{S_N}{N}|<\frac{1}{N^{1/4}}} \;&\leq\; \frac{f^{(4)}(0)}{4!}(\frac{S_N}{N})^4 + \frac{f^{(6)}(0)}{6!}(\frac{S_N}{N})^6 + O(\frac{1}{N^2})\\ &=\; \frac{1}{N^2}\left(\frac{f^{(4)}(0)}{4!}(\frac{S_N}{\sqrt{N}})^4 + \frac{f^{(6)}(0)}{N6!}(\frac{S_N}{\sqrt{N}})^6 + O(1)\right)\end{aligned}$$

Since the distribution of the ξ_i^k's satisfies (9) for every fixed $m \in \mathbb{N}$ we have

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left|\frac{S_N}{\sqrt{N}}\right|^m < \infty$$

(see for instance Theorem 2.9 in [22]) and (45) follows. Now we are left to deal with C_2^N which is nothing but $C_{1,1}^N$ up to a multiplicative constant. This ends the proof of 3).

4. Now we have the tools needed to prove that

$$\int_{\mathbb{R}} h_N(u) d\nu_N(u) \to \int_{\mathbb{R}} h(u) g_{\frac{1}{1-\beta}}(u) du.$$

Indeed, for any $a > 0$ we have

$$\left| \int_{\mathbb{R}} h_N(u) d\nu_N(u) - \int_{\mathbb{R}} h(u) g_{\frac{1}{1-\beta}}(u) du \right| \le D_1^N + D_2^N + D_3^N + D_4 \tag{46}$$

with

$$D_1^N = \left| \int_{\mathbb{R}} h_N(u) \mathbf{1}_{|u| \le a} d\nu_N(u) - \int_{\mathbb{R}} h(u) \mathbf{1}_{|u| \le a} d\nu_N(u) \right|,$$

$$D_2^N = \left| \int_{\mathbb{R}} h(u) \mathbf{1}_{|u| \le a} d\nu_N(u) - \int_{\mathbb{R}} h(u) \mathbf{1}_{|u| \le a} g_{\frac{1}{1-\beta}}(u) du \right|$$

$$D_3^N = \left| \int_{\mathbb{R}} h_N(u) \mathbf{1}_{|u| > a} d\nu_N(u) \right|, \quad \text{and} \quad D_4 = \left| \int_{\mathbb{R}} h(u) \mathbf{1}_{|u| > a} g_{\frac{1}{1-\beta}}(u) du \right|.$$

First of all let us notice that since $h \in L^1(g_{\frac{1}{1-\beta}}(u)du)$ we can make D_4 as small as we want by an appropriate choice of a. Next, according to Hölder's inequality, we have for $\alpha > 1$ given by Lemma 3 that

$$D_3^N \le \left(\int_{\mathbb{R}} |h_N(u)|^\alpha d\nu_N(u) \right)^{1/\alpha} \nu_N((-\infty, a] \cup [a, \infty)).$$

We already know that $\sup_{N \in \mathbb{N}} \int_{\mathbb{R}} |h_N|^\alpha d\nu_N < \infty$ while

$$\begin{aligned} \nu_N((-\infty, -a] \cup [a, \infty)) &= \frac{\mathbb{E} \mathbf{1}_{\left|\frac{S_N}{\sqrt{N}}\right| \ge a} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}}}{\mathbb{E} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}}} \\ &\le \frac{(\mathbb{E} e^{\frac{N\beta\kappa}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}})^{1/\kappa}}{\mathbb{E} e^{\frac{N\beta}{2}(\frac{S_N}{N})^2} \mathbf{1}_{|\frac{S_N}{N}| < \frac{1}{N^{1/4}}}} \mathbb{P}(\left| \frac{S_N}{\sqrt{N}} \right| \ge a) \end{aligned}$$

due to Hölder's inequality with $\kappa > 1$ such that $\beta\kappa < 1$. Since the ξ_i^k's satisfy (9) the distribution of $\frac{S_N}{\sqrt{N}}$ converges weakly to that of a Gaussian random variable. We can thus make $\mathbb{P}(|\frac{S_N}{\sqrt{N}}| \ge a)$ as small as we want for N large enough by an appropriate choice of a while the numerator and denominator of the fraction in the last display are already known to be uniformly bounded in N. Finally D_1^N and D_2^N can be made small for N large enough for any choice of a because of 1) and 2) in Lemma 3.

■

We conclude that

$$A_{1,2}^N = e^{\phi(\beta)} \left[1 + \frac{\lambda}{N} \left(\frac{\beta}{1-\beta} \right)^2 + o(\frac{1}{N}) \right] \left[\frac{\mu}{N} \frac{\beta}{(1-\beta)^2} + o(\frac{1}{N}) \right]$$

Collecting the latter display and (41) leads to

$$A_1^N = e^{\phi(\beta)} \left[1 + \frac{\lambda}{N} \left(\frac{\beta}{1-\beta} \right)^2 + \frac{\mu}{N} \frac{\beta}{(1-\beta)^2} + o(\frac{1}{N}) \right].$$

This ends the proof of 1) in Theorem 2.

3.3.2 Asymptotic control of A_2^N

First of all let us notice that due to the choice of d_0 we made we have

$$\mathbb{E} e^{\beta N f(\frac{S_N}{N})} \mathbf{1}_{\frac{1}{N^{1/4}} \leq |\frac{S_N}{N}| < d_0} \leq \mathbb{E} e^{\frac{\beta(1+\delta)}{2} N (\frac{S_N}{N})^2} \mathbf{1}_{\frac{1}{N^{1/4}} \leq |\frac{S_N}{N}| < d_0}. \tag{47}$$

We know from [1] that for every sequence $(b_N = N^\alpha)_{N\in\mathbb{N}}$ such that $\alpha \in (1/2, 1)$ the sequence $(S_N/b_N)_{N\in\mathbb{N}}$ satisfies a Large Deviations Principle in the scale b_N^2/N and with good rate function $J(x) = x^2/2$. Let us bound the right hand side of (47) by

$$\mathbb{E} e^{\frac{\beta(1+\delta)}{2} \frac{b_N^2}{N} (\frac{S_N}{b_N})^2} \mathbf{1}_{\frac{N^{3/4}}{b_N} \leq |\frac{S_N}{b_N}| < \frac{d_0 N}{b_N}} \leq \mathbb{E} e^{\frac{\beta(1+\delta)}{2} \frac{b_N^2}{N} (\frac{S_N}{b_N})^2}.$$

We know from (38) that $x \mapsto \frac{\beta(1+\delta)}{2} x^2 - \frac{1}{2} x^2$ attains its unique global maximum at 0. Thus, we can adapt the proof of Lemma 2 to show that there exists an $\eta > 1$ such that

$$\sup_{N\in\mathbb{N}} \mathbb{E} e^{\frac{\eta\beta(1+\delta)}{2} \frac{b_N^2}{N} (\frac{S_N}{b_N})^2} < \infty. \tag{48}$$

Hence, we can apply Varadhan's Lemma as stated in Theorem 4.3.1 in [11] for the particular choice $b_N = N^{3/4}$ and we obtain

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{\sqrt{N}} \log \mathbb{E} e^{\frac{\beta(1+\delta)}{2} \sqrt{N} (\frac{S_N}{N^{3/4}})^2} \mathbf{1}_{|\frac{S_N}{N^{3/4}}| \geq 1} &= \sup_{|x|\geq 1} \left\{ \frac{\beta(1+\delta)}{2} x^2 - \frac{1}{2} x^2 \right\} \\ &< 0. \end{aligned} \tag{49}$$

Combining (47) and (49) we get

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \log \mathbb{E} e^{\beta N f(\frac{S_N}{N})} 1_{\frac{1}{N^{1/4}} \leq \left|\frac{S_N}{N}\right| < d_0} < 0.$$

This proves 2) in Theorem 2.

3.3.3 Asymptotic control of A_3^N

According to Lemma 2 there exists a $\kappa > 1$ such that (36) holds. Thus, according to Varadhan's Lemma we have

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} e^{\beta N f(\frac{S_N}{N})} \mathbf{1}_{\left|\frac{S_N}{N}\right| \geq d} &= \sup_{|x|\geq d} \{\beta f(x) - I(x)\} \\ &< 0 \end{aligned}$$

whenever $d > 0$ and $\beta < \boldsymbol{\beta}_c$. This proves 3) in Theorem 2.

4 Proof of Theorem 1

We start this Section by giving the technical results needed to prove Lemmas A,B,C and D (Section 4.1). Next we proceed to the proper proof of these lemmas (Sections 4.2-4.5) which is related to the proof of Conditions (i-iii) in [8]. However what we want to prove is different from the results in [8]. As a consequence we need new arguments, which is the reason why (29)-(31) are proved in full details. Finally we prove Conditions (i,ii,iii) (Section 4.6). In order to keep the paper's size within reasonable limits we give the proof of only one of the technical results (Lemma 5) at the very end of the section.

4.1 Some technical results

For every $\sigma^i, \sigma^j \in \Sigma_N$ we denote by $\theta^{i,j} = N^{-1}(\sigma^i \cdot \sigma^j)$. The following proposition is a generalization of 1) in Theorem 2 that we will not prove.

Proposition 1 **a)** *For every $\beta_i \in (0,1)$ and $\sigma^1, \sigma^2, \sigma^3 \in \Sigma_N$ let us denote by*

$$\psi_N(\beta_1, \beta_2, \sigma^1, \sigma^2) = \log \mathbb{E} \left\{ \prod_{i=1}^{2} e^{\beta_i N f(\frac{\sigma^i \cdot \xi}{N})} \mathbf{1}_{\left|\frac{\sigma^i \cdot \xi}{N}\right| < \frac{1}{N^{1/4}}} \right\}, \tag{50}$$

and

$$\Lambda_N(\beta_1, \beta_2, \beta_3, \sigma^1, \sigma^2, \sigma^3) = \log \mathbb{E} \left\{ \prod_{i=1}^{3} e^{\beta_i N f(\frac{\sigma^i \cdot \xi}{N})} \mathbf{1}_{\left|\frac{\sigma^i \cdot \xi}{N}\right| < \frac{1}{N^{1/4}}} \right\}. \tag{51}$$

These functions depend on the σ^i's only through the $\theta^{i,j}$'s.
b) *Furthermore, if $\beta_1, \beta_2 < 1$ and β_1, β_2 and $\theta^{1,2}$ satisfy*

$$(1-\beta_1)(1-\beta_2) - (\theta^{1,2})^2 \beta_1 \beta_2 > 0 \tag{52}$$

then

$$\begin{aligned}\psi_N(\beta_1,\beta_2,\theta^{1,2}) &= \psi(\beta_1,\beta_2,\theta^{1,2}) + \frac{\lambda}{N}\left[\frac{\beta_1^2}{(1-\beta_1)^2}+\frac{\beta_2^2}{(1-\beta_2)^2}\right]+ \\ &+\frac{2\lambda}{N}\frac{\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)}+\frac{\mu}{N}\left[\frac{\beta_1}{(1-\beta_1)^2}+\frac{\beta_2}{(1-\beta_2)^2}\right]+ \\ &+O\Big(\frac{(\theta^{1,2})^2}{N}\Big)+o\Big(\frac{1}{N}\Big) \qquad (53)\end{aligned}$$

where

$$\psi(\beta_1,\beta_2,\theta^{1,2}) = -\frac{1}{2}\log\Big((1-\beta_1)(1-\beta_2)-(\theta^{1,2})^2\beta_1\beta_2\Big) \qquad (54)$$

and $o(1/N)$ is uniform in $\theta^{1,2}$ belonging to compact subsets of (52). If the β_i's and the $\theta^{i,j}$'s, $i,j=1,2,3$ satisfy $\beta_i<1$, (52) and

$$\begin{aligned}&(1-\beta_1)(1-\beta_2)(1-\beta_3)-(1-\beta_1)\beta_2\beta_3(\theta^{2,3})^2-(1-\beta_2)\beta_1\beta_3(\theta^{1,3})^2 \\ &\qquad -(1-\beta_3)\beta_1\beta_2(\theta^{1,2})^2-2\beta_1\beta_2\beta_3\theta^{1,2}\theta^{1,3}\theta^{2,3}>0\ , \qquad (55)\end{aligned}$$

then the expansion

$$\begin{aligned}&\Lambda_N(\beta_1,\beta_2,\beta_3,\theta^{1,2},\theta^{1,3},\theta^{2,3}) = \Lambda(\beta_1,\beta_2,\beta_3,\theta^{1,2},\theta^{1,3},\theta^{2,3}) \\ &\quad +\frac{\lambda}{N}\left[\frac{\beta_1^2}{(1-\beta_1)^2}+\frac{\beta_2^2}{(1-\beta_2)^2}+\frac{\beta_3^2}{(1-\beta_3)^2}\right] \\ &\quad +\frac{2\lambda}{N}\left[\frac{\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)}+\frac{\beta_1\beta_3}{(1-\beta_1)(1-\beta_3)}+\frac{\beta_2\beta_3}{(1-\beta_2)(1-\beta_3)}\right] \\ &\quad +\frac{\mu}{N}\left[\frac{\beta_1}{(1-\beta_1)^2}+\frac{\beta_2}{(1-\beta_2)^2}+\frac{\beta_3}{(1-\beta_3)^2}\right] \\ &\quad +\sum_{i,j=1,2,3} O\Big(\frac{(\theta^{i,j})^2}{N}\Big)+o\Big(\frac{1}{N}\Big) \qquad (56)\end{aligned}$$

holds uniformly on compacts, where

$$\begin{aligned}\Lambda(\beta_1,\beta_2,\beta_3,\theta^{1,2},\theta^{1,3},\theta^{2,3}) &= -\frac{1}{2}\log\Big((1-\beta_1)(1-\beta_2)(1-\beta_3)-(1-\beta_1)\beta_2\beta_3(\theta^{2,3})^2 \\ &-(1-\beta_2)\beta_1\beta_3(\theta^{1,3})^2-(1-\beta_3)\beta_1\beta_2(\theta^{1,2})^2-2\beta_1\beta_2\beta_3\theta^{1,2}\theta^{1,3}\theta^{2,3}\Big)\ . \qquad (57)\end{aligned}$$

c) *Finally, for every $\beta<1$ and every $\sigma^i\in\Sigma_N, i=1,\dots,4$, let*

$$\kappa_N(\beta,\sigma^1,\sigma^2,\sigma^3,\sigma^4) = \log\mathbb{E}\left\{\prod_{i=1}^4 e^{\beta N f(\frac{\sigma^i.\xi}{N})}\mathbf{1}_{\left|\frac{\sigma^i.\xi}{N}\right|<\frac{1}{N^{1/4}}}\right\}. \qquad (58)$$

This function depends on the σ^i's only through the $\theta^{i,j}$'s, $i,j=1,\dots,4$. Furthermore there exists a $\tau=\tau(\beta)>0$ small enough such that if $\max_{i,j=1,\dots,4}|\theta^{i,j}|<\tau$ then (52) and (55) with $\beta_i=\beta$ are satisfied and

$$\kappa_N(\beta,\sigma^1,\sigma^2,\sigma^3,\sigma^4) = \kappa(\beta,\sigma^1,\sigma^2,\sigma^3,\sigma^4) + \frac{16\lambda}{N}\frac{\beta^2}{(1-\beta)^2} + \frac{4\mu}{N}\frac{\beta}{(1-\beta)^2} + \sum_{i,j=1}^{4} O\Big(\frac{(\theta^{i,j})^2}{N}\Big) + o\Big(\frac{1}{N}\Big) \tag{59}$$

with $o(1/N)$ uniform in the $\theta^{i,j}$'s and

$$\begin{aligned}\kappa(\beta,\Theta) &= -\tfrac{1}{2}\log\ \Big[(1-\beta)^4 - \beta^2(1-\beta)^2||\Theta||_2^2 \\ &\quad -2\beta^3(1-\beta)\Big(\theta^{1,3}\theta^{1,2}\theta^{2,3} + \theta^{1,4}\theta^{1,2}\theta^{2,4} + \theta^{1,4}\theta^{1,3}\theta^{3,4} + \theta^{2,4}\theta^{2,3}\theta^{3,4}\Big) \\ &\quad +\beta^4\Big((\theta^{1,2})^2(\theta^{3,4})^2 + (\theta^{1,3})^2(\theta^{2,4})^2 + (\theta^{1,4})^2(\theta^{2,3})^2\Big) \\ &\quad -2\beta^4(\theta^{1,2}\theta^{1,4}\theta^{2,3}\theta^{3,4} + \theta^{1,2}\theta^{1,3}\theta^{2,4}\theta^{3,4} + \theta^{1,3}\theta^{2,4}\theta^{2,3}\theta^{1,4})\Big]\end{aligned} \tag{60}$$

where $\Theta = (\theta^{i,j})_{i,j=1,\ldots,4}$.

The point in Proposition 1 is that the asymptotic expansions hold true for values of the β_i's up to 1. This is required by the truncation method of Talagrand [25] that we will use in the sequel. It can only be obtained in presence of truncations on the value of the $|\sigma^i\cdot\xi|$'s. On the basis of Proposition 1, the proof of the following lemma is obtained following the proof of Lemma 1 in [8].

Lemma 4 *There exists an ε_0, with $0 < \epsilon_0 < \tau(\beta_0)$, such that for any three $\epsilon_{i,j}$ $i,j = 1,2,3$ satisfying $0 < \epsilon_{i,j} < \epsilon_0 < \tau(\beta_0)$ we have*

$$\lim_{N\to\infty}\sup_{0\le t\le t_0} N^2\,\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3}\Big[\big(\prod_{i,j=1,2,3}\mathbf{1}_{|\theta^{i,j}|<\varepsilon_{i,j}}\big)\mathbb{E}\Big(Re_N(t,\sigma^1)Re_N(t,\sigma^2)-\Delta\Gamma(t)\Big) \mathbb{E}\Big(Re_N(t,\sigma^1)Re_N(t,\sigma^3)-\Delta\Gamma(t)\Big)\mathbb{E}\Big(\textstyle\prod_{i=1}^3 e_N(t,\sigma^i)\Big)\Big] = 0, \tag{61}$$

and for every $0 < \varepsilon < \varepsilon_0$ there exists a $C < \infty$ such that

$$\lim_{N\to\infty}\sup_{0\le t\le t_0} N^2\,\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\,\mathbf{1}_{\max|\theta^{i,j}|<\varepsilon}\Big|\mathbb{E}\prod_{i=1}^{4}Re_N(t,\sigma^i)\Big|\mathbb{E}\Big[\prod_{i=1}^{4}e_N(t,\sigma^i)\Big] < C.$$

The main difficulty to overcome in the proof of Theorem 1, apart from the above mentioned control of the small values of Z_N, is the explosion of its second moment. This requires to truncate the Boltzmann weights. However, in contrast with [8], we need more sophisticated truncations to deal with the special form of the Boltzmann weights (24) considered here. For that purpose we fix E small enough to ensure that

$$\frac{\beta_0^2 t_0}{(1-\beta_0)(1-\beta_0-E)} < 1, \qquad 0 < \frac{E}{1-\beta_0-E} < 1. \tag{62}$$

Note that this choice of $E > 0$ is possible only under condition (12). Let us now introduce a first event truncating the Hamiltonian:

$$B_{N,t,\sigma,E} = \left\{ H_N(t,\sigma) < \frac{[t_0 N]}{2(1-\beta_0-E)} \right\}. \tag{63}$$

For every fixed integer N and $t \geq 0$ and every integer m such that $0 \leq m \leq [Nt]$ let us denote by $T_{N,t,m}$ the set of $\{-1,1\}-$valued $[Nt]-$tuples with exactly m 1's. To any $\vec{k} = \{k_1, \dots, k_{[Nt]}\} \in T_{N,t,m}$ we associate $\vec{k}_<$ (resp. $\vec{k}_>$) which is the set of indices i such that $k_i = 1$ (resp. $k_i = -1$). We now define new events truncating the "relevant" part of the Hamiltonian:

$$B_{N,t,m,\vec{k},\sigma,E} = \left\{ N \sum_{k \in \vec{k}_<} f\left(\frac{\sigma \cdot \xi^k}{N}\right) < \frac{[t_0 N]}{2(1-\beta_0-E)} \right\}, \tag{64}$$

and

$$\overline{B_{N,t,m,\vec{k},\sigma,E}} = \left\{ N \sum_{k \in \vec{k}_<} f\left(\frac{\sigma \cdot \xi^k}{N}\right) \geq \frac{[t_0 N]}{2(1-\beta_0-E)} \right\}. \tag{65}$$

Finally, we shall write

$$e_N(t,\sigma)\mathbb{T}_{B_{N,t,\sigma,E}} = \sum_{m=0}^{[Nt]} \sum_{\vec{k} \in T_{N,t,m}} \left(\prod_{k \in \vec{k}_<} e^{\beta_0 N f(\frac{\sigma \cdot \xi^k}{N})} \mathbf{1}_{\left|\frac{\sigma \cdot \xi^k}{N}\right| < \frac{1}{N^{1/4}}} \mathbf{1}_{B_{N,t,m,\vec{k},\sigma,E}} \right) e^{-m\phi_N(\beta_0)} \left(\prod_{k \in \vec{k}_>} \mathbf{1}_{\left|\frac{\sigma \cdot \xi^k}{N}\right| \geq \frac{1}{N^{1/4}}} \right) e^{-([Nt]-m)\phi_N(\beta_0)},$$

and

$$e_N(t,\sigma)\mathbb{T}_{\overline{B_{N,t,\sigma,E}}} = \sum_{m=0}^{[Nt]} \sum_{\vec{k} \in T_{N,t,m}} \left(\prod_{k \in \vec{k}_<} e^{\beta_0 N f(\frac{\sigma \cdot \xi^k}{N})} \mathbf{1}_{\left|\frac{\sigma \cdot \xi^k}{N}\right| < \frac{1}{N^{1/4}}} \mathbf{1}_{\overline{B_{N,t,m,\vec{k},\sigma,E}}} \right) e^{-m\phi_N(\beta_0)} \left(\prod_{k \in \vec{k}_>} \mathbf{1}_{\left|\frac{\sigma \cdot \xi^k}{N}\right| \geq \frac{1}{N^{1/4}}} \right) e^{-([Nt]-m)\phi_N(\beta_0)}.$$

By convention the products above are set to be 1 whenever they run over the empty set. One notice that for every $E, N, t > 0$ and $\sigma \in \Sigma_N$ we have

$$e_N(t,\sigma) = e_N(t,\sigma)\mathbb{T}_{\overline{B_{N,t,\sigma,E}}} + e_N(t,\sigma)\mathbb{T}_{B_{N,t,\sigma,E}}.$$

In words, "multiplicating" by $\mathbb{T}_{B_{N,t,\sigma,E}}$ amounts to perform a truncation only on the "relevant" factors of the Boltzmann weights. We obtain

Lemma 5 *Let E be fixed according to (62). For every $\varepsilon > 0$ there exist constants $K > 0$ and $h_1 = h_1(\varepsilon, E) > 0$ such that for all $N \geq N_1 = N_1(\epsilon, E)$*

$$\sup_{0 \leq t \leq t_0} \mathbb{E}\, \mathsf{E}_\sigma\, e_N(t,\sigma) \mathbb{T}_{\overline{B_{N,t,\sigma,E}}} \leq K e^{-h_1 N} \tag{66}$$

and

$$\sup_{0 \leq t \leq t_0} \mathbb{E}\, \mathsf{E}_{\sigma^1,\sigma^2}\, e_N(t,\sigma^1) \mathbb{T}_{B_{N,t,\sigma^1,E}} e_N(t,\sigma^2) \mathbb{T}_{B_{N,t,\sigma^2,E}} \mathbf{1}_{|\theta^{1,2}| > \varepsilon} \leq K e^{-h_1 N}. \tag{67}$$

Define now the event

$$\begin{aligned} A_{N,t,h,\varepsilon,E} \;=\; & \left\{ \mathsf{E}_\sigma\, e_N(t,\sigma) \mathbb{T}_{\overline{B_{N,t,\sigma,E}}} < e^{-\frac{hN}{2}} \right\} \\ & \bigcap \left\{ \mathsf{E}_{\sigma^1,\sigma^2}\, e_N(t,\sigma^1) \mathbb{T}_{B_{N,t,\sigma^1,E}} e_N(t,\sigma^2) \mathbb{T}_{B_{N,t,\sigma^2,E}} \mathbf{1}_{|\theta^{1,2}| > \varepsilon} < e^{-\frac{hN}{2}} \right\} . \end{aligned}$$

Lemma 5 implies immediately, via Chebichev's inequality, the following corollary

Corollary 1 *Let E be fixed by (62). For all $\epsilon > 0$ and all $N \geq N_1(\epsilon, E)$*

$$\mathbb{P}\Big(\bigcup_{k=0}^{[Nt_0]-1} \overline{A_{N,k/N,h_1,\epsilon,E}} \Big) \leq 2[Nt_0] K \exp(-h_1 N/2) \tag{68}$$

where h_1 and N_1 are from Lemma 5.

Moreover, we will need the following

Lemma 6 *Let E be fixed again according to (62). For every $\gamma > 0$ small enough there exist constants $K > 0$ and $h_2 = h_2(\gamma, E) > 0$ and $\varepsilon_1 = \varepsilon_1(\gamma) > 0$ such that for every $\varepsilon < \varepsilon_1$ and all $N \geq N_2 = N_2(\epsilon, \gamma, E)$*

$$\mathbb{E}\, \mathsf{E}_{\sigma^1,\sigma^2,\sigma^3} \left[\left(\prod_{i=2}^{3} e_N(t,\sigma^i) \right) \mathbf{1}_{|\theta^{1,2}| < \varepsilon, |\theta^{1,3}| < \varepsilon, |\theta^{2,3}| < \gamma} e_N(t,\sigma^1) \mathbb{T}_{\overline{B_{N,t,\sigma^1,E}}} \right] \leq K e^{-h_2 N}$$

and

$$\mathbb{E}\, \mathsf{E}_{\sigma^1,\sigma^2,\sigma^3} \left[\left(\prod_{i=1}^{3} e_N(t,\sigma^i) \mathbb{T}_{B_{N,t,\sigma^i,E}} \right) \mathbf{1}_{|\theta^{1,2}| < \varepsilon, |\theta^{1,3}| < \varepsilon, |\theta^{2,3}| > \gamma} \right] \leq K e^{-h_2 N}.$$

In contrast with [8], we do not need to truncate the increment of the Hamiltonian due to the presence of truncations in our Boltzmann weights (24). From now on, we fix $\gamma < \varepsilon_0$ where ε_0 is given by Lemma 4 and we fix $\varepsilon < \min(\varepsilon_0, \varepsilon_1(\gamma))$ with $\varepsilon_1(\gamma)$ given by Lemma 6.

4.2 Proof of Lemma A

We bound I^N by the sum of two terms $\mathrm{I}^N \leq \mathrm{I}^N_1 + \mathrm{I}^N_2$ where the factors $e_N(\frac{k}{N}, \sigma) \in \mathcal{F}^N_{\frac{k}{N}}$ are truncated:

$$\mathrm{I}^N_1 = \sum_{k=0}^{[Nt_0]-1} \mathbb{P}\Big(\Big| \mathsf{E}_\sigma \, Re_N(\frac{k}{N}, \sigma) e_N(\frac{k}{N}, \sigma) \mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma,E}}} \Big| > \frac{\alpha}{2} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big), \tag{69}$$

and

$$\mathrm{I}^N_2 = \sum_{k=0}^{[Nt_0]-1} \mathbb{P}\Big(\Big| \mathsf{E}_\sigma \, Re_N(\frac{k}{N}, \sigma) e_N(\frac{k}{N}, \sigma) \mathbb{T}_{B_{N,\frac{k}{N},\sigma,E}} \Big| > \frac{\alpha}{2} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big). \tag{70}$$

Combining Chebichev's inequality and (28) we obtain

$$\begin{aligned} |\mathrm{I}^N_1| \;&\leq\; \frac{2}{\alpha} \sum_{k=0}^{[Nt_0]-1} \mathbb{E}\Big(\mathsf{E}_\sigma \, |Re_N(\frac{k}{N}, \sigma)| e_N(\frac{k}{N}, \sigma) \mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma,E}}} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big) \\ &\leq\; \frac{2}{\alpha} \sum_{k=0}^{[Nt_0]-1} K e^{\sqrt{N}} \, \mathsf{E}_\sigma (e_N(\frac{k}{N}, \sigma) \mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}). \end{aligned} \tag{71}$$

According to Lemma 5 the latter quantity converges to 0 in $L^1(\mathbb{P})$ exponentially fast, hence $\mathrm{I}^N_1 \xrightarrow{\mathbb{P}} 0$.

Let us turn to I^N_2. By Chebichev's inequality with the fourth moment, the problem is reduced to the convergence in probability of

$$\sum_{k=0}^{[Nt_0]-1} \mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4} \prod_{i=1}^{4} \big[Re_N(\frac{k}{N}, \sigma^i) e_N(\frac{k}{N}, \sigma^i) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}} \big] \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big) = \mathrm{I}^N_{2,1,>\epsilon} + \mathrm{I}^N_{2,1,<\epsilon} \tag{72}$$

where $\mathrm{I}^N_{2,1,<\epsilon}$ is the preceding term when the summation $\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}$ runs over the set

$$L_\varepsilon = \big\{ (\sigma^1, \sigma^2, \sigma^3, \sigma^4) \in \Sigma_N^4 : \forall i, j \;\; |\theta^{i,j}| < \varepsilon \big\}$$

and $\mathrm{I}^N_{2,1,>\epsilon}$ when the summation runs over $\overline{L_\varepsilon}$. The absolute value $|\mathrm{I}^N_{2,1,>\epsilon}|$ is not greater than the sum of 6 terms like the left-hand side of (72) with $|Re_N(\frac{k}{N}, \sigma^i)|$ and where the expectation $\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}$ is taken over σ^i, $i = 1, \ldots, 4$, with $|\theta^{i,j}| > \epsilon$ for one among the six possible pairs of i, j. The term with e.g. $|\theta^{1,2}| > \epsilon$ is bounded by

$$\sum_{k=0}^{[Nt_0]-1} K^4 e^{4\sqrt{N}} Z_N(\frac{k}{N})^2 \, \mathsf{E}_{\sigma^1,\sigma^2} \Big(\mathbf{1}_{|\theta^{1,2}| \geq \varepsilon} \prod_{j=1}^{2} e_N(\frac{k}{N}, \sigma^j) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^j,E}} \Big). \tag{73}$$

In view of Corollary 1, it suffices to show the convergence to zero in probability of (73) where each term in the sums over k is multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}$. Actually, the terms in round bracket $\mathsf{E}_{\sigma^1,\sigma^2}(\cdot)$ are not greater than $e^{-h_1 N/2}$ on these events. Moreover, due to Doob's inequality (see e.g. VII.3.3 in [24]), $Z_N(\frac{k}{N})$ is uniformly bounded above on a set that can be made arbitrarily large. Combining these facts we conclude that $\mathrm{I}^N_{2,1,>\epsilon} \xrightarrow{\mathbb{P}} 0$.

Next, let us proceed with $\mathrm{I}^N_{2,1,<\epsilon}$. Contrary to $\mathrm{I}^N_{2,1,>\epsilon}$, here "truncations" by $B_{N,\frac{k}{N},\sigma^i,E}$ are obstacles to overcome: to get rid of the $\mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}$'s we show the convergence in probability to zero of

$$\sum_{j=1}^{4}\sum_{k=0}^{[Nt_0]-1} \mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\,\mathbf{1}_{L_\epsilon}\mathbb{E}\big[\prod_{i=1}^{4}|Re_N(\frac{k}{N},\sigma^i)|\big]\prod_{i=1}^{4}e_N(\frac{k}{N},\sigma^i)\mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma^j,E}}} \tag{74}$$

By Proposition 1 the expectation $\mathbb{E}[\prod_{i=1}^{4}|Re_N(\frac{k}{N},\sigma^i)|]$ with $|\theta^{i,j}|<\epsilon$ is bounded. Then (74) is not greater than

$$C\sum_{j=1}^{4}\sum_{k=0}^{[Nt_0]-1} \mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\,\mathbf{1}_{L_\epsilon}\prod_{i=1}^{4}e_N(\frac{k}{N},\sigma^i)\mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma^j,E}}} \tag{75}$$

for some constant $C>0$. Again, it is sufficient to consider the latter sum over k with each term multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}$. It is bounded by $4Ce^{-h_1N/2}\sum_{k=0}^{[Nt_0]-1}Z_N(\frac{k}{N})^3$. Since $Z_N(\frac{k}{N})$ is uniformly bounded above on a set that can be made arbitrarily large (75) converges to 0 in probability.

Finally, it remains to show the convergence in probability of $\mathbb{I}^N_{2,2,<\epsilon}$ without truncations. The $L^1(\mathbb{P})$-norm under consideration is equal to

$$\begin{aligned}\mathbb{E}\Big|\sum_{k=0}^{[Nt_0]-1}\mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\,\mathbf{1}_{L_\epsilon}\prod_{i=1}^{4}Re_N(\frac{k}{N},\sigma^i)e_N(\frac{k}{N},\sigma)\;\Big|\;\mathcal{F}^N_{\frac{k}{N}}\Big)\Big|&\\ \le\sum_{k=0}^{[Nt_0]-1}\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\,\mathbf{1}_{L_\epsilon}\Big|\mathbb{E}\Big(\prod_{i=1}^{4}Re_N(\frac{k}{N},\sigma^i)\Big)\Big|\mathbb{E}\prod_{i=1}^{4}e_N(\frac{k}{N},\sigma^i)&\end{aligned} \tag{76}$$

By (62) of Lemma 4 each term of the sum (76) is bounded by CN^{-2} with some constant $C>0$. Then the sum (76) is of the order $O(N^{-1})$. This completes the proof of Lemma A.

4.3 Proof of Lemma B

We have to show that for every $0\le t\le t_0$ and every η

$$\begin{aligned}\mathbb{I}^N=\sum_{k=0}^{[Nt]-1}\Big|\mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2}\Big[Re_N(\frac{k}{N},\sigma^1)Re_N(\frac{k}{N},\sigma^2)-\Delta\Gamma(\frac{k}{N})\Big]&\\ \times e_N(\frac{k}{N},\sigma^1)e_N(\frac{k}{N},\sigma^2)\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\eta}\;\Big|\;\mathcal{F}^N_{\frac{k}{N}}\Big)\Big|\xrightarrow{\mathbb{P}}0.&\end{aligned} \tag{77}$$

Let us bound $\mathbb{I}^N\le\mathbb{I}^N_1+\mathbb{I}^N_2$ where in the first term the Boltzmann factors $e_N(\frac{k}{N},\sigma^i)$ are truncated by $\mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}$ and in the second we include the remaining terms.

First, we show the convergence to zero of $\mathbb{I}^N_2$. Once again it suffice to analyze $\widetilde{\mathbb{I}}^N_2$ which is $\mathbb{I}^N_2$ with each term in the sum over k multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}$. We obtain

$$\widetilde{\mathbb{I}}_2^N \leq 3 \sum_{k=0}^{[Nt]-1} \big[K^2 e^{2\sqrt{N}} + 2 \sup_{0\le t\le t_0} \Gamma(t)\big] Z_N(\frac{k}{N})\, \mathsf{E}_\sigma \Big(e_N(\frac{k}{N},\sigma)\mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}\Big) \mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}$$

$$\leq 3e^{-h_1 N/2} \sum_{k=0}^{[Nt]-1} \big[K^2 e^{2\sqrt{N}} + 2 \sup_{0\le t\le t_0} \Gamma(t)\big] Z_N(\frac{k}{N}). \tag{78}$$

The bound (78) converges to zero in $L^1(\mathbb{P})$ exponentially fast, thus $\mathbb{I}_2^N \xrightarrow{\mathbb{P}} 0$. So, we are lead to study $\mathbb{I}_1^N$ that we split in its turn: $\mathbb{I}_1^N \leq \mathbb{I}_{1,1}^N + \mathbb{I}_{1,2}^N$ where

$$\mathbb{I}_{1,1}^N = \sum_{k=0}^{[Nt]-1} \Big| \mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1) Re_N(\frac{k}{N},\sigma^2) - \Delta\Gamma(\frac{k}{N})] e_N(\frac{k}{N},\sigma^1)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^1,E}} \times e_N(\frac{k}{N},\sigma^2)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^2,E}} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big)\Big| ,$$

$$\mathbb{I}_{1,2}^N = \sum_{k=0}^{[Nt]-1} \Big| \mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1) Re_N(\frac{k}{N},\sigma^2) - \Delta\Gamma(\frac{k}{N})] e_N(\frac{k}{N},\sigma^1)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^1,E}} \times e_N(\frac{k}{N},\sigma^2)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^2,E}} \mathbf{1}_{\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N})>\eta} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big)\Big| .$$

Let us show that $\mathbb{I}_{1,2}^N$ converges to zero in probability. We bound it by $|\mathbb{I}_{1,2}^N| \leq \mathbb{I}_{1,2,1}^N + 2\sup_{0\le t\le t_0}\Gamma(t)\mathbb{I}_{1,2,2}^N$ where

$$\mathbb{I}_{1,2,1}^N = \sum_{k=0}^{[Nt]-1} \Big| \mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2}[Re_N(\frac{k}{N},\sigma^1) Re_N(\frac{k}{N},\sigma^2)] e_N(\frac{k}{N},\sigma^1)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^1,E}} \times e_N(\frac{k}{N},\sigma^2)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^2,E}} \mathbf{1}_{\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N})>\eta} \Big| \mathcal{F}^N_{\frac{k}{N}}\Big) \Big| .$$

and

$$\mathbb{I}_{1,2,2}^N = \sum_{k=0}^{[Nt]-1} Z_N(\frac{k}{N})^2 \mathbb{P}\Big(|\Delta M_N(\frac{k}{N})| Z_N(\frac{k}{N}) > \eta \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big). \tag{79}$$

Since $Z_N(\frac{k}{N})$ is uniformly bounded above on a set that can be made arbitrarily large, Lemma A implies $\mathbb{I}_{1,2,2}^N \xrightarrow{\mathbb{P}} 0$. On the other hand, applying Cauchy-Schwarz inequality twice yields

$$|\mathbb{I}_{1,2,1}^N|^2 \leq \sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4} \prod_{i=1}^{4} [Re_N(\frac{k}{N},\sigma^i) e_N(\frac{k}{N},\sigma^i)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}] \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big) \times \sum_{k=0}^{[Nt]-1} \mathbb{P}\Big(|\Delta M_N(\frac{k}{N})| Z_N(\frac{k}{N}) > \eta \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big).$$

$$\leq (\mathrm{I}^N_{2,1,>\varepsilon} + \mathrm{I}^N_{2,1,<\varepsilon}) \times \mathrm{I}^N, \tag{80}$$

thus $\mathbb{I}^N_{1,2} \xrightarrow{\mathbb{P}} 0$.

Now let us consider $\mathbb{I}^N_{1,1}$ that we bound $\mathbb{I}^N_{1,1} \leq \mathbb{I}^N_{1,1,>\epsilon} + \mathbb{I}^N_{1,1,<\epsilon}$ where in the first term the expectation $\mathsf{E}_{\sigma^1,\sigma^2}$ is taken over σ^1, σ^2 with $|\theta^{1,2}| > \epsilon$ and in the second – over σ^1, σ^2 with $|\theta^{1,2}| < \epsilon$. Let us benefit from truncations in the analysis of the first term. As usual, by Corollary 1 instead of $\mathbb{I}^N_{1,1,>\epsilon}$ we show the convergence in probability only of $\widetilde{\mathbb{I}}^N_{1,1,>\epsilon}$ where each term in the sum over k is multiplied by $\mathbf{1}_{\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}}$. By definition of the $\mathrm{A}_{N,\frac{k}{N},h_1,\varepsilon,E}$'s

$$|\widetilde{\mathbb{I}}^N_{1,1,>\epsilon}| \leq [tN][K^2 e^{2\sqrt{N}} + 2 \sup_{0\leq t\leq t_0} \Gamma(t)] e^{-h_1 N/2},$$

hence $\mathbb{I}^N_{1,1>\epsilon} \xrightarrow{\mathbb{P}} 0$. Thus, we have to investigate $\mathbb{I}^N_{1,1,<\epsilon}$ which convergence in probability follows from the $L^1(\mathbb{P})$ convergence of

$$\begin{aligned} J_N &= \sum_{k=0}^{[Nt]-1} \Big|\mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2}\, 1_{|\theta^{1,2}|<\epsilon} [Re_N(\frac{k}{N},\sigma^1) Re_N(\frac{k}{N},\sigma^2) - \Delta\Gamma(\frac{k}{N})] \times \\ &\quad \times e_N(\frac{k}{N},\sigma^1) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^1,E}} e_N(\frac{k}{N},\sigma^2) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^2,E}} \Big| \mathcal{F}^N_{\frac{k}{N}}\Big)\Big| \\ &= \sum_{k=0}^{[Nt]-1} \Big|\, \mathsf{E}_{\sigma^1,\sigma^2}\, \mathbf{1}_{|\theta^{1,2}|<\epsilon} \chi_N(\theta^{1,2}) e_N(\frac{k}{N},\sigma^1) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^1,E}} e_N(\frac{k}{N},\sigma^2) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^2,E}} \Big| \end{aligned}$$

with the notation

$$\chi_N(\theta^{1,2}) = \mathbb{E}\Big(Re_N(\frac{k}{N},\sigma^1) Re_N(\frac{k}{N},\sigma^2) - \Delta\Gamma(\frac{k}{N})\Big).$$

Let us apply the Cauchy-Schwarz inequality:

$$\begin{aligned} \mathbb{E} J_N &\leq \sum_{k=0}^{[Nt]-1} \mathbb{E}\, \mathsf{E}_{\sigma^1} \big(e_N(\frac{k}{N},\sigma^1)\big)^{1/2} \\ &\quad \times [e_N(\frac{k}{N},\sigma^1) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^1,E}}]^{1/2} \Big| \mathsf{E}_{\sigma^2}\, \mathbf{1}_{|\theta^{1,2}|<\epsilon} \chi_N(\theta^{1,2}) e_N(\frac{k}{N},\sigma^2) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^2,E}} \Big| \\ &\leq \sum_{k=0}^{[Nt]-1} \Big(\mathbb{E}\, \mathsf{E}_{\sigma^1,\sigma^2,\sigma^3} \Big[\mathbf{1}_{|\theta^{1,2}|<\epsilon, |\theta|^{1,3}<\epsilon} \chi_N(\theta^{1,2}) \chi_N(\theta^{1,3}) \\ &\qquad \times \prod_{i=1}^{3} [e_N(\frac{k}{N},\sigma^i) \mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}]\Big]\Big)^{1/2} \end{aligned} \tag{81}$$

where we used the fact that $\mathbb{E}\, \mathsf{E}_{\sigma^1} e_N(\frac{k}{N},\sigma^1) = 1$. The expectation in (81) equals

$\mathbb{E}\,\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3}[\cdot] = L_1^N(\frac{k}{N}) + L_2^N(\frac{k}{N}) - L_3^N(\frac{k}{N})$ with

$$\begin{aligned}
L_1^N(\frac{k}{N}) &= \mathbb{E}\,\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3}\,\mathbf{1}_{|\theta^{1,2}|<\epsilon,|\theta^{1,3}|<\epsilon,|\theta^{2,3}|>\gamma}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3}) \\
&\quad\times\prod_{i=1}^{3}[e_N(\frac{k}{N},\sigma^i)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}] \\
L_2^N(\frac{k}{N}) &= \mathbb{E}\,\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3}\,\mathbf{1}_{|\theta^{1,2}|<\epsilon,|\theta^{1,3}|<\epsilon,|\theta^{2,3}|<\gamma}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3}) \\
&\quad\times\prod_{i=1}^{3}e_N(\frac{k}{N},\sigma^i) \\
|L_3^N(\frac{k}{N})| &\leq 3\mathbb{E}\,\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3}\,\mathbf{1}_{|\theta^{1,2}|<\epsilon,|\theta^{1,3}|<\epsilon,|\theta^{2,3}|<\gamma}\chi_N(\theta^{1,2})\chi_N(\theta^{1,3}) \\
&\quad\times\prod_{i=1}^{2}[e_N(\frac{k}{N},\sigma^i)]\mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma^3,E}}}
\end{aligned}$$

Since $|\theta^{1,2}|<\epsilon$, $|\theta^{1,3}|<\epsilon$, the expansions of Proposition 1 are valid for $\chi_N(\theta^{1,2})$ and $\chi_N(\theta^{1,3})$. In particular $|\chi_N(\theta^{1,2})\chi_N(\theta^{1,3})|<C$ for some constant C all $N\geq 1$ and all $\theta^{1,2}$, $\theta^{1,3}$ with $|\theta^{1,2}|<\epsilon$ and $|\theta^{1,3}|<\epsilon$. Then by Lemma 6

$$\begin{aligned}
\sup_{0\leq k\leq[Nt_0]-1}|L_1^N(\frac{k}{N})| &\leq Ce^{-h_2N} \\
\sup_{0\leq k\leq[Nt_0]-1}|L_3^N(\frac{k}{N})| &\leq 3Ce^{-h_2N}
\end{aligned}$$

Finally, recall that γ and ϵ were chosen smaller than ϵ_0 of Lemma (4). From the estimate (61) of Lemma 4 it follows that

$$\lim_{N\to\infty}\sup_{0\leq k\leq[Nt_0]-1}N^2L_2^N(\frac{k}{N})=0,$$

and Lemma B is proved.

4.4 Proof of Lemma C

We split $\mathrm{III}^N=\mathrm{III}_1^N+\mathrm{III}_2^N$ where in the first term III_1^N the Boltzmann factors $e_N(\frac{k}{N},\sigma)$ are truncated by $\mathbb{T}_{B_{N,\frac{k}{N},\sigma,E}}$ and in the second III_2^N by $\mathbb{T}_{\overline{B_{N,\frac{k}{N},\sigma,E}}}$. Proceeding along the line of (71) we get $\mathrm{III}_2^N\xrightarrow{\mathbb{P}}0$. Now, to complete the proof, we need to show the convergence in probability of III_1^N. We apply Hölder inequality twice to obtain

$$\begin{aligned}
\mathrm{III}_1^N &\le \sum_{k=0}^{[Nt]-1} \Big[\mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\prod_{i=1}^{4} Re_N(\frac{k}{N},\sigma^i)e_N(\frac{k}{N},\sigma^i)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}\,\Big|\,\mathcal{F}^N_{\frac{k}{N}}\Big)^{1/4} \\
&\quad\times\mathbb{P}\Big(|\Delta M_N(\frac{k}{N})|Z_N(\frac{k}{N})>\eta\,\Big|\,\mathcal{F}^N_{\frac{k}{N}}\Big)^{3/4}\Big] \\
&\le \Big[\sum_{k=0}^{[Nt]-1}\mathbb{E}\Big(\mathsf{E}_{\sigma^1,\sigma^2,\sigma^3,\sigma^4}\prod_{i=1}^{4} Re_N(\frac{k}{N},\sigma^i)e_N(\frac{k}{N},\sigma^i)\mathbb{T}_{B_{N,\frac{k}{N},\sigma^i,E}}|\mathcal{F}^N_{\frac{k}{N}}\Big)\Big]^{1/4} \\
&\quad\times\Big[\sum_{k=0}^{[Nt]-1}\mathbb{P}\Big(|\Delta M_N(\frac{k}{N})|Z_N(\frac{k}{N})>\eta\,\Big|\,\mathcal{F}^N_{\frac{k}{N}}\Big)\Big]^{3/4} \\
&= \Big[\mathrm{I}^N_{2,1,>\epsilon}+\mathrm{I}^N_{2,1,<\epsilon}\Big]^{1/4}\times\Big[\mathrm{I}^N\Big]^{3/4}\xrightarrow{\mathbb{P}}0.
\end{aligned}$$

Lemma C is proved.

4.5 Proof of Lemma D

Let us introduce some new processes

$$\mathcal{M}_N(t)=\sum_{k=0}^{[Nt]-1}\mathbb{E}\Big([\Delta M_N(\frac{k}{N})]^2\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1}\,\Big|\,\mathcal{F}^N_{\frac{k}{N}}\Big), \tag{82}$$

$$\mathcal{N}_N(t)=\sum_{k=0}^{[Nt]-1}\mathbb{E}\Big(\Delta M_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1}\,\Big|\,\mathcal{F}^N_{\frac{k}{N}}\Big), \tag{83}$$

and

$$\widetilde{Z_N}(t)=\prod_{k=0}^{[Nt]-1}\Big(1+\Delta M_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1}\Big). \tag{84}$$

The proof of Lemma D is worked out in two steps. First we prove that Z_N is bounded below by a functionnal of $\mathcal{M}_N$ and $\mathcal{N}_N$ on a set which probability can be made arbitrarily large. In a second step we prove that this latter process converges uniformly in probability to a deterministic path, a fact that leads to Lemma D.

First step: There exists a b $(0<b\le 1)$ such that $\Delta M_N\ge b-1>-1$. Let us recall that $Z_N(t)=\prod_{k=0}^{[Nt]-1}\Big(1+\Delta M_N(\frac{k}{N})\Big)$. Since $Z_N\ge 0$ we see that $\widetilde{Z_N}(t)\le Z_N(t)$ for every $t\ge 0$. We prove that

Lemma 7 *For every $N,a,\varepsilon>0$ we have*

$$\mathbb{P}\Big(\forall t\ge 0\ :\ Z_N(t)\ge e^{-a-\frac{1+\varepsilon}{2}b^{-(2+\varepsilon)}\mathcal{M}_N(t)+\mathcal{N}_N(t)}\Big)\ge 1-e^{-a\varepsilon}.$$

Proof It follows from the Taylor-Lagrange formula that for all positive λ and all $u \geq b-1$ we have

$$(1+u)^{-\lambda} \leq 1 - \lambda u + \frac{\lambda(1+\lambda)}{2} b^{-(\lambda+2)} u^2.$$

As a consequence, for every integers N, k we get

$$\begin{aligned}
&\mathbb{E}\Big(\Big(1+\Delta M_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1}\Big)^{-\lambda} \,\Big|\, \mathcal{F}^N_{\frac{k}{N}}\Big) \leq \\
&\quad \leq \; 1 - \lambda \mathbb{E}\Big(\Delta M_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1} \,\Big|\, \mathcal{F}^N_{\frac{k}{N}}\Big) + \\
&\qquad + \frac{\lambda(1+\lambda)}{2} b^{-(\lambda+2)} \mathbb{E}\Big([\Delta M_N(\frac{k}{N})]^2 \mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1} \,\Big|\, \mathcal{F}^N_{\frac{k}{N}}\Big) \\
&\quad \leq \; \exp\Big\{ -\lambda \mathbb{E}\Big(\Delta M_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1} \,\Big|\, \mathcal{F}^N_{\frac{k}{N}}\Big) + \\
&\qquad + \frac{\lambda(1+\lambda)}{2} b^{-(\lambda+2)} \mathbb{E}\Big([\Delta M_N(\frac{k}{N})]^2 \mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<1} \,\Big|\, \mathcal{F}^N_{\frac{k}{N}}\Big) \Big\}.
\end{aligned}$$

Thus for every $N, \lambda > 0$ the process

$$Y_{N,\lambda}(t) = \widetilde{Z_N}(t)^{-\lambda} \exp\Big\{ \lambda \mathcal{N}_N(t) - \frac{\lambda(1+\lambda)}{2} b^{-(\lambda+2)} \mathcal{M}_N(t) \Big\}$$

is a positive supermartingale w.r.t. the filtration $(\mathcal{F}^N_t)_{t\geq 0}$. Now, for every $a, \varepsilon > 0$, let us define the optional time T by

$$T = \inf\Big\{ t \geq 0 : \widetilde{Z_N}(t) < \exp\Big[-a - \frac{1+\varepsilon}{2} b^{-(2+\varepsilon)} \mathcal{M}_N(t) + \mathcal{N}_N(t) \Big] \Big\}$$

and let $s \in (0, \infty)$. From the Optional Sampling Theorem (see e.g. Chapter 1, Theorem 3.22 in [16]) we obtain

$$\begin{aligned}
1 \;&=\; \mathbb{E}Y_{N,\lambda}(0) \geq \mathbb{E}Y_{N,\lambda}(T \wedge s) \geq \mathbb{E}Y_{N,\lambda}(T)\mathbf{1}_{T\leq s} \\
&\geq\; \mathbb{E}\Big(\mathbf{1}_{T\leq s} \exp\Big\{ \lambda a + \frac{\lambda[(1+\varepsilon)b^{-(2+\varepsilon)} - (1+\lambda)b^{-(2+\lambda)}]}{2} \mathcal{M}_N(T) \Big\} \Big)
\end{aligned}$$

according to the definition of T. Taking now $\lambda = \varepsilon$ we obtain $\mathbb{P}(T \geq s) \geq 1 - \exp(-a\varepsilon)$. Since $\widetilde{Z_N} \leq Z_N$ we obtain the desired result by letting $s \to \infty$. ■

<u>*Second step:*</u> We now proceed to the proper proof of Lemma D. To this end let us introduce the function $F_\varepsilon(x) = \frac{e^{-c_\varepsilon x} - 1}{c_\varepsilon}$ with $c_\varepsilon = (1+\varepsilon)b^{-(2+\varepsilon)}$. Due to the particular form of F_ε for every real valued map $(x_s)_{s\geq 0}$ we have that $|\Delta F_\varepsilon(x_s)| = |F'_\varepsilon(x_s)||F_\varepsilon(\Delta x_s)|$. Moreover, if $(x_s)_{s\geq 0}$ is bounded below there exists a K_ε such that $|F_\varepsilon(x_s)| \leq K_\varepsilon |x_s|$. Let us introduce the process

$$X_s^{N,\varepsilon} = \mathcal{M}_N(s) - \frac{2}{c_\varepsilon}\mathcal{N}_N(s) - \Gamma(\beta_0, s)$$

and recall that according to Lemma 7 the event

$$\mathcal{A}_{a,\varepsilon}^N = \left\{\forall t \geq 0 \ : \ Z_N(t)^{-2} \leq e^{2a + c_\varepsilon \mathcal{M}_N(t) - 2\mathcal{N}_N(t)}\right\}$$

has probability at least $1 - e^{-a\varepsilon}$ for every $a, \varepsilon > 0$. We have

$$\begin{aligned}
& \max_{0\leq s\leq t_0} |F_\varepsilon(X_s^{N,\varepsilon})| \mathbf{1}_{\mathcal{A}_{a,\varepsilon}^N} \\
\leq & \ \Big(\sum_{k=0}^{[Nt_0]-1} |\Delta F_\varepsilon(X_{\frac{k}{N}}^{N,\varepsilon})| \Big) \mathbf{1}_{\mathcal{A}_{a,\varepsilon}^N} \\
\leq & \ K_\varepsilon \Big(\sum_{k=0}^{[Nt_0]-1} |\Delta X_{\frac{k}{N}}^{N,\varepsilon}| Z_N(\frac{k}{N})^2 Z_N(\frac{k}{N})^{-2} |F'_\varepsilon(X_{\frac{k}{N}}^{N,\varepsilon})| \Big) \mathbf{1}_{\mathcal{A}_{a,\varepsilon}^N} \\
\leq & \ K_\varepsilon e^{2a + c_\varepsilon \Gamma(\beta_0, t_0)} \mathbf{1}_{\mathcal{A}_{a,\varepsilon}^N} \times \\
& \times \sum_{k=0}^{[Nt_0]-1} \left| \mathbb{E}\Big(\left[(\Delta M_N(\frac{k}{N}))^2 - \Delta\Gamma(\beta_0, \frac{k}{N}) \right] (Z_N(\frac{k}{N}))^2 \mathbf{1}_{\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N}) < 1} \ \Big| \ \mathcal{F}_{\frac{k}{N}}^N \Big) \right| \\
& + \frac{2}{c_\varepsilon} K_\varepsilon e^{2a + c_\varepsilon \Gamma(\beta_0, t_0)} \mathbf{1}_{\mathcal{A}_{a,\varepsilon}^N} \times \\
& \times \sum_{k=0}^{[Nt_0]-1} Z_N(\frac{k}{N}) \left| \mathbb{E}\Big(\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N}) \mathbf{1}_{\Delta M_N(\frac{k}{N}) Z_N(\frac{k}{N}) < 1} \ \Big| \ \mathcal{F}_{\frac{k}{N}}^N \Big) \right| \\
& + K_\varepsilon e^{2a + c_\varepsilon \Gamma(\beta_0, t_0)} \mathbf{1}_{\mathcal{A}_{a,\varepsilon}^N} \times \\
& \times \sum_{k=0}^{[Nt_0]-1} |\Delta\Gamma(\beta_0, \frac{k}{N})| Z_N(\frac{k}{N})^2 \mathbb{P}(|\Delta M_N(\frac{k}{N})| Z_N(\frac{k}{N}) > 1 \ \Big| \ \mathcal{F}_{\frac{k}{N}}^N \Big). \qquad (85)
\end{aligned}$$

Once again, due to Doob's inequality, Z_N is uniformly bounded above on a set that can be made arbitrarily large. Combining (85) with Lemmas A,B and C yields $\max_{0\leq s\leq t_0} |F_\varepsilon(X_s^{N,\varepsilon})| \xrightarrow{\mathbb{P}} 0$ hence $\max_{0\leq s\leq t_0} |X_s^{N,\varepsilon}| \xrightarrow{\mathbb{P}} 0$ for every fixed $\varepsilon > 0$ when $N \to \infty$. Combined with Lemma 7 it yields that for every $u > 0$ there exists a $\delta > 0$ such that for every $N \geq N(u)$ we have $\mathbb{P}(\min_{0\leq t\leq t_0} Z_N(t) \geq \delta) \geq 1 - e^{-u}$.

4.6 Proof of Conditions (i,ii,iii)

Condition (ii) Combining Lemma A with Lemma D clearly leads to Condition (ii).

Condition (iii) Due to the fact that $\Delta M_N > -1$, Condition (iii) is equivalent to

$$\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Delta M_N(\frac{k}{N}) \mathbf{1}_{\Delta M_N(\frac{k}{N}) > 1} \ \Big| \ \mathcal{F}_{\frac{k}{N}}^N \Big) \xrightarrow{\mathbb{P}} 0.$$

According to Lemma D, we can always find a $\delta > 0$ to make the following

$$\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Delta M_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})>1} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big) \le$$
$$\le \frac{1}{\delta}\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big(\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})>\delta} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big)$$

hold true on a set of probability as large as we want. Condition (iii) follows.

Condition (i) Due to Lemma D and Doob's inequality we can choose δ_1 and δ_2 such that for every N, k we have $\delta_1 \le Z_N(\frac{k}{N}) \le \delta_2$ with probability as close to 1 as we want. Thus $\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\delta_1} \le \mathbf{1}_{|\Delta M_N(\frac{k}{N})|<1} \le \mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\delta_2}$, and Condition (ii) is proved once both

$$\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big((\Delta M_N(\frac{k}{N}))^2\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\delta_1} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big) \xrightarrow{\mathbb{P}} \Gamma(\beta_0, t) \tag{86}$$

and

$$\sum_{k=0}^{[Nt]-1} \mathbb{E}\Big((\Delta M_N(\frac{k}{N}))^2\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\delta_2} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big) \xrightarrow{\mathbb{P}} \Gamma(\beta_0, t) \tag{87}$$

are showed to hold. We only prove (86). According to Lemma A, it is equivalent to prove that

$$\sum_{k=0}^{[Nt]-1} \Big|\mathbb{E}\Big(\Big[(\Delta M_N(\frac{k}{N}))^2 - \Delta\Gamma(\beta_0, \frac{k}{N})\Big]\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\delta_1} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big)\Big|$$

converges in probability to 0. But this quantity is bounded by

$$\frac{1}{\delta_1^2}\sum_{k=0}^{[Nt]-1} \Big|\mathbb{E}\Big(\Big[(\Delta M_N(\frac{k}{N}))^2 - \Delta\Gamma(\beta_0, \frac{k}{N})\Big] Z_N(\frac{k}{N})^2\mathbf{1}_{\Delta M_N(\frac{k}{N})Z_N(\frac{k}{N})<\delta_1} \;\Big|\; \mathcal{F}^N_{\frac{k}{N}}\Big)\Big|$$

which converges to 0 in probability. Condition (i) follows.

4.7 Proof of Lemma 5

Due to the fact that the ξ_i^k's are independent and identically distributed we have

$$\mathbb{E}e_N(t,\sigma)\mathbb{T}_{\overline{B_{N,t,\sigma,E}}} = \sum_{m=0}^{[Nt]}\Bigg[\binom{[Nt]}{m}\Bigg(\mathbb{E}e^{\beta_0 H_N(\frac{m}{N},\sigma)}\Bigg(\prod_{k=1}^{m}\mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\Bigg)e^{-m\phi_N(\beta_0)}\mathbf{1}_{\overline{B_{N,\frac{m}{N},\sigma,E}}}\Bigg)$$
$$\mathbb{P}\left(\left|\frac{S_N}{N}\right| \ge \frac{1}{N^{1/4}}\right)^{[Nt]-m} e^{-([Nt]-m)\phi_N(\beta_0)}\Bigg],$$

adopting the convention that the product above is equal to 1 if $m = 0$. On the one hand, according to Theorem 2, there exists a constant $K > 0$ such that for every $1 \leq m \leq [Nt_0]$

$$
\begin{aligned}
\mathbb{E}e^{\beta_0 H_N(\frac{m}{N},\sigma)} \left(\prod_{k=1}^{m} \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\right) \mathbf{1}_{\overline{B_{N,\frac{m}{N},\sigma,E}}} e^{-m\phi_N(\beta_0)} \\
&= \mathbb{E}e^{-EH_N(\frac{m}{N},\sigma)} e^{(\beta_0+E)H_N(\frac{m}{N},\sigma)-m\phi_N(\beta_0)} \\
&\quad \mathbf{1}_{\overline{B_{N,\frac{m}{N},\sigma,E}}} \left(\prod_{k=1}^{m} \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\right) \\
&\leq e^{-\frac{[t_0 N]}{2(1-\beta_0-E)}-m\phi_N(\beta_0)} \mathbb{E}e^{(\beta_0+E)H_N(\frac{m}{N},\sigma)} \prod_{k=1}^{m} \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}} \\
&\leq K e^{-\frac{[t_0 N]E}{2(1-\beta_0-E)}-m\phi_N(\beta_0)} e^{-\frac{m}{2}\log(1-\beta_0-E)} \\
&\leq K e^{-\frac{[Nt_0]E}{2}\left(\frac{E}{1-\beta_0-E}-\log(1+\frac{E}{1-\beta_0-E})\right)}.
\end{aligned}
$$

This procedure is borrowed to Talagrand [25]. Since $0 < \frac{E}{1-\beta_0-E} < 1$ and $x - \log(1+x) > 0$ for every $0 < x < 1$ we can write that for some constant $K > 0$

$$
\mathbb{E}e^{\beta_0 H_N(\frac{m}{N},\sigma)-m\phi_N(\beta_0)} \left(\prod_{k=1}^{m} \mathbf{1}_{\left|\frac{\sigma\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\right) \mathbf{1}_{\overline{B_{N,\frac{m}{N},\sigma,E}}} \leq K e^{-Nh_i(E)} \tag{88}
$$

where $h_i(E) = \frac{t_0}{2}\left(\frac{E}{1-\beta_0-E} - \log(1+\frac{E}{1-\beta_0-E})\right) > 0$. On the other hand, due to Chernoff's bound there exists a constant $K > 0$ such that $\mathbb{P}(|S_N/N| \geq \frac{1}{N^{1/4}}) \leq 2e^{-NI(\frac{1}{N^{1/4}})} \leq 2Ke^{-\sqrt{N}}$. Together with (88) it leads to

$$
\mathbb{E}\,\mathsf{E}_\sigma\, e_N(t,\sigma)\mathbb{T}_{\overline{B_{N,t,\sigma,E}}} \;\leq\; K e^{-Nh_i(E)}
$$

for some constant $K > 0$, which proves (66). Next we turn to the proof of (67). According to the definition of $e_N(t,\sigma)\mathbb{T}_{B_{N,t,\sigma,E}}$ and due to the fact that the ξ_i^k's are independent and identically distributed we have

$$
\begin{aligned}
&\mathbb{E}e_N(t,\sigma^1)\mathbb{T}_{B_{N,t,\sigma^1,E}}e_N(t,\sigma^2)\mathbb{T}_{B_{N,t,\sigma^2,E}} \\
&\quad = \mathbb{E}\left(\sum_{m=0}^{[Nt]}\sum_{\vec{k}\in T_{N,t,m}}\prod_{k\in\vec{k}_<} e^{\beta_0 Nf(\frac{\sigma^1\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,t,m,\vec{k},\sigma^1,E}}e^{-m\phi_N(\beta_0)}\right. \\
&\qquad\qquad \left.\prod_{k\in\vec{k}_>}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|\geq\frac{1}{N^{1/4}}}e^{-([Nt]-m)\phi_N(\beta_0)}\right) \\
&\qquad \left(\sum_{l=0}^{[Nt]}\sum_{\vec{p}\in T_{N,t,l}}\prod_{p\in\vec{p}_<} e^{\beta_0 Nf(\frac{\sigma^2\cdot\xi^p}{N})}\mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^p}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,t,l,\vec{p},\sigma^2,E}}e^{-l\phi_N(\beta_0)}\right. \\
&\qquad\qquad \left.\prod_{p\in\vec{p}_>}\mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^p}{N}\right|\geq\frac{1}{N^{1/4}}}e^{-([Nt]-l)\phi_N(\beta_0)}\right) \\
&\quad \leq \sum_{m,l=0}^{[Nt]}\sum_{\substack{\vec{k}\in T_{N,t,m}\\ \vec{p}\in T_{N,t,l}}} P_{\vec{k}_<,\vec{p}_<}P_{\vec{k}_<,\vec{p}_>}P_{\vec{k}_>,\vec{p}_<}P_{\vec{k}_>,\vec{p}_>}
\end{aligned}
$$

where

$$
\begin{aligned}
P_{\vec{k}_<,\vec{p}_<} &= \mathbb{E}\prod_{k\in\vec{k}_<\cap\vec{p}_<} e^{\beta_0 Nf(\frac{\sigma^1\cdot\xi^k}{N})+\beta_0 Nf(\frac{\sigma^2\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,t,m,\vec{k}_<\cap\vec{p}_<,\sigma^1,E}} \\
&\qquad \mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,t,l,\vec{k}_<\cap\vec{p}_<,\sigma^2,E}}e^{-2|\vec{k}_<\cap\vec{p}_<|\phi_N(\beta_0)} \\
&= \mathbb{E}\prod_{k=1}^{|\vec{k}_<\cap\vec{p}_<|} e^{\beta_0 Nf(\frac{\sigma^1\cdot\xi^k}{N})+\beta_0 Nf(\frac{\sigma^2\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,t,m,\vec{k}_<\cap\vec{p}_<,\sigma^1,E}} \\
&\qquad \mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,t,l,\vec{k}_<\cap\vec{p}_<,\sigma^2,E}}e^{-2|\vec{k}_<\cap\vec{p}_<|\phi_N(\beta_0)} \\
&= \mathbb{E}e^{\beta_0 H_N(\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^1)+\beta_0 H_N(\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^2)-2|\vec{k}_<\cap\vec{p}_<|\phi_N(\beta_0)} \\
&\qquad \prod_{k=1}^{|\vec{k}_<\cap\vec{p}_<|}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^1,E}}\mathbf{1}_{B_{N,\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^2,E}},
\end{aligned}
$$

$$
P_{\vec{k}_<,\vec{p}_>} = \mathbb{E}\prod_{k\in\vec{k}_<\cap\vec{p}_>} e^{\beta_0 Nf(\frac{\sigma^1\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^k}{N}\right|\geq\frac{1}{N^{1/4}}}e^{-2|\vec{k}_<\cap\vec{p}_>|\phi_N(\beta_0)},
$$

$$
P_{\vec{k}_>,\vec{p}_<} = \mathbb{E}\prod_{k\in\vec{k}_>\cap\vec{p}_<}\mathbf{1}_{\left|\frac{\sigma^1\cdot\xi^k}{N}\right|\geq\frac{1}{N^{1/4}}}e^{\beta_0 Nf(\frac{\sigma^2\cdot\xi^k}{N})}\mathbf{1}_{\left|\frac{\sigma^2\cdot\xi^k}{N}\right|<\frac{1}{N^{1/4}}}e^{-2|\vec{k}_>\cap\vec{p}_<|\phi_N(\beta_0)},
$$

and

$$P_{\vec{k}_>,\vec{p}_>} = \mathbb{E}\prod_{k\in\vec{k}_>\cap\vec{p}_>}\mathbf{1}_{\left|\frac{\sigma^1.\xi^k}{N}\right|\geq\frac{1}{N^{1/4}}}\mathbf{1}_{\left|\frac{\sigma^2.\xi^k}{N}\right|\geq\frac{1}{N^{1/4}}}e^{-2|\vec{k}_>\cap\vec{p}_>|\phi_N(\beta_0)}.$$

We shall study these four terms separetely:
$\underline{P_{\vec{k}_<,\vec{p}_<}}$ We prove that for any $\sigma^1,\sigma^2\in\Sigma_N$, any $N,t>0$, any m,l such that $1\leq m,l\leq[Nt]$ and finally any $\vec{k}\in T_{N,t,m},\vec{p}\in T_{N,t,l}$ we have

$$P_{\vec{k}_<,\vec{p}_<}\leq e^{\frac{[t_0N]}{2}\frac{\beta_0^2\theta^2}{(1-\beta_0)(1-\beta_0-E)}}. \tag{89}$$

Let us denote by $\lambda(\theta)$ the lowest solution of the equation

$$(1-x)^2-\theta^2x^2=(1-\beta_0)^2, \tag{90}$$

i.e. the lowest root of the polynomial function

$$Q_\theta(x)=x^2(1-\theta^2)-2x+\beta_0(2-\beta_0).$$

Since for every $\theta\in[-1,1]$ we have $Q_\theta(\beta_0)\leq 0$ we necessarily have $\lambda(\theta)\leq\beta_0$. Let us also notice that due to (90) the condition (52) with $\beta_1=\beta_2=\lambda(\theta)$ that here reduces to

$$(1-\lambda(\theta))^2-\theta^2\lambda(\theta)^2>0$$

is necessarily fulfilled. Thus, applying Talagrand's idea once again we write

$$\begin{aligned}
P_{\vec{k}_<,\vec{p}_<} &\leq \mathbb{E}e^{\lambda(\theta)(H_N(\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^1)+H_N(\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^2))} \\
&\quad e^{(\beta_0-\lambda(\theta))(H_N(\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^1)+\beta_0H_N(\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^2))}e^{-2|\vec{k}_<\cap\vec{p}_<|\phi_N(\beta_0)} \\
&\quad \prod_{k=1}^{|\vec{k}_<\cap\vec{p}_<|}\mathbf{1}_{\left|\frac{\sigma^1.\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{\left|\frac{\sigma^2.\xi^k}{N}\right|<\frac{1}{N^{1/4}}}\mathbf{1}_{B_{N,\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^1,E}}\mathbf{1}_{B_{N,\frac{|\vec{k}_<\cap\vec{p}_<|}{N},\sigma^2,E}} \\
&\leq e^{(\beta_0-\lambda(\theta))\frac{[t_0N]}{1-\beta_0-E}}e^{|\vec{k}_<\cap\vec{p}_<|\psi_N(\lambda(\theta),\lambda(\theta),\theta)}e^{-2|\vec{k}_<\cap\vec{p}_<|\phi_N(\beta_0)}
\end{aligned} \tag{91}$$

According to Proposition 1 we get

$$\begin{aligned}
&[t_0N]\frac{(\beta_0-\lambda(\theta))}{1-\beta_0-E}\psi_N(\lambda(\theta),\lambda(\theta),\theta) \\
=\ &[t_0N]\frac{(\beta_0-\lambda(\theta))}{1-\beta_0-E}-\frac{|\vec{k}_<\cap\vec{p}_<|}{2}\left[\log((1-\lambda(\theta))^2-\lambda(\theta)^2\theta^2)+O(1/N)\right] \\
=\ &[t_0N]\frac{(\beta_0-\lambda(\theta))}{1-\beta_0-E}-\frac{|\vec{k}_<\cap\vec{p}_<|}{2}\left[\log((1-\beta_0)^2)+O(1/N)\right]
\end{aligned}$$

where $O(1/N)$ is uniform in θ. Meanwhile, due to the fact that $\lambda(\theta)$ solves (90) we have

$$\beta_0 - \lambda(\theta) \leq \frac{\lambda(\theta)^2\theta^2}{2-\lambda-\beta_0} \leq \frac{\beta_0^2\theta^2}{2(1-\beta_0)}.$$

This, together with (91) yield (89).
$\underline{P_{\vec{k}_<,\vec{p}_>}}$ Let $u>1$ be small enough that $\beta_0 u < \boldsymbol{\beta}_c$ and v be its Hölder conjugate. Due to Theorem 2 there exists a constant $K>0$ such that

$$\mathbb{E} \prod_{k\in\vec{k}_<\cap\vec{p}_>} e^{\beta_0 N f(\frac{\sigma^1.\xi^k}{N})} \mathbf{1}_{\left|\frac{\sigma^1.\xi^k}{N}\right|<\frac{1}{N^{1/4}}} \mathbf{1}_{\left|\frac{\sigma^2.\xi^k}{N}\right|\geq\frac{1}{N^{1/4}}} e^{-2|\vec{k}_<\cap\vec{p}_>|\phi_N(\beta_0)} \leq \left(Ke^{-\frac{\sqrt{N}}{v}}\right)^{|\vec{k}_<\cap\vec{p}_>|} \tag{92}$$

Naturally, the same holds true for $P_{\vec{k}_>,\vec{p}_<}$.
$\underline{P_{\vec{k}_<,\vec{p}_<}}$ Due to Cauchy-Schwarz inequality, we obtain in the same way that there exists a constant $K>0$ such that

$$P_{\vec{k}_>,\vec{p}_>} \leq \left(Ke^{-\frac{\sqrt{N}}{2}}\right)^{|\vec{k}_>\cap\vec{p}_>|}.$$

<u>Conclusion</u> Because of (12) we have

$$\frac{\beta_0^2 t_0}{(1-\beta_0)^2} < 1$$

Thus, according to (89) there exists a c, $0<c<1$ such that

$$\mathbb{E}e_N(t,\sigma^1)\mathbb{T}_{B_{N,t,\sigma^1,E}} e_N(t,\sigma^2)\mathbb{T}_{B_{N,t,\sigma^2,E}} \leq e^{N\frac{c}{2}\theta^2}.$$

Let us denote by I the rate function governing the large deviations of $\theta = \frac{\sigma^1.\sigma^2}{N}$ under the uniform measure on Σ_N^2. We know that $I(u) \geq \frac{1}{2}u^2$. Thus according to Varadhan's Lemma

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\, \mathsf{E}_{\sigma^1,\sigma^2}\, e_N(t,\sigma^1)\mathbb{T}_{B_{N,t,\sigma^1,E}} e_N(t,\sigma^2)\mathbb{T}_{B_{N,t,\sigma^2,E}} \mathbf{1}_{|\theta|>\varepsilon} \leq \\ \leq \sup_{|u|>\varepsilon}\left\{\frac{c}{2}u^2 - \frac{1}{2}u^2\right\} \\ = -h_{ii}(\varepsilon)\end{aligned}$$

where h_{ii} is obviously a positive function. Defining $h_1(\epsilon,E) = \min(h_i(E),h_{ii}(\varepsilon))$ ends the proof.

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