

# Convergence to equilibrium in Wasserstein distance for Fokker-Planck equations

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## Abstract

We describe conditions on non-gradient drift diffusion Fokker-Planck equations for its solutions to converge to equilibrium with a uniform exponential rate in Wasserstein distance. This asymptotic behaviour is related to a functional inequality, which links the distance with its dissipation and ensures a spectral gap in Wasserstein distance. We give practical criteria for this inequality and compare it to classical ones. The key point is to quantify the contribution of the diffusion term to the rate of convergence, which to our knowledge is a novelty.

**Key words:** Diffusion equations, Wasserstein distance, functional inequalities, spectral gap

## Introduction

In this work we consider the Fokker-Planck equation

$$\partial_t \mu_t = \nabla \cdot (\nabla \mu_t + \mu_t A), \quad t > 0, x \in \mathbb{R}^n \quad (1)$$

where  $A$  is given vector field on  $\mathbb{R}^n$ . The evolution preserves mass and positivity, and we are concerned with initial data  $\mu_0$  which are probability measures on  $\mathbb{R}^n$ , so that so are the solutions  $\mu_t = \mu(t, \cdot)$  at any time  $t > 0$ .

We are interesting in criteria ensuring uniform bounds on the long time behaviour of solutions.

To explain our main issue, let us start with the classical case when  $A = \nabla V$  such that  $\int e^{-V} dx = 1$ . The probability measure  $d\mu(x) = e^{-V(x)} dx$  is a stationary solution of the equation and it is interesting to know for which  $V$  all solutions  $\mu_t$  converge to  $\mu$  as  $t$  tends to infinity, in which sense and with a rate.

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There are various distances to measure the discrepancy between a solution of the equation and the stationary one : total variation (as in Meyn-Tweedie's approach),  $L^2$ -norm, relative entropy, Wasserstein distance. Perhaps, the simplest way of measuring the gap between a solution  $\mu_t$  and  $e^{-V}$  is the  $L^2$ -norm of the difference, namely, the quantity

$$G(t) = \int (\mu_t - e^{-V})^2 e^V dx = \int \left( \frac{\mu_t}{e^{-V}} - 1 \right)^2 e^{-V} dx.$$

Formally, by integration by parts,

$$G'(t) = 2 \int \left( \frac{\mu_t}{e^{-V}} - 1 \right) \partial_t \mu_t dx = -2 \int \left| \nabla \left( \frac{\mu_t}{e^{-V}} - 1 \right) \right|^2 e^{-V} dx, \quad t > 0.$$

Here  $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^n$ . In particular the quantity  $G(t)$  is non-increasing in time.

Assume now that the measure  $e^{-V}$  satisfies a Poincaré inequality with constant  $C > 0$ , that is,

$$\int \left( f - \int f e^{-V} \right)^2 e^{-V} dx \leq \frac{1}{C} \int |\nabla f|^2 e^{-V} dx \quad (2)$$

for all  $f$ . By choosing  $f = \mu_t/e^{-V} - 1$ , we obtain  $G'(t) \leq -2CG(t)$ . Hence

$$\int |\mu_t - e^{-V}|^2 e^V dx \leq e^{-2Ct} \int |\mu_0 - e^{-V}|^2 e^V dx, \quad t \geq 0 \quad (3)$$

by integration. In particular this ensures the strong convergence of  $\mu_t$  to  $e^{-V}$  in  $L^2(e^V)$  for any initial datum  $\mu_0$  in  $L^2(e^V)$ . In fact, (3) is equivalent to (2) by time-differentiating at  $t = 0$ .

Then simple criteria are known for a measure  $e^{-V}$  to satisfy the Poincaré inequality (2): for instance, (2) holds with constant  $C > 0$  if the Hessian matrix  $\nabla^2 V(x)$  is uniformly bounded by below by  $C \text{Id}_n$  (known as the Bakry-Émery criterion, see [ABC<sup>+</sup>00] for instance); more generally it holds for some  $C$  if  $V$  is convex, see for example [BBCG08]. The argument can also be performed for diverse convex functionals of the quantity  $\mu_t/e^{-V}$ , under the name of entropy method (see [AMTU01] for instance).

In fact the Poincaré inequality (2) implies the following stronger contraction property between any two solutions: if  $\mu_t$  and  $\nu_t$  are two solutions with initial data in  $L^2(e^V)$ , then we can apply (2) to  $f = (\mu_t - \nu_t)/e^{-V}$  to obtain the contraction property

$$\int |\mu_t - \nu_t|^2 e^V dx \leq e^{-2Ct} \int |\mu_0 - \nu_0|^2 e^V dx, \quad t \geq 0. \quad (4)$$

It implies (3) by letting  $\nu_0 = e^{-V}$ .

As a conclusion, the long time convergence estimate (3) is equivalent to the (seemingly stronger)  $L^2$ -contraction property (4) of two solutions, and to the Poincaré inequality (2).

Contraction results between solutions to (1) can also be measured in terms of Wasserstein distances. If  $\mu_1, \mu_2$  are two probability measures on  $\mathbb{R}^n$ , their Wasserstein distance is defined by

$$W_2(\mu_1, \mu_2) = \inf (\mathbb{E}|X - Y|^2)^{1/2},$$

where the infimum runs over all random variables  $X$  and  $Y$  with law respectively  $\mu_1$  and  $\mu_2$ . This distance metrizes a weak convergence (as opposed to the strong  $L^2$  convergence above), but has the advantage of being defined on the more natural space of probability measures on  $\mathbb{R}^n$ .

It is adapted to (1) since, by the Itô formula, a measure solution  $\mu_t$  to (1) can be seen as the law at time  $t$  of the process  $(X_t)_{t \geq 0}$  solution to the stochastic differential equation

$$dX_t = \sqrt{2} dB_t - \nabla V(X_t) dt. \quad (5)$$

Here  $(B_t)_{t \geq 0}$  is a standard Brownian motion in  $\mathbb{R}^n$  and the initial datum  $X_0$  is distributed according to  $\mu_0$ .

Let now  $\mu_0$  and  $\nu_0$  be two probability measures on  $\mathbb{R}^n$ , and consider  $(X_t)_{t \geq 0}$  (resp.  $(Y_t)_{t \geq 0}$ ) the solution to (5) starting from  $X_0$  of law  $\mu_0$  (resp.  $Y_0$  of law  $\nu_0$ ), both driven by the same Brownian motion. Then

$$\frac{d}{dt} |X_t - Y_t|^2 = -2 (\nabla V(X_t) - \nabla V(Y_t)) \cdot (X_t - Y_t).$$

Now, if  $V$  satisfies  $\nabla^2 V(x) \geq C \text{Id}_n$  for all  $x \in \mathbb{R}^n$  and for a  $C \in \mathbb{R}$ , that is,

$$(\nabla V(x) - \nabla V(y)) \cdot (x - y) \geq C |x - y|^2 \quad (6)$$

for all  $x, y \in \mathbb{R}^d$ , then

$$\frac{d}{dt} |X_t - Y_t|^2 \leq -2C |X_t - Y_t|^2.$$

This gives

$$\mathbb{E} |X_t - Y_t|^2 \leq e^{-2Ct} \mathbb{E} |X_0 - Y_0|^2$$

by integrating in time and taking the expectation. Then

$$W_2^2(\mu_t, \nu_t) \leq \mathbb{E} |X_t - Y_t|^2$$

since  $X_t$  and  $Y_t$  have respective laws  $\mu_t$  and  $\nu_t$ . Since moreover  $X_0$  and  $Y_0$  are any variable with respective laws  $\mu_0$  and  $\nu_0$  we can take the infimum over  $X_0$  and  $Y_0$  to obtain the following contraction between any two solutions :

$$W_2(\mu_t, \nu_t) \leq e^{-Ct} W_2(\mu_0, \nu_0), \quad t \geq 0. \quad (7)$$

Such a contraction estimate is a key estimate in the theory of gradient flows in the space of probability measures, an instance of which is (1) when  $A = \nabla V$  (see [AGS08]).

In particular, by choosing  $\mu_0$  as the stationary solution  $e^{-V}$  it implies the bound

$$W_2(\nu_t, e^{-V}) \leq e^{-Ct} W_2(\nu_0, e^{-V}), \quad t \geq 0 \quad (8)$$

for any initial condition  $\mu_0$ . For  $C > 0$  it ensures that  $e^{-V}$  is the only stationary state of (1) and quantifies the convergence of all solutions to it; it can be seen as a spectral gap in Wasserstein distance.

Of course (7) is a stronger statement than (8) since it enables to compare any two solutions, and not only a solution to the stationary one. But it asks for extremely strong assumptions on the drift: indeed, according to K.-T. Sturm and M. von Renesse, the uniform convexity condition (6) is in fact *equivalent* to (7); more generally when the vector field  $A$  is not necessary a gradient, then solutions of (1) satisfy (7) if and only if (6) holds with  $A$  instead of  $\nabla V$  (see [SvR05] and Section 4, and also [NPS11] for a duality proof of the sufficient condition).

The purpose of this work is twofold: First, to consider possibly non-gradient drifts  $A$ , which naturally appear for example in polymeric fluid flow or Wigner-Fokker-Planck equation (see

[JLBLO06] or [ACM10]). Such non gradient drifts forbid the gradient flow approach to (1), which holds only in the gradient case. Then, and overall, to give weaker conditions than (6) on the drift  $A$  for the uniform convergence estimate (8) to hold for solutions to (1). As for the  $L^2$ -norm and the Poincaré inequality, it will be described by a functional inequality called  $WJ$  inequality, which links the Wasserstein distance with its dissipation along the flow of the equation. As will be seen later on, an interesting fact is that it holds for potentials which are uniformly convex only at infinity. For that purpose we will use the diffusion term to overcome the possible degeneracy of the potential convexity in some region. We will see on examples how an a priori polynomial rate of convergence can simply be turned into an exponential rate by this method. To our knowledge this is the first quantitative use of the contribution of the diffusion term in measuring the convergence to equilibrium in Wasserstein distance.

In Section 1, we introduce the objects studied in the paper. In Section 2 we derive the Wasserstein distance dissipation along solutions to (1) when  $A$  is not necessarily a gradient. Then we introduce the  $WJ$  inequality which governs the uniform convergence (8). Section 3 is devoted to practical conditions to the  $WJ$  inequality and to its connections with classical functional inequalities as the Poincaré or logarithmic Sobolev inequalities. In Section 4 we consider a more general Fokker-Planck equation involving a non constant diffusion matrix: we formally derive a simple characterization of the contraction property (7) in terms of the coefficients of the equation.

Let us finish by some possible extension to nonlinear models. For example, contraction properties such as (7) also hold for nonlinear equations such as the granular media equation

$$\partial_t \mu_t = \nabla \cdot (\nabla \mu_t + \mu_t (\nabla V + \nabla W *_x \mu_t)), \quad t > 0, x \in \mathbb{R}^n$$

under hypothesis like (6) on the potentials  $V$  and  $W$  (see [CMV06]); here  $*_x$  stands for the convolution on  $\mathbb{R}^n$ . It is then natural to hope that we can go beyond this strict convexity assumption using our approach. In the forthcoming paper [BGG] we will precisely show that the method is sufficiently robust to extend this results to non-uniformly convex potentials.

## 1 Framework

We consider the Fokker-Planck equation starting from a probability measure  $\mu_0$ ,

$$\partial_t \mu_t = \nabla \cdot (\nabla \mu_t + \mu_t A) = \nabla \cdot (\mu_t (\nabla \log \mu_t + A)), \quad t > 0, x \in \mathbb{R}^n \quad (9)$$

where  $A$  is a  $\mathcal{C}^1$  function on  $\mathbb{R}^n$  and  $\nabla \cdot G$  is the divergence of a vector field  $G$ .

The existence of a non-explosive solution can be proven under simple conditions on  $A$ . For instance, if there exist  $a$  and  $b$  such that

$$x \cdot A(x) \geq -a|x|^2 - b$$

for all  $x$ , then for any initial datum  $\mu_0$  in the space  $\mathcal{P}_2(\mathbb{R}^n)$  of probability measures  $\rho$  on  $\mathbb{R}^n$  such that  $\int |x|^2 d\rho(x) < \infty$  there exists a continuous curve  $(\mu_t)_{t \geq 0}$  of probability measures such that (9) holds in the sense of distributions. One can also classically prove that for any  $t > 0$  a solution  $\mu_t$  has a smooth positive density with respect to the Lebesgue measure. See [Str08], [NPS11] and the references therein for instance.

Itô's formula implies that the law  $(\mu_t)_{t \geq 0}$  of the Markov process

$$dX_t = \sqrt{2}dB_t - A(X_t)dt, \quad (10)$$

where  $X_0$  has law  $\mu_0$  and  $(B_t)_{t \geq 0}$  is a Brownian motion on  $\mathbb{R}^n$ , is a solution to (9). Equation (9) is also called the Kolmogorov forward equation.

We assume that there exists a positive and smooth stationary solution  $e^{-V}$  of (9), which is a probability measure and where  $V$  is a  $C^2$  map on  $\mathbb{R}^n$ . Letting  $F = A - \nabla V$ , equation (9) reads

$$\partial_t \mu_t = \nabla \cdot (\mu_t (\nabla \log \mu_t + \nabla V + F)). \quad (11)$$

Here the vector field  $F$  satisfies  $\nabla \cdot (e^{-V} F) = 0$ , which is a necessary and sufficient condition for  $e^{-V}$  to be a stationary solution.

The generator  $L^*$  defined by  $L^* f = \Delta f + \nabla \cdot (f(\nabla V + F))$  for  $f$  a  $C^2$  map on  $\mathbb{R}^n$  is the dual operator in  $L^2(dx)$  of  $L$  defined by  $Lf = \Delta f - \nabla f \cdot (\nabla V + F)$ . Moreover  $L$  is the infinitesimal generator of the Markov semigroup  $(P_t)_{t \geq 0}$  defined by

$$P_t f(x) = E_x(f(X_t))$$

for any smooth function  $f$ ; here  $(X_t)_{t \geq 0}$  is the Markov process, solution of the stochastic differential equation (10), such that  $X_0 = x$ . In other words, the function  $P_t f$  solves the partial differential equation

$$\partial_t u = Lu, \quad (12)$$

with initial datum  $f$ .

If  $\mu_t$  is a solution to (11) then  $\varphi_t = e^V \mu_t$  satisfies the PDE

$$\partial_t \varphi_t = \Delta \varphi_t - \nabla \varphi_t \cdot (\nabla V - F). \quad (13)$$

Conversely, if  $\varphi_t$  is a smooth positive solution to (13) with initial datum  $\varphi_0$  such that  $\int \varphi_0 e^{-V} dx = 1$ , then

$$\mu_t = e^{-V} \varphi_t$$

for  $t \geq 0$  is a positive probability density which solves (11) with the initial datum  $\varphi_0 e^{-V}$ . The diffusion operator  $L^\top f = \Delta f - \nabla f \cdot (\nabla V - F)$  can now be seen as the infinitesimal generator of a Markov semigroup denoted  $(P_t^\top)_{t \geq 0}$ . It is the dual of  $L$  in  $L^2(d\mu)$ , where  $d\mu = e^{-V} dx$ , that is, for smooth functions  $f$  and  $g$

$$\int f Lg d\mu = \int g L^\top f d\mu.$$

Moreover, the measure  $d\mu = e^{-V} dx$  is an invariant measure for both generators  $L$  and  $L^\top$ , that is, for any  $f$

$$\int L^\top f d\mu = \int Lf d\mu = 0.$$

When  $A = \nabla V$  (or equivalently  $F = 0$ ), then (11) is the usual Fokker-Planck equation whereas the dual form (12) is the general Ornstein-Uhlenbeck equation. In that case  $L^\top = L$  and  $L$  is symmetric in  $L^2(d\mu)$ . We say that  $\mu$  is reversible and when  $\int e^{-V} dx < +\infty$ , up to a normalization constant it gives an explicit invariant probability measure.

The discrepancy between probability measures will mainly be estimated in terms of the Wasserstein distance: for two measures  $\mu$  and  $\nu$  in  $\mathcal{P}_2(\mathbb{R}^n)$  it is defined by

$$W_2(\mu, \nu) = \inf \left( \int_{\mathbb{R}^{2n}} |x - y|^2 d\pi(x, y) \right)^{1/2},$$

where the infimum runs over all probability measures  $\pi$  on  $\mathbb{R}^n \times \mathbb{R}^n$  with marges  $\mu$  and  $\nu$ , that is, for any bounded functions  $f$  and  $g$  on  $\mathbb{R}^n$

$$\int_{\mathbb{R}^{2n}} (f(x) + g(y)) d\pi(x, y) = \int_{\mathbb{R}^n} f d\mu + \int_{\mathbb{R}^n} g d\nu$$

(see [AGS08] or [Vil09] for example). This definition is of course the same as the one given in the introduction in terms of random variables.

Brenier's Theorem gives an explicit expression of the Wasserstein distance: if  $\mu$  is absolutely continuous with respect to the Lebesgue measure then there exists a convex function  $\varphi$  such that  $\nabla\varphi\#\mu = \nu$ , that is,

$$\int_{\mathbb{R}^n} g d\nu = \int_{\mathbb{R}^n} g(\nabla\varphi) d\mu$$

for every bounded test function  $g$ ; moreover

$$W_2^2(\mu, \nu) = \int_{\mathbb{R}^n} |x - \nabla\varphi(x)|^2 d\mu(x).$$

The Legendre transform will be useful for the next sections: for a map  $\varphi : \mathbb{R}^n \mapsto \mathbb{R} \cup \{\infty\}$  it is the map  $\varphi^* : \mathbb{R}^n \mapsto \mathbb{R} \cup \{\infty\}$  defined by

$$\varphi^*(q) = \sup_{x \in \mathbb{R}^n} \{q \cdot x - \varphi(x)\}.$$

If  $\mu$  and  $\nu$  are probability densities in  $\mathcal{P}_2(\mathbb{R}^n)$  such that  $\nabla\varphi\#\mu = \nu$ , then  $\nabla\varphi^*\#\nu = \mu$ .

## 2 Convergence in Wasserstein distance

Convergence in Wasserstein distance is related to its time-derivative, which was studied by L. Ambrosio, N. Gigli and G. Savaré in [AGS08, Th. 8.4.7] (see also [Vil09, Th. 23.9]).

If  $F$  is a  $C^1$  map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  then we let  $\nabla F$  be its Jacobian matrix and  $\nabla^S F = (\nabla F + \nabla F^T)/2$  its symmetric part.

For a probability measure  $\mu$  and a probability density  $h$  with respect to  $\mu$  we let

$$H(\nu|\mu) = \int h \log h d\mu, \quad I(\nu|\mu) = \int \frac{|\nabla h|^2}{h} d\mu \tag{14}$$

respectively be the entropy and the Fisher information of  $\nu = h\rho$  with respect to  $\mu$ .

**Theorem 2.1** ([AGS08]) *Assume that  $V, F$  are such that  $\int |F|^4 d\mu < \infty$ .*

*Let  $\nu_t$  be a measure solution of (11) with initial condition having a smooth density  $\nu_0$  such that*

$$\int \nu_0^2 e^V dx < \infty, \quad H(\nu_0|\mu) < \infty. \tag{15}$$

Then the map  $t \mapsto W_2(\nu_t, \mu)$  is absolutely continuous and for almost every  $t \geq 0$

$$\frac{1}{2} \frac{d}{dt} W_2^2(\nu_t, \mu) = \int (\nabla \psi_t - x) \cdot (\nabla \log \nu_t + A) d\nu_t \quad (16)$$

where for every  $t \geq 0$ ,  $\nabla \psi_t \# \nu_t = \mu$ .

Let us first give a direct and formal proof of this result. Brenier's Theorem implies that

$$W_2^2(\nu_t, \mu) = \int |\nabla \varphi_t(x) - x|^2 d\mu(x)$$

for all  $t \geq 0$ , where  $\nabla \varphi_t \# \mu = \nu_t$ . Then by formal time-differentiation

$$\frac{1}{2} \frac{d}{dt} W_2^2(\nu_t, \mu) = \int (\nabla \varphi_t(x) - x) \cdot \partial_t \nabla \varphi_t d\mu.$$

Now for all  $g$  the time-derivative of  $\int g(\nabla \varphi_t) d\mu = \int g d\nu_t$  is

$$\int \nabla g(\nabla \varphi_t) \cdot \partial_t \nabla \varphi_t d\mu = \int g \partial_t \nu_t.$$

For  $g(x) = \frac{|x|^2}{2} - \varphi_t^*(x)$ , which satisfies  $\nabla g(\nabla \varphi_t(x)) = \nabla \varphi_t(x) - x$  by Legendre transform properties, this gives

$$\frac{1}{2} \frac{d}{dt} W_2^2(\nu_t, \mu) = \int \left( \frac{|x|^2}{2} - \varphi_t^* \right) \partial_t \nu_t.$$

An integration by parts implies (16) with  $\psi_t = \varphi_t^*$ .

Observe moreover that  $\mu$  is an invariant measure with respect to the generator  $L$ , so

$$\frac{1}{2} \frac{d}{dt} W_2^2(\nu_t, \mu) = \int \left[ L \left( \frac{|x|^2}{2} - \psi_t^* \right) + L \left( \frac{|x|^2}{2} - \psi_t^* \right) (\nabla \psi_t^*) \right] d\mu. \quad (17)$$

This form will be useful for the rest of the paper.

**Proof**

◁ It is a direct application of [Vil09, Theorem 23.9] and we now check its assumptions.

First, the vector field  $\xi_t = \nabla \log \nu_t + \nabla V + F$  is locally Lipschitz since the solution  $\nu_t$  has a smooth and positive density on  $(0, \infty)$ . Let us now check that

$$\int_{t_1}^{t_2} \int |\xi_t|^2 d\nu_t dt < \infty$$

for every  $0 < t_1 < t_2$ . Indeed

$$\int |\xi_t|^2 d\nu_t = \int \left| \nabla \log \frac{\nu_t}{\mu} + F \right|^2 d\nu_t \leq 2 I(\nu_t | \mu) + 2 \int |F|^2 d\nu_t.$$

On the one hand, since  $\nabla \cdot (F e^{-V}) = 0$ ,

$$\int_{t_1}^{t_2} I(\nu_t | \mu) dt \leq \int_0^{t_2} I(\nu_t | \mu) dt = H(\nu_0 | \mu) - H(\nu_{t_2} | \mu) \leq H(\nu_0 | \mu).$$

As for the other term, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int |F|^2 d\nu_t &= \int |F|^2 \frac{\nu_t}{\mu} d\mu \leq \left( \int |F|^4 d\mu \right)^{1/2} \left( \int \left( \frac{\nu_t}{\mu} \right)^2 d\mu \right)^{1/2} \\ &\leq \left( \int |F|^4 d\mu \right)^{1/2} \left( \int \left( \frac{\nu_0}{\mu} \right)^2 d\mu \right)^{1/2}. \end{aligned}$$

The last two bounds imply

$$\int_{t_1}^{t_2} \int |\xi_t|^2 d\nu_t dt \leq 2H(\nu_0|\mu) + 2(t_2 - t_1) \left( \int |F|^4 d\mu \int \nu_0^2 e^V dx \right)^{1/2} < \infty.$$

▷

**Remark 2.2** *In the gradient flow case when  $F = 0$ , then (15) can be replaced by the sole condition  $H(\nu_0|\mu) < \infty$ , which is the classical assumption (see [AGS08]). Let us also notice that the coupled conditions  $\int |F|^4 d\mu < \infty$ ,  $\int \nu_0^2 e^V < \infty$  can be modified by using the Hölder or the Young inequality instead of the Cauchy-Schwarz inequality, and for instance be replaced by  $\int e^{F^2} d\mu < \infty$ ,  $H(\nu_0|\mu) < \infty$ .*

**Corollary 2.3** *Assume that  $d\mu = e^{-V} dx$  is a probability measure with  $\nabla \cdot (e^{-V} F) = 0$  and make the same hypotheses as in Theorem 2.1. Assume moreover that*

$$W_2^2(\nu, \mu) \leq \frac{1}{C} \int (x - \nabla\psi) \cdot (\nabla \log \nu + A) d\nu \quad (18)$$

for all  $\nu$ , where  $\nabla\psi \# \nu = \mu$ . Then

$$W_2(\nu_t, \mu) \leq e^{-Ct} W_2(\nu_0, \mu), \quad t \geq 0$$

for any solution  $(\nu_t)_t$  to (11) starting from a probability density  $\nu_0$  satisfying (15).

**Proof**

◁ It is a consequence of Theorem 2.1 since the map  $t \mapsto W_2(\nu_t, \mu)$  is absolutely continuous. ▷

If  $\psi$  is a  $C^2$  function then by (17) the inequality (18) becomes

$$W_2^2(\nu, \mu) \leq \frac{1}{C} \int [\Delta\varphi + \Delta\varphi^*(\nabla\varphi) - 2n + (A(\nabla\varphi) - A) \cdot (\nabla\varphi - x)] d\mu \quad (19)$$

where  $\varphi = \psi^*$  and  $\nabla\varphi \# \mu = \nu$ . This motivates the following definition:

**Definition 2.4** *We say that the couple  $(\mu, A)$ , where  $\mu$  is a probability measure and  $A$  is a vector field, satisfies a WJ inequality with constant  $C > 0$  if*

$$W_2(\nu, \mu) \leq \sqrt{\frac{1}{C} J(\nu|(\mu, A))} \quad (20)$$

for every probability measure  $\nu$ ; here

$$J(\nu|(\mu, A)) = \int [\Delta\varphi + \Delta\varphi^*(\nabla\varphi) - 2n + (A(\nabla\varphi) - A) \cdot (\nabla\varphi - x)] d\mu$$

where  $\nabla\varphi\#\mu = \nu$ . We implicitly assume in the definition that  $J(\nu|\mu, A)$  is well defined and non-negative.

For simplicity, if  $d\mu = e^{-V}dx$  and  $A = \nabla V$ , or equivalently  $F = 0$ , then  $J(\nu|\mu, A)$  is denoted  $J(\nu|\mu)$  and we say that the probability measure  $\mu$  satisfies a *WJ inequality*.

As pointed out above one can write

$$J(\nu|\mu, A) = \int (x - \nabla\psi) \cdot (\nabla \log \nu + A) d\nu$$

where  $\nabla\psi\#\nu = \mu$ ; this will be a useful formulation.

This definition is general, and by Corollary 2.3 the *WJ inequality* governs the uniform exponential convergence of solutions to (9) towards the equilibrium  $e^{-V}$  in the case when  $\nabla \cdot (e^{-V}F) = 0$ . In this case the quantity  $J(\nu_t|e^{-V}, A)$  is the dissipation of the squared Wasserstein distance  $W_2^2(\nu_t, e^{-V})$  between a solution  $\nu_t$  and the equilibrium  $e^{-V}$ .

A simple but key observation is the following

**Lemma 2.5** *If  $\varphi$  is a  $C^2$  strictly convex function on  $\mathbb{R}^n$  then for all  $x$*

$$\Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n \geq 0,$$

*and is 0 if and only if the Hessian matrix  $\nabla^2\varphi(x)$  at  $x$  is the identity matrix.*

**Proof**

$\triangleleft$  Given  $x \in \mathbb{R}^n$  we write  $\nabla^2\varphi(x)$  as  $ODO^*$  where  $O$  is orthonormal,  $D = \text{diag}(d_1, \dots, d_n)$  and  $d_i$  are the positive eigenvalues of  $\nabla^2\varphi(x)$ .

Observe that  $\nabla\varphi^*(\nabla\varphi(x)) = x$ , and then

$$\nabla^2\varphi^*(\nabla\varphi(x))\nabla^2\varphi(x) = \text{Id}_n.$$

This leads to

$$\nabla^2\varphi^*(\nabla\varphi(x)) = (\nabla^2\varphi(x))^{-1} = O D^{-1} O^*.$$

Then

$$\Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n = \sum_{i=1}^n d_i + \sum_{i=1}^n \frac{1}{d_i} - 2n = \sum_{i=1}^n \left( \sqrt{d_i} - \frac{1}{\sqrt{d_i}} \right)^2 \geq 0,$$

with equality if and only if the  $d_i$  are all equal to 1.  $\triangleright$

If  $A$  is *monotone*, that is, if

$$(A(x) - A(y)) \cdot (x - y) \geq 0$$

for all  $x, y$ , then by Lemma 2.5 the quantity  $J(\nu|\mu, A)$  is non-negative for all  $\nu = \nabla\varphi\#\mu$ .

We do not know whether this is also the case in the general case. Observe that along the evolution of the Fokker-Planck equation, the dissipation of the relative entropy to the steady state, and more generally of relative  $\varphi$ -entropies with  $\varphi$  convex, is non-negative; this is however not always the case for the Fisher information, as pointed out by B. Helffer (see [ABC<sup>+</sup>00]).

**Remark 2.6** *In this work we focus on the estimate (8) in the Euclidean Wasserstein distance and give simple necessary and sufficient conditions (weaker than strictly positive curvature) on the drift for (8) to hold for any initial condition  $\mu_0$ .*

*Let us stress that in our study, it is important that there is no (larger than 1) multiplicative constant on the right-hand side of (8). Indeed, there are various ways to get convergence result of the form*

$$W_2(\mu_t, \mu) \leq K e^{-Ct} W_2(\mu_0, \mu) \quad (21)$$

*for a constant  $K$  larger than 1. Let us mention two different approaches.*

*i. Suppose that  $\mu$  satisfies a logarithmic Sobolev inequality with constant  $C$ , that is*

$$H(f\mu|\mu) \leq \frac{1}{2C} I(f\mu|\mu) \quad (22)$$

*for all probability densities  $f$  with respect to  $\mu$ . This inequality can be proved in infinite negative curvature cases and is equivalent to the exponential decay of the entropy*

$$H(\mu_t|\mu) \leq e^{-2C(t-t_0)} H(\mu_{t_0}|\mu).$$

*Recall then that such a logarithmic Sobolev inequality implies a Talagrand inequality with constant  $C$ , that is,*

$$W_2(\nu, \mu) \leq \sqrt{\frac{2}{C} H(\nu|\mu)} \quad (23)$$

*for all  $\nu$  (see for example [OV00]). Hence*

$$W_2(\mu_t, \mu) \leq K(C, t_0) e^{-Ct} \sqrt{\text{Ent}_\mu(\mu_{t_0})} \leq \tilde{K}(V, C, t_0) e^{-Ct} W_2(\mu, \mu_0)$$

*for all  $t$ . The last inequality follows from a regularization argument derived from a Harnack type inequality under regularity assumptions on  $V$  (see [Wan04]).*

*ii. Another approach relies on the study of the contraction in a Wasserstein distance for a twisted metric, equivalent to the Euclidean one, so that such a contraction result will lead to convergence in the Euclidean Wasserstein distance as in (21), with a  $K > 1$ . This has been successfully done for the kinetic Fokker-Planck equation in a perturbation of the Gaussian case (infinite curvature case) in [BGM10] using the simplest of coupling (same Brownian motion for the two different dynamics, as in the introduction). Recently, A. Eberle [Ebe11] has used reflection coupling to establish contraction results in a twisted metric for a reversible Fokker-Planck equation under lower negative curvature and sufficient quadratic growth condition at infinity.*

## 3 The $WJ$ inequality

### 3.1 Sufficient conditions

We first begin with the following straightforward consequence of Lemma 2.5:

**Proposition 3.1** *If  $\mu$  is a probability measure on  $\mathbb{R}^n$  and  $A$  such that*

$$\nabla^S A \geq C \text{Id}_n \quad (24)$$

*with  $C > 0$ , uniformly on  $\mathbb{R}^n$ , then  $(\mu, A)$  satisfies a  $WJ(C)$  inequality.*

In particular the standard Gaussian measure  $\gamma$  on  $\mathbb{R}^n$  satisfies a  $WJ$  inequality with constant 1 and the constant 1 is optimal. Observe indeed that

$$J(\nu|\gamma) = \int (\Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n)d\gamma(x) + W_2^2(\nu, \gamma)$$

for all  $\nu = \nabla\varphi\#\gamma$ . Hence it is always larger than  $W_2^2(\nu, \gamma)$  by Lemma 2.5; moreover it is equal to  $W_2^2(\nu, \gamma)$  if and only if the non-negative term  $\Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n$  is 0 for almost every  $x$ , that is, if and only if  $\nabla^2\varphi(x) = 0$ , by Lemma 2.5, that is, if and only if  $\nu$  is a translation of  $\gamma$ .

For uniformly convex potentials  $V$ , or more generally under (24), the  $WJ$  inequality for  $(\mu, A)$  is obtained without using the non-negative contribution  $\Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n$  in  $J$ , which stems from the diffusion term. Let us now see how the diffusion term enables to obtain a  $WJ$  inequality in non-uniformly convex cases and even non-convex cases.

A first idea is to use the entropy  $H$ . It is related to  $W_2$  and  $J$  as follows.

**Lemma 3.2** *Let  $d\mu = e^{-V} dx$  be a probability measure and  $F$  a vector field such that  $\nabla \cdot (e^{-V} F) = 0$ . If  $\nabla^2 V \geq \lambda_1 \text{Id}_n$  and  $\nabla^S F \geq \lambda_2 \text{Id}_n$  for some  $\lambda_1, \lambda_2 \in \mathbb{R}$ , uniformly in  $\mathbb{R}^n$ , then*

$$H(\nu|\mu) + \left(\frac{\lambda_1}{2} + \lambda_2\right)W_2^2(\nu, \mu) \leq J(\nu|(\mu, A)) \quad (25)$$

for every probability measure  $\nu$ .

**Proof**

◁ We follow the proof of Theorem 1 in [CE02]. Let  $\nu$  be a probability on  $\mathbb{R}^n$  with a smooth positive density  $f$  with respect to  $\mu$ . If  $\nabla\psi\#\nu = \mu$  then, by change of variables,

$$f e^{-V} = e^{-V(\nabla\psi)} \det(\nabla^2\psi).$$

Then

$$\begin{aligned} \int f \log f d\mu &= \int [V - V(\nabla\psi) + \log \det(\nabla^2\psi)] f d\mu \leq \int [V - V(\nabla\psi) + \Delta(\psi - \frac{|x|^2}{2})] f d\mu \\ &= \int [V - V(\nabla\psi) + \nabla V \cdot (\nabla\psi - x)] f d\mu - \int (\nabla\psi - x) \cdot \nabla f d\mu \end{aligned}$$

by convexity and integration by parts. Moreover

$$J(\nu|(\mu, A)) = \int (x - \nabla\psi) \cdot \nabla f d\mu + \int (x - \nabla\psi) \cdot F f d\mu$$

and

$$\int (x - \nabla\psi) \cdot F(\nabla\psi) d\nu = \int (\nabla\psi^* - x) \cdot F d\mu = - \int (\psi^* - \frac{|x|^2}{2}) \nabla \cdot (e^{-V} F) = 0$$

since  $\nabla\psi\#\nu = \mu$  and  $\nabla \cdot (e^{-V} F) = 0$ . Hence

$$H(\nu|\mu) = \int f \log f d\mu \leq \int [V - V(\nabla\psi) + \nabla V \cdot (\nabla\psi - x) - (F - F(\nabla\psi)) \cdot (x - \nabla\psi)] d\nu + J(\nu, (A, \mu)).$$

Now, by a Taylor expansion,

$$V - V(\nabla\psi) + \nabla V \cdot (\nabla\psi - x) = - \int_0^1 (1-t) (\nabla\psi - x) \cdot [\nabla^2 V(x + t(\nabla\psi - x)) (\nabla\psi - x)] dt \leq - \frac{\lambda_1}{2} |\nabla\psi - x|^2$$

and

$$-(F - F(\nabla\psi)) \cdot (x - \nabla\psi) = - \int_0^1 (\nabla\psi - x) \cdot [\nabla^S F(x + t(\nabla\psi - x))(\nabla\psi - x)] dt \leq -\lambda_2 |\nabla\psi - x|^2.$$

This concludes the argument by combining the two expressions.  $\triangleright$

**Remark 3.3** When  $F = 0$ , then inequality (25) has been derived in [OV00] in the proof of the HWI inequality

$$H(\nu|\mu) \leq W_2(\nu, \mu) \sqrt{I(\nu|\mu)} - \frac{\lambda_1}{2} W_2^2(\nu, \mu),$$

where  $I(\nu|\mu)$  is the Fisher information of  $\nu$  with respect to  $\mu$ , defined in (14). It implies the HWI inequality since

$$J(\nu|\mu) = \int (x - \nabla\psi) \cdot \nabla \log \frac{\nu}{\mu} d\nu \leq \sqrt{\int |x - \nabla\psi|^2 d\nu} \sqrt{\int \left| \nabla \log \frac{\nu}{\mu} \right|^2 d\nu} = W_2(\nu, \mu) \sqrt{I(\nu|\mu)}$$

by the Cauchy-Schwarz inequality; here  $\nabla\psi \# \nu = \mu$ .

Moreover, again for  $F = 0$ , inequality (25) appears in [AGS08] as a fundamental inequality in the general theory of gradient flows, see [AGS08, Th. 4.0.4] for instance.

As in (23), a measure  $\mu$  is said to satisfy a (transportation) Talagrand inequality with constant  $C > 0$ , denoted  $WH(C)$ , if

$$W_2(\nu, \mu) \leq \sqrt{\frac{2}{C} H(\nu|\mu)}$$

for all measure  $\nu$  absolutely continuous with respect to  $\mu$ .

**Proposition 3.4** Assume that the measure  $\mu = e^{-V}$  satisfies a  $WH(C)$  inequality and that  $\nabla^2 V(x) \geq \lambda_1 \text{Id}_n$ ,  $\nabla^S A(x) \geq \lambda_2 \text{Id}_n$  with  $\lambda_1, \lambda_2 \in \mathbb{R}$  and all  $x$ . Then it satisfies a  $WJ((C + \lambda_1 + 2\lambda_2)/2)$  inequality if  $C > -\lambda_1 - 2\lambda_2$ .

For instance, when  $A = \nabla V$  and  $\lambda_1 = C > 0$ , this result includes the uniformly convex case as in Proposition 3.1, with the right constant  $C$ . Larger classes of measures  $\mu$  satisfying a  $WH$  inequality are described in [GL10], including for example potentials  $V$  which are the sum of a uniformly convex and of a bounded function.

**Proof**

$\triangleleft$  By Lemma 3.2 and assumptions,

$$-J(\nu|(\mu, A)) + \left(\frac{\lambda_1}{2} + \lambda_2\right) W_2^2(\nu, \mu) \leq -H(\nu|\mu) \leq -\frac{C}{2} W_2^2(\nu, \mu)$$

for any measure  $\nu$ . This concludes the argument.  $\triangleright$

For a non-gradient drift, there is a strong assumption on the measure  $\mu$  in Proposition 3.4, which is not always easy to be checked since  $\mu$  may not be explicit. We can replace it by another criterium, which asks for weaker assumptions on  $\mu$ , for instance:

**Proposition 3.5** *Let  $A$  be a  $C^1$  monotone map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  for which there exist two constants  $R \geq 0$  and  $K > 0$  such that*

$$\nabla^S A(x) \geq K$$

*for all  $|x| \geq R$ , and let  $d\mu = e^{-V}$  be a probability measure on  $\mathbb{R}^n$ , with  $V$  continuous.*

*Then  $(\mu, A)$  satisfies a  $WJ$  inequality with constant  $C = C(V, R, K)$ .*

**Remark 3.6** *The constant  $C$  given by the proof depends on  $V$  only through its minima and maxima on the ball of center 0 and radius  $3R$ . Observe that the proof requires only  $V$  to be bounded on this ball, and that any ball of center 0 and radius  $> R$  would work.*

The proof consists in overcoming the lack of convexity near the origin by using the diffusion term. It will be given in Section 5.

Let us see the influence of the diffusion term on the rate of convergence to equilibrium on a simple example, for instance for the potential  $V(x) = x^4$  on  $\mathbb{R}$  and  $F = 0$ . By Proposition 3.4 or 3.5, the measure  $e^{-V}$  satisfies a  $WJ(C)$  inequality, whence solutions  $\mu_t$  to the Fokker-Planck equation (9) converge exponentially fast to it, according to  $W_2^2(\mu_t, e^{-V}) \leq e^{-Ct} W_2^2(\mu_0, e^{-V})$ . On the other hand, without diffusion, the solution at time  $t$  to  $\partial_t \mu_t = \nabla \cdot (\mu_t \nabla V)$  is the distribution of the points  $x(t)$  initially at  $x(0)$  drawn according to  $\mu_0$  and evolving according to  $x'(t) = -x^3$ . This solves into  $x(t)^2 = x(0)^2 / (1 + 2tx(0)^2)$ , so that the solution  $\mu_t$  converges to the unique steady state  $\delta_0$  according to

$$W_2^2(\mu_t, \delta_0) = \int \frac{x^2}{1 + 2tx^2} d\mu_0(x) \sim \frac{1}{t}$$

for large  $t$ .

## 3.2 Tensorization and perturbation

Fundamental properties of functional inequalities lie in the range of stability: non dependence on the dimension, which enables to consider problems in infinite dimension, and stability by perturbation, which enables to reach more general potentials.

The following two results are important to extend the practical conditions we just derived. The first one concerns the tensorization : namely, the product of measures satisfying a  $WJ$  inequality also satisfies a  $WJ$  inequality.

**Proposition 3.7 (Tensorization)** *Suppose that the measures and drifts  $(\mu_i, A_i)_{1 \leq i \leq n}$  satisfy a  $WJ(C_i)$  inequality on  $\mathbb{R}^{n_i}$  respectively. Then  $(\otimes_{i=1}^n \mu_i, A)$  with  $A(x) = (A_i(x_i))_{1 \leq i \leq n}$  for  $x = (x_i)_i$  on the product space satisfies a  $WJ$  inequality with constant  $\min_i C_i$ .*

### **Proof**

$\triangleleft$  Let us assume for simplicity of notation that  $n_i = 1$  for all  $i$ , and let us denote  $d\mu^n(x) = \otimes_{i=1}^n d\mu_i(x_i) dx_i$ . For  $x \in \mathbb{R}^n$  we let  $\hat{x}_i \in \mathbb{R}^{n-1}$  have the same coordinate than  $x$ , but the  $i$ -th coordinate  $x_i$ , which is removed.

Let now  $\varphi$  be a  $C^2$  strictly convex function on  $\mathbb{R}^n$ . Noticing that all its restrictions  $x_i \mapsto \varphi(\hat{x}_i, x_i)$  are also  $C^2$  strictly convex functions on  $\mathbb{R}$ , and using the  $WJ$  inequality for each  $\mu_i$  we

get

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\varphi(x) - x|^2 d\mu^n(x) &= \sum_{i=1}^n \int_{\mathbb{R}^{n-1}} \otimes_{j \neq i} d\mu_j(\hat{x}_i) \int_{\mathbb{R}} |\partial_i\varphi(x) - x_i|^2 d\mu_i(x_i) \\
&\leq \frac{1}{\min_i C_i} \sum_{i=1}^n \int \otimes_{j \neq i} d\mu_j(\hat{x}_i) \int (A_i(\partial_i\varphi(x)) - A_i(x_i))(\partial_i\varphi(x) - x_i) d\mu_i(x_i) \\
&\quad + \frac{1}{\min_i C_i} \sum_{i=1}^n \int \otimes_{j \neq i} d\mu_j(\hat{x}_i) \int \left( \partial_{ii}^2\varphi(x) + \frac{1}{\partial_{ii}^2\varphi(x)} - 2 \right) d\mu_i(x_i) \\
&\leq \frac{1}{\min_i C_i} \int_{\mathbb{R}^n} \sum_{i=1}^n (A_i(\partial_i\varphi(x)) - A_i(x_i))(\partial_i\varphi(x) - x_i) d\mu^n(x) \\
&\quad + \frac{1}{\min_i C_i} \int_{\mathbb{R}^n} \sum_{i=1}^n \left( \partial_{ii}^2\varphi(x) + \frac{1}{\partial_{ii}^2\varphi(x)} - 2 \right) d\mu^n(x).
\end{aligned}$$

Now, in the first term,

$$\sum_{i=1}^n (A_i(\partial_i\varphi(x)) - A_i(x_i))(\partial_i\varphi(x) - x_i) = (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x).$$

In the second term we fix  $x \in \mathbb{R}^n$  and, in the notation of Lemma 2.5, we write  $\nabla^2\varphi(x) = ODO^*$  where  $O$  is orthonormal and  $D = \text{diag}(d_1, \dots, d_n)$ . Then

$$\sum_{i=1}^n \partial_{ii}^2\varphi(x) = \text{tr}(\nabla^2\varphi(x)) = \sum_{i=1}^n d_i.$$

Moreover  $\partial_{ii}\varphi(x) = \sum_{j=1}^n O_{ij}^2 d_j$  with  $\sum_{i=1}^n O_{ij}^2 = 1$ , and  $x \mapsto x^{-1}$  is convex on  $\{x > 0\}$ , so by the Jensen inequality

$$\sum_{i=1}^n \frac{1}{\partial_{ii}^2\varphi(x)} = \sum_{i=1}^n \frac{1}{\sum_{j=1}^n O_{ij}^2 d_j} \leq \sum_{i=1}^n \sum_{j=1}^n O_{ij}^2 \frac{1}{d_j} = \sum_{j=1}^n \sum_{i=1}^n O_{ij}^2 \frac{1}{d_j} = \sum_{j=1}^n \frac{1}{d_j}$$

since also  $\sum_{j=1}^n O_{ji}^2 = 1$ . Hence

$$\sum_{i=1}^n \left( \partial_{ii}^2\varphi(x) + \frac{1}{\partial_{ii}^2\varphi(x)} - 2 \right) \leq \sum_{i=1}^n \left( d_i + \frac{1}{d_i} - 2 \right) = \Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n$$

as in the proof of Lemma 2.5. This concludes the proof.  $\triangleright$

Let us come back to the PDE motivation of the WJ inequality : letting  $d\mu_i(x_i) = e^{-V_i(x_i)} dx_i$  for each  $i$ , then  $\nabla \cdot ((A - \nabla V)(x) e^{-V(x)}) = 0$  on the product space with  $V(x) = \sum_{i=1}^n V_i(x_i)$  for  $x = (x_i)_i$  as soon as  $\nabla \cdot ((A_i - \nabla V_i)(x_i) e^{-V_i(x_i)}) = 0$  on  $\mathbb{R}^{n_i}$  for each  $i$ . Hence  $e^{-V} dx = \otimes_{i=1}^n e^{-V_i(x_i)} dx_i$  is indeed a stationary measure of the corresponding PDE on the product space if so is each  $e^{-V_i(x_i)} dx_i$  on  $\mathbb{R}^{n_i}$ .

The second result is about the perturbation of the measure  $\mu$ . For classical functional inequalities, such as Poincaré inequality,  $WH$  or logarithmic Sobolev inequality, perturbations by bounded potentials are allowed (see for example [ABC<sup>+</sup>00, GL10]). Here we have to be more restrictive, not only on the perturbation term but also on the initial measure satisfying a  $WJ$  inequality.

**Proposition 3.8 (Perturbation)** *Suppose that the measure and drift  $(\mu, A)$  satisfy a WJ(C) inequality and that for an  $\alpha \leq 0$*

*i.  $(A(y) - A(x)) \cdot (y - x) \geq \alpha|y - x|^2$  for all  $x, y$ .*

*Consider a map  $T$  on  $\mathbb{R}^n$  such that  $e^{-T}d\mu$  is a probability measure and for a  $K \geq 0$*

*ii.  $|T(x)| \leq K$  for all  $x$ ,*

*and a map  $B$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  such that for a  $\beta \in \mathbb{R}$*

*iii.  $(B(y) - B(x)) \cdot (y - x) \geq \beta|x - y|^2$  for all  $x, y$ .*

*If  $-\beta e^{2K} - \alpha(e^{2K} - 1) < C$ , then  $(e^{-T}\mu, A+B)$  satisfies a WJ inequality with constant  $Ce^{-2K} + \beta + \alpha(1 - e^{-2K})$ .*

Note that

**Proof**

$\triangleleft$  Let  $\tilde{\mu} = e^{-T}\mu$ , and let  $\varphi$  be a  $C^2$  strictly convex map on  $\mathbb{R}^n$ . Then

$$\begin{aligned} \int |\nabla\varphi(x) - x|^2 d\tilde{\mu}(x) &\stackrel{ii.}{\leq} e^K \int |\nabla\varphi(x) - x|^2 d\mu \\ &\stackrel{WJ}{\leq} \frac{e^K}{C} \int (A(\nabla\varphi) - A) \cdot (\nabla\varphi - x) d\mu \\ &\quad + \frac{e^K}{C} \int (\Delta\varphi + \Delta\varphi^*(\nabla\varphi) - 2n) d\mu. \end{aligned} \tag{26}$$

Since  $\Delta\varphi(x) + \Delta\varphi^*(\nabla\varphi(x)) - 2n \geq 0$  by Lemma 2.5, the second integral on the right-hand side of (26) is bounded by

$$\frac{e^{2K}}{C} \int (\Delta\varphi + \Delta\varphi^*(\nabla\varphi) - 2n) d\tilde{\mu}$$

by *ii.* Moreover, by *i.* and *ii.*, we write the first integral on the right-hand side of (26) as

$$\begin{aligned} &\int [(A(\nabla\varphi) - A) \cdot (\nabla\varphi - x) - \alpha|\nabla\varphi - x|^2] d\mu + \alpha \int |\nabla\varphi - x|^2 d\mu \\ &\leq e^K \int (A(\nabla\varphi) - A) \cdot (\nabla\varphi - x) d\tilde{\mu} - \alpha(e^K - e^{-K}) \int |\nabla\varphi - x|^2 d\tilde{\mu}. \end{aligned}$$

Then, by *iii.*, we bound the first integral on the above right-hand side by

$$\int ((A+B)(\nabla\varphi) - (A+B)(x)) \cdot (\nabla\varphi - x) d\tilde{\mu} - \beta \int |\nabla\varphi - x|^2 d\tilde{\mu}.$$

This concludes the proof by collecting all terms and using the positivity conditions on the coefficients.  $\triangleright$

Typically  $A = \nabla V$  with  $\mu = e^{-V}$  and the bounded perturbation is given by  $B = \nabla T$ . Note also that one can adapt the proof above to give a variant of this result for  $\alpha > 0$ .

### 3.3 Necessary conditions

We conclude this section by comparing the  $WJ$  inequality for a measure  $\mu = e^{-V}$  and a drift  $A$  with more classical inequalities.

We first prove that a  $WJ$  inequality implies a Poincaré inequality:

**Proposition 3.9** *If  $(\mu, A)$  satisfies a  $WJ(C)$  inequality then  $\mu$  satisfies a Poincaré inequality with the same constant  $C$ , that is, for every smooth function  $f$*

$$\int \left( f - \int f d\mu \right)^2 d\mu \leq \frac{1}{C} \int |\nabla f|^2 d\mu.$$

**Proof**

◁ Let  $f$  be a smooth map on  $\mathbb{R}^n$  and  $\varphi$  be defined by

$$\varphi(x) = \frac{|x|^2}{2} + \varepsilon f(x)$$

for small  $\varepsilon$ . Then for all  $x$  the Hessian matrices of  $\varphi$  and  $f$  and their respective eigenvalues  $d_i$  and  $f_i$  for  $1 \leq i \leq n$  satisfy

$$\nabla^2 \varphi(x) = \text{Id}_n + \varepsilon \nabla^2 f(x), \quad d_i = 1 + \varepsilon f_i.$$

Hence, as in Lemma 2.5,

$$\begin{aligned} \Delta \varphi^*(\nabla \varphi(x)) + \Delta \varphi(x) - 2n &= \sum_{i=1}^n \left( \frac{1}{d_i} + d_i - 2 \right) = \sum_{i=1}^n \left( \frac{1}{1 + \varepsilon f_i} + 1 + \varepsilon f_i - 2 \right) \\ &= \varepsilon^2 \sum_{i=1}^n f_i^2 + o(\varepsilon^2) = \varepsilon^2 \|\nabla^2 f\|_{HS} + o(\varepsilon^2). \end{aligned}$$

Moreover

$$\nabla V(\nabla \varphi(x)) - \nabla V(x) = \varepsilon \nabla^2 V(x) \nabla f(x) + o(\varepsilon).$$

Hence, for this map  $\varphi$ , the  $WJ$  inequality now reads

$$\varepsilon^2 \int [\|\nabla^2 f(x)\|_{HS} + \nabla f(x) \cdot \nabla^2 V(x) \nabla f(x)] d\mu(x) + o(\varepsilon^2) \geq \varepsilon^2 C \int |\nabla f(x)|^2 d\mu$$

where  $\|M\|_{HS}$  is the Hilbert-Schmidt norm of a matrix  $M$ . Letting  $\varepsilon \rightarrow 0$ , we recover the well-known integral  $\Gamma_2$  criterion (see for example [ABC<sup>+</sup>00, Prop. 5.5.4]), which is equivalent to the Poincaré inequality with constant  $C$ . ▷

We now turn to the  $WI$  inequality in the particular case  $A = \nabla V$ .

An inequality looking like  $WJ$  has been introduced in [OV00] and studied in [GLWY09, GLWW09] for its equivalence to deviation inequalities for integral functional of Markov processes: thus it has high practical interest. We say that a probability measure  $\mu$  satisfies a  $WI$  inequality with constant  $C > 0$  (called  $LSI + T(C)$  in [OV00]) if for every probability measure  $\nu$  absolutely continuous with respect to  $\mu$

$$W_2(\nu, \mu) \leq \frac{1}{C} \sqrt{I(\nu|\mu)}.$$

Here  $I(\nu|\mu)$  is the Fisher information of  $\nu$  with respect to  $\mu$  defined in (14).

If  $\mu$  satisfies a  $WJ(C)$  inequality then

$$W_2^2(\nu, \mu) \leq \frac{1}{C} J(\nu|\mu) \leq \frac{1}{C} W_2(\nu, \mu) \sqrt{I(\nu|\mu)}$$

by the Cauchy-Schwarz inequality, as in Remark 3.3:

**Proposition 3.10** *A  $WJ$  inequality implies a  $WI$  inequality with the same constant.*

Let us now examine the link with the Talagrand (23) and logarithmic Sobolev (22) inequalities.

**Corollary 3.11** *1) A  $WJ$  inequality implies a  $WH$  inequality with the same constant.*

*2) Assume that the probability measure  $\mu = e^{-V} dx$  satisfies a  $WJ(C)$  inequality and  $\nabla^2 V \geq \rho \text{Id}_n$ , for some  $\rho \in \mathbb{R}$ . Then  $\mu$  satisfies a logarithmic Sobolev with constant  $C \left(1 + \frac{\max(0, -\rho)}{2C}\right)^{-2}$ .*

**Proof**

$\triangleleft$  1) By [GLWW09, Theorem 2.4], a  $WI$  inequality implies a  $WH$  inequality with the same constant, so that the result comes from Proposition 3.10.

2) By [OV00, Theorem 2], the following  $HWI$  inequality holds: for all  $\nu$

$$H(\nu|\mu) \leq \sqrt{I(\nu|\mu)} W_2(\nu, \mu) + \frac{\rho_-}{2} W_2^2(\nu, \mu) = W_2(\nu, \mu) \left( \sqrt{I(\nu|\mu)} + \frac{\rho_-}{2} W_2(\nu, \mu) \right).$$

Here  $\rho_- = \max(0, -\rho)$ . As a  $WJ(C)$  inequality implies both  $WH(C)$  and  $WI(C)$  inequalities, we get

$$H(\nu|\mu) \leq \sqrt{\frac{2}{C} H(\nu|\mu)} \left(1 + \frac{\rho_-}{2C}\right) \sqrt{I(\nu|\mu)}$$

which ends the proof.

$\triangleright$

Observe that in the uniformly convex case when  $\rho > 0$ , then  $\mu$  classically satisfies all  $WJ$ ,  $WI$ ,  $WH$  and logarithmic Sobolev inequalities with constant  $C$ . Moreover, under the assumption 2) with  $C \geq \max(\rho, 0)$ , then [OV00, Corollary 3.2] ensures a log Sobolev inequality with constant  $C(2 - \rho/C)^{-1}$ : for instance for  $\rho \leq 0$  it is worse than our constant (since then  $C \geq |\rho|/2$ ).

Remark also, by [GLWY09] and [OV00], that a  $WI(C)$  or a  $WH(C)$  inequality imply a Poincaré inequality with the same constant, hence providing an alternative proof to Proposition 3.9.

Observe finally that the general bound

$$W_2(\nu_0, \mu) - W_2(\nu_t, \mu) \leq t^{1/2} H(\nu_0|\mu)^{1/2}$$

was obtained in [CG06, Remark 4.9] for all  $t$  and solutions  $(\nu_t)_t$  to (1), hence directly proving that a uniform decay of the Wasserstein distance as in (8) implies a  $WH$  inequality with constant

$$\sup_{t>0} 2 \frac{(1 - e^{-Ct})^2}{t} = 2C \sup_{x>0} \frac{(1 - e^{-x})^2}{x} \sim 0.8C \text{ instead of } C, \text{ which is optimal.}$$

We do not know whether a logarithmic Sobolev inequality, which implies a  $WI$  inequality, also implies a  $WJ$  inequality, or whether the converse holds without the curvature condition of Corollary 3.11.

## 4 Contraction in Wasserstein distance

We saw in the introduction that the condition

$$(A(y) - A(x)) \cdot (y - x) \geq C|y - x|^2$$

for all  $x, y$  is a sufficient condition for measure solutions  $(\mu_t)_t$  and  $(\nu_t)_t$  to (9) to satisfy the contraction property

$$W_2(\mu_t, \nu_t) \leq e^{-Ct} W_2(\mu_0, \nu_0), \quad t \geq 0. \quad (27)$$

In fact it is a necessary condition. Indeed K.-T. Sturm and M. von Renesse have proven in [SvR05] that a Riemannian manifold has Ricci curvature bounded from below by  $C \in \mathbb{R}$  if and only if solutions to the heat equation satisfy (27), where  $W_2$  is defined by means of the Riemannian distance. This result has been extended by F.-Y. Wang to include an additional drift, and our case in particular (see [Wan04, Theorem 5.6.1]) :

**Theorem 4.1 ([SvR05])** *The following assertions are equivalent :*

1) *For all initial conditions  $\mu_0$  and  $\nu_0$  in  $\mathcal{P}_2(\mathbb{R}^n)$ , for all  $t \geq 0$ ,*

$$W_2(\mu_t, \nu_t) \leq e^{-Ct} W_2(\mu_0, \nu_0),$$

where  $\mu_t$  (resp.  $\nu_t$ ) are solutions of (9) starting from  $\mu_0$  (resp.  $\nu_0$ ).

1') *For all  $x, y \in \mathbb{R}^n$  and all  $t \geq 0$ ,*

$$W_2(\mu_t, \nu_t) \leq e^{-Ct} |x - y|,$$

where  $\mu_t$  (resp.  $\nu_t$ ) are solutions of (9) starting from  $\delta_x$  (resp.  $\delta_y$ ).

2) *For all  $x, y \in \mathbb{R}^n$  the vector field  $A$  satisfies*

$$(A(y) - A(x)) \cdot (y - x) \geq C|y - x|^2.$$

Consider now a general Fokker-Planck equation

$$\partial_t \mu_t = \nabla \cdot (\nabla(\mu_t D(x)) + \mu_t A(x)), \quad t > 0, x \in \mathbb{R}^n. \quad (28)$$

where  $D(x)$  is a positive symmetric matrix. Then one can equip  $\mathbb{R}^n$  with a metric, given by  $D(x)$ , and for which (28) reads as (9) (for another  $A$ ), and then use Wang's result to characterize (27), for the distance  $W_2$  associated with this metric, in terms of  $D$  and  $A$ .

We now use tools introduced in Section 2 and formally derive a simple characterization in terms of  $D$  and  $A$  of the contraction property (27) between solutions to (28), now for the usual distance  $W_2$  associated with the Euclidean norm on  $\mathbb{R}^n$ . As a particular case, it gives a formal proof of Theorem 4.1.

Here  $D(x)$  will be a non-negative symmetric matrix and, if  $M(x) = (M_{ij}(x))$  is an  $(n, n)$  matrix, then  $\nabla M$  is the vector in  $\mathbb{R}^n$  with  $j$ -th coordinate equal to  $\sum_{i=1}^n \partial_i M_{ij}(x)$ .

In the whole section matrices are meant to have size  $(n, n)$ . If  $M = (M_{ij})$  and  $N = (N_{ij})$  are two matrices we let  $M : N = \sum_{i,j=1}^n M_{ij} N_{ij}$ ,  $\|M\|_{HS}^2 = M : M$  be the squared Hilbert-Schmidt norm of  $M$  and  $M^*$  be its transposed matrix.

**Proposition 4.2** *Formally, in the above notation, if*

$$(A(x) - A(y)) \cdot (x - y) \geq C|x - y|^2 + \inf \{ \|\sigma(x) - \sigma(y)\|_{HS}^2; \sigma(x)\sigma(x)^* = D(x), \sigma(y)\sigma(y)^* = D(y) \} \quad (29)$$

for all  $x, y$ , then

$$W_2(\mu_t, \nu_t) \leq e^{-Ct} W_2(\mu_0, \nu_0) \quad (30)$$

for all solutions  $(\mu_t)_t$  and  $(\nu_t)_t$  to (28).

*It is a necessary condition as soon as moreover  $D(x)$  is a positive matrix for all  $x$ .*

The infimum is in fact a minimum as will be seen in the proof. The condition (29) is natural in view of the following interpretation of (28):

Let  $\mu_0$  and  $\nu_0$  be given in  $\mathcal{P}_2(\mathbb{R}^n)$ . For each  $x$  choose any matrix  $\sigma(x)$  such that  $\sigma(x)\sigma(x)^* = D(x)$  and assume that the stochastic differential equation

$$dX_t = \sqrt{2} \sigma(X_t) dB_t - A(X_t) dt$$

has a global solution  $(X_t)_{t \geq 0}$  (resp.  $(Y_t)_{t \geq 0}$ ), for a Brownian motion  $(B_t)_{t \geq 0}$  and an initial datum  $X_0$  (resp.  $Y_0$ ) with law  $\mu_0$  (resp.  $\nu_0$ ), such that  $\mathbb{E}|X_0 - Y_0|^2 = W_2^2(\mu_0, \nu_0)$ .

Assuming that

$$(A(x) - A(y)) \cdot (x - y) \geq C|x - y|^2 + \|\sigma(x) - \sigma(y)\|_{HS}^2$$

for all  $x, y$  and this choice of  $\sigma(x)$ , one can adapt the coupling argument given in the introduction to prove that

$$\mathbb{E}|X_t - Y_t|^2 \leq e^{-2Ct} \mathbb{E}|X_0 - Y_0|^2 = e^{-2Ct} W_2^2(\mu_0, \nu_0).$$

Then, by the Itô formula and since  $\sigma(x)\sigma(x)^* = D(x)$ , the law  $\mu_t$  of  $X_t$  is a solution to (28) with initial datum  $\mu_0$ , and analogously for the law  $\nu_t$  of  $Y_t$ . Hence they satisfy

$$W_2^2(\mu_t, \nu_t) \leq \mathbb{E}|X_t - Y_t|^2 \leq e^{-2Ct} W_2^2(\mu_0, \nu_0).$$

Then we can repeat the argument for any choice of  $\sigma(x)$  and, under uniqueness of solutions to (28), conclude that (29) is a sufficient condition to the contraction property (30).

The **proof of Proposition 4.2** goes in several steps:

1. Time derivative of the Wasserstein distance between solutions.

**Lemma 4.3** *In the above notation, if  $(\mu_t)_t$  and  $(\nu_t)_t$  are two solutions to (28), then for  $t \geq 0$*

$$\frac{1}{2} \frac{d}{dt} W_2^2(\mu_t, \nu_t) = -J(\nu_t | \mu_t)$$

where

$$J(\nu | \mu) = \int \left[ D(x) : (\nabla^2 \varphi(x) - I) + D(\nabla \varphi(x)) : ((\nabla^2 \varphi(x))^{-1} - I) + (A(\nabla \varphi(x)) - A(x)) \cdot (\nabla \varphi(x) - x) \right] d\mu(x)$$

if  $\nu = \nabla \varphi \# \mu$ .

**Proof**

◁ The operator  $L^*$  defined by  $L^*\mu = \nabla \cdot (\nabla(\mu D) + \mu A)$  is the dual in  $L^2(dx)$  of  $L$  defined by  $Lf = D : \nabla^2 f - A \cdot \nabla f$ . Hence, by adapting the derivation of (17) to any two solutions,

$$\frac{1}{2} \frac{d}{dt} W_2^2(\nu_t, \mu_t) = \int L\left(\frac{|x|^2}{2} - \varphi_t\right) d\mu_t + \int L\left(\frac{|x|^2}{2} - \varphi_t^*\right) d\nu_t$$

if  $\nabla\varphi_t \# \mu_t = \nu_t$ . This leads to the lemma by observing that

$$L\left(\frac{|x|^2}{2} - \varphi_t\right) = D : (I - \nabla^2\varphi_t) - A \cdot (x - \nabla\varphi_t)$$

and  $\nabla^2\varphi^*(\nabla\varphi(x)) = (\nabla^2\varphi(x))^{-1}$ . ▷

2. If for all  $x$  the matrix  $\sigma(x)$  is such that  $\sigma(x)\sigma(x)^* = D(x)$ , then

$$\begin{aligned} J(\nu|\mu) &= \int \left[ \sigma(x)\sigma(x)^* : (\nabla^2\varphi(x) - I) + \sigma(\nabla\varphi(x))\sigma(\nabla\varphi(x))^* : ((\nabla^2\varphi(x))^{-1} - I) \right. \\ &\quad \left. + \|\sigma(x) - \sigma(y)\|_{HS}^2 \right] d\mu(x) \\ &\quad + \int \left[ (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x) - \|\sigma(x) - \sigma(y)\|_{HS}^2 \right] d\mu(x) \end{aligned}$$

if  $\nu = \nabla\varphi \# \mu$ .

With this decomposition, the following lemma ensures the sufficient part of Proposition 4.2:

**Lemma 4.4** *If  $m, n$  and  $S$  are matrices with  $S$  symmetric and non-negative, then*

$$mm^* : (S - I) + nn^* : (S^{-1} - I) + \|m - n\|_{HS}^2 \geq 0,$$

and is 0 if and only if  $n = Sm$ .

**Proof**

◁ With the usual convention of repeated indexes,

$$mm^* : (S - I) + nn^* : (S^{-1} - I) + \|m - n\|_{HS}^2 = m_{ij}m_{kj}S_{ik} + n_{ij}n_{kj}S_{ik}^{-1} - 2m_{ij}n_{ij}.$$

Now the symmetric matrix  $S$  can be written as  $ODO^*$  where  $O$  is orthogonal and  $D = \text{diag}(s_1, \dots, s_n)$  with all  $s_i > 0$ . Then

$$m_{ij}m_{kj}S_{ik} = m_{ij}m_{kj}O_{ip}S_pO_{pk}^* = O_{pi}^*m_{ij}O_{pk}^*m_{kj}S_p = (O^*m)_{pj}^2s_p$$

and analogously with  $n$  and  $S^{-1} = OD^{-1}O^*$ . Moreover

$$m_{ij}n_{ij} = \delta_{ik}m_{ij}n_{kj} = O_{ip}O_{pk}^*m_{ij}n_{kj} = O_{pi}^*m_{ij}O_{pk}^*n_{kj} = (O^*m)_{pj}(O^*n)_{pj},$$

so collecting the three terms we are left with the quantity

$$(O^*m)_{pj}^2s_p + (O^*n)_{pj}^2(s_p)^{-1} - 2(O^*m)_{pj}(O^*n)_{pj} = \left( (O^*m)_{pj}(s_p)^{1/2} - (O^*n)_{pj}(s_p)^{-1/2} \right)^2 \quad (31)$$

which is non-negative. This proves the first part of the lemma.

If now this quantity is 0, then  $R_{pj} := (O^*n)_{pj} - (O^*m)_{pj}s_p$  is 0 for all  $p, j$ , so that

$$n_{ij} - (Sm)_{ij} = \sum_{qp} O_{ip} O_{pq}^* n_{qj} - \sum_{p,r} O_{ip} s_p O_{pr}^* m_{rj} = \sum_p O_{ip} [(O^*n)_{pj} - (O^*m)_{pj}s_p] = 0$$

for all  $i, j$ . Hence  $n = Sm$ .

Conversely, if  $n = Sm$ , then  $(OR)_{ij} = \sum_p O_{ip} R_{pj} = 0$  for all  $i, j$ , that is,  $OR = 0$ , whence  $R_{pj} = 0$  for all  $p, j$ . Finally  $mm^* : (S - I) + nn^* : (S^{-1} - I) + \|m - n\|_{HS}^2 = 0$  by (31).  $\triangleright$

**3.** We now give a second general result on symmetric matrices:

**Lemma 4.5** *Let  $M$  and  $N$  be non-negative symmetric matrices. Then there exist matrices  $m$  and  $n$  such that  $mm^* = M$ ,  $nn^* = N$  and*

$$\|m - n\|_{HS}^2 = \inf \{ \|m' - n'\|_{HS}^2; m'm'^* = M, n'n'^* = N \}.$$

*If moreover  $M$  is invertible, then any minimizer  $(m, n)$  is such that  $nm^{-1}$  is symmetric.*

**Proof**

$\triangleleft$  Let us first observe that the set of admissible  $(m', n')$  is closed, and bounded since for such  $m'$

$$\sum_{i,j} m'_{ij}{}^2 = \sum_i M_{ii},$$

and similarly for  $n'$ . Moreover the map  $(m', n') \mapsto \|m' - n'\|_{HS}$  is continuous, which ensures the existence of a minimizer.

Let now  $(m, n)$  be such a minimizer. First observe that

$$(mO)(mO)^* = mOO^*m^* = M$$

for all orthogonal matrix  $O$ . Hence

$$\|m - n\|_{HS}^2 \leq \|mO - n\|_{HS}^2,$$

that is

$$\|m\|_{HS}^2 + \|n\|_{HS}^2 - 2m : n \leq \|mO\|_{HS}^2 + \|n\|_{HS}^2 - 2(mO) : n$$

or

$$m : n \geq (mO) : n \tag{32}$$

since  $\|mO\|_{HS}^2 = \|m\|_{HS}^2$  for an orthogonal matrix  $O$ , as can be seen by direct computation.

Let now  $1 \leq a, b \leq n$  be fixed, and let  $O$  be the orthogonal matrix with coefficients  $O_{ii} = 1$  for  $i \neq a, b$ ,  $O_{aa} = O_{bb} = \cos \varepsilon$  and  $O_{ba} = -O_{ab} = \sin \varepsilon$ . For this matrix  $O$ , the relation (32) reads

$$\sin \varepsilon \sum_i (n_{ia}m_{ib} - n_{ib}m_{ia}) \leq (\cos \varepsilon - 1) \sum_i (n_{ia}m_{ia} - n_{ib}m_{ib})$$

Letting  $\varepsilon$  tend to  $0+$  and  $0-$ , we obtain

$$\sum_i n_{ia}m_{ib} = \sum_i n_{ib}m_{ia}.$$

Now fix  $1 \leq c, d \leq n$ , multiply this identity by  $(m^{-1})_{bc}(m^{-1})_{ad}$  and sum over  $a$  and  $b$  to obtain  $(nm^{-1})_{cd} = (nm^{-1})_{dc}$ . This ensures that  $nm^{-1}$  is symmetric.  $\triangleright$

4. We can now prove the necessary part in Proposition 4.2. Let  $a, b \in \mathbb{R}^n$  be fixed, and  $\mu_0 = \delta_a$ . By Lemma 4.5 with  $M = D(a)$  and  $N = D(b)$ , there exist  $m, n$  such that  $mm^* = D(a)$ ,  $nn^* = D(b)$  and  $nm^{-1}$  is symmetric. Observe moreover that

$$\|m - n\|_{HS}^2 = \inf \{ \|\sigma(a) - \sigma(b)\|_{HS}^2; \sigma(a)\sigma(a)^* = D(a), \sigma(b)\sigma(b)^* = D(b) \}.$$

Then we let  $\varphi$  be the map defined on  $\mathbb{R}^n$  by

$$\varphi(x) = \frac{1}{2}(nm^{-1}x) \cdot x + (b - nm^{-1}a) \cdot x.$$

It is a  $C^2$  map, and strictly convex since  $\nabla^2\varphi(x)$  is for all  $x$  the positive symmetric matrix  $nm^{-1}$ . Moreover  $\nabla\varphi(a) = b$ , so  $\nabla\varphi\#\mu_0 = \delta_b$  which we take as the second initial datum  $\nu_0$ .

For these initial data  $\mu_0$  and  $\nu_0$ , the contraction property

$$W_2(\mu_t, \nu_t) \leq e^{-Ct}W_2(\mu_0, \nu_0)$$

gives

$$C W_2(\mu_0, \nu_0)^2 \leq J(\nu_0|\mu_0)$$

by time-differentiation at  $t = 0$  and Lemma 4.3, that is, here,

$$C|b-a|^2 \leq \left[ mm^* : (\nabla^2\varphi(a) - I) + nn^* : ((\nabla^2\varphi(a))^{-1} - I) + \|m-n\|_{HS}^2 \right] + (A(b) - A(a)) \cdot (b-a) - \|m-n\|_{HS}^2.$$

Since  $\nabla^2\varphi(a) = nm^{-1}$ , the term in square brackets is 0 by Lemma 4.4, and we finally obtain

$$(A(b) - A(a)) \cdot (b-a) \geq C|b-a|^2 + \inf \{ \|\sigma(a) - \sigma(b)\|_{HS}^2; \sigma(a)\sigma(a)^* = D(a), \sigma(b)\sigma(b)^* = D(b) \}.$$

This concludes the proof of Proposition 4.2.

## 5 Proof of Proposition 3.5

We first state a general result on the map  $A$ :

**Lemma 5.1** *Let  $A$  be a  $C^1$  monotone map on  $\mathbb{R}^n$  for which there exist two constants  $R$  and  $K > 0$  such that  $\nabla^S A(x) \geq K$  for all  $|x| \geq R$ . Then*

$$(A(x) - A(y)) \cdot (x - y) \geq \frac{K}{3}|x - y|^2$$

if  $|x| \geq 2R$  or  $|y| \geq 2R$ .

**Proof**

$\triangleleft$  Let  $x$  and  $y$  be fixed in  $\mathbb{R}^n$  with  $|y| \geq 2R$ , and let us first write

$$\begin{aligned} (A(x) - A(y)) \cdot (x - y) &= \int_0^1 \nabla^S A(y + t(x - y)) (x - y) \cdot (x - y) dt \\ &= r \int_0^r \nabla^S A(y + s\theta) \theta \cdot \theta ds \end{aligned}$$

for  $x = y + r\theta$  with  $r(=|x-y|) \geq 0$  and  $\theta \in \mathbb{S}^{n-1}$ .

1. If  $\{y + t(x-y); 0 \leq t \leq 1\} \cap \{z \in \mathbb{R}^n; |z| \leq R\} = \emptyset$ , then

$$\int_0^1 \nabla^S A(y + t(x-y)) (x-y) \cdot (x-y) dt \geq K|x-y|^2 \geq K \frac{|x-y|^2}{3}.$$

2. If  $\{y + t(x-y); 0 \leq t \leq 1\} \cap \{z \in \mathbb{R}^n; |z| \leq R\} \neq \emptyset$ , then let  $0 \leq r_- \leq r_+$  such that

$$\{y + s\theta; s \geq 0\} \cap \{z \in \mathbb{R}^d; |z| \leq R\} = [r_-\theta, r_+\theta].$$

Observe that

$$r_- = |y - (y + r_-\theta)| \geq \inf\{|y-z|; |z| \leq R\} = |y| - R$$

and

$$r_+ \leq \sup\{|y-z|; |z| \leq R\} = |y| + R$$

with  $|y| \geq 2R$ , so that

$$r_- \geq \frac{r_+}{3}.$$

2.1. If  $r_- \leq r \leq r_+$ , then

$$\int_0^r \nabla^S A(y + s\theta) \theta \cdot \theta ds \geq \int_0^{r_-} \nabla^S A(y + s\theta) \theta \cdot \theta ds \geq Kr_- \geq K \frac{r_+}{3} \geq K \frac{r}{3}.$$

2.2. If  $r_+ \leq r$ , then

$$\begin{aligned} \int_0^r \nabla^S A(y + s\theta) \theta \cdot \theta ds &\geq \int_0^{r_-} \nabla^S A(y + s\theta) \theta \cdot \theta ds + \int_{r_+}^r \nabla^S A(y + s\theta) \theta \cdot \theta ds \\ &\geq Kr_- + K(r - r_+) \geq K\left(\frac{r_+}{3} + r - r_+\right) = K\left(\frac{r}{3} + \frac{2(r - r_+)}{3}\right) \geq K \frac{r}{3}. \end{aligned}$$

This concludes the argument, all cases being covered.  $\triangleright$

We now turn to the **proof of Proposition 3.5**. Let  $\varphi$  be a given strictly convex function on  $\mathbb{R}^n$ . Let us recall that for the Hessian operator

$$\nabla^2 \varphi^* (\nabla \varphi(x)) = (\nabla^2 \varphi(x))^{-1}$$

and in particular

$$\Delta \varphi^* (\nabla \varphi(x)) = \text{trace}(\nabla^2 \varphi(x))^{-1}$$

Let  $X$  be the subset of  $\mathbb{R}^n$  defined by

$$X = \{x \in \mathbb{R}^n, |x| \leq 2R, |\nabla \varphi(x)| \leq 2R\}.$$

1. First of all, by monotonicity of  $A$  and Lemma 5.1,

$$\begin{aligned} &\int_{\mathbb{R}^n} (A(\nabla \varphi(x)) - A(x)) \cdot (\nabla \varphi(x) - x) e^{-V(x)} dx \\ &\geq \int_{\mathbb{R}^n \setminus X} (A(\nabla \varphi(x)) - A(x)) \cdot (\nabla \varphi(x) - x) e^{-V(x)} dx \geq \frac{K}{3} \int_{\mathbb{R}^n \setminus X} |\nabla \varphi(x) - x|^2 e^{-V(x)} dx. \end{aligned}$$

2. On the other hand, for  $\theta \in \mathbb{S}^{n-1}$  we let  $R_\theta = \sup\{r \geq 0, r\theta \in X\}$ . In particular  $R_\theta \theta \in X$  and  $R_\theta \leq 2R$ . Then we let  $r_\theta \in [R_\theta, 3R]$  such that

$$|\nabla\varphi(r_\theta \theta) - r_\theta \theta| = \inf\{|\nabla\varphi(r \theta) - r \theta|, R_\theta \leq r \leq 3R\}.$$

In particular

$$|\nabla\varphi(r_\theta \theta)| \leq |\nabla\varphi(r_\theta \theta) - r_\theta \theta| + |r_\theta \theta| \leq |\nabla\varphi(R_\theta \theta) - R_\theta \theta| + |r_\theta \theta| \leq 2R + 2R + 3R = 7R$$

since  $|\nabla\varphi(R_\theta \theta)| \leq 2R$  and  $|R_\theta \theta| \leq 2R$  for  $R_\theta \theta \in X$ .

Then, for  $r \theta \in X$  with  $0 \leq r \leq R_\theta \leq r_\theta$ , let us write

$$\nabla\varphi(r\theta) - r\theta = \nabla\varphi(r_\theta\theta) - r_\theta\theta + \int_{r_\theta}^r [\nabla^2\varphi(s\theta) - I] \theta ds.$$

Letting  $H = \nabla^2\varphi(s\theta)$  for notational convenience, we decompose as

$$[H - I] \theta = [H^{\frac{1}{2}} - H^{-\frac{1}{2}}] H^{\frac{1}{2}} \theta$$

so that

$$\begin{aligned} \left| \int_r^{r_\theta} [H - I] \theta dt \right|^2 &\leq \left( \int_r^{r_\theta} |H^{\frac{1}{2}} - H^{-\frac{1}{2}}| |H^{\frac{1}{2}} \theta| ds \right)^2 \\ &\leq \int_r^{r_\theta} |H^{\frac{1}{2}} - H^{-\frac{1}{2}}|^2 e^{-V(s\theta)} ds \int_r^{r_\theta} |H^{\frac{1}{2}} \theta|^2 e^{+V(s\theta)} ds. \end{aligned}$$

by the Hölder inequality. But

$$\begin{aligned} |H^{\frac{1}{2}} - H^{-\frac{1}{2}}|^2 &= \sup_x \frac{|[H^{\frac{1}{2}} - H^{-\frac{1}{2}}]x|^2}{|x|^2} = \sup_x \frac{([H - 2I + H^{-1}]x) \cdot x}{|x|^2} \\ &\leq \text{trace}(H - 2I + H^{-1}) = \Delta\varphi(s\theta) - 2n + (\Delta\varphi^*)(\nabla\varphi(s\theta)). \end{aligned}$$

since the eigenvalues of  $H - 2I + H^{-1}$  are non-negative. Moreover

$$|H^{\frac{1}{2}} \theta|^2 = (H^{\frac{1}{2}} \theta) \cdot (H^{\frac{1}{2}} \theta) = H \theta \cdot \theta.$$

Hence

$$\begin{aligned} |\nabla\varphi(r\theta) - r\theta|^2 &\leq 2|\nabla\varphi(r_\theta\theta) - r_\theta\theta|^2 \\ &\quad + 2 \int_r^{r_\theta} (\Delta\varphi(s\theta) - 2n + \Delta\varphi^*(\nabla\varphi(s\theta))) e^{-V(s\theta)} ds \int_r^{r_\theta} (H \theta) \cdot \theta e^{+V(s\theta)} ds \end{aligned}$$

where

$$\int_r^{r_\theta} (H \theta) \cdot \theta ds = (\nabla\varphi(r_\theta\theta) - \nabla\varphi(r\theta)) \cdot \theta \leq |\nabla\varphi(r_\theta\theta)| + |\nabla\varphi(r\theta)| \leq 9R$$

for  $r \theta \in X$ . Hence

$$\begin{aligned} \int_{X, |x| \leq 2R} |\nabla\varphi(x) - x|^2 e^{-V(x)} dx &= \int_{\mathbb{S}^{n-1}} \int_0^{R_\theta} r^{n-1} |\nabla\varphi(r\theta) - r\theta|^2 e^{-V(r\theta)} dr d\theta \\ &\leq 2 \int_{\mathbb{S}^{n-1}} \int_0^{R_\theta} r^{n-1} |\nabla\varphi(r_\theta\theta) - r_\theta\theta|^2 e^{-V(r\theta)} dr d\theta \end{aligned}$$

$$+18R e^{\sup\{V(x); |x|\leq 2R\}} \int_{\mathbb{S}^{n-1}} \int_0^{R\theta} r^{n-1} \int_r^{r\theta} (\Delta\varphi(s\theta) - 2n + \Delta\varphi^*(\nabla\varphi(s\theta))) e^{-V(s\theta)} ds dr d\theta.$$

But

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \int_0^{R\theta} r^{n-1} \int_r^{r\theta} (\Delta\varphi(s\theta) - 2n + \Delta\varphi^*(\nabla\varphi(s\theta))) e^{-V(s\theta)} ds dr d\theta \\ & \leq \int_{\mathbb{S}^{n-1}} \int_0^{R\theta} \int_r^{r\theta} s^{n-1} (\Delta\varphi(s\theta) - 2n + \Delta\varphi^*(\nabla\varphi(s\theta))) e^{-V(s\theta)} ds dr d\theta \\ & \leq 2R \int_{\mathbb{S}^{n-1}} \int_0^{3R} s^{n-1} (\Delta\varphi(s\theta) - 2n + \Delta\varphi^*(\nabla\varphi(s\theta))) e^{-V(s\theta)} ds d\theta \\ & = 2R \int_{|x|\leq 3R} (\Delta\varphi(x) - 2n + \Delta\varphi^*(\nabla\varphi(x))) e^{-V(x)} dx. \end{aligned}$$

Hence

$$\begin{aligned} & \int_{X, |x|\leq 2R} |\nabla\varphi(x) - x|^2 e^{-V(x)} dx \\ & \leq 2 e^{-\inf\{V(x); |x|\leq 2R\}} \frac{(2R)^n}{n} \int_{\mathbb{S}^{n-1}} |\nabla\varphi(r\theta) - r\theta|^2 d\theta \\ & \quad + 18R e^{\sup\{V(x); |x|\leq 2R\}} 2R \int_{|x|\leq 3R} (\Delta\varphi(x) - 2n + \Delta\varphi^*(\nabla\varphi(x))) e^{-V(x)} dx. \quad (33) \end{aligned}$$

Moreover, by Lemma 5.1 and the definition of  $r_\theta$ ,

$$\begin{aligned} & \int_{|x|\leq 3R} (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x) e^{-V(x)} dx \\ & \geq \int_{2R\leq |x|\leq 3R} (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x) e^{-V(x)} dx \\ & \geq \frac{K}{3} \int_{2R\leq |x|\leq 3R} |\nabla\varphi(x) - x|^2 e^{-V(x)} dx \geq \frac{K}{3} e^{-\sup\{V(x); |x|\leq 3R\}} \int_{2R\leq |x|\leq 3R} |\nabla\varphi(x) - x|^2 dx \\ & = \frac{K}{3} e^{-\sup\{V(x); |x|\leq 3R\}} \int_{2R}^{3R} r^{n-1} \int_{\mathbb{S}^{n-1}} |\nabla\varphi(r\theta) - r\theta|^2 dr d\theta \\ & \geq \frac{K}{3} e^{-\sup\{V(x); |x|\leq 3R\}} \int_{2R}^{3R} r^{n-1} \int_{\mathbb{S}^{n-1}} |\nabla\varphi(r\theta) - r\theta|^2 dr d\theta. \end{aligned}$$

Hence there exists a constant  $C$  such that

$$C \int_{\mathbb{S}^{n-1}} |\nabla\varphi(r\theta) - r\theta|^2 d\theta \leq \int_{|x|\leq 3R} (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x) e^{-V(x)} dx. \quad (34)$$

It follows from (33) and (34) that

$$\begin{aligned} C \int_{X, |x|\leq 2R} |\nabla\varphi(x) - x|^2 e^{-V(x)} dx & \leq \int_{|x|\leq 3R} (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x) e^{-V(x)} dx \\ & \quad + \int_{|x|\leq 3R} (\Delta\varphi(x) - 2n + \Delta\varphi^*(\nabla\varphi(x))) e^{-V(x)} dx. \end{aligned}$$

Moreover

$$\int_{|x|\leq 3R} (A(\nabla\varphi(x)) - A(x)) \cdot (\nabla\varphi(x) - x) e^{-V(x)} dx \geq \frac{K}{3} \int_{2R\leq |x|\leq 3R} |\nabla\varphi(x) - x|^2 e^{-V(x)} dx,$$

so that

$$C \int_X |\nabla\varphi(x) - x|^2 e^{-V(x)} dx \leq \int_{|x| \leq 3R} (A(\nabla\varphi(x) - A(x)) \cdot (\nabla\varphi(x) - x) e^{-V(x)} dx + \int_{|x| \leq 3R} (\Delta\varphi(x) - 2n + \Delta\varphi^*(\nabla\varphi(x))) e^{-V(x)} dx.$$

Finally the last two integrands are non-negative maps, so we can bound from above these last two integrals on the set  $\{|x| \leq 3R\}$  by the corresponding integrals on the whole  $\mathbb{R}^n$ .

**3.** We conclude the proof of Proposition 3.5 by adding the estimates in **1.** and **2.**

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