Optimal reflection of diffusions and barrier options pricing under constraints

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Abstract

We introduce a new class of control problems in which the gain depends on the solution of a stochastic differential equation reflected at the boundary of a bounded domain, along directions which are controlled by a bounded variation process. We provide a PDE characterization of the associated value function. This study is motivated by applications in mathematical finance where such equations are related to the pricing of barrier options under portfolio constraints.

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1 Introduction

This paper is motivated by a previous work [2] where a new class of parabolic PDE with Neumann and Dirichlet conditions is introduced. The starting point of [2] is the problem of hedging a barrier option under portfolio constraints. It shows that the super-hedging price is a viscosity solution of an equation of the form

$$\begin{cases} \min\left\{-\mathcal{L}\varphi\;,\min\limits_{e\in E}\mathcal{H}^e\varphi\right\}=0 & \text{on} \quad [0,T)\times\mathcal{O} \\ \min\left\{\varphi\;,\;\min\limits_{e\in E}\mathcal{H}^e\varphi\right\}=0 & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi-\hat{g}=0 & \text{on} \quad \{T\}\times\bar{\mathcal{O}}\;. \end{cases} \tag{1.1}$$

Here, $\mathcal{O}$ is an open domain of $\mathbb{R}^d$ outside of which the option is desactivated, E is a compact subset of $\mathbb{R}^\ell$ which depends on the constraints imposed on the portfolio, $\mathcal{L}\varphi = \frac{\partial}{\partial t}\varphi + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^* D^2\varphi\right]$ is the Dynkin operator of the diffusion which models the evolution of the risky assets, $\mathcal{H}^e\varphi := \delta(\cdot,e)\varphi - \langle\gamma(\cdot,e),D\varphi\rangle$ for some (oblique) inward direction $\gamma(x,e)$ and $\hat{g}$ is a "smoothed" version of the payoff of the option which satisfies $\min\limits_{e\in E}\mathcal{H}^e\hat{g}\geq 0$ (see [2] for details and Section 4 below for an example).

When the solution φ of the above equation is positive, the spacial boundary condition reduces to $\min\limits_{e\in E}\mathcal{H}^e\varphi = 0$ on $[0,T)\times\partial\mathcal{O}$, and, in particular cases, see [13] and [14], the constraint $\mathcal{H}^e\varphi \geq 0$ on the parabolic boundary of $[0,T)\times\mathcal{O}$ propagates in the domain, which allows to simplify the above equation in

$$\begin{cases} -\mathcal{L}\varphi = 0 & \text{on} \quad [0,T)\times\mathcal{O} \\ \min_{e\in E}\mathcal{H}^e\varphi = 0 & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - \hat{g} = 0 & \text{on} \quad \{T\}\times\bar{\mathcal{O}}\,. \end{cases} \tag{1.2}$$

When E is a singleton $\{e_0\}$, such equations formally admit a Feynman-Kac representation of the form

$$\mathbb{E}\left[e^{-\int_t^T \delta(X(s),e_0)dL(s)}\hat{g}(X(T))\right] \tag{1.3}$$

where L is a non-decreasing process such that (X,L) solves on $[t,T]$

$$\begin{aligned} X(s) &= x + \textstyle\int_t^s \sigma(X(r))dW(r) + \int_t^s \gamma(X(r),e_0)dL(r) \\ X(s) \in \bar{\mathcal{O}} \quad &\text{and} \quad L(s) = \textstyle\int_t^s \mathbf{1}_{\{X(r)\in\partial\mathcal{O}\}}\,dL(r) \;\;,\; t\leq s\leq T\,, \end{aligned} \tag{1.4}$$

for a given standard Brownian motion W. Thus, in this particular case, the price of the barrier option is, at least formally, given by the expectation of a functional depending on the solution of a stochastic differential equation which is reflected at the boundary of $\mathcal{O}$ along the direction $\gamma(\cdot,e_0)$. This phenomenon was already observed in [13] in a particular setting and can be easily explained when $\hat{g}\geq 0$ and $\hat{g}$ is non-decreasing on $\mathcal{O}$, see Remark 4.4 below.

By analogy, (1.2) should be associated to a control problem of the form

$$\sup_{\epsilon\in\mathcal{E}}\mathbb{E}\left[e^{-\int_t^T \delta(X^\epsilon(s),\epsilon(s))dL^\epsilon(s)}\hat{g}(X^\epsilon(T))\right] \tag{1.5}$$

where (X^ϵ,L^ϵ) is the solution on $[t,T]$ of

$$\begin{aligned} X^\epsilon(s) &= x + \textstyle\int_t^s \sigma(X^\epsilon(r))dW(r) + \int_t^s \gamma(X^\epsilon(r),\epsilon(r))dL^\epsilon(r) \\ X^\epsilon(s) \in \bar{\mathcal{O}} \quad &\text{and} \quad L^\epsilon(s) = \textstyle\int_t^s \mathbf{1}_{\{X^\epsilon(r)\in\partial\mathcal{O}\}}\,dL^\epsilon(r) \;\;,\; t\leq s\leq T \end{aligned} \tag{1.6}$$

and $\mathcal{E}$ is a suitable set of adapted processes with values in E. The difference with (1.3) is that the direction of reflection is now controlled by the process $\epsilon\in\mathcal{E}$.

This naturally leads to the introduction of a new class of control problems of the form (1.5), which, to the best of our knowledge, have not been studied so far.

In this paper, we first show that (1.6) admits a strong solution in the case where $\mathcal{O}$ is bounded, $|\gamma|=1$ and $(\mathcal{O},\gamma)$ satisfies a uniform exterior cone condition:

$$\bigcup_{0\leq\lambda\leq r} B\left(x-\lambda\gamma(x,e),\lambda r\right)\subset\mathcal{O}^c \quad \text{for all } (x,e)\in\partial\mathcal{O}\times\mathbb{R}^\ell\,. \tag{1.7}$$

There is a huge literature on reflected SDEs and we refer to [7] for an overview of mains results. In the case where (X,ϵ) is the solution of a SDE with Lipschitz coefficients, the existence of a strong solution under the exterior sphere condition (1.7) is easily deduced from [6]. Indeed, it suffices to consider the extended system (X, ϵ) reflected at the boundary of $\mathcal{O} \times \tilde{E}$ for some open ball $\tilde{E} = B(0, \tilde{r})$ which contains the compact set E along a smooth direction $\tilde{\gamma}$ such that $\tilde{\gamma} = (\gamma, 0)$ on $\mathcal{O} \times E$ and $\tilde{\gamma} = (\gamma, -e/\tilde{r})/\sqrt{2}$ on $\mathcal{O} \times \partial\tilde{E}$. This system satisfies the exterior sphere condition of [6]. Since ϵ takes values in E, the reflection does not operate on this component and we deduce the existence of a solution to (1.6) from the results of [6]. However, this formulation is quite restrictive and we are interested by a more general class of controls.

We therefore come back to the initial deterministic Skorokhod problem and follow the steps of [6] which are inspired by [11]. The existence to the Skorokhod problem with directions of reflection controlled by a continuous function ϵ with bounded variations is deduced from [6] by using the above arguments which consists in considering an extended system. We then use suitable estimates on a family of test functions introduced in [5] to prove the existence of a solution to (1.6) in our general setting. Moreover, by considering SDEs with random coefficients, we are able to incorporate another control on the direction which takes the form of an Itô process, see Section 2.

We then introduce a control problem which generalizes (1.5) and prove that its value function is a viscosity solution of an equation of the form (1.2), for which we provide a comparison result. In the case where $\gamma(x, e)$ does not depend on e, it essentially follows from the results of [5]. In this paper, we propose a new set of conditions which is more adapted to our setting and does not seem to be covered by the existing literature, see Section 3.4 below.

In the last section, we discuss the link between (1.5) and the pricing of barrier options under portfolio constraints. In a particular setting, we prove that (1.5) coincides with the super-hedging price of the option, when (1.2) admits a sufficiently smooth solution. This generalizes previous results of [13]. When E is reduced to a singleton, this leads to a natural Monte-Carlo approach for its estimation. We let the discussion of more general cases for further researches.

Notations. Given $E \subset \mathbb{R}^m$, $m \geq 1$ and $E_i \subset \mathbb{R}^{m_i}$, $m_i \geq 1$ for $i \leq I$, we denote by $C^{k_1,\ldots,k_I}(E_1 \times \cdots \times E_I, E)$ (resp. $C_b^{k_1,\ldots,k_I}(E_1 \times \cdots \times E_I, E)$) the set of continuous maps φ from $E_1 \times \cdots \times E_I$ into E that admit continuous (resp. bounded) derivatives up to order k_i in their i-th component x_i. We omit k_i when it is equal to 0 and only write $C^{k_1}(E_1 \times \cdots \times E_I, E)$ when $k_1 = k_2 = \ldots = k_I$. We omit E when $E = \mathbb{R}$, and, in this case, we denote by $D_{x_i}\varphi$ and $D^2_{x_i}\varphi$ the (partial) Jacobian and Hessian matrix with respect to x_i. We simply write $D\varphi$ and $D^2\varphi$ for $D_{x_2}\varphi$ and $D^2_{x_2}\varphi$ if $I = 2$. For $T > 0$, we define $\mathrm{BV}([0, T], E)$ as the set of continuous maps from $[0, T]$ into E with a bounded total variation. For $\epsilon \in \mathrm{BV}([0, T], E)$, we set $|\epsilon| := \sum_{i \leq m} |\epsilon^i|$ where $|\epsilon^i|(t)$ is the total variation of ϵ^i on $[0, t]$, $t \geq 0$. We write E^c for $\mathbb{R}^m \setminus E$, ∂E and $\bar{E}$ denote the boundary and the closure of E, $\mathbb{R}^m_+ = [0, \infty)^m$, $\mathbb{R}^m_- = -\mathbb{R}^m_+$. The Euclidean norm of $x = (x^1, \ldots, x^m) \in \mathbb{R}^m$ is denoted by $|x|$, $B(x, r)$ is the open ball centered on x with radius r, $\langle \cdot, \cdot \rangle$ is the natural scalar product on $\mathbb{R}^m$. We denote by $\mathbb{M}^m$ the set of square matrices of dimension m and we extend the definition of $|\cdot|$ to $\mathbb{M}^m$ by identifying $\mathbb{M}^m$ to $\mathbb{R}^{m\times m}$. For $x \in \mathbb{R}^m$, $\mathrm{diag}\,[x]$ is the diagonal matrix of $\mathbb{M}^m$ whose i-th diagonal element is x^i, $\mathrm{Tr}\,[M]$ is the trace of $M \in \mathbb{M}^m$, M^* its transposition. All inequalities between random variables have to be taken in the a.s. sense.

2 SDEs with controlled reflecting directions

The aim of this section is to construct a stochastic differential equation wich is reflected at the boundary of some bounded open set $\mathcal{O} \subset \mathbb{R}^d$, $d \geq 1$, along a direction which is controlled by an adapted continuous process with bounded variations taking values in a compact subset E of $\mathbb{R}^\ell$, $\ell \geq 1$. We follow the arguments of [6] and start with the resolution of the associated (deterministic) Skorokhod problem.

2.1 The Skorokhod problem with controlled reflecting directions

For sake of completeness, we first recall one of the main results of [6] which provides a solution to the Skorokhod problem for oblique reflection on general bounded sets.

Theorem 2.1 (Dupuis and Ishii [6]) *Fix $\gamma \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ with $|\gamma| = 1$. Assume that there exists some $r \in (0,1)$ such that*

$$\bigcup_{0\leq\lambda\leq r} B\left(x - \lambda\gamma(x), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } x \in \partial\mathcal{O}\ . \tag{2.1}$$

Then, for all $\psi \in C([0,T], \mathbb{R}^d)$ satisfying $\psi(0) \in \bar{\mathcal{O}}$, there exists $(\phi, \eta) \in C([0,T], \bar{\mathcal{O}}) \times \mathrm{BV}([0,T], \mathbb{R}_+)$ such that

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s))d\eta(s)\ ,\ \eta(t) = \int_0^t \mathbf{1}_{\{\phi(s)\in\partial\mathcal{O}\}} d|\eta|(s)\ \ ,\ t \leq T\ .$$

Moreover, $(\phi(t), \eta(t)) \in \sigma(\psi(s), s \leq t)$ for all $t \leq T$, and uniqueness holds if $\psi \in \mathrm{BV}([0,T], \mathbb{R}^d)$.

Proof. See Theorem 4.8 and the discussion after Corollary 5.2 in [6]. □

We now fix an open bounded set $\mathcal{O} \subset \mathbb{R}^d$, a compact set $E \subset \mathbb{R}^\ell$ and γ satisfying

$$\gamma \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^d)\ ,\ |\gamma| = 1 \tag{2.2}$$

$$\exists\, r \in (0,1) \text{ s.t. } \bigcup_{0\leq\lambda\leq r} B\left(x - \lambda\gamma(x,e), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } (x,e) \in \partial\mathcal{O} \times \mathbb{R}^\ell\ . \tag{2.3}$$

We then deduce from Theorem 2.1 the following result.

Corollary 2.1 *Let the conditions (2.2) and (2.3) hold. Then, for all $\psi \in \mathrm{BV}([0,T], \mathbb{R}^d)$ satisfying $\psi(0) \in \bar{\mathcal{O}}$ and $\epsilon \in \mathrm{BV}([0,T], E)$, there exists a unique couple $(\phi, \eta) \in C([0,T], \bar{\mathcal{O}}) \times \mathrm{BV}([0,T], \mathbb{R}_+)$ such that*

$$\phi(t) = \psi(t) + \int_0^t \gamma(\phi(s), \epsilon(s))d\eta(s) \quad \text{and} \quad \eta(t) = \int_0^t \mathbf{1}_{\{\phi(s)\in\partial\mathcal{O}\}} d|\eta|(s)\ ,\ t \leq T\ . \tag{2.4}$$

Moreover, $(\phi(t), \eta(t)) \in \sigma((\psi(s), \epsilon(s)), s \leq t)$ for all $t \leq T$

Proof. This is an immediate consequence of Theorem 2.1. Since ϵ is valued in a compact set, it suffices to apply the above result to an extended fictitious reflected system (ψ, ϵ). We detail the proof for completeness. Fix $\tilde{r} > 0$ so that $\tilde{E} := B(0, \tilde{r})$ strictly contains E. Fix $\zeta \in C^2(\mathbb{R}^\ell, [0,1])$ such that $\zeta(e) = 0$ for $e \in E$ and $\zeta(e) = 1$ for $e \in \partial\tilde{E}$ and set, on $\mathbb{R}^{d+\ell}$, $\tilde{\gamma}(x,e) = (\gamma(x,e), -e\zeta(e)/\tilde{r})/|(\gamma(x,e), -e\zeta(e)/\tilde{r})|$. Since $|\gamma| = 1$, $|(\gamma(x,e), -e\zeta(e)/\tilde{r})| \geq 1$ and $\tilde{\gamma} \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^{d+\ell})$. Moreover, $|(\gamma(x,e), -e\zeta(e)/\tilde{r})|^2 \leq 2$ on the closure of $\mathcal{O} \times \tilde{E}$, $|(\gamma(x,e), -e\zeta(e)/\tilde{r})|^2 = 2$ if $e \in \partial\tilde{E}$, and $B(e + \lambda e/\tilde{r}, \lambda r) \cap \tilde{E} = \emptyset$ for all $e \in \partial\tilde{E}$ and $\lambda > 0$, recall that $r < 1$. We then deduce from (2.3) that for $(x,e) \in \partial(\mathcal{O} \times \tilde{E})$ and $\lambda \in [0, r/\sqrt{2}]$

$$|(y,f) - ((x,e) - \lambda\tilde{\gamma}(x,e))|^2 \leq \lambda^2(r/\sqrt{2})^2 \quad \Rightarrow \quad (y,f) \notin \mathcal{O} \times \tilde{E} \; .$$

We can therefore apply Theorem 2.1 to the couple (ψ, ϵ) reflected at the boundary of $\mathcal{O} \times \tilde{E}$. Since ϵ does not reach the boundary of $\tilde{E}$, this leads to the required result.

□

2.2 The stochastic Skorokhod problem with controlled reflecting direction

We now consider some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a d-dimensional standard Brownian motion W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ the natural filtration induced by W, satisfying the usual conditions, and assume that $\mathcal{F} = \mathcal{F}_T$. Given two uniformly Lipschitz functions μ and σ from $\mathbb{R}^d$ into $\mathbb{R}^d$ and $\mathbb{M}^d$ respectively, it is shown in [6] that, under the condition (2.1), there exists a unique couple (X, L) of $\mathbb{F}$-adapted continuous processes such that L is real valued, has bounded variations and

$$\begin{aligned} X(t) &= x + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(X(s))dL(s) \\ X(t) &\in \bar{\mathcal{O}} \quad \text{and} \quad L(t) = \int_0^t \mathbf{1}_{\{X(s) \in \partial\mathcal{O}\}} \, d|L|(s) \;\; , \; t \leq T \; . \end{aligned} \tag{2.5}$$

The aim of this section is to extend this result to the case where μ and σ are random, and γ is controlled by some continuous bounded variation process ϵ taking values in the compact set E, we refer to Remark 2.2 and 2.3 below for comments on this a-priori strong regularity assumption on the control ϵ.

In the following, given two subsets E_1 and E_2 of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, $m_1, m_2 \geq 1$, we denote by $L_{\mathbb{F}}(E_1, E_2)$ the set of measurable maps

$$f \quad : \quad (\omega, t, x) \in \Omega \times [0,T] \times E_1 \longrightarrow f_t(\omega, x) \in E_2$$

such that $t \mapsto f_t(\cdot, x)$ is progressively measurable for each $x \in E_1$, and

$$|f_t(\omega, x) - f_t(\omega, y)| \leq K|x - y| \;\; \forall \, x, y \in E_1 \;\; d\mathbb{P}(\omega) - a.s.$$

for some $K > 0$ independent of $(t, \omega) \in [0,T] \times \Omega$. In the sequel, we shall only write $f_t(x)$ for $f_t(\omega, x)$.

We denote by $\mathrm{BV}_{\mathbb{F}}(E_2)$ the set of E_2-valued continuous adapted processes with bounded variations. For ease of notations, we write $\mathcal{E}$ for $\mathrm{BV}_{\mathbb{F}}(E)$ and we set

$$\mathcal{E}^b_m := \{\epsilon \in \mathcal{E} \ : \ |\epsilon|(T) \le m \ \mathbb{P}-\text{a.s.}\} \quad , \ m > 0 \ .$$

In the rest of this section, we fix $(\mu, \sigma) \in L_{\mathbb{F}}(\mathbb{R}^d, \mathbb{R}^d \times \mathbb{M}^d)$ and assume that the conditions (2.2) and (2.3) hold. Our first result extends Theorem 5.1 in [6].

Lemma 2.1 *Let X be a continuous semimartingale with values in $\bar{\mathcal{O}}$. Fix $m > 0$ and $\epsilon \in \mathcal{E}^b_m$. Assume that Y is a continuous semimartingale with values in $\bar{\mathcal{O}}$ satisfying for $0 \le t_0 \le t \le T$*

$$Y(t) \ = \ X(t_0) + \int_{t_0}^t \mu_s(X(s))ds + \int_{t_0}^t \sigma_s(X(s))dW(s) + \int_{t_0}^t \gamma(Y(s), \epsilon(s))dL(s) \ ,$$

where L is an element of $\mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ such that

$$L(t) = \int_{t_0}^t \mathbf{1}_{\{Y(s)\in\partial\mathcal{O}\}} \ d|L|(s) \quad , \quad t_0 \le t \le T \ .$$

Let X' be an other continuous semimartingales with values in $\bar{\mathcal{O}}$ and assume that (Y', L') satisfies the same properties as (Y, L) with X' in place of X. Then, there is a constant $C_m > 0$ such that

$$\mathbb{E}\left[\sup_{t_0\le s\le t} |\Delta Y(s)|^4\right] \ \le \ C_m \mathbb{E}\left[|\Delta X(t_0)|^4 + \int_{t_0}^t \sup_{t_0\le s\le u} |\Delta X(s)|^4 du\right] \quad , \quad t_0 \le t \le T$$

where ΔY and ΔX stand for $Y - Y'$ and $X - X'$.

In order to prove Lemma 2.1, we shall appeal to the following technical result. It is a simple extension of Theorem 3.2 in [6] which is based on Theorem 4.1 in [5].

Lemma 2.2 *Given $\theta \in (0, 1)$ there exists a family of functions $(f_\varepsilon)_{\varepsilon>0}$ in $C^2(\bar{\mathcal{O}} \times \bar{\mathcal{O}} \times E)$ and a constant $K > 0$ independent of $\varepsilon > 0$ such that, for all $(y, y', e) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}} \times E$,*

$$\frac{|y-y'|^2}{\varepsilon} \le f_\varepsilon(y, y', e) \le K\left(\varepsilon + \frac{|y-y'|^2}{\varepsilon}\right) \tag{2.6}$$

$$\langle \gamma(y, e), D_y f_\varepsilon(y, y', e)\rangle \le K\frac{|y-y'|^2}{\varepsilon} \quad \text{if} \quad \langle y'-y, \gamma(y, e)\rangle \ge -\theta|y-y'| \ , \tag{2.7}$$

$$\langle \gamma(y', e), D_{y'} f_\varepsilon(y, y', e)\rangle \le K\frac{|y-y'|^2}{\varepsilon} \quad \text{if} \quad \langle y-y', \gamma(y', e)\rangle \ge -\theta|y-y'| \ , \tag{2.8}$$

$$|D_y f_\varepsilon(y, y', e) + D_{y'} f_\varepsilon(y, y', e)| \vee |D_e f_\varepsilon(y, y', e)| \le K\frac{|y-y'|^2}{\varepsilon} \ , \tag{2.9}$$

$$|D_y f_\varepsilon(y, y', e)| \vee |D_{y'} f_\varepsilon(y, y', e)| \le K\frac{|y-y'|}{\varepsilon} \ , \tag{2.10}$$

$$D^2_{(y,y')} f_\varepsilon(y, y', e) \le \frac{C}{\varepsilon}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + K\frac{|y-y'|^2}{\varepsilon} I_{2d} \ . \tag{2.11}$$

Moreover, there is $h \in C^2(\bar{\mathcal{O}} \times E)$ with non-negative values such that

$$\langle D_y h(y, e), \gamma(y, e)\rangle \ge 1 \quad \text{for all } (y, e) \in \partial\mathcal{O} \times E \ . \tag{2.12}$$

Proof. This follows from the proof of Theorem 4.1 in [5]. Since it is long, we only provide the main arguments. Let $g : (p,x) \in \mathbb{R}^d \times \mathbb{R}^d$ be as in Lemma 4.4 of [5]. In particular, it satisfies

$$|D_x g(p,x)| \le C\, |p|^2 \ , \tag{2.13}$$

for some $C > 0$. Let $\psi \in C^2(\mathbb{R})$ be a real non-decreasing function such that $\psi(t) = t$ for $t \ge 2$, $\psi(t) = 1$ for $t \le 1/2$ and $\psi(t) \ge t$ for all $t \in [1/2, 2]$. For $\varepsilon > 0$, we then define

$$f_\varepsilon(x,y,e) := \varepsilon \tilde{g}\left(\frac{x-y}{\varepsilon}, x, e\right) \ , \ (x,y,e) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^\ell \ ,$$

with

$$\tilde{g}(p,x,e) := \psi(g(p,\gamma(x,e)) \ , \ (p,x,e) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^\ell \ .$$

All the estimates, except the one on $|D_e f_\varepsilon(y,y',e)|$, follows directly from the property of g stated in Lemma 4.4 of [5] as in the proof of Theorem 4.1 in [5] (pp 1136-1137) for fixed values of e. Here, the constant K can be taken independent of e because E is bounded. The estimate on $|D_e f_\varepsilon(y,y',e)|$ follows from (2.13), the smoothness condition on γ and the boundedness of $\mathcal{O}$ and E. The existence of the function h follows from Theorem 3.2 in [6], see (3.20) of this paper. It suffices to repeat the argument of the proof of Corollary 2.1, i.e. consider a fictitious extended reflected system (x,e). □

Remark 2.1 Observe that given $\theta \in (0,1)$ such that $\theta^2 > 1-r^2$, we can find $\delta \in (0,r)$ for which $\langle y'-y, \gamma(y,e)\rangle \ge -\theta|y-y'|$ for all $e \in E$, $y \in \partial\mathcal{O}$ and $y' \in \bar{\mathcal{O}}$ such that $|y-y'| \le \delta$. This follows from (2.3) and the observation that $\langle y'-y, \gamma(y,e)\rangle \le -\theta|y-y'|$, $|y-y'| \le \delta$ and $|\gamma| = 1$ implies that

$$|y' - (y - \lambda\gamma(y,e))|^2 \ \ \le \ \ |y'-y|^2 - 2\lambda\theta|y-y'| + \lambda^2 = \lambda^2(1-\theta^2) \le \lambda^2 r^2$$

for $\lambda := |y-y'|/\theta \le \delta/(1-r^2)^{\frac{1}{2}}$ with δ small enough so that $\lambda \le r$.

Proof of Lemma 2.1. As in [6], we first observe that we can restrict to the case where $|Y-Y'| \le \delta$ where δ is defined as in Remark 2.1 for $\theta := (1+\sqrt{1-r^2})/2$. Indeed, since $\mathcal{O}$ is bounded, there is $\tilde{r} > 0$ such that $B(0,\tilde{r}/2) \supset \bar{\mathcal{O}}$ and if τ is the first time after t_0 when $|Y-Y'| \ge \delta$ then

$$\mathbb{E}\left[\sup_{t_0 \le s \le T} |\Delta Y(s)|^4\right] \le \frac{\tilde{r}^4}{\delta^4}\mathbb{E}\left[\sup_{t_0 \le s \le \tau} |\Delta Y(s)|^4\right] \ .$$

From now on, we therefore assume that $|Y-Y'| \le \delta$. For ease of notations, we also restrict to the case where $t_0 = 0$, the general case is handled similarly.
Recall from Lemma 2.2 the definitions of h and f_ε for θ defined as above. We fix $\varepsilon, \lambda > 0$ and define the smooth function $\tilde{f}_\varepsilon$ on $\bar{\mathcal{O}} \times \bar{\mathcal{O}} \times E$ by

$$\tilde{f}_\varepsilon(y,y',e) := e^{-\lambda(h(y,e)+h(y',e))} f_\varepsilon(y,y',e) \ . \tag{2.14}$$

Fix $\bar{K} > 0$. Set

$$A_t \quad := \quad \int_0^t e^{-\bar{K}|\epsilon|(s)} \left(\left| D_e \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) \right| - \bar{K} \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) \right) d|\epsilon|(s)$$

and $\beta_s := e^{-\bar{K}|\epsilon|(s)} e^{-\lambda(h(Y(s),\epsilon(s))+h(Y'(s),\epsilon(s)))}$. Since, by the estimates of Lemma 2.2,

$$\begin{aligned} & e^{-\bar{K}|\epsilon|(s)} \left| D_e \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) \right| \\ & \leq \beta_s \left(2\lambda \sup_{(y,e)\in\bar{\mathcal{O}}\times E} |D_e h(y,e)| + 1 \right) K \left(\varepsilon + \frac{|Y(s) - Y'(s)|^2}{\varepsilon} \right) \end{aligned}$$

and

$$e^{-\bar{K}|\epsilon|(s)} \bar{K} \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) \geq \beta_s \bar{K} \frac{|Y(s) - Y'(s)|^2}{\varepsilon}$$

we can find $C > 0$, independent of λ and ε, such that $A_t \leq \lambda C \varepsilon$ for $\bar{K}$ large enough with respect to K, λ and $|D_e h|$.

Thus, applying Itô's Lemma to $\xi := (e^{-\bar{K}|\epsilon|(t)} \tilde{f}_\varepsilon(Y(t), Y'(t), \epsilon(t)))_{t \leq T}$ leads to

$$\xi_t \quad \leq \quad \xi_0 + T\lambda C \varepsilon + G_t + G'_t + H_t \tag{2.15}$$

where

$$\begin{aligned} G_t \quad &:= \quad \int_0^t e^{-\bar{K}|\epsilon|(s)} \langle D_y \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)), \gamma(Y(s), \epsilon(s)) \rangle dL(s) \\ G'_t \quad &:= \quad \int_0^t e^{-\bar{K}|\epsilon|(s)} \langle D_{y'} \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)), \gamma(Y'(s), \epsilon(s)) \rangle dL'(s) \ , \end{aligned}$$

and

$$\begin{aligned} H_t \quad := \quad & \int_0^t e^{-\bar{K}|\epsilon|(s)} \langle D_y \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)), \mu_s(X(s)) - \mu_s(X'(s)) \rangle ds \\ + \quad & \int_0^t e^{-\bar{K}|\epsilon|(s)} \langle D_y \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)), [\sigma_s(X(s)) - \sigma_s(X'(s))] dW_s \rangle \\ + \quad & \int_0^t e^{-\bar{K}|\epsilon|(s)} \langle D_{y'} \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) + D_y \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)), \mu_s(X'(s)) \rangle ds \\ + \quad & \int_0^t e^{-\bar{K}|\epsilon|(s)} \langle D_{y'} \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) + D_y \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)), \sigma_s(X'(s)) dW_s \rangle \\ + \quad & \frac{1}{2} \int_0^t e^{-\bar{K}|\epsilon|(s)} \mathrm{Tr} \left[D^2_{(y,y')} \tilde{f}_\varepsilon(Y(s), Y'(s), \epsilon(s)) a_s(X(s), X'(s)) \right] ds \end{aligned}$$

with

$$a_s(x, x') = \begin{bmatrix} \sigma_s(x)\sigma_s(x)^* & \sigma_s(x)\sigma_s(x')^* \\ \sigma_s(x')\sigma_s(x)^* & \sigma_s(x')\sigma_s(x')^* \end{bmatrix}$$

where * denotes the transposition.

Now, observe that the estimates (2.6), (2.7) and (2.12) of Lemma 2.2, Remark 2.1 and the assumption $|Y - Y'| \leq \delta$ imply that

$$
\begin{aligned}
G_t &= \int_0^t \beta_s \langle D_y f_\varepsilon(Y(s), Y'(s), \epsilon(s)), \gamma(Y(s), \epsilon(s))\rangle dL(s) \\
&- \lambda \int_0^t \beta_s f_\varepsilon(Y(s), Y'(s), \epsilon(s)) \langle D_y h(Y(s), \epsilon(s)), \gamma(Y(s), \epsilon(s))\rangle dL(s) \\
&\leq (K - \lambda) \int_0^t \beta_s \frac{|\Delta Y(s)|^2}{\varepsilon} dL(s) .
\end{aligned}
$$

Similarly,

$$
G'_t \leq (K - \lambda) \int_0^t \beta_s \frac{|\Delta Y(s)|^2}{\varepsilon} dL'(s) .
$$

Taking $\lambda = K$, it then follows from (2.15) that

$$
\xi_t \leq \xi_0 + T\lambda C\varepsilon + H_t . \tag{2.16}
$$

Moreover, it follows from Doob's inequality, the estimates of Lemma 2.2, the a.s. Lipschitz continuity of μ and σ, the fact that Y, Y', X and X' are bounded, and the inequality $\alpha^2\beta^2 \leq \alpha^4 + \beta^4$, $\alpha, \beta \in \mathbb{R}$, that

$$
\mathbb{E}\left[\sup_{s \leq t} H_s^2\right] \leq C' \, \mathbb{E}\left[\int_0^t \frac{e^{-2\bar{K}|\epsilon|(s)}}{\varepsilon^2} \left(\varepsilon^4 + |\Delta Y(s)|^4 + |\Delta X(s)|^4\right) ds\right]
$$

where C' is a positive constant which does not depend on ε. Since $|\epsilon|(T) \leq m$, it follows from (2.16) and the left hand-side of (2.6) of Lemma 2.2 that

$$
\mathbb{E}\left[\sup_{s \leq t} |\Delta Y(s)|^4\right] \leq C_m \left(\varepsilon^4 + |\Delta X(0)|^4 + \int_0^t \mathbb{E}\left[\sup_{r \leq s} |\Delta Y(r)|^4 + \sup_{r \leq s} |\Delta X(r)|^4\right] ds\right)
$$

where C_m is a positive constant independent of ε. The required result is then obtained by sending $\varepsilon \longrightarrow 0$ and using Gronwall's Lemma. □

We can now provide the main result of this section, which ensures the strong existence and uniqueness of a SDE with random coefficients and controlled reflecting directions.

Theorem 2.2 *Fix $\epsilon \in \mathcal{E}$, $t \in [0, T]$ and ξ a $\mathcal{F}_t$-measurable random variable with values in $\bar{\mathcal{O}}$. Then, there exists a unique continuous adapted process (X, L) such that $L \in \mathrm{BV}_{\mathbb{F}}(\mathbb{R}_+)$ and*

$$
\begin{aligned}
X(s) &= \xi + \int_t^s \mu_r(X(r))dr + \int_t^s \sigma_r(X(r))dW(r) + \int_t^s \gamma(X(r), \epsilon(r))dL(r) \\
L(s) &= \int_t^s \mathbf{1}_{\{X(r) \in \partial\mathcal{O}\}} \, d|L|(r) \quad , \quad t \leq s \leq T .
\end{aligned} \tag{2.17}
$$

Proof. Observe that Lemma 4.7 in [6] can be easily extended to our setting by appealing to the arguments already used in the proof of Corollary 2.1. The existence and uniqueness when $|\epsilon|(T)$ is uniformly bounded then follows from Corollary 2.1, Lemma 2.1 and the same arguments as in [6], see the discussion after their Corollary 5.2, or as in the proof of Proposition 4.1 in [11]. In the case where $|\epsilon|(T)$ is not uniformly bounded, we use a localization argument. For each $m \geq 1$, we define $\tau_m := \inf\{s \geq t : |\epsilon|(s) \geq m\}$ and let (X^m, L^m) be the unique solution of (2.17) associated to $\epsilon^m(\cdot) := \epsilon(\cdot \wedge \tau_m)$. We then define (X, L) by

$$(X, L)(s) \quad := \quad (X^1, L^1)(s)\mathbf{1}_{t \leq s \leq \tau_1} + \sum_{m \geq 2} (X^m, L^m)(s)\mathbf{1}_{\tau_{m-1} < s \leq \tau_m} .$$

It solves (2.17) associated to ϵ. The same argument provides uniqueness. □

Remark 2.2 The presence of the control ϵ in γ plays a similar role as the time dependence in non-linear Neumann type boundary conditions of the form $L(t, x, u, Du) = 0$ in the viscosity literature. To the best of our knowledge the papers dealing with such a time dependence impose rather strong regularity conditions. The less stringent seem to appear in [3] where, for fixed (x, u, p), the map $t \mapsto L(t, x, u, p)$ is absolutely continuous with respect to the Lebesgue measure, see condition (H6) of this paper. In particular, $t \mapsto L(t, x, u, p)$ has bounded variations. It is therefore not surprising to retrieve such a condition in the definition of the set of controls $\mathcal{E}$.

Remark 2.3 Let (a, b) be a predictable process with values in $\mathbb{M}^\ell \times \mathbb{R}^\ell$ satisfying

$$\int_0^t (|b(s)| + |a(s)|^2) < \infty \;\; \mathbb{P} - \text{a.s.}$$

and assume that the process Z defined on $[t, T]$ by

$$Z(s) := z + \int_t^s b(r)dr + \int_t^s a(r)dW(r)$$

takes values in a compact set F of $\mathbb{R}^\ell$. Then, it follows from Theorem 2.2 that existence and uniqueness holds for

$$\begin{aligned} X(s) &= x + \int_t^s \mu_r(X(r))dr + \int_t^s \sigma_r(X(r))dW(r) + \int_t^s \tilde{\gamma}(X(r), Z(r), \epsilon(r))dL(r) \\ L(s) &= \int_t^s \mathbf{1}_{\{X(r) \in \partial\mathcal{O}\}} \, d|L|(r) \;\; , \quad t \leq s \leq T \end{aligned}$$

when $\tilde{\gamma} \in C^2(\mathbb{R}^d \times \mathbb{R}^\ell \times \mathbb{R}^\ell, \mathbb{R}^d)$ satisfies

$$\bigcup_{0 \leq \lambda \leq r} B\left(x - \lambda\tilde{\gamma}(x, z, e), \lambda r\right) \subset \mathcal{O}^c \quad \text{for all } (x, z, e) \in \partial\mathcal{O} \times \mathbb{R}^{2\ell} ,$$

for some $r \in (0, 1)$. This is easily checked by arguing as in the proof of Corollary 2.1, i.e. introduce the fictitious reflected system (X, Z) and apply Theorem 2.2. This allows us to introduce a new control on the direction of reflection which corresponds to an Itô process.

3 Optimal control

As in the previous section, we consider a bounded open set $\mathcal{O} \subset \mathbb{R}^d$ and $\gamma \in C^2(\mathbb{R}^{d+\ell}, \mathbb{R}^d)$ such that $|\gamma| = 1$ and (2.3) holds.

3.1 Definitions and assumptions

We fix a compact subset A of $\mathbb{R}^\ell$ and denote by $\mathcal{A}$ the set of predictable processes with values in A.
Let μ and σ be two continuous maps on $\mathbb{R}^d \times A$ with values in $\mathbb{R}^d$ and $\mathbb{M}^d$ respectively. We assume that both are Lipschitz with respect to their first variable uniformly in the other one, so that $(\mu^\alpha, \sigma^\alpha)$ defined by

$$(\mu^\alpha_t, \sigma^\alpha_t)(\cdot) := (\mu, \sigma)(\cdot, \alpha(t)) \ , \ t \le T$$

belongs to $L_{\mathbb{F}}(\mathbb{R}^d; \mathbb{R}^d \times \mathbb{M}^d)$ for all $\alpha \in \mathcal{A}$. It then follows from Theorem 2.2 that, for all $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, their exists a unique solution $(X^{\alpha,\epsilon}_{t,x}, L^{\alpha,\epsilon}_{t,x})$ to (2.17) associated to $(\mu^\alpha, \sigma^\alpha)$ with initial conditions given by $(X^{\alpha,\epsilon}_{t,x}, L^{\alpha,\epsilon}_{t,x})(t) = (x, 0)$.

The aim of this section is to provide a PDE characterization for the control problem

$$v(t, x) \ := \sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}} J(t, x; \alpha, \epsilon) \tag{3.1}$$

where

$$\begin{aligned} J(t, x; \alpha, \epsilon) \ &:= \ \mathbb{E}\left[\beta^{\alpha,\epsilon}_{t,x}(T) g\left(X^{\alpha,\epsilon}_{t,x}(T)\right) + \int_t^T \beta^{\alpha,\epsilon}_{t,x}(s) f\left(X^{\alpha,\epsilon}_{t,x}(s), \alpha(s)\right) ds\right] \ , \\ \beta^{\alpha,\epsilon}_{t,x}(s) \ &:= \ e^{-\int_t^s \rho(X^{\alpha,\epsilon}_{t,x}(r),\epsilon(r)) dL^{\alpha,\epsilon}_{t,x}(r)} \ , \end{aligned}$$

and ρ, g, f are continuous real valued maps on $\bar{\mathcal{O}} \times E$, $\bar{\mathcal{O}}$ and $\bar{\mathcal{O}} \times A$ respectively. In order to ensure that J is well defined, we assume that $\rho \ge 0$. We also assume that
(i) g is Lipschitz continuous,
(ii) f is Lipschitz continuous in its first variable, uniformly in its second one,
(iii) ρ is C^1 with Lipschitz first derivative in its first variable, uniformly in the second one, and Lipschitz in its second variable, uniformly in the first one.

3.2 Dynamic programming

We first provide some useful estimates on $X^{\alpha,\epsilon}_{t,x}$ and J which will be used to derive the dynamic programming principle of Lemma 3.2 below.

Proposition 3.1 *For each $m > 0$, there is a constant $C_m > 0$ such that for all $(\alpha, \epsilon) \in \mathcal{A} \times \mathcal{E}^b_m$, $t \le t' \le T$ and $x, x' \in \bar{\mathcal{O}}$, we have:*

$$\mathbb{E}\left[\sup_{t'\le s\le T} |X^{\alpha,\epsilon}_{t,x}(s) - X^{\alpha,\epsilon}_{t',x'}(s)|^4\right]^{\frac{1}{4}} \ \le \ C_m \left(|x - x'| + |t' - t|^{\frac{1}{4}}\right) \ , \tag{3.2}$$

$$\mathbb{E}\left[\sup_{t\le s\le t'} |X^{\alpha,\epsilon}_{t,x}(s) - x|^4\right]^{\frac{1}{4}} + \mathbb{E}\left[L^{\alpha,\epsilon}_{t,x}(t')^2\right]^{\frac{1}{2}} \ \le \ C_m \, |t' - t|^{\frac{1}{4}} \ , \tag{3.3}$$

$$\mathbb{E}\left[\sup_{t'\le s\le T} |\ln(\beta^{\alpha,\epsilon}_{t,x}(s)) - \ln(\beta^{\alpha,\epsilon}_{t',x'}(s))|\right] \ \le \ C_m \left(|x - x'| + |t' - t|^{\frac{1}{4}}\right) . \tag{3.4}$$

Proof. We write (X, L, β) and (X', L', β') for $(X^{\alpha,\epsilon}_{t,x}, L^{\alpha,\epsilon}_{t,x}, \beta^{\alpha,\epsilon}_{t,x})$ and $(X^{\alpha,\epsilon}_{t',x'}, L^{\alpha,\epsilon}_{t',x'}, \beta^{\alpha,\epsilon}_{t',x'})$.
1. It follows from Lemma 2.1 and Gronwall's Lemma that

$$\mathbb{E}\left[\sup_{t'\le s\le T}|X(s)-X'(s)|^4\right] \quad\le\quad C_m\,\mathbb{E}\left[|X(t')-x'|^4\right]$$

where $C_m>0$ denotes a generic constant independent of (t,t',x,x'). Choosing some large $\bar K>0$, applying Itô's Lemma to $(e^{-\bar K|\epsilon|(t)}\tilde f_\varepsilon(X(t),y,\epsilon(t)))_{t\le T}$, $y\in\bar{\mathcal{O}}$ and $\tilde f_\varepsilon$ defined as in (2.14), and using the same arguments as in Lemma 2.1 (the terms corresponding to A_t and G_t are treated similarly, the term corresponding to H_t is bounded by using the fact that the integrands are bounded by C/ε for some $C>0$ by Lemma 2.2), leads to

$$\mathbb{E}\left[\sup_{t\le s\le t'}|X(s)-y|^4\right]\le C_m\left(|t'-t|+|x-y|^4\right). \tag{3.5}$$

This proves (3.2) and the bound for the first term in (3.3).
2. We now provide the bound for the second term in (3.3). Let h be defined as in Lemma 2.2. Applying Itô's Lemma to $h(X,\epsilon)-h(x,\epsilon)$ and using (2.12) leads to:

$$\begin{aligned}
0\le L(t') \;&\le\; \int_t^{t'}\langle D_xh(X(s),\epsilon(s)),\gamma(X(s),\epsilon(s))\rangle dL(s)\\
&=\; h(X(t'),\epsilon(t'))-h(x,\epsilon(t'))\\
&-\; \int_t^{t'}\left(\langle D_xh(X(s),\epsilon(s)),\mu_s(X(s))\rangle+\frac{1}{2}\mathrm{Tr}[D_x^2h(X(s),\epsilon(s))\sigma_s\sigma_s^*(X(s))]\right)ds\\
&-\; \int_t^{t'}\langle D_xh(X(s),\epsilon(s)),\sigma_s(X(s))dW_s\rangle\\
&-\; \int_t^{t'}\langle D_eh(X(s),\epsilon(s))-D_eh(x,\epsilon(s)),d\epsilon(s)\rangle
\end{aligned}$$

where, by the Lipschitz continuity of D_eh,

$$\left|\int_t^{t'}\langle D_eh(X(s),\epsilon(s))-D_eh(x,\epsilon(s)),d\epsilon(s)\rangle\right| \quad\le\quad C\sup_{t\le s\le t'}|X(s)-x|\,|\epsilon|(T)\,,$$

for some $C>0$ which depends only on h. Since, $|\epsilon|(T)\le m$, the bound for the second term in (3.3) then follows from the Lipschitz continuity of the coefficients, the previous estimates and the boundedness of $\mathcal{O}$ and E.
3. We finally prove (3.4). Since $|\gamma|=1$ and $\rho|\gamma|^2$ is bounded, we have for $s\in[t',T]$

$$\begin{aligned}
&|\ln\beta(s)-\ln\beta'(s)|\\
&=|\int_t^s(\rho|\gamma|^2)(X(s),\epsilon(s))dL(s)-\int_{t'}^s(\rho|\gamma|^2)(X'(s),\epsilon(s))dL'(s)\,|\\
&\le|\int_t^s\langle(\rho\gamma)(X(s),\epsilon(s)),\gamma(X(s),\epsilon(s))\rangle dL(s)-\int_{t'}^s\langle(\rho\gamma)(X(s),\epsilon(s)),\gamma(X'(s),\epsilon(s))\rangle dL'(s)\,|\\
&+|\int_{t'}^s\langle(\rho\gamma)(X(s),\epsilon(s))-(\rho\gamma)(X'(s),\epsilon(s)),\gamma(X'(s),\epsilon(s))\rangle dL'(s)|\\
&\le|\int_{t'}^s\langle\rho\gamma(X(s),\epsilon(s)),\gamma(X(s),\epsilon(s))\rangle dL(s)-\int_{t'}^s\langle\rho\gamma(X(s),\epsilon(s)),\gamma(X'(s),\epsilon(s))\rangle dL'(s)\,|\\
&+C\left(L(t')+\sup_{t'\le s\le T}|X(s)-X'(s)|L'(T)\right)
\end{aligned}$$

for some $C > 0$ independent of (s, x, x', t, t'). If we assume that $\rho \in C^{2,1}(\mathbb{R}^{d+\ell}, \mathbb{R})$, then applying Itô's Lemma to $\langle X - X', \gamma(X, \epsilon)\rho(X, \epsilon)\rangle$ on $[t', T]$ as in 2. and using the Lipschitz continuity of the coefficients and the bound on $|\epsilon|$ allows to show that

$$\mathbb{E}\left[\sup_{t'\le s\le T} \left| \int_{t'}^{s} \langle \rho\gamma(X(s), \epsilon(s)), \gamma(X(s), \epsilon(s))\rangle dL(s) - \int_{t'}^{s} \langle \rho\gamma(X(s), \epsilon(s)), \gamma(X'(s), \epsilon(s))\rangle dL'(s) \right|\right]$$
$$\le C_m\, \mathbb{E}\left[\sup_{t'\le s\le T} |X(s) - X'(s)|^2\right]^{\frac{1}{2}} \left(1 + \mathbb{E}\left[L(T)^2\right]^{\frac{1}{2}}\right)$$

where C_m depends on ρ only through the bounds on $|\rho|$, on the first and second derivatives in its first variable and on the first derivative in its second variable. Thus, by Cauchy-Schwartz inequality,

$$\mathbb{E}\left[\sup_{t'\le s\le T} |\ln\beta(s) - \ln\beta'(s)|\right]$$
$$\le C_m \left(\mathbb{E}\left[L(t')\right] + \mathbb{E}\left[\sup_{t'\le s\le T} |X(s) - X'(s)|^2\right]^{\frac{1}{2}} \left(\mathbb{E}\left[(L'(T))^2\right]^{\frac{1}{2}} + \mathbb{E}\left[(L(T))^2\right]^{\frac{1}{2}} + 1\right)\right) ,$$

for some $C_m > 0$ as above. In view of the previous estimates, the result follows for ρ smooth enough. Since the estimate of (3.3) clearly does not depend on ρ, this result is easily extended to the general case by a standard approximation argument. □

Remark 3.1 It follows from the pathwise uniqueness result of Theorem 2.2 and standard arguments, see e.g. Theorem 5.3.19 and Theorem 5.4.20 of [10], that $X^{\alpha,\varepsilon}_{t,x}$ is a strong Markov process.

Lemma 3.1 *Fix $m > 0$ and set*

$$v_m(t,x) := \sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}^b_m} J(t,x;\alpha,\epsilon)\ \ ,\ (t,x)\in[0,T]\times\bar{\mathcal{O}}\ .$$

Then, there is $C_m > 0$ such that

$$|J(t,x;\alpha,\epsilon) - J(t',x';\alpha,\epsilon)| + |v_m(t,x) - v_m(t',x')| \le C_m \left(|t-t'|^{\frac{1}{4}} + |x - x'|\right)$$

for all $(t,t',x,x') \in [0,T]^2 \times \bar{\mathcal{O}}^2$ and $(\alpha,\epsilon) \in \mathcal{A}\times\mathcal{E}^b_m$. Moreover, $v = \lim_{m\to\infty} \uparrow v_m = \sup_{m>0} v_m$ on $[0,T]\times\bar{\mathcal{O}}$ and v is lower semi-continuous.

Proof. Since

$$|v_m(t,x) - v_m(t',x')| \le \sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}^b_m} |J(t,x;\alpha,\epsilon) - J(t',x';\alpha,\epsilon)|\ ,$$

the first assertion follows from the uniform estimates of Proposition 3.1, the Lipschitz continuity assumptions on the parameter g, f and the fact that $\rho \ge 0$ so that $\beta^{\alpha,\epsilon}_{t,x} \le 1$ for all $(t,x) \in [0,T]\times\bar{\mathcal{O}}$ and $(\alpha,\epsilon) \in \mathcal{A}\times\mathcal{E}$. Clearly $(v_m)_{m>0}$ is non-decreasing and $v \ge \sup_{m>0} v_m$. Thus, it remains to prove that $v \le \sup_{m>0} v_m$, the lower semi-continuous of v will then follow from the continuity of each v_m. To see this fix, $(t,x) \in [0,T)\times\bar{\mathcal{O}}$, $(\alpha,\epsilon) \in \mathcal{A}\times\mathcal{E}$ and set $\tau_m := \inf\{s \in [t,T]\ :\ |\epsilon|(s) \ge m\}$ and $\epsilon_m := \epsilon(\cdot \wedge \tau_m)$, $m > 0$.

Since $\tau_m \to \infty$, we have $\beta_{t,x}^{\alpha,\epsilon_m} f(X_{t,x}^{\alpha,\epsilon_m}, \alpha) \to \beta_{t,x}^{\alpha,\epsilon} f(X_{t,x}^{\alpha,\epsilon}, \alpha)$ $dt \times d\mathbb{P}$-a.e. on $[t,T]$ and $(X_{t,x}^{\alpha,\epsilon_m}, \beta_{t,x}^{\alpha,\epsilon_m})(T) \to (X_{t,x}^{\alpha,\epsilon}, \beta_{t,x}^{\alpha,\epsilon})(T)$ $\mathbb{P}-$ a.s. as $m \to \infty$. By dominated convergence and the continuity of g, we then deduce that $J(t,x;\alpha,\epsilon_m) \to J(t,x;\alpha,\epsilon)$. This implies that, for each $\varepsilon > 0$, we can find $m > 0$ such that $v(t,x) - \varepsilon \le v_m(t,x)$ and therefore $v(t,x) \le \sup_{m>0} v_m(t,x)$. □

We can now prove the following dynamic programming principle.

Lemma 3.2 *Fix $(t,x) \in [0,T] \times \bar{\mathcal{O}}$. For all $[t,T]$-valued stopping time θ, we have*

$$v(t,x) = \sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}} \mathbb{E}\left[\beta_{t,x}^{\alpha,\epsilon}(\theta) v\left(\theta, X_{t,x}^{\alpha,\epsilon}(\theta)\right) + \int_t^\theta \beta_{t,x}^{\alpha,\epsilon}(s) f\left(X_{t,x}^{\alpha,\epsilon}(s), \alpha(s)\right) ds\right] .$$

Proof. Fix $(t_0,x_0) \in [0,T) \times \bar{\mathcal{O}}$ (the case $t_0 = T$ is trivial). The fact that $v(t_0,x_0)$ is bounded from above by

$$\sup_{(\alpha,\epsilon)\in\mathcal{A}\times\mathcal{E}} \mathbb{E}\left[\beta_{t_0,x_0}^{\alpha,\epsilon}(\theta) v\left(\theta, X_{t_0,x_0}^{\alpha,\epsilon}(\theta)\right) + \int_{t_0}^\theta \beta_{t_0,x_0}^{\alpha,\epsilon}(s) f\left(X_{t_0,x_0}^{\alpha,\epsilon}(s), \alpha(s)\right) ds\right]$$

follows from the Markov feature of our model, see Remark 3.1. We now prove the converse inequality.

1. Fix $m > 0$. Let $(B_n)_{n\ge1}$ be a partition of $[0,T] \times \bar{\mathcal{O}}$ and $(t_n,x_n)_{n\ge1}$ be a sequence such that $(t_n,x_n) \in B_n$ for each $n \ge 1$. By definition, we can find $\xi^n := (\alpha^n, \epsilon^n) \in \mathcal{A} \times \mathcal{E}_m^b$ such that

$$J(t_n,x_n;\xi^n) \ge v_m(t_n,x_n) - \varepsilon/3 \ , \tag{3.6}$$

where $\varepsilon > 0$ is a fix parameter. Moreover, by the uniform continuity of v_m and $J(\cdot;\xi)$ for $\xi \in \mathcal{A} \times \mathcal{E}_m^b$, see Lemma 3.1, we can choose $(B_n,t_n,x_n)_{n\ge1}$ in such a way that

$$|J(\cdot;\xi^n) - J(t_n,x_n;\xi^n)| + |v_m - v_m(t_n,x_n)| \le \varepsilon/3 \quad \text{on } B_n \ . \tag{3.7}$$

2. Given $\xi \in \mathcal{A} \times \mathcal{E}_m^b$ and θ a stopping time with values in $[t_0,T]$, we define $\bar{\xi} \in \mathcal{A} \times \mathcal{E}_m^b$ by

$$\bar{\xi}(t) \ := \ \xi(t)\mathbf{1}_{t<\theta} + \mathbf{1}_{t\ge\theta} \sum_{n\ge1} \xi^n(t)\ \mathbf{1}_{\{(\theta, X_{t_0,x_0}^{\xi}(\theta))\in B_n\}} \ .$$

Using successively the Markov feature of our model, see Remark 3.1, (3.7), (3.6), (3.7) again and the fact that $\rho \ge 0$ (which implies that $\beta_{t_0,x_0}^{\cdot}(\theta) \le 1$), we deduce that, for all

$\xi = (\alpha, \epsilon) \in \mathcal{A} \times \mathcal{E}^b$,

$$
\begin{aligned}
J(t_0, x_0; \bar{\xi}) &\geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) J(\theta, X^{\xi}_{t_0,x_0}(\theta); \bar{\xi}) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right] \\
&= \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) \sum_{n\geq 1} J(\theta, X^{\xi}_{t_0,x_0}(\theta); \xi^n) \mathbf{1}_{\{(\theta, X^{\xi}_{t_0,x_0}(\theta)) \in B_n\}}\right] \\
&+ \mathbb{E}\left[\int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right] \\
&\geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) \sum_{n\geq 1} J(t_n, x_n; \xi^n) \mathbf{1}_{\{(\theta, X^{\xi}_{t_0,x_0}(\theta)) \in B_n\}}\right] \\
&+ \mathbb{E}\left[\int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right] - \varepsilon/3 \\
&\geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) \left(\sum_{n\geq 1} v_m(t_n, x_n; \xi^n) \mathbf{1}_{\{(\theta, X^{\xi}_{t_0,x_0}(\theta)) \in B_n\}}\right)\right] \\
&+ \mathbb{E}\left[\int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right] - 2\varepsilon/3 \\
&\geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) v_m(\theta, X^{\xi}_{t_0,x_0}(\theta)) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right] - \varepsilon\ .
\end{aligned}
$$

By arbitrariness of $\varepsilon > 0$, this shows that

$$
v(t_0, x_0) \geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) v_m(\theta, X^{\xi}_{t_0,x_0}(\theta)) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right]\ . \tag{3.8}
$$

Since $v_m \to v$ as $m \to \infty$ by Lemma 3.1, it follows by dominated convergence that, for all $\xi \in \mathcal{A} \times \mathcal{E}^b$,

$$
v(t_0, x_0) \geq \mathbb{E}\left[\beta^{\xi}_{t_0,x_0}(\theta) v(\theta, X^{\xi}_{t_0,x_0}(\theta)) + \int_t^{\theta} \beta^{\xi}_{t_0,x_0}(s) f(X^{\xi}_{t_0,x_0}(s), \alpha(s)) ds\right]\ .
$$

The same localization argument as in the proofs of Theorem 2.2 and Lemma 3.1 then imply that the above inequality actually holds for all $\xi \in \mathcal{A} \times \mathcal{E}$. □

3.3 PDE characterization for the optimal control problem

In this section, we show that v is a solution of

$$
\mathcal{K}\varphi = 0
$$

where

$$
\mathcal{K}\varphi := \begin{cases} \min_{a\in A} \left(-\mathcal{L}^a\varphi - f(\cdot, a)\right) = 0 & \text{on} \quad [0,T) \times \mathcal{O} \\ \min_{e\in E} \mathcal{H}^e\varphi = 0 & \text{on} \quad [0,T) \times \partial\mathcal{O} \\ \varphi - g = 0 & \text{on} \quad \{T\} \times \bar{\mathcal{O}} \end{cases}
$$

and, for a smooth function φ on $[0,T] \times \bar{\mathcal{O}}$ and $(a,e) \in A \times E$, we set

$$
\begin{aligned}
\mathcal{L}^a\varphi &:= \frac{\partial}{\partial t}\varphi + \langle \mu(\cdot, a), D\varphi \rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma(\cdot, a)\sigma(\cdot, a)^* D^2\varphi\right] \\
\mathcal{H}^e\varphi &:= \rho(\cdot, e)\varphi - \langle \gamma(\cdot, e), D\varphi \rangle\ .
\end{aligned}
$$

3.3.1 Definitions

Since v may not be smooth, we need to consider the above equation in the viscosity sense. Moreover, the boundary conditions may not be satisfied in a strong sense and, as usual, we have to consider a relaxed version, see e.g. [4]. We therefore introduce the operator $\mathcal{K}_+$ and $\mathcal{K}_-$ defined as

$$\mathcal{K}_+\varphi \;:=\; \begin{cases} \mathcal{K}\varphi & \text{on} \quad [0,T]\times\mathcal{O} \\ \max\left\{\min_{a\in A} -\mathcal{L}^a\varphi - f(\cdot,a)\;,\; \min_{e\in E}\mathcal{H}^e\varphi\right\} & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \varphi - g & \text{on} \quad \{T\}\times\partial\mathcal{O} \end{cases}$$

and

$$\mathcal{K}_-\varphi \;:=\; \begin{cases} \mathcal{K}\varphi & \text{on} \quad [0,T]\times\mathcal{O} \\ \min\left\{\min_{a\in A} -\mathcal{L}^a\varphi - f(\cdot,a)\;,\; \min_{e\in E}\mathcal{H}^e\varphi\right\} & \text{on} \quad [0,T)\times\partial\mathcal{O} \\ \min\left\{\varphi - g\;,\; \min_{e\in E}\mathcal{H}^e\varphi\right\} & \text{on} \quad \{T\}\times\partial\mathcal{O}\;. \end{cases}$$

Definition 3.1 *We say that a lower-semicontinuous (resp. upper-semicontinuous) function w on $[0,T]\times\bar{\mathcal{O}}$ is a viscosity supersolution (resp. subsolution) of*

$$\mathcal{K}\varphi = 0 \tag{3.9}$$

if for all $\varphi\in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ and all $(t,x)\in[0,T)\times\bar{\mathcal{O}}$ which realizes a local minimum (resp. maximum) of $w-\varphi$ equal to 0, we have $\mathcal{K}_+\varphi\geq 0$ (resp. $\mathcal{K}_-\varphi\leq 0$) We say that a locally bounded function w is a (discontinuous) viscosity solution of (3.9) *if w_* (resp. w^*) is a supersolution (resp. subsolution) of* (3.9) *where*

$$\begin{aligned} w^*(t,x) &:= \limsup_{(t',x')\to(t,x),\,(t',x')\in D} w(t',x') \\ w_*(t,x) &:= \liminf_{(t',x')\to(t,x),\,(t',x')\in D} w(t',x')\;\;,\;\;(t,x)\in[0,T]\times\bar{\mathcal{O}}\;, \end{aligned}$$

with $D:=[0,T)\times\mathcal{O}$.

Remark 3.2 Take $E=\tilde{K}_1:=\tilde{K}\cap\partial B(0,1)$ where $\tilde{K}$ is the domain of the support function δ of a closed convex set $K\subset\mathbb{R}^\ell$, i.e.

$$\delta(e) \;:=\; \sup_{y\in K}\langle y,e\rangle\;\;,\;e\in\mathbb{R}^\ell\;,$$

and assume that $\rho(x,e)=\delta(e)$ and $\gamma(x,e)=e$ on $\partial\mathcal{O}\times E$. Then, for $\varphi\in C^1(\bar{\mathcal{O}},(0,\infty))$, the constraint

$$\min_{e\in E}\mathcal{H}^e\varphi = \min_{e\in E}\left(\delta(e)\varphi - \langle e, D\varphi\rangle\right)\geq 0$$

means that $D\varphi/\varphi\in K$, see e.g. [12]. In this case, the term $\mathcal{H}^e\varphi\geq 0$ can be assimilated to a constraint on the gradient of the logarithm of the solution at the boundary of $\mathcal{O}$. A similar constraint appears in [2], but in the whole domain.

Remark 3.3 Assume that $\mathcal{O}$ is C^2 and that σ satisfies the non-characteristic boundary condition

$$\min_{a\in A} |\sigma(x,a)\xi| > 0 \quad \text{for all} \quad x \in \partial\mathcal{O} \text{ and } \xi \in \mathbb{R}^d \setminus \{0\} \;. \tag{3.10}$$

Then, it follows from the same arguments as in 2. of the proof of Proposition 6.3 of [2] that w is a supersolution of $\mathcal{K}_+\varphi = 0$ only if it is a supersolution of $\bar{\mathcal{K}}_+\varphi = 0$ where

$$\bar{\mathcal{K}}_+\varphi \;\; := \;\; \begin{cases} \mathcal{K}_+\varphi & \text{on} \quad ([0,T]\times\mathcal{O}) \cup (\{T\}\times\bar{\mathcal{O}}) \\ \min_{e\in E} \mathcal{H}^e\varphi & \text{on} \quad [0,T)\times\partial\mathcal{O} \end{cases}$$

Similarly, it follows from the same arguments as in 2. of Proposition 6.6 in [2] that w is a subsolution of $\mathcal{K}_-\varphi = 0$ only if it is a subsolution of $\bar{\mathcal{K}}_-\varphi = 0$ where

$$\bar{\mathcal{K}}_-\varphi \;\; := \;\; \begin{cases} \mathcal{K}_-\varphi & \text{on} \quad ([0,T]\times\mathcal{O}) \cup (\{T\}\times\bar{\mathcal{O}}) \\ \min_{e\in E} \mathcal{H}^e\varphi & \text{on} \quad [0,T)\times\partial\mathcal{O} \;. \end{cases}$$

3.3.2 Super and subsolution properties

Proposition 3.2 *The function v_* is a viscosity supersolution of* (3.9)*.*

Proof. The fact that $v_* \geq g$ on $\{T\}\times\bar{\mathcal{O}}$ is a direct consequence of the lower-semicontinuity of v, see Lemma 3.1. Fix $(t_0,x_0) \in [0,T)\times\bar{\mathcal{O}}$ and $\varphi \in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ such that

$$0 \;=\; (v_*-\varphi)(t_0,x_0) = \min_{[0,T]\times\bar{\mathcal{O}}} (v_*-\varphi) \;.$$

1. We first assume that $(t_0,x_0) \in [0,T)\times\partial\mathcal{O}$ and that

$$\max\left\{ \min_{a\in A} -\mathcal{L}^a\varphi(t_0,x_0) - f(x_0,a) \;,\; \min_{e\in E} \mathcal{H}^e\varphi(t_0,x_0) \right\} =: -2\varepsilon < 0$$

and work toward a contradiction. Under the above assumption, we can find $(a_0,e_0) \in A\times E$ and $\delta \in (t_0, T-t_0)$ for which

$$\max\left\{-\mathcal{L}^{a_0}\varphi - f(\cdot,a_0) \;,\; \mathcal{H}^{e_0}\varphi\right\} \leq -\varepsilon \tag{3.11}$$

on $\bar{B}_0 \cap \bar{D}_0$ where $B_0 := B(t_0,\delta)\times B(x_0,\delta)$ and $D_0 := (t_0-\delta, t_0+\delta)\times\mathcal{O}$. Observe that we can assume, without loss of generality, that (t_0,x_0) achieves a strict local minimum so that

$$\inf_{\partial_p B_0 \cap \bar{D}_0} (v_*-\varphi) =: \zeta > 0 \;, \tag{3.12}$$

where $\partial_p B_0 = ([t_0-\delta, t_0+\delta]\times\partial B(x_0,\delta)) \cup (\{t_0+\delta\}\times B(x_0,\delta))$. Let $(t_k,x_k)_{k\geq 1}$ be a sequence in $B_0 \cap D_0$ satisfying

$$(t_k,x_k) \;\longrightarrow\; (t_0,x_0) \;\text{ and }\; v(t_k,x_k) \;\longrightarrow\; v_*(t_0,x_0) \quad \text{as } k \longrightarrow \infty$$

so that

$$\eta_k := v(t_k,x_k) - \varphi(t_k,x_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty\ . \tag{3.13}$$

Let us write (X^k, L^k, β^k) for $(X^{a_0,e_0}_{t_k,x_k}, L^{a_0,e_0}_{t_k,x_k}, \beta^{a_0,e_0}_{t_k,x_k})$ where (a_0,e_0) is viewed as an element of $\mathcal{A}\times\mathcal{E}$. Set

$$\theta^k := \inf\left\{s \ge t_k \ :\ (s, X^k(s)) \notin B_0\right\} \quad , \quad \vartheta^k := \inf\left\{s \ge t_k \ :\ X^k(s) \notin \mathcal{O}\right\}\ .$$

It then follows from Itô's Lemma, (3.11) and (3.12) that

$$\begin{aligned} v(t_k,x_k) &\le \eta_k + \mathbb{E}\left[\beta^k(\theta^k)v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s),a_0)ds\right] \\ &- \mathbb{E}\left[\zeta \mathbf{1}_{\theta^k<\vartheta^k} + \left(\beta^k(\theta^k)\zeta + \varepsilon L^k(\theta^k)\right)\mathbf{1}_{\theta^k\ge\vartheta^k}\right] \end{aligned}$$

where we used the fact that $\beta^k(\theta^k) = 1$ on $\{\theta^k < \vartheta^k\}$. Let $c > 0$ be such that $|\rho| \le c$ on $\bar{\mathcal{O}} \times E$ and observe that

$$\nu := \inf_{\ell\in[0,\infty)} e^{-c\ell}\zeta + \varepsilon\ell > 0\ .$$

It follows that

$$v(t_k,x_k) \le \eta_k - \zeta\wedge\nu + \mathbb{E}\left[\beta^k(\theta^k)v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s),a_0)ds\right]$$

which leads to a contradiction to Lemma 3.2 for k large enough, recall (3.13).

2. The case where $(t_0,x_0) \in [0,T)\times\mathcal{O}$ is treated similarly. We assume that

$$\min_{a\in A} -\mathcal{L}^a\varphi(t_0,x_0) - f(x_0,a) =: -2\varepsilon < 0\ ,$$

and repeat the above argument with δ small enough so that $B(x_0,\delta) \subset \mathcal{O}$ and therefore $\theta^k < \vartheta^k$ (so that the reflection does not operate on $[t_0,\theta^k]$). □

Proposition 3.3 *The function v^* is a viscosity subsolution of* (3.9).

Proof. Fix $(t_0,x_0) \in [0,T)\times\bar{\mathcal{O}}$ and $\varphi \in C^{1,2}([0,T]\times\bar{\mathcal{O}})$ such that

$$0 = (v^* - \varphi)(t_0,x_0) = \max_{[0,T]\times\bar{\mathcal{O}}} (v^*-\varphi)\ .$$

The case where $(t_0,x_0) \in [0,T)\times\bar{\mathcal{O}}$ is treated by similar arguments as in the proof of Proposition 3.2, see also below. We therefore assume that $t_0 = T$.

1. We first consider the case where $x_0 \in \partial\mathcal{O}$. We assume that

$$\min\left\{\varphi - g\ ,\ \min_{e\in E}\mathcal{H}^e\varphi\right\} =: 2\varepsilon > 0\ .$$

Set $\phi(t,x) = \varphi(t,x) + \sqrt{T-t}$ so that $(\partial/\partial t)\phi(t,x) \to -\infty$ as $t \to T$ and observe that (T,x_0) also achieves a maximum for $v^* - \phi$. Without loss of generality, we can therefore assume that $(\partial/\partial t)\varphi(t,x) \to -\infty$ as $t \to T$ and that we can find $\delta \in (t_0, T-t_0)$ for which

$$\min\left\{\min_{a\in A} -\mathcal{L}^a\varphi - f(\cdot,a)\ ,\ \varphi - g\ ,\ \min_{e\in E}\mathcal{H}^e\varphi\right\} \ge \varepsilon \tag{3.14}$$

on $\bar{B}_0 \cap \bar{D}_0$ where $B_0 := [t_0 - \delta, T) \times B(x_0, \delta)$ and $D_0 := (t_0 - \delta, T) \times \mathcal{O}$. Observe that we can assume, without loss of generality, that (t_0, x_0) achieves a strict local maximum so that

$$\max_{\partial_p B_0 \cap \bar{D}_0} (v^* - \varphi) =: -\zeta < 0 \;, \tag{3.15}$$

where $\partial_p B_0 = ([t_0 - \delta, T] \times \partial B(x_0, \delta)) \cup (\{T\} \times B(x_0, \delta))$. Let $(t_k, x_k)_{k \geq 1}$ be a sequence in $B_0 \cap D_0$ satisfying

$$(t_k, x_k) \;\longrightarrow\; (t_0, x_0) \;\; \text{and} \;\; v(t_k, x_k) \;\longrightarrow\; v^*(t_0, x_0) \quad \text{as } k \longrightarrow \infty$$

so that

$$\eta_k := v(t_k, x_k) - \varphi(t_k, x_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \;. \tag{3.16}$$

Let us write (X^k, L^k, β^k) for $(X^{\alpha,\epsilon}_{t_k,x_k}, L^{\alpha,\epsilon}_{t_k,x_k}, \beta^{\alpha,\epsilon}_{t_k,x_k})$ where (α, ϵ) is a given element of $\mathcal{A} \times \mathcal{E}$. Set

$$\theta^k := \inf \left\{ s \geq t_k \;:\; (s, X^k(s)) \notin B_0 \right\} \quad , \quad \vartheta^k := \inf \left\{ s \geq t_k \;:\; X^k(s) \notin \mathcal{O} \right\} \;.$$

It follows from Itô's Lemma, (3.14), (3.15) and the identity $v(T, \cdot) = g$ that

$$\begin{aligned} v(t_k, x_k) \;\; &\geq \;\; \eta_k + \mathbb{E}\left[\beta^k(\theta^k) v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s), \alpha(s)) ds \right] \\ &\quad + \;\; \mathbb{E}\left[\zeta \mathbf{1}_{\theta^k < \vartheta^k} + \left(\beta^k(\theta^k)(\zeta \wedge \varepsilon) + \varepsilon L^k(\theta^k) \right) \mathbf{1}_{\theta^k \geq \vartheta^k} \right] \;. \end{aligned}$$

Arguing as in 1. of the proof of Proposition 3.2, this implies that

$$\begin{aligned} v(t_k, x_k) \;\; &\geq \;\; \eta_k + \zeta \wedge \nu \\ &\quad + \;\; \mathbb{E}\left[\beta^k(\theta^k) v(\theta^k, X^k(\theta^k)) + \int_{t_k}^{\theta^k} \beta^k(s) f(X^k(s), \alpha(s)) ds \right] \end{aligned}$$

for some $\nu > 0$ independent of (α, ϵ). By arbitrariness of (α, ϵ) and (3.16), this leads to a contradiction to Lemma 3.2 for k large enough.
2. The case where $x_0 \in \mathcal{O}$ is treated similarly, it suffices to take δ small enough so that $B(x_0, \delta) \subset \mathcal{O}$ and therefore $\theta^k < \vartheta^k$. □

3.4 A comparison result

A lot of work has been done so far on comparison results for quasilinear second-order parabolic PDEs with non-linear or oblique derivative Neumann condition, see e.g. [1], [9], [3] or [5] and the references therein. However, as in the three first above papers, they usually require additional smoothness conditions on $\mathcal{O}$ or, as in [5], do not allow for non-linearities at the boundary.
In this section, we provide a comparison theorem for (3.9) in the case where there exist $\bar{e}$ and $\underline{e}$ in E such that

$$\underline{e} \in \arg\min\{\rho(x, e) \;,\; e \in E\} \;,\; \bar{e} \in \arg\max\{\rho(x, e) \;,\; e \in E\} \;\; \forall\, x \in \partial\mathcal{O} \;, \tag{3.17}$$

and additional conditions on the directions of reflection are imposed:
1. As in Section 7.B of [4], we first make an uniform exterior ball assumption in the direction γ:

$$\exists\, b>0 \text{ s.t. } \quad B(x-b\gamma(x,e),b)\cap\mathcal{O} \;=\; \emptyset \quad \text{for all } \;(x,e)\in\partial\mathcal{O}\times E\;. \tag{3.18}$$

2. We then assume that there is a $C^2(\bar{\mathcal{O}})$ function $\hat{h}$ such that

$$\langle \gamma(x,e)\;,\;D\hat{h}(x)\rangle \geq 1 \;\text{ for all }\; x\in\partial\mathcal{O} \quad\text{and}\quad e\in E\;. \tag{3.19}$$

3. The direction $\gamma(\cdot,\bar{e})$ satisfies

$$\inf_{e\in E}\langle \gamma(x,e)\;,\;\gamma(x,\bar{e})\rangle > 0 \;\text{ for all }\; x\in\partial\mathcal{O}\;. \tag{3.20}$$

Remark 3.4 The condition (3.19) holds in the case where E is a singleton, see (2.12). When $\partial\mathcal{O}$ is C^2, i.e. the algebraic distance d to $\partial\mathcal{O}$ is C^2, and

$$\min_{e\in E}\langle \gamma(x,e)\;,\;Dd(x)\rangle \geq \varepsilon \;\text{ for all }\; x\in\partial\mathcal{O}\;,$$

for some $\varepsilon>0$, then we can choose $\hat{h}=\varepsilon^{-1}d$. This imposes a restriction on the direction of reflection with respect to the unit normal inward vector at $x\in\partial\mathcal{O}$.

Under these conditions, we can state the following comparison theorem for super- and subsolutions of (3.9).

Proposition 3.4 *Assume that* (3.17)*,* (3.18)*,* (3.19) *and* (3.20) *hold. Let u (resp. w) be a bounded upper-semicontinuous viscosity subsolution (resp. lower-semicontinuous viscosity supersolution) of* (3.9)*. Then, $u\leq w$ on $[0,T]\times\bar{\mathcal{O}}$.*

Proof. We argue by contradiction and assume that $\max_{\bar{D}}(u-w)>0$, with $D:=[0,T)\times\mathcal{O}$. We can then find $\varepsilon>0$ small enough and $(t_0,x_0)\in\bar{D}$ such that

$$\max_{\bar{D}}(\tilde{u}-\tilde{w}-2\varepsilon H) = (\tilde{u}-\tilde{w}-2\varepsilon H)(t_0,x_0) =: \eta > 0 \tag{3.21}$$

where $\tilde{u}(t,x)=e^{\kappa t}u(t,x)$, $\tilde{w}(t,x)=e^{\kappa t}w(t,x)$ and $H(t,x):=e^{-\kappa t-\hat{h}(x)}$ where $\hat{h}$ is defined as in (3.19) and $\kappa>0$ is a constant parameter such that

$$-\mathcal{L}^a H \geq 0 \quad\text{on } \bar{D} \;\text{ for all } a\in A\;. \tag{3.22}$$

We first asume that

$$u(t_0,x_0)\geq 0\;. \tag{3.23}$$

The case $u(t_0,x_0)<0$ will be treated in 4. below.
Given $\lambda\in\mathbb{N}$, we next define

$$\Phi_\lambda(t,x,y) := \tilde{u}(t,x)-\tilde{w}(t,y)-\Psi_\lambda(t,x,y)$$

where

$$
\begin{aligned}
\Psi_\lambda(t,x,y) &:= \varepsilon(H(t,x)+H(t,y)) + \rho(x_0,\underline{e})u(t_0,x_0)\langle\gamma(x_0,\underline{e})\ ,\ x-y\rangle \\
&\quad +\frac{\lambda}{2}|x-y|^2 + |t-t_0|^2 + |x-x_0|^4\ ,
\end{aligned}
$$

for some $\zeta > 0$.

Let $(t_\lambda, x_\lambda, y_\lambda)$ be a global maximum point for Φ_λ on $\bar{D}$. Using standard arguments, one easily checks that

$$
(t_\lambda,x_\lambda)\to(t_0,x_0)\ ,\ \lambda|x_\lambda-y_\lambda|^2\to 0\ ,\ (\tilde{u}(t_\lambda,x_\lambda),\tilde{w}(t_\lambda,y_\lambda))\to(\tilde{u}(t_0,x_0),\tilde{w}(t_0,x_0)) \tag{3.24}
$$

as $\lambda \to \infty$, see e.g. Lemma 3.1 and Proposition 3.7 in [4].

Moreover, Ishii's Lemma, see Theorem 8.3 in [4], implies that we can find $p_{\lambda,1}, p_{\lambda,2} \in \mathbb{R}$ and two symmetric matrices $X_{\alpha,\lambda}$ and $Y_{\alpha,\lambda}$, depending on a parameter $\alpha > 0$, such that

$$
\begin{aligned}
&(p_{\lambda,1}, D_x\Psi_\lambda(t_\lambda,x_\lambda,y_\lambda), X_{\alpha,\lambda}) \in \bar{\mathcal{P}}^{2,+}_{\bar{\mathcal{O}}}\tilde{u}(t_\lambda,x_\lambda) \\
&(p_{\lambda,2}, -D_y\Psi_\lambda(t_\lambda,x_\lambda,y_\lambda), Y_{\alpha,\lambda}) \in \bar{\mathcal{P}}^{2,-}_{\bar{\mathcal{O}}}\tilde{w}(t_\lambda,y_\lambda)
\end{aligned} \tag{3.25}
$$

and

$$
p_{\lambda,1} - p_{\lambda,2} = 2(t_\lambda - t_0) - \kappa\varepsilon(H(t_\lambda,x_\lambda)+H(t_\lambda,y_\lambda)) \tag{3.26}
$$

$$
\begin{pmatrix} X_{\alpha,\lambda} & 0 \\ 0 & -Y_{\alpha,\lambda} \end{pmatrix} \le (A_\lambda + B_\lambda) + \alpha\,(A_\lambda+B_\lambda)^2 \tag{3.27}
$$

where

$$
\begin{aligned}
A_\lambda &:= \varepsilon\begin{pmatrix} D^2H(t_\lambda,x_\lambda) & 0 \\ 0 & D^2H(t_\lambda,y_\lambda)\end{pmatrix} + 12(x_\lambda-x_0)\otimes(x_\lambda-x_0) \\
B_\lambda &:= \lambda\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}\ ,
\end{aligned}
$$

see [4] for the notations $\bar{\mathcal{P}}^{2,+}_{\bar{\mathcal{O}}}$ and $\bar{\mathcal{P}}^{2,-}_{\bar{\mathcal{O}}}$.

1. Assume that $x_\lambda \in \partial\mathcal{O}$. Fix $e \in E$. Since $y_\lambda \in \bar{\mathcal{O}}$, it follows from (3.18) that $|x_\lambda - b\gamma(x_\lambda,e) - y_\lambda|^2 \ge b^2$. Since $|\gamma| = 1$, this implies

$$
2\langle\gamma(x_\lambda,e)\ ,\ y_\lambda - x_\lambda\rangle \ge -b^{-1}|x_\lambda-y_\lambda|^2\ . \tag{3.28}
$$

Then, it follows from the definition of $\underline{e}$, the fact that $|\gamma| = 1$, the assumptions $\rho \ge 0$, (3.23), (3.17), (3.19), (3.24) and (3.28) that

$$
\begin{aligned}
&\rho(x_\lambda,e)u(t_\lambda,x_\lambda) - \langle\gamma(x_\lambda,e)\ ,\ D_x\Psi_\lambda(t_\lambda,x_\lambda,y_\lambda)\rangle \\
&= (\rho(x_0,e)-\rho(x_0,\underline{e}))u(t_0,x_0) + \rho(x_0,\underline{e})u(t_0,x_0)(1-\langle\gamma(x_0,e)\ ,\ \gamma(x_0,\underline{e})\rangle) \\
&+O(\lambda^{-1}) - \langle\gamma(x_\lambda,e)\ ,\ \lambda(x_\lambda-y_\lambda) - \varepsilon D\hat{h}(x_\lambda)H(t_\lambda,x_\lambda)\rangle \\
&\ge O(\lambda^{-1}) + \varepsilon H(t_0,x_0)\ .
\end{aligned}
$$

Arguing as above, using the inequalities $\rho \geq 0$, $u(t_0, x_0) \geq w(t_0, x_0)$ and observing that $\langle \gamma(y_\lambda, \underline{e})\ ,\ \gamma(x_0, \underline{e}) \rangle \to 1$, we also deduce that, if $y_\lambda \in \partial\mathcal{O}$,

$$\begin{aligned}
&\rho(y_\lambda, \underline{e}) w(t_\lambda, y_\lambda) - \langle \gamma(y_\lambda, \underline{e})\ ,\ -D_y \Psi_\lambda(t_\lambda, x_\lambda, y_\lambda) \rangle \\
&\leq \rho(x_0, \underline{e})(w(t_0, x_0) - u(t_0, x_0)) - \varepsilon H(t_0, x_0) + O(\lambda^{-1}) \\
&\leq -\varepsilon H(t_0, x_0) + O(\lambda^{-1})\ ,
\end{aligned}$$

2. We now assume that, up to a subsequence, $t_\lambda = T$ for all $\lambda \in \mathbb{N}$. By 1. and the fact that $H(t_0, x_0) > 0$, we must have $u(t_\lambda, x_\lambda) \leq g(x_\lambda)$ and $g(y_\lambda) \leq w(t_\lambda, y_\lambda)$. Since g is continuous, we deduce from (3.24) that $u(t_0, x_0) \leq w(t_0, w_0)$ which contradicts (3.21), recall that $H > 0$.

3. The rest of the proof is standard. We first observe that $\tilde{u}$ and $\tilde{w}$ are viscosity super- and subsolutions of $\tilde{\mathcal{K}}_+ \varphi = 0$ and $\tilde{\mathcal{K}}_- \varphi = 0$ where $\tilde{\mathcal{K}}_+$ and $\tilde{\mathcal{K}}_-$ are defined as $\mathcal{K}_+$ and $\mathcal{K}_-$ with $\mathcal{L}^a$ replaced by $\tilde{\mathcal{L}}^a$ defined by

$$\tilde{\mathcal{L}}^a \varphi = -\kappa\varphi + \mathcal{L}^a \varphi\ .$$

In view of 1., 2. and $H(t_0, x_0) > 0$, we may find a_λ in the compact set A such that, after possibly passing to a subsequence,

$$\begin{aligned}
0 &\geq \kappa \tilde{u}(t_\lambda, x_\lambda) - p_{\lambda,1} - \langle \mu(t_\lambda, x_\lambda, a_\lambda), D_x \Psi_\lambda(t_\lambda, x_\lambda, y_\lambda) \rangle \\
&- \frac{1}{2} \mathrm{Tr}\left[\sigma\sigma^*(t_\lambda, x_\lambda, a_\lambda) X_{\eta,\lambda}\right] - f(t_\lambda, x_\lambda, a_\lambda) \\
0 &\leq \kappa \tilde{w}(t_\lambda, y_\lambda) - p_{\lambda,2} - \langle \mu(t_\lambda, y_\lambda, a_\lambda), -D_y \Psi_\lambda(t_\lambda, x_\lambda, y_\lambda) \rangle \\
&- \frac{1}{2} \mathrm{Tr}\left[\sigma\sigma^*(t_\lambda, y_\lambda, a_\lambda) Y_{\eta,\lambda}\right] - f(t_\lambda, y_\lambda, a_\lambda)\ .
\end{aligned}$$

Taking the difference of these two equations, using (3.21) and (3.22), the Lipschitz continuity of the coefficients and the fact that A and $\mathcal{O}$ are bounded, (3.26) and (3.27) leads to

$$\begin{aligned}
\kappa\eta + O(\lambda^{-1}) &\leq \kappa\left(\tilde{u}(t_\lambda, x_\lambda) - \tilde{w}(t_\lambda, y_\lambda)\right) \\
&\leq O\left(|t_\lambda - t_0| + \lambda |x_\lambda - y_\lambda|^2 + |x_\lambda - x_0|^2 + C_\lambda \alpha\right)
\end{aligned}$$

where $C > 0$ is independent of λ and α and C_λ depends only on λ. Sending $\alpha \to 0$ and then $\lambda \to \infty$ thus leads to a contradiction, recall (3.24).

4. The case $u(t_0, x_0) < 0$ is treated similarly. It suffices to consider the test function

$$\begin{aligned}
\Psi_\lambda(t, x, y) &:= \varepsilon(H(t, x) + H(t, y)) + \tilde{b}^{-1} \rho(x_0, \bar{e}) u(t_0, x_0) \langle \gamma(x_0, \tilde{e})\ ,\ x - y \rangle \\
&+ \frac{\lambda}{2}|x - y|^2 + |t - t_0|^2 + |x - x_0|^4\ ,
\end{aligned}$$

where $\bar{e}$ is defined in (3.17), $\tilde{b} > 0$ and $\tilde{e} \in E$ satisfy

$$\min_{e \in E} \langle \gamma(x_0, \bar{e})\ ,\ \gamma(x_0, e) \rangle = \langle \gamma(x_0, \bar{e})\ ,\ \gamma(x_0, \tilde{e}) \rangle = \tilde{b}$$

recall (3.20). With this modification, the arguments of 1. becomes

$$\begin{aligned}
&\rho(x_\lambda, e)u(t_\lambda, x_\lambda) - \langle \gamma(x_\lambda, e)\ ,\ D_x\Psi_\lambda(t_\lambda, x_\lambda, y_\lambda)\rangle \\
&= (\rho(x_0, e) - \rho(x_0, \bar{e}))u(t_0, x_0) + \rho(x_0, \bar{e})u(t_0, x_0)(1 - \tilde{b}^{-1}\langle \gamma(x_0, e)\ ,\ \gamma(x_0, \tilde{e})\rangle) \\
&+O(\lambda^{-1}) - \langle \gamma(x_\lambda, e)\ ,\ \lambda(x_\lambda - y_\lambda) - \varepsilon D\hat{h}(x_\lambda)H(t_\lambda, x_\lambda)\rangle \\
&\geq O(\lambda^{-1}) + \varepsilon H(t_0, x_0)
\end{aligned}$$

in the case where $x_\lambda \in \partial\mathcal{O}$, and

$$\begin{aligned}
&\rho(y_\lambda, \bar{e})w(t_\lambda, y_\lambda) - \langle \gamma(y_\lambda, \bar{e})\ ,\ -D_y\Psi_\lambda(t_\lambda, x_\lambda, y_\lambda)\rangle \\
&\leq \rho(x_0, \bar{e})(w(t_0, x_0) - u(t_0, x_0)) \\
&+\rho(x_0, \bar{e})u(t_0, x_0)(1 - \tilde{b}^{-1}\langle \gamma(x_0, \bar{e})\ ,\ \gamma(x_0, \tilde{e})\rangle) - \varepsilon H(t_0, x_0) + O(\lambda^{-1}) \\
&\leq -\varepsilon H(t_0, x_0) + O(\lambda^{-1})
\end{aligned}$$

in the case where $y_\lambda \in \partial\mathcal{O}$. The rest of the proof is similar. □

Remark 3.5 Observe that the right-hand side part of the condition (3.17) and (3.20) are only used in step 4. of the above proof to treat the case $u(t_0, x_0) < 0$. It is therefore not required if $u \geq 0$ on $[0, T) \times \partial\mathcal{O}$. Similarly, it can be dropped if $w \geq 0$ on $[0, T) \times \partial\mathcal{O}$ since, in this case, (3.21) also implies that $u(t_0, x_0) \geq 0$.

Remark 3.6 Assume that

$$\mu(x, a) = \mathrm{diag}\,[x]\,\bar{\mu}(x, a)\ ,\ \sigma(x, a) = \mathrm{diag}\,[x]\,\bar{\sigma}(x, a) \quad \text{on } \mathbb{R}^d_+ \times A$$

and

$$\gamma(x, e) = \mathrm{diag}\,[x]\,\bar{\gamma}(x, e) \quad \text{on } (\partial\mathcal{O} \cap (0, \infty)^d) \times E$$

with $\bar{\mu}$, $\bar{\sigma}$ and $\bar{\gamma}$ such that μ, σ and γ satisfy the general assumptions of this section. Then, the process $X^{\alpha,\epsilon}_{t,x}$ takes values in $(0, \infty)^d$ whenever $x \in (0, \infty)^d$. It is therefore natural to consider the PDE $\mathcal{K}\varphi = 0$ on $[0, T] \times (\bar{\mathcal{O}} \cap (0, \infty)^d)$, with a notion of viscosity solution similar to the one of Definition 3.1 with $\mathcal{O}$, $\partial\mathcal{O}$ and $\bar{\mathcal{O}}$ replaced by $\mathcal{O}^* := \mathcal{O} \cap (0, \infty)^d$, $\partial\mathcal{O}^* := \partial\mathcal{O} \cap (0, \infty)^d$ and $\bar{\mathcal{O}}^* := \bar{\mathcal{O}} \cap (0, \infty)^d$.
The proof of Proposition 3.2 and Proposition 3.3 are easily adapted to this context. We therefore obtain that v is a viscosity solution of $\mathcal{K}\varphi = 0$ on $[0, T] \times \bar{\mathcal{O}}^*$. Moreover, the proof of the comparison principle of Proposition 3.4 can also be extended. It suffices to add an additional penalty function of the form $k \sum_{i \leq d} |x^i|^{-1}$, with $k \to \infty$, as in [2].

Remark 3.7 The smoothness assumptions on ρ and γ are only used either to construct $(X^{\alpha,\epsilon}_{t,x}, L^{\alpha,\epsilon}_{t,x})$ or to prove the dynamic programming principle of Lemma 3.2. We shall see through an example in Section 4.3 below how they can be relaxed.

4 Application to the pricing of barrier options under constraints

As already stated in the introduction, the main motivation comes from applications in mathematical finance. More precisely, [2] provides a PDE characterization of the super-hedging price of barrier options under portfolio constraints which is very similar to the equation $\mathcal{K}\varphi = 0$ up to an additional term inside the domain $\mathcal{O}$ which imposes a constraint on the gradient of the logarithm of the solution.
The aim of this section is to show that the super-hedging price of barrier options under portfolio constraints can actually admit a dual formulation in terms of an optimal control problem for a reflected diffusion in which the direction of reflection is controlled. Due to the additional term which appears in the PDE of [2], we can not expect this result to be general and we shall restrict to a Black and Scholes type model, see below.

In order to simplify the presentation, we shall work under quite restrictive conditions, assuming for instance that the equation $\mathcal{K}\varphi = 0$ admits a sufficiently smooth solution for a suitable choice of parameters. The general case is left for further research.

4.1 Problem formulation

We briefly present the hedging problem. Details can be found in [2] and the references contained in this paper.

We consider a financial market which consists of one non-risky asset, whose price process is normalized to unity, and d risky assets $S_{t,x} = (S^i_{t,x})_{i \le d}$ which solve on $[t, T]$

$$S_{t,x}(s) = x + \int_t^s \text{diag}\left[S_{t,x}(r)\right] \Sigma \, dW(r)$$

where Σ is a d-dimensionnal invertible matrix. A financial strategy is described by a d-dimensional predictable process $\pi = (\pi^1,...,\pi^d)$ (viewed as a line vector) satisfying the integrability condition

$$\int_0^T |\pi(s)|^2 ds \ < \ \infty \qquad \mathbb{P} - \text{a.s.} \tag{4.1}$$

where $\pi^i(s)$ is the proportion of wealth invested at time s in the risky asset $S^i_{t,x}$. To an initial capital $y \in \mathbb{R}$ and a financial strategy π, we associate the induced wealth process $Y^{\pi}_{t,y}$ which solves on $[t, T]$

$$Y(s) \ = \ y + \int_t^s Y(r)\pi(r)\text{diag}\left[S_{t,x}(r)\right]^{-1} dS_{t,x}(r) = y + \int_t^s Y(r)\pi(r)\Sigma \, dW(r) \; . \tag{4.2}$$

In this paper, we restrict to the case where the proportion invested in the risky asset are constrained to be bounded from below. Given $m^i > 0$, $i \le d$, we set

$$K \ := \ \prod_{i=1}^{d} [-m^i, \infty)$$

and denote by Π_K the set of financial strategies π and satisfying

$$\pi \in K \ \ dt \times d\mathbb{P} - \text{a.e.} \tag{4.3}$$

We consider an up-and-out type option. More precisely, we take $\mathcal{O}$ such that

$$\mathcal{O}^* := \mathcal{O} \cap (0,\infty)^d \quad = \quad \left\{ x \in (0,\infty)^d \ : \ \sum_{i=1}^d x^i < \kappa \right\} \ , \ \kappa > 0 \ .$$

The "pay-off" of the barrier option is a continuous map g defined on $\mathbb{R}^d_+$ satisfying

$$g \geq 0 \ \text{ on } \mathcal{O}^* \quad \text{ and } \quad g = 0 \ \text{ on } \partial\mathcal{O}^* := \partial\mathcal{O} \cap (0,\infty)^d \ . \tag{4.4}$$

In order to apply the general results of [2], we assume that the map $\hat{g}$ defined by

$$\hat{g}(x) \quad = \quad \sup_{y \in \mathbb{R}^d_-} e^{-\delta(y)} g(x^1 e^{y^1}, \ldots, x^d e^{y^d}) \ , \ x \in \bar{\mathcal{O}}^* := \bar{\mathcal{O}} \cap (0,\infty)^d$$

is continuous. Here, δ is the support function of K, see Remark 3.2. We also assume that $\hat{g}$ is almost everywhere differentiable on $\bar{\mathcal{O}}^*$ and we denote by $D\hat{g}$ its gradient, when it is well defined.

Remark 4.1 One easily checks that

$$\hat{g}(x) \quad = \quad \sup_{y \in \mathbb{R}^d_-} e^{-\delta(y)} \hat{g}(x^1 e^{y^1}, \ldots, x^d e^{y^d}) \ , \ x \in \bar{\mathcal{O}}^* \ ,$$

see [2], which implies

$$\inf \left\{ \delta(e)\hat{g}(x) - \langle e \ , \ \text{diag}\,[x]\, D\hat{g}(x) \rangle, \ e \in \tilde{K}_1 \right\} \geq 0$$

for all $x \in \bar{\mathcal{O}}^*$ where $D\hat{g}$ is well defined. Here, $\tilde{K}_1 := \mathbb{R}^d_- \cap \partial B(0,1)$ is the set of unit elements of the domain of δ, see Remark 3.2.

The option pays $g(S_{t,x}(T))$ at T if and only if $S_{t,x}$ does not exit $\mathcal{O}^*$ before T. Since $S_{t,x}$ has positive components, this corresponds to the situation where

$$\tau_{t,x} \quad := \quad \inf\{ s \in [t,T] \ : \ S_{t,x}(s) \notin \mathcal{O} \} > T \ ,$$

with the usual convention $\inf \emptyset = \infty$.

The super-replication cost of the barrier option is then defined as the minimal initial dotation y such that $Y^\pi_{t,y}(T) \geq g(S_{t,x}(T)) \mathbf{1}_{T < \tau_{t,x}}$ for some suitable strategy $\pi \in \Pi_K$. This leads to the introduction of the value function defined on $[0,T] \times \bar{\mathcal{O}}^*$ by

$$w(t,x) \quad := \quad \inf \left\{ y \in \mathbb{R} \ : \ Y^\pi_{t,y}(T) \geq g(S_{t,x}(T)) \mathbf{1}_{T < \tau_{t,x}} \quad \text{for some } \pi \in \Pi_K \right\} \ . \tag{4.5}$$

4.2 PDE characterization

We define $\mathcal{L}$ as $\mathcal{L}^0$ with $A = \{0\}$, $\mu = 0$, $\sigma(x, \cdot) = \text{diag}\,[x]\,\Sigma$ and $f = 0$. The next result is a consequence of [2].

Theorem 4.1 (Bentahar and Bouchard [2]) *The value function w is the unique viscosity solution in the class of bounded functions on $[0, T] \times (\bar{\mathcal{O}} \cap \mathbb{R}^d_+)$ of $\mathcal{G}\varphi = 0$ where $\mathcal{G}\varphi$ equals*

$$\begin{cases} \min\left\{-\mathcal{L}\varphi(t,x)\ ,\ \min\limits_{e\in\tilde{K}_1} \left(\delta(e)\varphi(t,x) - \langle e, \text{diag}\,[x]\, D\varphi(t,x)\rangle\right)\right\} & on \quad [0,T)\times\mathcal{O}^* \\ \min\left\{\varphi\ ,\ \min\limits_{e\in\tilde{K}_1} \left(\delta(e)\varphi(t,x) - \langle e, \text{diag}\,[x]\, D\varphi(t,x)\rangle\right)\right\} & on \quad [0,T)\times\partial\mathcal{O}^* \\ \varphi - \hat{g} & on \quad \{T\}\times\bar{\mathcal{O}}^*\ . \end{cases}$$

In the above theorem, the notion of viscosity solution has to be taken in the classical sense. When the equation (4.6)-(4.7)-(4.8) below admits a sufficiently smooth solution, the above equation can be simplified as follows.

Proposition 4.1 *Assume that there is a bounded non-negative $C^{1,3}([0,T)\times\mathcal{O}^*)\cap C^{0,1}([0,T)\times\bar{\mathcal{O}}^*)\cap C([0,T]\times\bar{\mathcal{O}}^*)$ function ψ such that $\partial\psi/\partial t \in C^{0,1}([0,T)\times\bar{\mathcal{O}}^*)$ and satisfying*

$$-\mathcal{L}\psi(t,x) = 0 \ \ on\ [0,T)\times\mathcal{O}^* \tag{4.6}$$

$$\min_{e\in\tilde{K}_1} \left(\delta(e)\psi(t,x) - \langle e, \text{diag}\,[x]\, D\psi(t,x)\rangle\right) = 0 \ \ on\ [0,T)\times\partial\mathcal{O}^* \tag{4.7}$$

$$\psi = \hat{g} \ \ on\ \{T\}\times\bar{\mathcal{O}}^* \tag{4.8}$$

$$\lim_{\substack{(t',x')\to(T,x)\\ (t',x')\in[0,T)\times\mathcal{O}^*}} D\psi(t',x') = D\hat{g}(x) \ \ almost\ everywhere\ on\ \bar{\mathcal{O}}^*\ . \tag{4.9}$$

Then, $\psi = w$ on $[0,T)\times\mathcal{O}^*$ and ψ is the unique bounded solution to (4.6)-(4.7)-(4.8) on $[0,T]\times\bar{\mathcal{O}}^*$.

Proof. In view of Theorem 4.1, it suffices to show that ψ is a solution of $\mathcal{G}\varphi = 0$. Clearly, it is a subsolution. Since $\psi \geq 0$, the supersolution property holds if, in addition to (4.6)-(4.7)-(4.8), we have

$$\min_{e\in\tilde{K}_1} \left(\delta(e)\psi(t,x) - \langle e, \text{diag}\,[x]\, D\psi(t,x)\rangle\right) \geq 0 \quad \text{on} \quad [0,T)\times\mathcal{O}^*\ . \tag{4.10}$$

To see this, observe that (4.6) implies that each component $\phi^k := (D\psi)^k$ of $D\psi$ solves on $[0,T)\times\mathcal{O}^*$

$$-\frac{\partial}{\partial t}\phi^k(t,x) - \frac{1}{2}\text{Tr}\left[\text{diag}\,[x]\,\Sigma\Sigma'\text{diag}\,[x]\, D^2\phi^k(t,x)\right] - \langle D\phi^k(t,x)^*\text{diag}\,[x]\,\Sigma\ ,\ \Sigma^k\rangle = 0$$

where Σ^k denotes the k-th line of Σ. Applying Itô's Lemma to $\langle e, \text{diag}\,[S_{t,x}]\, D\psi(\cdot, S_{t,x})\rangle$, $e \in \tilde{K}_1$ and $(t,x) \in [0,T)\times\mathcal{O}^*$, and using (4.9), we deduce that

$$\begin{aligned} \langle e, \text{diag}\,[x]\, D\psi(t,x)\rangle &= \mathbb{E}\left[\langle e, \text{diag}\,[S_{t,x}(\tau_{t,x})]\, D\psi(\tau_{t,x}, S_{t,x}(\tau_{t,x}))\rangle \mathbf{1}_{\tau_{t,x}<T}\right] \\ &+ \ \mathbb{E}\left[\langle e, \text{diag}\,[S_{t,x}(T)]\, D\hat{g}(S_{t,x}(T))\rangle \mathbf{1}_{\tau_{t,x}\geq T}\right]\ . \end{aligned}$$

Since by (4.6) and (4.8)

$$\psi(t,x) = \mathbb{E}\left[\psi(\tau_{t,x}, S_{t,x}(\tau_{t,x}))\mathbf{1}_{\tau_{t,x}<T} + \hat{g}(S_{t,x}(T))\mathbf{1}_{\tau_{t,x}\geq T}\right] \ ,$$

it follows from (4.7) and Remark 4.1 that

$$\delta(e)\psi(t,x) - \langle e, \mathrm{diag}\,[x]\, D\psi(t,x)\rangle \geq 0$$

which, by arbitrariness of e, provides the required result. □

4.3 Dual formulation

The equation (4.6)-(4.7)-(4.8) is very similar to $\mathcal{K}\varphi = 0$ with $E = \tilde{K}_1$ and

$$\rho(x,e) := \delta(e)/|\mathrm{diag}\,[x]\, e| \ , \ \gamma(x,e) = \mathrm{diag}\,[x]\, e/|\mathrm{diag}\,[x]\, e| \ .$$

However, the gradient of $\mathrm{diag}\,[x]\, e/|\mathrm{diag}\,[x]\, e|$ may blow up near $\partial(0,\infty)^d$ and it is not possible to consider a smooth extension of γ on $\mathbb{R}^{2d}$ (even on $\mathbb{R}^d_+ \times \tilde{K}_1$).
In order to surround this difficulty, we use the following construction. First we define $\mathcal{O}$ as

$$\mathcal{O} := \{x \in \mathbb{R}^d \ : \ \sum_{i=1}^{d} |x^i| < \kappa\}$$

so that $\mathcal{O}^* = \{x \in (0,\infty)^d \ : \ \sum_{i=1}^d x^i < \kappa\}$. Let $r \in (0,1/2)$ be such that $B(0,2r) \subset \mathcal{O}$. Then, given a non-increasing $C^2(\mathbb{R},[0,1])$ function ϕ such that $\phi(y) = 1$ if $y \leq 1$ and $\phi(y) = 0$ if $y \geq 3/2$, we set, for $n \geq 1$,

$$z_n(e) \ := \ \left(e^i\phi(ne^i+2) - (1-\phi(ne^i+2))\right)_{i\leq d} \ .$$

Observe that $z_n(e) = e$ on $E_n := \{e \in \tilde{K}_1 \ : \ e^i \leq -n^{-1}\ \forall\, i \leq d\}$, $z_n(e) \in (-\infty,-1/(2n)]^d$ for all $e \in \mathbb{R}^d$, and

$$|\mathrm{diag}\,[x]\, e| \geq r/(2n) := \eta_n \quad \text{for } (x,e) \in B(0,r)^c \times (-\infty,-1/(2n)]^d \ . \tag{4.11}$$

We then set (with $1_d = (1,\ldots,1) \in \mathbb{R}^d$)

$$\begin{aligned}
\bar{\gamma}_n(x,e) \ &:= \ \mathrm{diag}\,[x]\, z_n(e)\left(1-\phi\left(\frac{3}{2}|\mathrm{diag}\,[x]\, e|/\eta_n\right)\right) - 1_d\phi\left(\frac{3}{2}|\mathrm{diag}\,[x]\, e|/\eta_n\right)\\
\gamma_n(x,e) \ &:= \ \bar{\gamma}_n(x,e)/|\bar{\gamma}_n(x,e)| \ .
\end{aligned}$$

Using (4.11), one easily checks that $\gamma_n \in C^2(\mathbb{R}^{2d},\mathbb{R}^d)$. Moreover, $\gamma_n(x,e) = \gamma(x,e) = \mathrm{diag}\,[x]\, e/|\mathrm{diag}\,[x]\, e|$ on $B(0,r)^c \times E_n$ and (2.3) holds for $(\mathcal{O},\gamma_n)$.

For $\epsilon \in \mathcal{E}_0 := \cup_{n\geq 1}\ \mathrm{BV}_{\mathbb{F}}(E_n)$ and $(t,x) \in [0,T] \times \bar{\mathcal{O}}^*$, we can then define $(X^\epsilon_{t,x}, L^\epsilon_{t,x}) := (X^{0,\epsilon}_{t,x}, L^{0,\epsilon}_{t,x})$ as in Section 3 with $\mu = 0$, $\sigma(x,a) = \mathrm{diag}\,[x]\,\Sigma$ and γ defined as above. Clearly, $X^\epsilon_{t,x}$ takes values in $(0,\infty)^d$.

We next define ρ on $\mathbb{R}^d \times \tilde{K}_1$ as

$$\rho(x,e) \ = \ (\delta(e)/|\mathrm{diag}\,[x]\, e|)(1-\phi(|x|/r+1/2))$$

so that ρ is continuous on $\mathbb{R}^d \times \tilde{K}_1$, satisfies the assumption of Section 3 as a function on $\mathbb{R}^d \times E_n$, for all $n \geq 1$, and

$$\rho(x,e) \quad = \quad \delta(e)/|\mathrm{diag}\,[x]\,e| \quad \text{on} \ \ \partial\mathcal{O}^* \times (\cup_{n\geq 1} E_n) \ .$$

With this construction, we can now consider the control problem

$$v(t,x) \quad := \quad \sup_{\epsilon \in \mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T)\right)\right] \ , \ (t,x) \in [0,T] \times \bar{\mathcal{O}}^* \ .$$

Proposition 4.2 *The function v is a bounded viscosity solution on $[0,T] \times \bar{\mathcal{O}}^*$ of* (4.6)-(4.7)-(4.8)*.*

Proof. For $n \geq 1$ and $(t,x) \in [0,T] \times \bar{\mathcal{O}}^*$, set

$$v_n(t,x) \quad := \quad \sup_{\epsilon \in \mathcal{E}_n} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T)\right)\right]$$

where $\mathcal{E}_n := \mathrm{BV}_{\mathbb{F}}(E_n)$. It follows from the previous discussion that we can apply Lemma 3.2 to v_n. Since, $v = \sup_{n\geq 1} v_n = \lim_{n\to\infty} \uparrow v_n$, a monotone convergence argument shows that the dynamic programming principle of Lemma 3.2 holds for v. Following the arguments used in Proposition 3.2 and Proposition 3.3, and using the continuity of ρ and γ on $(B(0,r)^c \cap (0,\infty)^d) \times \tilde{K}_1 \supset \partial\mathcal{O}^* \times \tilde{K}_1$, we deduce that v is a viscosity solution of $\mathcal{K}\varphi = 0$ on $[0,T] \times \bar{\mathcal{O}}^*$, see Remark 3.6. Since

$$\delta(e)y - \langle e, \mathrm{diag}\,[x]\,p\rangle \geq 0 \quad \Leftrightarrow \quad |\mathrm{diag}\,[x]\,e|^{-1}\left(\delta(e)y - \langle e, \mathrm{diag}\,[x]\,p\rangle\right) \geq 0$$

for $(x,e,y,p) \in \partial\mathcal{O}^* \times \tilde{K}_1 \times \mathbb{R} \times \mathbb{R}^d$, this implies that v is a viscosity solution on $[0,T] \times \bar{\mathcal{O}}^*$ of (4.6)-(4.7)-(4.8). □

In view of Proposition 4.1, we finally obtain the main result of this section which provides a dual formulation for the super-hedging price w.

Theorem 4.2 *Let the conditions of Proposition 4.1 holds. Then, for all $(t,x) \in [0,T] \times \mathcal{O}^*$,*

$$w(t,x) = \sup_{\epsilon \in \mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T)\right)\right] \ . \tag{4.12}$$

Remark 4.2 It follows from Theorem 4.1, Proposition 4.2 and Theorem 7.1 in [2] that

$$w(t,x) \geq \sup_{\epsilon \in \mathcal{E}_0} \mathbb{E}\left[e^{-\int_t^T \rho(X^\epsilon_{t,x}(s),\epsilon(s))dL^\epsilon_{t,x}(s)} \hat{g}\left(X^\epsilon_{t,x}(T)\right)\right]$$

even if the conditions of Proposition 4.1 are not satisfied.

Remark 4.3 When $d = 1$, we retrieve the results of [13], see also [14]. In this case, $\mathcal{E}_0 = \{-1\}$ and the right hand-side quantity in (4.12) can be computed by using Monte-Carlo methods.

Remark 4.4 It follows from [2], that w admits the dual formulation

$$w(t,x) = \sup_{\vartheta \in \Theta} \mathbb{E}^{\vartheta}\left[e^{-\int_t^T \delta(\vartheta(s))ds}\hat{g}\left(S_{t,x}(T)\right)\mathbf{1}_{\tau_{t,x}>T}\right]$$

where Θ denotes the set of bounded adapted processes with values in $\mathbb{R}^d_-$ and $\mathbb{E}^{\vartheta}$ is the expectation operator under the equivalent probability measure $\mathbb{Q}^{\vartheta}$ under which the process W^{ϑ} defined by

$$W^{\vartheta}(t) \;=\; W(t) - \int_0^t \Sigma^{-1}\vartheta(s)ds \quad t \le T\;,$$

is a Brownian motion. Since

$$S_{t,x}(s) \;=\; x + \int_t^s \mathrm{diag}\left[S_{t,x}(r)\right]\Sigma dW^{\vartheta}(r) + \int_t^s \mathrm{diag}\left[S_{t,x}(r)\right]\vartheta(r)dr\;,$$

and W^{ϑ} has the same law under $\mathbb{Q}^{\vartheta}$ than W under $\mathbb{P}$, this is, at least formally, equivalent to:

$$w(t,x) = \sup_{\vartheta \in \Theta} \mathbb{E}\left[e^{-\int_t^T \delta(\vartheta(s))ds}\hat{g}\left(S^{\vartheta}_{t,x}(T)\right)\mathbf{1}_{\tau^{\vartheta}_{t,x}>T}\right]$$

with $S^{\vartheta}_{t,x}$ now defined as the solution of

$$S^{\vartheta}_{t,x}(s) \;=\; x + \int_t^s \mathrm{diag}\left[S^{\vartheta}_{t,x}(r)\right]\Sigma dW(r) + \int_t^s \mathrm{diag}\left[S^{\vartheta}_{t,x}(r)\right]\vartheta(r)dr\;.$$

A formal change of variable ($\vartheta = |\tilde{\vartheta}|\tilde{\vartheta}/|\mathrm{diag}\left[S^{\tilde{\vartheta}}_{t,x}\right]\tilde{\vartheta}|$) then leads to

$$w(t,x) = \sup_{\tilde{\vartheta} \in \Theta} \mathbb{E}\left[e^{-\int_t^T |\tilde{\vartheta}(s)|\rho(S^{\tilde{\vartheta}}_{t,x}(s),\tilde{\vartheta}(s))ds}\hat{g}\left(S^{\tilde{\vartheta}}_{t,x}(T)\right)\mathbf{1}_{\tau^{\tilde{\vartheta}}_{t,x}>T}\right] \tag{4.13}$$

where

$$S^{\tilde{\vartheta}}_{t,x}(s) = x + \int_t^s \mathrm{diag}\left[S^{\tilde{\vartheta}}_{t,x}(r)\right]\Sigma dW(r) + \int_t^s |\tilde{\vartheta}(s)|\gamma(S^{\tilde{\vartheta}}_{t,x}(r),\tilde{\vartheta}(r))dr\;,$$

$\rho(x,e) = \delta(e)/|\mathrm{diag}\,[x]\,e|$, $\gamma(x,e) = \mathrm{diag}\,[x]\,e/|\mathrm{diag}\,[x]\,e|$, $\tilde{\tau}^{\vartheta}_{t,x}$ is the first exit time of $S^{\tilde{\vartheta}}_{t,x}$ from $\mathcal{O}^*$ and we use the convention $0/0 = 0$.

For very large values of $|\tilde{\vartheta}|$, the process $S^{\tilde{\vartheta}}$ is "essentially" reflected in the direction $\gamma(S^{\tilde{\vartheta}}_{t,x},\tilde{\vartheta})$.
Moreover, since $\hat{g} \ge 0$, we should seek for a control $\tilde{\vartheta}$ such that $\tau^{\tilde{\vartheta}}_{t,x} > T$, i.e. which "causes reflection" of $S^{\tilde{\vartheta}}$ at least at the boundary $\partial\mathcal{O}^*$. The "reflection" should also be optimal so that the right hand-side of (4.13) is maximal. If $d = 1$ and $\hat{g}$ is non-decreasing on $\mathcal{O}^*$, the action of $\tilde{\vartheta}$ should then be minimal since it decreases the value of $S^{\tilde{\vartheta}}_{t,x}(T)$ and $\rho(x,e) > 0$ if $e \neq 0$. Thus, at the limit, the process should be reflected only at the boundary $\partial\mathcal{O}^*$. This phenomenon, which was already observed in [13] in the one dimensional case, naturally leads to the formulation (4.12).

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