Congested traffic dynamics, weak flows and very degenerate elliptic equations

Lorenzo Brasco[12], Guillaume Carlier[23], Filippo Santambrogio[2]

Abstract: Starting from a model of traffic congestion, we introduce a minimal-flow–like variational problem whose solution is characterized by a very degenerate elliptic PDE. We precisely investigate the relations between these two problems, which can be done by considering some weak notion of flow for a related ODE. We also prove regularity results for the degenerate elliptic PDE, which enables us in some cases to apply the DiPerna-Lions theory.

Résumé: Partant d'un problème de transport congestionné, nous introduisons un problème vartiationnel vectoriel dont la solution est caractérisée par une EDP elliptique très dégénérée. Nous étudions précisément les relations entre ces deux problèmes grâce à une notion faible de flot pour une certaine EDO. Nous établissons aussi des résultats de régularité pour l'EDP dégénérée, ce qui nous permet dans certains cas d'utiliser la théorie de DiPerna-Lions.

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1 Introduction

Traffic congestion issues have received a lot of attention from engineers since the 50's mainly in network models (see [21], [4] and the references therein). In such (finite-dimensional) network models, congestion effects are captured through the fact that the travel time of each arc of the network is an increasing function of the flow on this arc. In [21], Wardrop defined a concept of equilibrium for such congested networks that has been very popular since. Roughly speaking, a Wardrop equilibrium is a flow configuration that satisfies natural mass preservation constraints (Kirchhoff's law and compatibility with the given distribution of sources and sinks) and such that every actually used (i.e. where the flow is positive) path connecting a source and destination should be a shortest path (taking into account the congestion effects). Recently, in [7], a model of continuous congested traffic equilibrium has been proposed as well as a generalization of Wardrop's equilibrium

[1]Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo, 5, 56127 Pisa, ITALY, brasco@mail.dm.unipi.it

[2]CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, Pl. de Lattre de Tassigny, 75775 Paris Cedex 16, FRANCE, carlier@ceremade.dauphine.fr, filippo@ceremade.dauphine.fr

[3]DMA, École Normale Supérieure de Paris, 45, rue d'Ulm, 75230 Paris Cedex 05, FRANCE

to a continuous setting. In this model, an equilibrium is a probability measure over a set of paths that gives full mass to geodesics for a metric that itself depends on the measure due to congestion effects.

One aim of the present paper is to construct such equilibria as measures supported in some sense on the integral curves of some non-autonomous vector field (not regular in general). For the sake of completeness and to motivate what follows, we will briefly explain the model and some of the results of [7]: there, a domain $\Omega \subset \mathbb{R}^N$ and two probability measures μ_0 and μ_1 on $\overline{\Omega}$ are given (distribution of sources and sinks or residents and services, say, in a urban region). In the framework of [7], an equilibrium is in fact a probability measure Q on $C([0,1];\overline{\Omega})$ that solves the following variational problem:

$$\inf_{Q \in \mathcal{Q}^p(\mu_0,\mu_1)} \int_\Omega H(i_Q(x))\, dx \tag{1.1}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is some convex increasing function with a $p-$th power growth at infinity, $\mathcal{Q}^p(\mu_0,\mu_1)$ is the set of probability measures on $C([0,1];\overline{\Omega})$ concentrated on absolutely continuous curves satisfying compatibility conditions with the distributions of sources and sinks (i.e. $(e_0)_\# Q = \mu_0$ and $(e_1)_\# Q = \mu_1$, where the maps $e_t : C([0,1];\overline{\Omega}) \to \overline{\Omega}$ are the evaluation maps at time t) and such that i_Q is an L^p function, where i_Q is the *traffic intensity* associated to Q defined by

$$\int_{\overline{\Omega}} \varphi(x)\, di_Q(x) := \int_{C([0,1];\overline{\Omega})} \left(\int_0^1 \varphi(\gamma(t))|\gamma'(t))|\, dt \right) dQ(\gamma), \quad \text{for every } \varphi \in C(\overline{\Omega}).$$

In this formulation, i_Q represents the total cumulated traffic and H is defined by $H(0) = 0$, $H'(i) = g(i)$ where g is an increasing function that models the congestion effect: that is, roughly speaking, if the intensity of traffic is i_Q then the congested metric is $g(i_Q)$. Once again, we refer to [7] for more details and in particular the existence of a solution to (1.1) as soon as $\mathcal{Q}^p(\mu_0,\mu_1) \neq \emptyset$ and the precise sense in which the Euler-Lagrange equation of (1.1) corresponds to the fact that Q is a Wardrop equilibrium (i.e. Q-a.e. γ is a geodesic for the metric $g(i_Q)$, a metric which, by the way, is typically given by an L^q function only, with $q = p/(p-1)$, so that one has to properly define distances and geodesics in such a non-continuous setting, and this is one of the main issues solved by [7]).

As already mentioned, one aim of this paper is to construct solutions of (1.1). A first ingredient to achieve this goal, is to introduce a minimal-flow–like problem and to relate it to the scalar problem (1.1) as follows. First, for $Q \in \mathcal{Q}^p(\mu_0,\mu_1)$, define the vector-valued measure σ_Q by:

$$\int_{\overline{\Omega}} \varphi(x)\, d\sigma_Q(x) := \int_{C([0,1];\overline{\Omega})} \left(\int_0^1 \langle \varphi(\gamma(t)), \gamma'(t) \rangle\, dt \right) dQ(\gamma), \quad \text{for every } \varphi \in C(\overline{\Omega}, \mathbb{R}^N),$$

i.e. sort of a vector version of i_Q. It is immediate to check that $|\sigma_Q| \leq i_Q$ so that $\sigma_Q \in L^p(\Omega, \mathbb{R}^N)$ and that

$$\text{div}\sigma_Q = \mu_0 - \mu_1, \quad \sigma_Q \cdot \nu = 0 \text{ on } \partial\Omega,$$

in the sense of distributions. Since H is increasing, this implies that the infimum of (1.1) is larger than that of the minimal flow problem:

$$\inf_{\sigma\in L^p(\Omega;\mathbb{R}^N)}\left\{\int_\Omega \mathcal{H}(\sigma)\ dx\ :\ \mathrm{div}\sigma=\mu_0-\mu_1,\ \sigma\cdot\nu=0 \text{ on } \partial\Omega\right\}, \tag{1.2}$$

where $\mathcal{H}(\sigma):=H(|\sigma|)$. In the sequel, problem (1.2) will be often referred to as the "vector" problem while problem (1.1) as the "scalar" problem.

If, conversely, σ solves (1.2) and if we are able to construct $Q\in\mathcal{Q}^p(\mu_0,\mu_1)$ such that $i_Q=|\sigma|$, then Q will be a solution to (1.1). Heuristically (i.e. ignoring regularity issues) a natural candidate Q is given by $Q:=\int\delta_{X(.\,,x)}\,d\mu_0(x)$ where $X(.\,,x)$ is the flow of the non-autonomous ODE:

$$\partial_t X(t,x)=\widehat{\sigma}(X(t,x),t),\ X(0,x)=x,\ \ \widehat{\sigma}(x,t):=\frac{\sigma(x)}{(1-t)\mu_0(x)+t\mu_1(x)}, \tag{1.3}$$

with σ solving (1.2), according to a deformation argument which essentially dates back to Moser (see [19]) and which has also been exploited by Evans and Gangbo in the context of mass transportation problems (see [14]) . If $\widehat{\sigma}$ is Lipschitz, this flow can be defined in a classical sense and the situation is relatively easy to understand. This leads us to the study of the regularity of $\widehat{\sigma}$ and hence of σ: unfortunately, we will see that requiring σ to be Lipschitz will be unrealistic for the models of traffic congestion we are interested in. Formally (see section 2 for details and precise assumptions), by duality, the solution of (1.2) is $\sigma=\nabla\mathcal{H}^*(\nabla u)$ where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and u solves the PDE:

$$\begin{cases}\mathrm{div}\nabla\mathcal{H}^*(\nabla u) &=& \mu_0-\mu_1, & \text{in } \Omega,\\ \nabla\mathcal{H}^*(\nabla u)\cdot\nu &=& 0, & \text{on } \partial\Omega.\end{cases} \tag{1.4}$$

Hence, the question immediately becomes a question on regularity properties for the solutions of this equation.

For instance, if one takes $\mathcal{H}(\sigma)=|\sigma|^p/p$, then it is easy to see that we have $\nabla\mathcal{H}^*(z)=|z|^{q-2}z$, so that (1.4) simply becomes a homogeneous Neumann problem for the q-Laplacian operator. This degenerate elliptic equation has been widely studied in literature and in general one cannot hope for better results than $C^{1,\alpha}$ regularity for u (i.e. $\sigma\in C^{0,\alpha}$, see for instance [11, 17]).

Yet, the situation in the cases which are motivated by traffic congestion is even worse. Indeed, let us recall that $H'=g$ where g is the congestion function relating the metric to the traffic intensity. It is therefore natural to have $g(0)>0$: the metric is positive even if there is no traffic, so that the radial function $\mathcal{H}$ is not differentiable at 0 and then its subdifferential at 0 contains a ball. By duality, this implies $\nabla\mathcal{H}^*=0$ on this ball which makes (1.4) very degenerate. A reasonable model of congestion is $g(t)=\lambda+t^{p-1}$ for $t\geq 0$, with $p>1$ and $\lambda>0$, so that

$$\mathcal{H}(\sigma)=\frac{1}{p}|\sigma|^p+\lambda|\sigma|,\ \mathcal{H}^*(z)=\frac{1}{q}(|z|-\lambda)_+^q,\ \text{with } q=\frac{p}{p-1}. \tag{1.5}$$

In this very degenerate case, one will not look for the regularity of u but only of $\sigma=\nabla\mathcal{H}^*(\nabla u)$. Regularity for this term should not be astonishing, as far as one notices that Ω can be, roughly speaking, divided into two zones, one where $\sigma=0$, the other where the equation is less degenerate (but obviously the two regions are not open sets and one has to make rigorous this idea). Assuming μ_0 and μ_1 have Lipschitz densities bounded from below by positive constants, if one can prove Sobolev regularity of $\nabla\mathcal{H}^*(\nabla u)$ as well as an L^∞ bound on ∇u for the PDE (1.4), then one can define a flow for (1.3) in the sense of the DiPerna-Lions theory. Such regularity results have, in our opinion, their own interest and have not been treated a lot in the literature. Lipschitz regularity of u is almost classical since it is clear that estimates on high values of $|\nabla u|$ should only involve the region where the equation is strongly elliptic. Anyway, they are not evident especially if one deals with global results up to $\partial\Omega$ or with variable coefficients. We will refer to a paper of Fonseca, Fusco and Marcellini ([15]), where the results we need are proven for variational purposes. Section 5 will just explain how to use these results with the regularity assumptions we have. On the contrary, Section 4 will give a detailed proof of the Sobolev regularity of $\nabla\mathcal{H}^*(\nabla u)$. The proof is mainly based on simple variants of usual schemes for the $p-$Laplacian and follows what is done in a paper by Carstensen and Müller, [9], where similar results are proven for relaxations of certain nonconvex functionals. The main differences between our proof and that in [9] lie in some pointwise inequalities on the operator which appears in the divergence, so that we are able to deal with some more general growth cases. Moreover, we will explicitly address in the Sobolev proof the use of the L^∞ result of Section 5, which allows to deal quite easily with one of the terms which appear in the estimates.

Let us also mention the recent work [20], where continuity of the same term $\nabla\mathcal{H}^*(\nabla u)$ is proved in dimension 2. Actually this kind of regularity results is not necessary for the purpose of our paper, i.e. defining the flow of $\widehat{\sigma}$, but still finds applications in congestion, proving for instance the continuity of the equilibrium metrics ([7, 20]) .

In general, as explained in subsection 3.2, when very little regularity is available on the velocity field $\widehat{\sigma}$, it is still possible to relate (1.1) to (1.2) and (1.3) by using the notion of superposition solutions and the superposition principle (see [2]).

The plan of the paper is as follows: Section 2 is devoted to a precise characterization of the minimal flow problem. In Section 3, different notions of flows for (1.3) are considered and the precise connection between the scalar problem (1.1) to (1.2) and (1.3) is given as well as the proof of the equality of the values of (1.1) and (1.2) by using the concept of superposition solutions. Then, focusing on the case of (1.5), we prove Sobolev regularity of $\nabla\mathcal{H}^*(\nabla u)$ in Section 4 and we address Lipschitz regularity of u in Section 5 for the degenerate PDE (1.4).

2 Minimal flow model

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a smooth (in a sense made precise later on) boundary and let $\mu_0, \mu_1 \in \mathscr{P}(\overline{\Omega})$ be two given probability measures over its closure. We consider the following minimization problem

$$\inf_{\sigma\in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(\sigma(x))\ dx \ :\ \operatorname{div}\sigma = \mu_0 - \mu_1,\ \sigma\cdot\nu = 0 \text{ on } \partial\Omega \right\}, \tag{2.1}$$

where $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}$ satisfies:

(i) $\mathcal{H}$ is a strictly convex radially symmetric function, with $\mathcal{H}(0) = 0$;

(ii)
$$a|\sigma|^p \le \mathcal{H}(\sigma) \le b(|\sigma|^p + 1),\ \sigma \in \mathbb{R}^N,$$
for some $p \in (1,\infty)$ and a, b positive constants;

(iii) $\mathcal{H}$ is differentiable in $\mathbb{R}^N \setminus \{0\}$ and there exists a positive constant c such that
$$|\nabla\mathcal{H}(\sigma)| \le c(|\sigma|^{p-1} + 1),\ \sigma \in \mathbb{R}^N \setminus \{0\}.$$

Example 1. Taking $\mathcal{H}(\sigma) = |\sigma|$, then (2.1) becomes the *continuous transportation model*

$$\inf_{\sigma\in\mathcal{M}(\Omega;\mathbb{R}^N)} \left\{\|\sigma\|_{L^1}\ :\ \mathrm{div}\sigma = \mu_0 - \mu_1,\ \sigma\cdot\nu = 0 \text{ on } \partial\Omega\right\},$$

which is nothing but an equivalent formulation of the Monge's problem, with cost equal to the distance (see [3]).

Example 2. Another interesting case, more related to the case of congested dynamics, is given by the choice $H(\sigma) = |\sigma|^2$, for which the minimal value (2.1) is given by (see [8] for the details)

$$C(\mu_0,\mu_1) = \begin{cases} \|\mu_0 - \mu_1\|^2_{X^*}, & \text{if } \mu_0 - \mu_1 \in X^* \\ +\infty, & \text{otherwise.} \end{cases}$$

where X^* indicates the dual of the Hilbert space $X = W^{1,2}_\diamond(\Omega) = \{\varphi \in W^{1,2}(\Omega)\ :\ \int_\Omega \varphi = 0\}$, equipped with the scalar product

$$\langle\varphi,\psi\rangle_X = \int_\Omega \langle\nabla\varphi,\nabla\psi\rangle\, dx.$$

Even for this simple problem with quadratic cost, it is only thanks to the results in the present paper that one gets a rigorous equivalence between the "vector" problem used in [8] and the models suggested by Beckmann ([3]) which are better interpreted with a "scalar" construction.

In what follows, we will mainly confine our analysis to the case in which

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + \lambda|\sigma|,\ \sigma \in \mathbb{R}^N, \tag{2.2}$$

with $p \in (1,2]$ and $\lambda > 0$ a positive constant. The reasons for the restriction on the exponent p are twofold: on the one hand, the scalar problem of [7] is fully understood under the extra assumption $p < N/(N-1)$ (i.e. $p < 2$ in two dimensions, which is the most relevant case in applications); on the other hand we will see extra difficulties arise concerning elliptic regularity whenever we are in the singular case $q = p/(p-1) < 2$.

Theorem 2.1. *Suppose that the infimum in* (2.1) *is finite and let* σ_0 *be its unique optimizer, then there exists* $\varphi_0 \in W^{1,q}(\Omega)$ *such that*

$$\sigma_0 = \nabla \mathcal{H}^*(\nabla \varphi_0), \tag{2.3}$$

and φ_0 *is a weak solution of*

$$\begin{cases} \operatorname{div} \nabla \mathcal{H}^*(\nabla u) & = & \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla \mathcal{H}^*(\nabla u) \cdot \nu & = & 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\mathcal{H}^*$ *is the Legendre transform of* $\mathcal{H}$ *and* $q = p/(p-1)$.

Proof. We first observe that problem (2.1) consists in minimizing a strictly convex and coercive functional on L^p subject to a convex and closed constraint: then an optimizer σ_0 exists and must be unique. It is well known that problem (2.1) has a dual formulation, given by the convex analysis formula (see for instance [13])

$$\sup \left\{ \int_\Omega \varphi \, d(\mu_1 - \mu_0) - \int_\Omega \mathcal{H}^*(\nabla \varphi) \, dx \right\} = \inf \left\{ \int_\Omega \mathcal{H}(\sigma) \ : \ \operatorname{div} \sigma = \mu_0 - \mu_1, \ \sigma \cdot \nu = 0 \right\}.$$

Due to the superlinear growth and the strict convexity of $\mathcal{H}$, we get that $\mathcal{H}^* \in C^1$ and it verifies the following growth conditions

$$B(|z|^q - 1) \leq \mathcal{H}^*(z) \leq A|z|^q,$$

where $q = p/(p-1)$, then using the Direct Methods of the Calculus of Variations it is not difficult to show that the dual problem admits at least a solution φ_0 belonging to $W^{1,q}_\diamond(\Omega)$, where

$$W^{1,q}_\diamond = \left\{ \varphi \in W^{1,q}(\Omega) \ : \ \int_\Omega \varphi(x) \, dx = 0 \right\}.$$

We observe further that the Euler-Lagrange equation of

$$\mathcal{F}(\varphi) = \int_\Omega \mathcal{H}^*(\nabla \varphi(x)) \, dx - \int_\Omega \varphi(x) \, d(\mu_1 - \mu_0),$$

is given by (2.4), so that φ_0 solves it, in distributional sense. Moreover, φ_0 and σ_0 verifies

$$\int_\Omega \mathcal{H}(\sigma_0) = \int_\Omega \varphi_0(\mu_1 - \mu_0) - \int_\Omega \mathcal{H}^*(\nabla \varphi_0) = \int_\Omega \nabla \varphi_0 \cdot \sigma_0 - \int_\Omega \mathcal{H}^*(\nabla \varphi_0),$$

where we have used the fact that $\operatorname{div} \sigma_0 = \mu_0 - \mu_1$ and $\sigma_0 \cdot \nu = 0$. The previous can be written as

$$\int_\Omega \mathcal{H}(\sigma_0) + \int_\Omega \mathcal{H}^*(\nabla \varphi_0) = \int_\Omega \nabla \varphi_0 \cdot \sigma_0,$$

which, by means of the so called Legendre reciprocity formula, implies that

$$\sigma_0(x) \in \partial \mathcal{H}^*(\nabla \varphi_0(x)), \quad \text{for } \mathscr{L}^N\text{-a.e. } x \in \Omega.$$

Using the fact that $\mathcal{H}^* \in C^1$, we obtain that actually the subgradient set $\partial \mathcal{H}^*$ is made of just an element, namely the gradient $\nabla \mathcal{H}^*$, concluding the proof. $\square$

3 Different meanings and equivalences

In this section we discuss how to connect the "scalar" problem on measures on paths to the "vector" problem on fields with prescribed divergence: in which sense and when they are equivalent and how to pass from one minimizer to the other.

3.1 Cauchy-Lipschitz flow

Let us consider a non-autonomous vector field $\mathbf{v} : [0,1]\times\Omega \to \mathbb{R}^N$ such that $\mathbf{v}\cdot\nu = 0$, where ν stands for the outer normal vector to $\partial\Omega$. It is well-known that if $\mathbf{v}$ is sufficiently smooth, say Lipschitz with respect to the spatial variable, then for every μ_0 the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\mu(t,x) + \mathrm{div}_x(\mathbf{v}(t,x)\mu(t,x)) = 0, & (t,x)\in[0,1]\times\Omega, \\ \mu(0,x) = \mu_0(x), & x\in\Omega, \end{cases} \tag{3.1}$$

is given by

$$\mu(t,\cdot) = (X(t,\cdot))_{\#}\mu_0, \tag{3.2}$$

where $X : [0,1]\times\overline{\Omega} \to \overline{\Omega}$ is the *flow* of $\mathbf{v}$, that is X is the map that to every $(s,x)\in[0,1]\times\overline{\Omega}$ assigns the position at time s of the curve γ satisfying

$$\begin{cases} \gamma'(s) = \mathbf{v}(s,\gamma(s)) \\ \gamma(0) = x \end{cases} \tag{3.3}$$

This is a particular case of the *method of characteristics* which basically says that the solution of (3.1) is given by the evolution, through the flow of $\mathbf{v}$, of the initial measure μ_0 (see [2] for a clarifying exposition of this theory).

We now take two probability measures μ_0 and μ_1 on Ω, absolutely continuous w.r.t to $\mathscr{L}^N$ and having density given by f_0 and f_1, respectively.

Using the above remarks on ODEs and the continuity equation, we now illustrate our general strategy to prove the equivalence between the two problems

$$\inf_{Q\in\mathcal{Q}^p(\mu_0,\mu_1)} \int_\Omega H(i_Q)\,dx \quad \text{and} \quad \inf_{\sigma\in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(\sigma)\,dx \;:\; \mathrm{div}\sigma = f_0 - f_1,\ \sigma\cdot\nu = 0 \right\}. \tag{3.4}$$

We already know that in general the value of the vector minimization problem (right hand side of (3.4)) is less than or equal to the value of the scalar one. The key point is to show that, given the optimizer σ of the vector problem, we can construct a $Q\in\mathcal{Q}^p(\mu_0,\mu_1)$ such that $|\sigma| = i_Q$. Then (3.4) is a straightforward consequence of the monotonicity assumptions on H.

As we already mentioned, the main idea will be the use of the deformation argument due to Moser and used later by Evans and Gangbo: for the moment we make the further assumption that f_0 and f_1 are Lipschitz continuous and bounded from below, that is $f_0, f_1 \geq c > 0$ on Ω.

If σ is the unique solution of the convex optimization problem (2.1), we construct the non-autonomous vector field

$$\widehat{\sigma}(t,x) = \frac{\sigma(x)}{(1-t)f_0(x) + tf_1(x)}, \ (t,x) \in [0,1] \times \Omega. \tag{3.5}$$

The latter will not have any Lipschitz continuity property in general, unless the optimizer σ itself is regular: anyway, if we assume that one can prove $\sigma \in \mathrm{Lip}(\Omega)$, then the flow $X : [0,1] \times \Omega \to \Omega$ of $\widehat{\sigma}$ is well-defined and we can take μ_t as in (3.2). In this way, we have obtained the solution of (3.1), with $\mathbf{v} = \widehat{\sigma}$ and initial datum f_0. Moreover, the same Cauchy problem is solved by the linear interpolating curve

$$\rho_t(\cdot) := (1-t)f_0(\cdot) + tf_1(\cdot), \tag{3.6}$$

which implies, due to well-posedness of (3.1), that ρ_t and μ_t must coincide. This in turn yields that

$$(X(1,\cdot))_{\#}f_0 = f_1, \tag{3.7}$$

which ensures that $X(1,\cdot)$ transports μ_0 on μ_1. If we now consider the probability measure concentrated on the flow, i.e. $Q = \int \delta_{X(\cdot,x)}\, d\mu_0(x)$, then thanks to (3.7) this is admissible and it is not difficult to see that $i_Q = |\sigma|$ (we will give all the details in Theorem 3.2 below), which finally implies that the minimum of the two problems coincide. Moreover, this construction provides a transport map (that is $X(1,\cdot)$) from μ_0 to μ_1, whose transport "rays" (i.e. integral curves) evidently do not cross and which is monotone on transport rays (as a consequence of Cauchy-Lipschitz Theorem).

Remark 1. We point out that in this setting, where everything is sufficiently smooth, property (3.7) can be proved at a Lagrangian level, without mentioning the well-posedness of the continuity equation: indeed one can use a trick of Dacorogna and Moser (see [10]) to show that the quantity

$$h(t,x) = \det \nabla_x X(t,x)[(1-t)f_0(X(t,x)) + tf_1(X(t,x))],$$

is actually constant in time. Then using the fact that $X(0,x) = x$ we get that

$$f_0(x) = f_1(X(1,x)) \det \nabla_x X(1,x),$$

which in turn implies (3.7) by means of the area formula.

Anyway, recalling the optimality condition for σ provided by Theorem 2.1, the reader can easily convince himself that our choice for the function H rules out any possibility of Lipschitz regularity for σ. So the previous construction of Q is purely formal: we will see in the next subsections how (and in what sense) one can still construct a flow X and make this construction a rigorous one.

Remark 2. On the contrary, when one takes $\mathcal{H}(z) = |z|^2$, standard elliptic theory allows to prove Lipschitz regularity for σ and this concept of Cauchy-Lipschitz flows may be used.

3.2 Superposition of flows

For a general vector field $\mathbf{v}$ under very mild assumptions, the most general meaning that we can give to the flow of $\mathbf{v}$ is in terms of the so-called *superposition principle*, that we now explain in some details. As far as we can see, this provides a very weak concept of flow (a probabilistic one, let's say), which anyway is strong enough to still give sense to the construction of the previous subsection.

Definition 1. Let $Q \in \mathscr{P}(C([0,1];\overline{\Omega}))$ be concentrated on the absolutely continuous solutions of (3.3), in the sense that

$$\int_{C([0,1];\overline{\Omega})} \left| \gamma(t) - \gamma(0) - \int_0^t \mathbf{v}(s,\gamma(s))\ ds \right|\ dQ(\gamma) = 0, \quad \text{for every } t \in [0,1]. \tag{3.8}$$

If we define the curve of measures μ_t^Q through

$$\int_{\overline{\Omega}} \varphi(x)\, d\mu_t^Q(x) := \int_{C([0,1];\overline{\Omega})} \varphi(\gamma(t))\, dQ(\gamma), \quad \text{for every } \varphi \in C(\overline{\Omega}), \tag{3.9}$$

then this curve μ_t^Q is called *superposition solution* of Problem (3.1): μ_t^Q is actually a distributional solution of the continuity equation, with initial datum $\mu_0 = \mu_0^Q$.

Remark 3. It is not hard to see that when $\mathbf{v}$ is smooth, formula (3.9) is exactly equivalent to (3.2). Indeed in this case, for every $x \in \Omega$, there exists a unique curve $X(\cdot, x)$ solving (3.3), so that $Q = \int Q^x\, d\mu_0(x)$ with Q^x a Dirac mass concentrated on this curve, that is $Q^x = \delta_{X(\cdot,x)}$, and (3.9) now becomes

$$\int_{\overline{\Omega}} \varphi(x)\ d\mu_t^Q(x) = \int_{\overline{\Omega}} \varphi(X(x,t))\ d\mu_0(x) = \int_{\overline{\Omega}} \varphi(x)\ d(X(t,\cdot))_{\#}\mu_0(x).$$

In this way, we can think of the concept of superposition solutions as a *probabilistic* version of the method of characteristics.

The most valuable fact of this theory is that, under suitable integrability conditions of the vector field $\mathbf{v}$, every positive measure-valued distributional solution of (3.1) can be realized as a superposition solution: a proof can be found in [2] (Theorem 12).

Theorem 3.1 (Superposition principle)**.** *Let μ_t be a positive measure-valued solution of the continuity equation*

$$\frac{\partial}{\partial t}\mu_t + \operatorname{div}(\mathbf{v}\mu_t) = 0,$$

with the vector field $\mathbf{v}$ satisfying the following condition

$$\int_0^1 \int_\Omega |\mathbf{v}(t,x)|\ d\mu_t(x)\ dt < +\infty, \tag{3.10}$$

then μ_t is a superposition solution.

Using the concept of superposition solution, it is now a straightforward fact to provide a rigorous proof of the equivalence between the two problems in (3.4).

Theorem 3.2. *Let $\mu_0, \mu_1 \in \mathscr{P}(\Omega)$ having L^p density w.r.t. to $\mathscr{L}^N$, given by f_0 and f_1, respectively. Then equality of values of the two problems in* (3.4) *holds true.*

Proof. As before, we take the minimizer σ of the vector problem and we consider the non-autonomous vector field defined by (4.15). We point out that the L^p assumption on the densities has been chosen in order to guarantee finiteness of the infima of both problems (see [7]). With this choice of $\widehat{\sigma}$, the linear interpolating curve $\mu_t = (1-t)\mu_0 + t\mu_1$ is a positive measure-valued distributional solution of the continuity equation

$$\frac{\partial}{\partial t}\mu_t + \operatorname{div}(\widehat{\sigma}\mu_t) = 0,$$

with initial datum μ_0. Moreover, $\widehat{\sigma}$ satisfies hypothesis (3.10), so that μ_t is a superposition solution: this means that there exists a probability measure $Q \in \mathscr{P}(C([0,1];\overline{\Omega}))$ such that (3.8) holds and

$$\int_\Omega \varphi(x)\ d\mu_t(x) = \int_\Omega \varphi(x)\ d\mu_t^Q(x), \quad \text{for every } \varphi \in C(\overline{\Omega}),$$

with μ_t^Q given by (3.9) (observe that in the Cauchy-Lipschitz case, this amounted to say that ρ_t defined by (3.6) had to coincide with the solution given by (3.2)). This Q is admissible, that is $Q \in \mathcal{Q}(\mu_0, \mu_1)$ and moreover, using Fubini Theorem and the disintegration $Q = \int Q^x\, d\mu_0(x)$, we get

$$\begin{aligned}
\int_{\overline{\Omega}} \varphi(x)\, di_Q(x) &= \int_C \int_0^1 \varphi(\gamma(t))|\gamma'(t)|\, dt\, dQ(\gamma) = \int_0^1 \int_C \varphi(\gamma(t))|\gamma'(t)|\, dQ(\gamma)\, dt \\
&= \int_0^1 \int_\Omega \int_C \varphi(\gamma(t))|\gamma'(t)|\, dQ^x(\gamma)\, d\mu_0(x)\, dt \\
&= \int_0^1 \int_\Omega \varphi(x)|\widehat{\sigma}(t,x)|\, d\mu_t(x)\, dt = \int_0^1 \int_\Omega \varphi(x)|\sigma(x)|\, dx\, dt,
\end{aligned}$$

so that

$$\int_\Omega \varphi(x)\ di_Q(x) = \int_\Omega \varphi(x)|\sigma(x)|\ dx, \quad \text{for every } \varphi \in C(\overline{\Omega}).$$

This clearly implies that $i_Q = |\sigma|$ and thus $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$ and it solves the scalar problem in (3.4), concluding the proof. □

Notice that the regularity of the curves which are charged by the measure Q corresponding to a superposition solution is very poor. On the contrary, if one knows that $\mathbf{v}$ is continuous, these curves are C^1 and they solve their ODE in a classical sense. The forthcoming paper [20] will prove a C^0 result in two spatial dimensions for the vector field we are interested in. Obviously, continuity without Lipschitz continuity or similar conditions is not sufficient for ensuring any kind of uniqueness result. We will see in a while that some kind of uniqueness may be recovered by an intermediate concept of solution.

3.3 DiPerna-Lions flow

As far as now, we have seen that everything goes well if we face a Lipschitz vector field $\mathbf{v}$ and that we can at least prove equality of the minima if, instead, $\mathbf{v}$ is only integrable. In the latter case, it is not evident to add anything else to this equality and in particular one has no real clue to construct a minimizer for the scalar problem from a minimizer for the vector one. The problem is mainly linked to the lack of uniqueness. We will see in this section an intermediate concept, for vector fields which are not Lipschitz but much better than just integrable.

If $\mathbf{v}(t,\cdot) \in W^{1,1}(\Omega)$ and the vector field has bounded divergence, we can enforce the conclusion of Theorem 3.2 and guarantee that the optimal Q associated to the optimizer σ is actually concentrated on a uniquely defined flow X (possibly in a.e. sense), trasporting μ_0 to μ_1.

In fact in this setting, it is still possible to give sense to formula (3.2), through the DiPerna-Lions theory of flows of weakly differentiable vector fields: we recall the following fundamental result (see Theorem III.2 of [12]; the same results are also presented in [2] where the language is more similar to ours).

Theorem 3.3. *Let $\mathbf{v} \in L^1([0,1]; W^{1,1}(\Omega))$ and such that $\mathrm{div}_x \mathbf{v} \in L^1([0,1]; L^\infty(\Omega))$. Then there exists a unique $X \in C^0([0,1]\times[0,1]; L^1(\Omega;\mathbb{R}^N))$ which leaves $\overline{\Omega}$ invariant and such that:*

(i) if we set $A(t) = \int_0^t \|\mathrm{div}_x \mathbf{v}(\tau,\cdot)\|_\infty \, d\tau$, then

$$e^{-|A(t)-A(s)|}\mathscr{L}^N \le (X(t,s,\cdot))_\# \mathscr{L}^N \le e^{|A(t)-A(s)|}\mathscr{L}^N, \quad \textit{for every } t \in [0,1];$$

(ii) X satisfies the group property

$$X(t_3,t_1,x) = X(t_3,t_2,X(t_2,t_1,x)), \quad \textit{for } \mathscr{L}^N\textit{-a.e. } x \in \Omega, \quad \textit{for every } t_1 < t_2 < t_3 \in [0,1];$$

(iii) for every $s \ge 0$ and for $\mathscr{L}^N$-a.e. $x \in \Omega$, X is an absolutely continuous integral solution of (3.3)*, that is*

$$X(t,s,x) = x + \int_s^t \mathbf{v}(r, X(r,s,x)) \, dr, \quad \textit{for } \mathscr{L}^N\textit{-a.e. } x \in \Omega,\ t \ge s.$$

Moreover, if $\mu_0 = \rho_0 \mathscr{L}^N$ with $\rho_0 \in L^p(\Omega)$, then for every $s \in [0,1)$

$$\mu(t,\cdot) = X(t,s,\cdot)_\# \mu_0, \ \ s \le t \in [0,1],$$

is the unique renormalized solution in $C^0([s,1]; L^p(\Omega))$ of the continuity equation, with initial datum $\mu(s,x) = \mu_0(x)$.

Definition 2. We recall that μ is said to be a *renormalized solution* of the continuity equation if there holds

$$\frac{\partial}{\partial t}\beta(\mu) + \mathbf{v}\cdot\nabla_x \beta(\mu) + (\mathrm{div}_x \mathbf{v})\mu\beta'(\mu) = 0, \quad \text{in } (0,1)\times\Omega, \tag{3.11}$$

in the sense of distributions, for every $\beta \in C^1(\mathbb{R})$.

Clearly, every renormalized solution is a distributional solution (just take $\beta \equiv 1$ in (3.11)), while in general the converse does not hold true. It is a remarkable fact of the DiPerna-Lions theory that when $\mathbf{v}$ has a Sobolev regularity in x, then $\mathbf{v}$ has the *renormalization property*, that is every distributional solution is actually a renormalized one. Moreover, renormalized solutions are the right class in which existence, uniqueness and stability of solutions to the continuity equation can be proved: *this is crucial for our construction.* Indeed, as already observed in the subsection on Cauchy-Lipschitz flow, well-posedness of the continuity equation guarantees that the flow at time 1 transports μ_0 on μ_1, so that the measure Q associated to σ is admissible.

Finally, we just point out that the renormalization property can be proved also for vector fields with BV regularity (with respect to the space variable), as shown by Ambrosio ([1]): some L^∞ bounds on the divergence of the vector field are again essential.

Due to the previous facts, the rest of the paper is devoted to provide Sobolev and L^∞ estimates for the optimizer σ under the following assumptions:

(i) $\mu_i = f_i \mathscr{L}^N$, with $f_i \in \text{Lip}\ (\Omega)$ and $f_i \geq c > 0$, for $i = 0, 1$;

(ii) Ω open connected bounded subset of $\mathbb{R}^N$ having smooth boundary.

In fact with these assumptions, the vector field $\widehat{\sigma}$ given by (4.15) is well-defined and satisfies the hypotheses of DiPerna-Lions Theorem, once we know that $\sigma \in W^{1,r} \cap L^\infty$, for some $r \geq 1$. Indeed, the Sobolev regularity of $\widehat{\sigma}$ is equivalent to that of σ, once f_0 and f_1 are Lipschitz. For the condition on the divergence one may see that we have

$$\operatorname{div} \widehat{\sigma} = \frac{\operatorname{div} \sigma}{\rho_t} - \frac{\langle \sigma, \nabla \rho_t \rangle}{\rho_t^2}.$$

Lipschitz regularity and lower bounds on $\rho_t = (1-t) f_0 + t f_1$ (i.e. on f_0 and f_1) and L^∞ on σ seem compulsory for getting the assumption on the divergence of $\widehat{\sigma}$.

Moreover, (i) guarantees that then $(1-t)\mu_0 + t\mu_1$ is a renormalized solution of (3.1) and so it must coincide with $X_\# \mu_0$.

We will achieve these results strongly relying on the optimality condition for σ provided by Theorem 2.1, which ensures that $\sigma = \nabla \mathcal{H}^*(\nabla u)$, where $u \in W^{1,q}(\Omega)$ is a distributional solution of the degenerate elliptic equation

$$\operatorname{div}(\nabla \mathcal{H}^*(\nabla u)) = f_0 - f_1, \tag{3.12}$$

under homogeneous Neumann boundary conditions.

4 Sobolev regularity of the vector field

In order to apply the DiPerna-Lions theory, first of all we have to show that σ is weakly differentiable: we will indeed show that $\sigma \in W^{1,r}(\Omega)$, for a suitable r. Observe that, in general, if one looks at the solution u itself, no more than $C^{0,1}$ regularity should be expected for equation (3.12). Indeed, with the choice

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + |\sigma|,\ \sigma \in \mathbb{R}^N,$$

we get

$$\nabla\mathcal{H}^*(z) = (|z|-1)_+^{q-1}\frac{z}{|z|},\ z \in \mathbb{R}^N,$$

so that every 1−Lipschitz function is a solution of the homogeneous equation. Moreover, we have

$$\frac{(|\nabla u|-1)_+^{q-1}}{|\nabla u|}|\xi|^2 \le \langle D^2\mathcal{H}^*(\nabla u)\xi,\xi\rangle \le (q-1)(|\nabla u|-1)_+^{q-2}|\xi|^2,\ \xi \in \mathbb{R}^N, \tag{4.1}$$

that is the ellipticity constants degenerate in the region $\{|\nabla u| \le 1\}$.

We will confine our analysis to the non-singular case $q \ge 2$, which is anyway relevant for the applications to minimization problems in traffic congestion.

First of all, we need the following pointwise inequalities. This is the main point where the precise structure of $\mathcal{H}^*$ plays a role.

Lemma 4.1. *For every $q \ge 2$, let us define the following vector field*

$$G(z) = |\nabla\mathcal{H}^*(z)|^{\frac{p}{2}}\frac{z}{|z|} = (|z|-1)_+^{\frac{q}{2}}\frac{z}{|z|},\ z \in \mathbb{R}^N. \tag{4.2}$$

Then for every $z, w \in \mathbb{R}^N$ we get

$$\langle \nabla\mathcal{H}^*(z) - \nabla\mathcal{H}^*(w), z-w\rangle \ge \frac{4}{q^2}|G(z)-G(w)|^2, \tag{4.3}$$

$$|\nabla\mathcal{H}^*(z) - \nabla\mathcal{H}^*(w)| \le (q-1)\left(|G(z)|^{\frac{q-2}{q}} + |G(w)|^{\frac{q-2}{q}}\right)|G(z)-G(w)|. \tag{4.4}$$

Proof. We first observe that if $\max\{|z|,|w|\} \le 1$, then (4.3) and (4.4) are trivially true. Secondly, in the case $\min\{|z|,|w|\} \le 1$, supposing for example that $|w| \le 1$ and $|z| > 1$, using Cauchy-Schwarz inequality we get

$$\langle \nabla\mathcal{H}^*(z), z-w\rangle = \frac{(|z|-1)_+^{q-1}}{|z|}\langle z, z-w\rangle \ge (|z|-1)_+^{q-1}|z| - (|z|-1)_+^{q-1} = (|z|-1)_+^q,$$

which proves (4.3), while (4.4) is easily seen to be true in this case, too.

Let us now suppose that $|z| > 1$ and $|w| > 1$. Now, we recall the inequality (see [17])

$$\langle |s|^{q-2}s - |t|^{q-2}t, s-t\rangle \ge \frac{4}{q^2}\left||s|^{\frac{q-2}{2}}s - |t|^{\frac{q-2}{2}}t\right|^2,\ s,t \in \mathbb{R}^N, \tag{4.5}$$

and we see that if we are able to prove the following

$$\left\langle |s|^{q-2}s-|t|^{q-2},(|s|+1)\frac{s}{|s|}-(|t|+1)\frac{t}{|t|}\right\rangle \geq \langle |s|^{q-2}s-|t|^{q-2}t,s-t\rangle, \tag{4.6}$$

then choosing

$$s=(|z|-1)_+\frac{z}{|z|},\ t=(|w|-1)_+\frac{w}{|w|},$$

and using (4.6) in combination with (4.5), we obtain (4.3). So, we are left to prove inequality (4.6): one sees that this is equivalent to

$$|s|^{q-1}+|t|^{q-1}-\langle s,t\rangle\left[\frac{|s|^{q-2}}{|t|}+\frac{|t|^{q-2}}{|s|}\right]\geq 0,$$

which is just a simple consequence of Cauchy-Schwarz inequality $\langle s,t\rangle \leq |s||t|$.

In order to prove (4.4), it is enough to start from the inequality (see [17])

$$||s|^{q-2}s-|t|^{q-2}t|\leq (q-1)(|s|^{\frac{q-2}{2}}+|t|^{\frac{q-2}{2}})\left||s|^{\frac{q-2}{2}}s-|t|^{\frac{q-2}{2}}t\right|,$$

which is valid for every $t,s\in\mathbb{R}^N$ and then take s and t as before. □

Notice that the pointwise estimates proved in the lemma above are not of the same kind of those that are used in [9], which involved $|\nabla\mathcal{H}^*(z)-\nabla\mathcal{H}^*(w)|^2$.

4.1 Interior Sobolev estimates

For the sake of clarity, we will first prove, thanks to an adaption of an argument used by Bojarski and Iwaniec (see [5]) for the $p-$Laplacian operator, a local result that we will modify later in order to arrive up to $\partial\Omega$.

Theorem 4.2. *Let $\Omega\subset\mathbb{R}^N$ be an open set and take $f\in W^{1,p}(\Omega)$, with $p=q/(q-1)$. If $u\in W^{1,q}(\Omega)$ is a local weak solution of*

$$-\mathrm{div}\,(\nabla\mathcal{H}^*(\nabla u))=f,\ \ in\ \Omega, \tag{4.7}$$

then we get $\mathcal{G}\in W^{1,2}_{\mathrm{loc}}(\Omega)$, where the function $\mathcal{G}$ is defined by

$$\mathcal{G}(x):=G(\nabla u(x))=(|\nabla u(x)|-1)_+^{\frac{q}{2}}\frac{\nabla u(x)}{|\nabla u(x)|},\ \ x\in\Omega. \tag{4.8}$$

More precisely, for every $\Sigma\subset\subset\Omega$ there exists a constant $C=C(N,q)$ such that

$$\|\nabla\mathcal{G}(x)\|^2_{L^2(\Sigma)}\leq\frac{C}{\mathrm{dist}\,(\Sigma,\partial\Omega)^2}\|\nabla u\|^q_{L^q(\Omega)}+C\,\|\nabla f\|^p_{L^p(\Omega)}.$$

Proof. We fix two subsets compactly contained in Ω, that is $\Sigma \subset\subset \Sigma_0 \subset\subset \Omega$ and such that $0 < h_0 = \mathrm{dist}\,(\Sigma, \partial\Omega) = 2\,\mathrm{dist}\,(\Sigma_0, \partial\Omega)$: we aim to prove that $\mathcal{G} \in W^{1,2}(\Sigma)$, using integrated difference quotients. First of all, we observe that

$$\int_\Omega \langle \nabla \mathcal{H}^*(\nabla u(x)), \nabla \varphi(x)\rangle\, dx = \int_\Omega f(x)\,\varphi(x)\, dx, \quad \text{for every } \varphi \in W^{1,q}_0(\Omega).$$

In particular, for every h such that $|h| < h_0/2$, taking a $\varphi \in W^{1,q}_0(\Sigma_0)$, we get that

$$\int_\Omega \langle \nabla \mathcal{H}^*(\nabla u(x + h\omega)), \nabla \varphi(x)\rangle\, dx = \int_\Omega f(x + h\omega)\varphi(x)\, dx,$$

for any direction $\omega \in S^{N-1}$. Hence subtracting and dividing by h, we obtain

$$\int_\Omega \langle \delta_{h,\omega} \nabla \mathcal{H}^*(\nabla u), \nabla \varphi\rangle\, dx = \int_\Omega \delta_{h,\omega} f\, \varphi\, dx, \tag{4.9}$$

for every $\varphi \in W^{1,q}_0(\Sigma_0)$, where we have used the notation $\delta_{h,\omega} g(x)$ for $(g(x + h\omega) - g(x))/h$.

We now want to exploit (4.9) for a suitable choice of the test function φ, in order to obtain $W^{1,2}$ estimates on $\mathcal{G}$. At this end, let us take a smooth cut-off function $\zeta \in C^1_0(\Sigma_0)$, such that: $0 \le \zeta \le 1$, $\zeta \equiv 1$ on Σ and $\|\nabla \zeta\|_\infty \le C(\mathrm{dist}\,(\Sigma, \partial\Omega))^{-1}$. Then we choose the test function $\varphi = \zeta^2\, \delta_{h,\omega} u$ for every pair $(h, \omega) \in \mathbb{R} \times S^{N-1}$ such that $|h| < h_0/2$: observe that with this choice, this is an admissible test function in (4.9). We now develop φ and use Cauchy-Schwarz inequality, getting

$$\begin{aligned}\int_\Omega \langle \delta_{h,\omega} \nabla \mathcal{H}^*(\nabla u), \delta_{h,\omega} \nabla u\rangle\, \zeta^2\, dx \le{}& 2 \int_\Omega |\delta_{h,\omega} \nabla \mathcal{H}^*(\nabla u)|\; \zeta\, |\nabla \zeta|\, |\delta_{h,\omega} u|\, dx \\ &+ \int_\Omega \zeta^2\, |\delta_{h,\omega} f|\, |\delta_{h,\omega} u|\, dx.\end{aligned}$$

An application of the pointwise inequalities (4.3) and (4.4) yields

$$\begin{aligned}\int_\Omega |\delta_{h,\omega} \mathcal{G}|^2 \zeta^2\, dx \le{}& C \int_\Omega \left(|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}} + |\mathcal{G}|^{\frac{q-2}{q}}\right) |\delta_{h,\omega} \mathcal{G}|\, \zeta\, |\nabla \zeta| |\delta_{h,\omega} u|\, dx \\ &+ \left(\int_\Omega \zeta^p |\delta_{h,\omega} f|^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega \zeta^q |\delta_{h,\omega} u|^q\, dx\right)^{\frac{1}{q}}\end{aligned}$$

where the constant C depends on q only. By means of Young's inequality, we get for every $\varepsilon > 0$

$$\begin{aligned}\left(|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}} + |\mathcal{G}|^{\frac{q-2}{q}}\right) |\delta_{h,\omega} \mathcal{G}|\, \zeta\, |\nabla \zeta| |\delta_{h,\omega} u| \le{}& \varepsilon\, |\delta_{h,\omega} \mathcal{G}|^2\, \zeta^2 \\ &+ \frac{1}{\varepsilon} \left(|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}} + |\mathcal{G}|^{\frac{q-2}{q}}\right)^2 |\nabla \zeta|^2 |\delta_{h,\omega} u|^2,\end{aligned}$$

so that choosing ε small enough, the term on the right-hand side containing $\delta_{h,\omega} \mathcal{G}$ can be absorbed by the term on the left hand-side. Up to now, we have shown

$$\begin{aligned}\int_\Omega |\delta_{h,\omega} \mathcal{G}|^2 \zeta^2\, dx \le{}& C \int_\Omega \left(|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}} + |\mathcal{G}|^{\frac{q-2}{q}}\right)^2 |\nabla \zeta|^2 |\delta_{h,\omega} u|^2\, dx \\ &+ \frac{1}{p} \int_\Omega \zeta^p |\delta_{h,\omega} f|^p\, dx + \frac{1}{q} \int_\Omega \zeta^q |\delta_{h,\omega} u|^q\, dx.\end{aligned}$$

A simple application of Hölder's inequality to the first term on the right-hand side, yields

$$\begin{aligned}\int_\Omega |\delta_{h,\omega}\mathcal{G}|^2\,\zeta^2\,dx \le C\left(\int_{\Sigma_0}\left(|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}}+|\mathcal{G}|^{\frac{q-2}{q}}\right)^{\frac{2q}{q-2}}\,dx\right)^{\frac{q-2}{q}}\left(\int_\Omega |\nabla\zeta|^q|\delta_{h,\omega}u|^q\,dx\right)^{\frac{2}{q}}\\ +\frac{1}{p}\int_\Omega \zeta^p|\delta_{h,\omega}f|^p\,dx+\frac{1}{q}\int_\Omega \zeta^q|\delta_{h,\omega}u|^q\,dx.\end{aligned}\tag{4.10}$$

It is now sufficient to observe that

$$\left(\int_{\Sigma_0}\left(|\mathcal{G}|^{\frac{q-2}{q}}+|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}}\right)^{\frac{2q}{q-2}}\,dx\right)^{\frac{q-2}{q}}\le 2\left(\int_\Omega |\mathcal{G}|^2\,dx\right)^{\frac{q-2}{q}},$$

so that inserting the latter in (4.10), we easily get

$$\begin{aligned}\int_\Omega |\mathcal{G}_h-\mathcal{G}|^2\zeta^2\,dx \le \frac{C}{\operatorname{dist}(\Sigma,\partial\Omega)^2}\left(\int_\Omega |\mathcal{G}|^2\,dx\right)^{\frac{q-2}{q}}\left(\int_{\Sigma_0}|\delta_{h,\omega}u|^q\,dx\right)^{\frac{2}{q}}\\ +\frac{1}{p}\int_\Omega \zeta^p|\delta_{h,\omega}f|^p\,dx+\frac{1}{q}\int_\Omega \zeta^q|\delta_{h,\omega}u|^q\,dx.\end{aligned}$$

Finally, we just observe that, by means of the characterization of Sobolev spaces in terms of integrated difference quotients (see [6]), we have

$$\int_{\Sigma_0}|\delta_{h,\omega}u|^q\,dx\le C_N\int_\Omega|\nabla u|^q\,dx,\quad\text{and}\quad \int_{\Sigma_0}|\delta_{h,\omega}f|^p\,dx\le C_N\int_\Omega|\nabla f|^p\,dx,$$

and moreover by the very definition of $\mathcal{G}$ we have

$$\int_\Omega|\mathcal{G}|^2\,dx\le\int_\Omega|\nabla u|^q\,dx,$$

so that in the end we get

$$\int_\Sigma|\delta_{h,\omega}\mathcal{G}|^2\,dx\le\frac{C}{\operatorname{dist}(\Sigma,\Omega)^2}\int_\Omega|\nabla u|^q\,dx+C\int_\Omega|\nabla f|^p\,dx,$$

that is $\mathcal{G}$ has a square-integrable weak derivative along the direction given by $\omega\in S^{N-1}$. □

Remark 4. We observe that as an easy consequence of Theorem 4.2 and Sobolev Imbedding Theorems, we get a gain of integrability for ∇u: indeed, in the case $N>2$, we get $\mathcal{G}\in L^{2^*}_{\text{loc}}(\Omega)$ and then

$$\int_\Omega(|\nabla u(x)|-1)_+^{\frac{qN}{N-2}}\,dx=\int_\Omega|\mathcal{G}(x)|^{\frac{2N}{N-2}}\,dx<+\infty,$$

which ensures that

$$\nabla u\in L^{q\frac{N}{N-2}}_{\text{loc}}(\Omega),\tag{4.11}$$

while if $N=2$ we get that $\mathcal{G}$ (and so $|\nabla u|$) is in every L^s_{loc}, with $s<\infty$. Moreover, in the case $q>N-2$, then we can assure that $u\in C^{0,\alpha}_{\text{loc}}(\Omega)$, with $\alpha=1-(N-2)/q$.

Going back to our vector field $\sigma = \mathcal{H}^*(\nabla u)$, Theorem 4.2 easily implies the following.

Corollary 4.3. *Under the assumptions of Theorem 4.2, we get*

$$\sigma = \nabla\mathcal{H}^*(\nabla u) = |\mathcal{G}|^{\frac{q-2}{q}}\mathcal{G} \in W^{1,r}_{\mathrm{loc}}(\Omega), \tag{4.12}$$

for suitable exponents $r = r(N,q)$ given by

$$r(N,q) = \begin{cases} 2, & \text{if } N = q = 2, \\ \text{any value } < 2, & \text{if } N = 2,\ q > 2, \\ \frac{Nq}{(N-1)q+2-N}, & \text{if } N > 2. \end{cases}$$

Proof. The case $q = 2$ is clearly trivial, in fact in this case $\sigma = \mathcal{G} \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Let us begin with the case $N > 2$: using inequality (4.4) with $z = \nabla u(x + h\,\omega)$ and $w = \nabla u(x)$, we get

$$|\delta_{h,\omega}\sigma| \le (q-1)\left(|\mathcal{G}_h(x)|^{\frac{q-2}{q}} + |\mathcal{G}(x)|^{\frac{q-2}{q}}\right)|\delta_{h,\omega}\mathcal{G}|.$$

and we already observed that $\mathcal{G} \in L^{2^*}_{\mathrm{loc}}(\Omega)$, so that $|\mathcal{G}|^{\frac{q-2}{q}} \in L^{\frac{2^*q}{(q-2)}}_{\mathrm{loc}}(\Omega)$ and the right hand side in the previous inequality belongs to $L^r_{\mathrm{loc}}(\Omega)$, with r being given by the relation

$$\frac{1}{r} = \frac{(q-2)}{2^*q} + \frac{1}{2}.$$

This clearly implies that we can control the integrated difference quotients

$$\int |\delta_{h,\omega}\sigma(x)|^r\, dx,$$

thus proving the assertion. Finally, when $N = 2$ and $q > 2$, we can proceed as before, taking into account the fact that $\mathcal{G} \in L^s_{\mathrm{loc}}(\Omega)$ for every $s < \infty$. Notice also that, should $\mathcal{G}$ be bounded, one would automatically get $\sigma \in W^{1,2}_{\mathrm{loc}}(\Omega)$. $\square$

Remark 5. The same arguments in the proof of Theorem 4.2 may obviously be applied to the case of uniformly elliptic equations, such as $-\mathrm{div}(\nabla\mathcal{K}(\nabla u)) = f$ with

$$\lambda|\xi|^2 \le \langle D^2\mathcal{K}(z)\xi, \xi\rangle \le \Lambda|\xi|^2.$$

In this case they provide the well-known local $W^{2,2}$ regularity estimates, under the sole assumption that $f \in L^2$ (see [16], Theorem 8.8). This last observation could suggest that actually we asked for a stronger regularity assumption on f, than what is really needed: as one can easily guess, this is strongly linked to the degeneracy of our operator $\nabla\mathcal{H}^*$. Actually, in non-degenerate equations, when we arrive to the term

$$\int \delta_{h,\omega} f\, \delta_{h,\omega} u\, dx,$$

we can pass all the increments on the function u, that is we can use the trick

$$\int \delta_{h,\omega} f\, \delta_{h,\omega} u\, dx = -\int \left(\int_0^1 f(x + th\omega)\, dt \right) \delta_{h,\omega} \nabla u\, dx,$$

thus getting something that may be estimated again by the L^2 norm of $\delta_{h,\omega}\nabla u$ (but to the power of one, while at the left hand side it is to the power of two). Yet, here this is no more useful, since $\mathcal{G}$ is not invertible as a function of ∇u: this is why we asked for higher regularity on f, which could somehow shock the experienced reader. It is worthwhile noticing that even in the case of the $q-$Laplacian, where the corresponding quantity $\mathcal{G}$ is given by $|\nabla u|^{\frac{q-2}{2}} \nabla u$, the difference quotients technique seems to work only with a Sobolev assumption on f and it is not clear (and interesting to be investigated) if the following implication holds

$$f \in L^s \overset{?}{\Longrightarrow} |\nabla u|^{\frac{q-2}{2}} \nabla u \in W^{1,2}(\Omega),$$

for a suitable exponent $s \geq p = q'$. We end up recalling that even with a very regular datum f, in the case of the $q-$Laplacian with $q > 2$, the weak differentiability of ∇u can be guaranteed only in a fractional sense: once you prove that $\mathcal{G} \in W^{1,2}$, this is just a consequence of the fact that

$$|\nabla u(x) - \nabla u(y)| \leq C \left| |\nabla u(x)|^{\frac{q-2}{2}} \nabla u(x) - |\nabla u(y)|^{\frac{q-2}{2}} \nabla u(y) \right|^{\frac{2}{q}},$$

so that $\nabla u \in W^{\frac{2}{q}-\varepsilon,q}$, for every $\varepsilon > 0$. Again, such a kind of result fails in the case of our operator $\nabla \mathcal{H}^*$.

4.2 Sobolev estimates up to the boundary

Under suitable assumptions on the domain Ω and on the boundary datum, Theorem 4.2 can be enforced, thus obtaining a global $W^{1,2}$ estimate. This is exactly the content of the next result.

Theorem 4.4. *Let us suppose that Ω is a $C^{3,1}$ domain and take $f \in W^{1,p}_\diamond(\Omega)$, with $p = q/(q-1)$. If $u \in W^{1,q}_\diamond(\Omega)$ is a weak solution of the following Neumann boundary problem*

$$\begin{cases} -\operatorname{div}(\nabla \mathcal{H}^*(\nabla u)) &= f, & \text{in } \Omega, \\ \nabla \mathcal{H}^*(\nabla u) \cdot \nu &= 0, & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

then we get $\mathcal{G} \in W^{1,2}(\Omega)$, where $\mathcal{G}$ is defined by (4.2).

Proof. Let $x_0 \in \partial\Omega$ and V be a neighborhood of x_0. We set $B^+ = \{x = (x', x_N) : |x| < 1,\ x_N \geq 0\}$, and observe that our assumptions on Ω are more than sufficient to guarantee the existence of a diffeomorphism Ψ sending B^+ on $V^+ = V \cap \overline{\Omega}$ and the flat part of ∂B^+ on $V \cap \partial\Omega$ (we will see later why we need higher regularity on Ω). Let us set $B^- = \mathcal{R}B^+$, $\mathcal{R}$ being the reflection with respect to the hyperplane $\{x_N = 0\}$, then we define

$$\widehat{u}(y) = \begin{cases} u(\Psi(y)), & \text{if } y \in B^+, \\ u(\Psi(\mathcal{R}y)), & \text{if } y \in B^-, \end{cases}$$

$$\widehat{f}(y) = \begin{cases} f(\Psi(y))|\det D\Psi(y)|, & \text{if } y \in B^+, \\ f(\Psi(\mathcal{R}y))|\det D\Psi(\mathcal{R}y)|, & \text{if } y \in B^-, \end{cases}$$

and observe that $\widehat{u} \in W^{1,q}(B)$ and $\widehat{f} \in W^{1,p}(B)$, with $B := B^+ \cup B^-$. Moreover, we will use the fact that $u \in W^{1,\infty}(\Omega)$ (see next Section, Theorem 5.2), so that the same is true for $\widehat{u}$, that is $\widehat{u} \in W^{1,\infty}(B)$. This an important and peculiar point of the proof, that we did not find in other strategies in the literature: we need L^∞ regularity on the gradient, to prove Sobolev regularity of a nonlinear function of the gradient itself. We now want to find the equation satisfied by $\widehat{u}$ in the unit ball: using the change of variables $x = \Psi(y)$ and the fact that u satisfies

$$\int_\Omega \langle \nabla \mathcal{H}^*(\nabla u(x)), \nabla \varphi(x)\rangle\, dx = \int_\Omega f(x)\varphi(x)\, dx, \quad \text{for every } \varphi \in W^{1,p}(\Omega),$$

it is easy to see that $\widehat{u}$ satisfies

$$\int_{B^+} \langle A(y, \nabla \widehat{u}(y)), \nabla \varphi(y)\rangle\, dy = \int_{B^+} \widehat{f}(y)\varphi(y)\, dy, \quad \text{for every } \varphi \in W^{1,p}_0(B),$$

where the function A is defined by

$$A(y,p) = |\det D\Psi(y)|\, \nabla \mathcal{H}^*(p\,[D\Psi(y)]^{-1})\left([D\Psi(y)]^{-1}\right)^t, \ (y,p) \in B^+ \times \mathbb{R}^N.$$

On the other hand, using the change of variables $x = \overline{\Psi}(y) := \Psi(\mathcal{R}y)$ we obtain

$$\int_{B^-} \langle \overline{A}(\nabla \widehat{u}(y)), \nabla \varphi(y)\rangle\, dy = \int_{B^-} \widehat{f}(y)\varphi(y)\, dy, \quad \text{for every } \varphi \in W^{1,p}_0(B),$$

with $\overline{A}$ given by

$$\overline{A}(y,p) = |\det D\overline{\Psi}(y)|\, \nabla \mathcal{H}^*(p\,[D\overline{\Psi}(y)]^{-1})\left([D\overline{\Psi}(y)]^{-1}\right)^t, \ (y,p) \in B^- \times \mathbb{R}^N.$$

Observe that we have

$$|\det D\overline{\Psi}(y)| = |\det D\Psi(\mathcal{R}y)| \quad \text{and} \quad [D\overline{\Psi}(y)]^{-1} = \mathcal{R}[D\Psi(\mathcal{R}y)]^{-1},$$

then let us assume for a moment the existence of $\mathcal{O} \in C^{1,1}(B^+; \mathbb{R}^{N\times N})$ such that for every $y \in B^+$, $\mathcal{O}(y)$ is an orthogonal matrix verifying

$$[D\Psi(y',0)]^{-1} = \mathcal{R}[D\Psi(y',0)]^{-1}\mathcal{O}(y',0), \ y' \in \partial B^+ \cap \partial B^-. \tag{4.14}$$

Setting for simplicity

$$\widehat{M}(y) = \begin{cases} [D\Psi(y)]^{-1}, & \text{if } y \in B^+ \\ \mathcal{R}[D\Psi(\mathcal{R}y)]^{-1}\mathcal{O}(\mathcal{R}y), & \text{if } y \in B^- \end{cases}$$

the previous discussion, together with asssumption (4.14) and the fact that

$$\nabla \mathcal{H}^*(z\,\mathcal{O}) = \nabla \mathcal{H}^*(z)\,\mathcal{O},$$

tells us that $\widehat{u}$ is a weak solution in B of the equation

$$-\mathrm{div} H(y,\nabla\widehat{u}) = \widehat{f}, \tag{4.15}$$

where the operator H is defined by

$$H(y,p) = |\det \widehat{M}(y)|^{-1}\,\nabla\mathcal{H}^*(p\,\widehat{M}(y))\widehat{M}(y)^t,\ \ (y,p)\in B\times\mathbb{R}^N.$$

Observe that having assumed (4.14), is crucial to obtain that $H(\cdot,p)$ is continuous across the hyperplane $\{x_N = 0\}$, which in turn implies that $H(\cdot,p)$ is Lipschitz.

So let us verify the existence of such a matrix field $\mathcal{O}$: by polar decomposition, we know that $OU = D\Psi$, with O orthogonal and U symmetric and positive definite. This implies that $O = D\Psi\,U^{-1}$ and $[D\Psi]^t D\Psi = U^2$, that is

$$\mathcal{O} = D\Psi([D\Psi]^t D\Psi)^{-\frac{1}{2}}, \tag{4.16}$$

which is our candidate for the $C^{1,1}$ matrix field. We now have to make an explicit choice for the diffeomorphism Ψ, in order to obtain that this $\mathcal{O}$ further verifies (4.14): we can suppose, up to a translation, that $x_0 = 0$ and moreover that, up to a rotation, the set $\overline{V}\cap\partial\Omega$ can be represented as the graph of a $g\in C^{3,1}$ defined on $\{|y|\le 1\ :\ y_N = 0\}$, that is $\overline{V}\cap\partial\Omega = \{(y',g(y'))\ :\ |y'|\le 1\}$. Then we see that taking Ψ of the following form

$$\Psi(y',y_N) = (y',g(y')) - y_N(\nabla g(y'),-1),$$

we get that Ψ is diffeomorphism between the interior of B^+ and $V\cap\Omega$ (up to redefine the neighborhood V, without changing its intersection with $\partial\Omega$). Moreover, we have the following expression for the Jacobian matrix

$$D\Psi(y) = \begin{bmatrix} \mathrm{Id}_{N-1} - y_N\,D^2 g(y') & \nabla g(y') \\ & \\ -(\nabla g(y'))^t & 1 \end{bmatrix}$$

where Id_{N-1} stands for the $(N-1)\times(N-1)$ identity matrix. Then it is easily seen that with the choice (4.16), property (4.14) is equivalent to require that

$$[D\Psi(y',0)]^t D\Psi(y',0) = (D\Psi(y',0)\mathcal{R})^2,$$

and this is a straightforward consequence of the structure of $D\Psi$. We point out that despite the hypothesis on $\partial\Omega$ of being $C^{3,1}$, the diffeomorphism we have provided is only of class $C^{2,1}$ (we use the gradient of g in the definition of Ψ). This need of an extra - somehow unnatural - regularity on Ω is in common with [9] : on the one hand we need a diffeomorphism having a Jacobian matrix with a special structure (condition (4.14)), which asks for one extra degree of regularity on the boundary, and on the other hand, at some point of the proof, we need this diffeomorphism to have a $C^{1,1}$ Jacobian matrix and not only Lipschitz. This asks for the other extra degree of regularity; it could seem strange but we feel that it corresponds to the fact that also f is supposed to be more regular than what is usually required, and this is due to the degeneracy of the equation. Anyway, we stress that the matrix field $\widehat{M}$ that we get in the end is piecewise $C^{1,1}$ (on the two sides of the boundary) and globally no more than $C^{0,1}$ (since the reflection does not respect this kind of higher regularity). This lack of global $C^{1,1}$ regularity will be fixed by means of global boundedness of $\nabla\widehat{u}$.

We now aim to show that $\widehat{\mathcal{G}}(y) = G(\nabla\widehat{u}(y)\,\widehat{M}(y)) \in W^{1,2}_{\mathrm{loc}}(B)$, then this will clearly imply that $\mathcal{G} \in W^{1,2}$ in a neighborhood of x_0, thus concluding the proof. Let us begin with some manipulations: in order to simplify the notations, we set $\mathfrak{d}(y) = |\det\widehat{M}(y)|^{-1}$, then we begin applying (4.3), so that as in Theorem 4.2 we obtain

$$\left|\delta_{h,\omega}\widehat{\mathcal{G}}\right|^2 \leq \langle \delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}), \delta_{h,\omega}(\nabla\widehat{u}\,\widehat{M})\rangle,$$

where as always $\delta_{h,\omega}$ denotes the incremental ratio in the direction $\omega \in S^{N-1}$. Then with some algebraic manipulations, the right-hand side can we re-written as

$$\begin{aligned}\langle \delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}), \delta_{h,\omega}(\nabla\widehat{u}\,\widehat{M})\rangle &= \langle \delta_{h,\omega}\left(\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})\widehat{M}^t\right), \delta_{h,\omega}\nabla\widehat{u}\rangle \\ &\quad - \langle \nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})(\delta_{h,\omega}\widehat{M}^t), \delta_{h,\omega}\nabla\widehat{u}\rangle - \langle \delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}), \nabla\widehat{u}\,\delta_{h,\omega}\widehat{M}\rangle,\end{aligned}$$

and multiplying by $\mathfrak{d}_h$ we obtain

$$\mathfrak{d}_h|\delta_{h,\omega}\widehat{\mathcal{G}}|^2 \leq \langle \delta_{h,\omega}H, \delta_{h,\omega}\nabla\widehat{u}\rangle - \langle \nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})(\delta_{h,\omega}\mathfrak{d}\,\widehat{M}^t), \delta_{h,\omega}\nabla\widehat{u}\rangle - \mathfrak{d}_h\langle \delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}), \nabla\widehat{u}\,\delta_{h,\omega}\widehat{M}\rangle.$$

We can now as always take a smooth cut-off function ζ supported in some smaller ball $B' \subset B$, multiply the previous inequality by ζ^2 and then integrate over $\operatorname{supp}\zeta$, so that we get

$$\begin{aligned}\int |\delta_{h,\omega}\widehat{\mathcal{G}}|^2\,dy &\leq \int |\delta_{h,\omega}H|\,|\nabla\zeta|\,\zeta\,|\delta_{h,\omega}\widehat{u}|\,dy - \int \langle \nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})\delta_{h,\omega}(\mathfrak{d}\,\widehat{M}), \delta_{h,\omega}\nabla\widehat{u}\rangle\,\zeta^2\,dy \\ &\quad + \int |\mathfrak{d}_h||\delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})|\,|\nabla\widehat{u}|\,|\delta_{h,\omega}\widehat{M}|\,\zeta^2\,dy + \int |\delta_{h,\omega}\widehat{f}|\,|\delta_{h,\omega}\widehat{u}|\zeta^2\,dy := \sum_{i=1}^4 \mathcal{I}_i,\end{aligned}$$

where we have used the fact that $\widehat{u}$ is a solution of (4.15) and $\mathfrak{d} \in L^\infty$ and $\mathfrak{d} \geq c > 0$. For simplicity, we now discuss separately the estimates of every integral:

Estimate for $\mathcal{I}_1$ We would like to use the basic inequality (4.4): we first observe that

$$\delta_{h,\omega}H(y,\nabla\widehat{u}) = \left[\nabla\mathcal{H}^*(\nabla\widehat{u}(y+h\omega)\,\widehat{M}(y+h\omega))\right]\,\delta_{h,\omega}(\mathfrak{d}\,\widehat{M}^t) + \left[\mathfrak{d}(y)\widehat{M}^t(y)\right]\,\delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}),$$

so that

$$\begin{aligned}\mathcal{I}_1 &\leq \int \left|\nabla\mathcal{H}^*(\nabla\widehat{u}(y+h\omega)\,\widehat{M}(y+h\omega))\right|\,|\delta_{h,\omega}(\mathfrak{d}\,\widehat{M}^t)|\,|\nabla\zeta|\,\zeta\,|\delta_{h,\omega}\widehat{u}|\,dy \\ &\quad + \int \left|\mathfrak{d}\,\widehat{M}^t\right|\,|\delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})|\,|\nabla\zeta|\,\zeta\,|\delta_{h,\omega}\widehat{u}|\,dy,\end{aligned}$$

and it is easily seen that the first term does not present any problem, thanks to the fact that $\widehat{u} \in W^{1,q}$, $\mathfrak{d}\,\widehat{M}^t \in W^{1,\infty}$ and $\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}) \in L^p$. On the contrary, the second integral is a kind of term that has already been estimated in the proof of Theorem 4.2, once one takes care of the fact that $\mathfrak{d}\,\widehat{M}^t \in W^{1,\infty}$: indeed, in this case it is only left to estimate

$$\int |\delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})|\,|\nabla\zeta|\,\zeta\,|\delta_{h,\omega}\widehat{u}|\;dy,$$

and it is now sufficient to apply (4.4), so that the previous integral can be majorized by

$$\int_\Omega \left(|\widehat{\mathcal{G}}_{h,\omega}|^{\frac{q-2}{q}} + |\widehat{\mathcal{G}}|^{\frac{q-2}{q}}\right)\,|\delta_{h,\omega}\widehat{\mathcal{G}}|\,\zeta\,|\nabla\zeta||\delta_{h,\omega}\widehat{u}|\;dx$$

then one can use the $\varepsilon-$Young inequality in order to absorbe a term of the kind $\int|\delta_{h,\omega}\widehat{\mathcal{G}}|^2\zeta^2$, and finally one is left with an integral

$$\int_\Omega \left(|\mathcal{G}_{h,\omega}|^{\frac{q-2}{q}} + |\mathcal{G}|^{\frac{q-2}{q}}\right)^2\,|\nabla\zeta|^2|\delta_{h,\omega}u|^2\;dx,$$

which can be easily estimated just as in the proof of Theorem 4.2.

Estimate for $\mathcal{I}_2$ This is the most delicate integral: indeed, we have to integrate by parts in order to avoid the difference quotients of ∇u. As a drawback, this will let appear second-order difference quotients of $\mathfrak{d}\,\widehat{M}$, which in principle can not be easily managed, due to the fact that this is only a Lipschitz function: anyway, thanks to our construction, we know that $\mathfrak{d}\,\widehat{M}$ is a function which is $C^{1,1}$ out of $\{x_N = 0\}$ and globally $C^{0,1}$: this means that its second-order difference quotients $\delta^2_{h,\omega}(\mathfrak{d}\,\widehat{M})$ (see below for their definition) are uniformly bounded if one stays out of a strip of size h around the hyperplane $\{x_N = 0\}$ and are bounded by Ch^{-1} in that strip (whose measure is of the order of h), which implies $\int\left|\delta^2_{h,\omega}(\mathfrak{d}\,\widehat{M})\right| \le C$.

First of all, we separate the two terms $\nabla\widehat{u}(y+h\omega)$ and $\nabla\widehat{u}(y)$ and we perform the change of variable $z = y + h\omega$ in the first integral, thus obtaining

$$\begin{aligned}\mathcal{I}_2 = {} & \frac{1}{h}\int\langle\nabla\mathcal{H}^*(\nabla\widehat{u}(y-h\omega)\,\widehat{M}(y-h\omega))\delta_{-h,\omega}(\mathfrak{d}\,\widehat{M}),\nabla\widehat{u}(y)\rangle\,\zeta^2(y-h\omega)\,dy\\ & + \frac{1}{h}\int\langle\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})\delta_{h,\omega}(\mathfrak{d}\,\widehat{M}),\nabla\widehat{u}(y)\rangle\,\zeta^2\,dy,\end{aligned}$$

which can be recast into

$$\begin{aligned}\mathcal{I}_2 = {} & \int\langle\mathcal{H}^*(\nabla\widehat{u}(y-h\omega)\,\widehat{M}(y-h\omega))\,\delta^2_{h,\omega}(\mathfrak{d}\,\widehat{M}),\nabla\widehat{u}\rangle\zeta^2(y-h\omega)\,dy\\ & + \int\langle\mathcal{H}^*(\nabla\widehat{u}(y-h\omega)\,\widehat{M}(y-h\omega))\,\delta_{h,\omega}(\mathfrak{d}\,\widehat{M}),\nabla\widehat{u}\rangle\,\delta_{-h,\omega}\zeta^2\,dy + \int\langle\delta_{-h,\omega}\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M}),\nabla\widehat{u}\rangle\,\zeta^2(y)\,dy,\end{aligned}$$

where

$$\delta^2_{h,\omega}(\mathfrak{d}\,\widehat{M}) = \frac{\mathfrak{d}(y+h\omega)\,M(y+h\omega) + \mathfrak{d}(y-h\omega)\,M(y-h\omega) - 2\mathfrak{d}(y)\,M(y)}{h^2}.$$

We now observe that the last two terms can be easily estimated as already seen (see the discussion for $\mathcal{I}_1$ and $\mathcal{I}_3$), while the first contains second-order differential quotients of $\mathfrak{d}\,\widehat{M}$: since all the other factors in the integral are bounded (because $\nabla\widehat{u}$ is bounded), this integral may be estimated with $C\int\left|\delta^2_{h,\omega}(\mathfrak{d}\,\widehat{M})\right|$, and this integral is indeed bounded.

Estimate for $\mathcal{I}_3$ Using the fact that $\widehat{M}$ is Lipschitz, together with estimate (4.4), yields

$$I_4\le C\,\mathrm{lip}(\widehat{M})\int\left|\delta_{h,\omega}\nabla\mathcal{H}^*(\nabla\widehat{u}\,\widehat{M})\right||\nabla\widehat{u}|\,\zeta^2\,dy\le\int\left|\delta_{h,\omega}\widehat{\mathcal{G}}\right|\left(|\widehat{\mathcal{G}}|^{\frac{q-2}{q}}+|\widehat{\mathcal{G}}|^{\frac{q-2}{q}}\right)|\nabla\widehat{u}|\,\zeta^2\,dy,$$

and this can be estimated as before (see the estimation for $\mathcal{I}_1$), absorbing the difference quotients of $\widehat{\mathcal{G}}$ in the left-hand side, thanks to Young's inequality.

Estimate for I_4 This is clearly the easy part: it is sufficient to use Hölder's inequality and the fact that $\widehat{f}\in W^{1,p}$ and $\widehat{u}\in W^{1,q}$. □

Finally, we get the desired global Sobolev estimate on the optimizer σ: the proof is a straightforward extension of that of Corollary 4.3.

Corollary 4.5. *Under the assumptions of Theorem 4.4, the conclusions of Corollary 4.3 are global.*

5 L^∞ estimates for the gradient

The proof of the Lipschitz regularity result will easily follow from results of Fonseca-Fusco-Marcellini [15] that we now recall (under slightly stronger assumptions which will be sufficient for our purpose). Let U be an open subset of $\mathbb{R}^N$ and $L:\ (y,\xi)\in U\times\mathbb{R}^N\mapsto L(y,\xi)$ be continuous and bounded from below and such that:

- there is a $C>0$ such that

$$L(y,\xi)\le C(1+|\xi|^q),\ \forall(y,\xi)\in U\times\mathbb{R}^N,\tag{5.1}$$

- there exists $R>0$ and $\nu>0$ such that $L(y,.)$ is $C^2(\mathbb{R}^N\setminus\overline{B}_R)$ for every $y\in U$ and

$$\langle D^2_{\xi,\xi}L(y,\xi)\lambda,\lambda\rangle\ge\nu(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2,\ \forall(y,\xi,\lambda)\in U\times\mathbb{R}^N\setminus\overline{B}_R\times\mathbb{R}^N,\tag{5.2}$$

- for $|\xi|>R$, $y\in U\mapsto L_\xi(y,\xi)$ is weakly differentiable with

$$|D_xL_\xi(y,\xi)|\le C(1+|\xi|^{q-1}),\ \forall(y,\xi)\in U\times\mathbb{R}^N\setminus\overline{B}_R.\tag{5.3}$$

For $v \in W^{1,q}_{\rm loc}(U)$ and $A \subset\subset U$ open, set

$$F(v, A) := \int_A L(y, \nabla v(y))dy.$$

Then $v \in W^{1,q}_{\rm loc}(U)$ is said to be a local minimizer of F in U whenever $F(v, B_r(x_0)) \leq F(w, B_r(x_0))$ as soon as $B_r(x_0) \subset\subset U$ and $w \in v + W^{1,q}_0(B_r(x_0))$. The following result of Fonseca-Fusco-Marcellini [15] will be extremely useful to derive the Lipschitz estimate on the solutions of (4.13) .

Theorem 5.1. *Under the assumptions above, if $v \in W^{1,q}_{\rm loc}(U)$ is a local minimizer of F in U then it is locally Lipschitz. More precisely there is a constant $C_0 = C_0(q, C, \nu, R)$ such that $||\nabla v||^q_{L^\infty(B_{r/2}(x_0))} \leq C_0(1 + \fint_{B_r(x_0)} |\nabla v|^q)$ as soon as $B_r(x_0) \subset\subset U$.*

We are now in position to prove the second regularity result we needed to apply DiPerna-Lions theory, i.e. the L^∞ estimate for ∇u. The following theorem will be actually applied, for the sake of this paper's applications, to the case of a $C^{3,1}$ domain with $f = \mu_0 - \mu_1$ being a Lipschitz function.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,1}$ domain. Given $\alpha \in (0, 1]$ and $f \in C^{0,\alpha}$ with zero-mean, every solution $u \in W^{1,q}_\diamond(\Omega)$ of the Neumann boundary value problem* (4.13) *is a Lipschitz function.*

Proof. Interior Lipschitz regularity directly follows from Theorem 5.1 (recalling $q \geq 2$ and (4.1)). To prove Lipschitz regularity up to the boundary, we proceed as in the proof of Theorem 4.4 and define $\widehat{u}$, $\widehat{f}$ and $\widehat{M}$ on B as previously. Then, let us introduce the function $\widehat{\varphi}$ solving the Poisson equation $\Delta\widehat{\varphi} = \widehat{f}$ with $\widehat{\varphi} = 0$ on ∂B. By standard elliptic regularity $\widehat{\varphi} \in C^{2,\alpha}$ and in particular $\nabla\widehat{\varphi}$ is Lipschitz. The utility of introducing $\widehat{\varphi}$ is the following : the result in Theorem 5.1 concerns minimizers of functionals of the form $\int L(y, \nabla u)dy$ with no explicit dependence on u; actually, our solution u is the minimizer of the functional $\int \mathcal{H}^*(\nabla u) - fu$; when we pass to $\widehat{u}$ the structure stays variational but we need to deal with the term in $\widehat{f}\widehat{u}$. This term may be absorbed in the other one, if we integrate by parts, since $\int_B \widehat{f}(v - \widehat{u}) = -\int_B \nabla\widehat{\varphi} \cdot \nabla(v - \widehat{u})$ for every $v \in \widehat{u} + W^{1,q}_0(B)$. Hence, let us then define for every $(y, \xi) \in B \times \mathbb{R}^N$:

$$L(y, \xi) := |\det \widehat{M}(y)|^{-1}\mathcal{H}^*(\widehat{M}(y)\ \xi) + \nabla\widehat{\varphi} \cdot \xi \ \text{ and }\ F(v) := \int_B L(y, \nabla v(y))dy,\ \forall v \in W^{1,q}(B).$$

Since F is convex and $\widehat{u}$ satisfies (4.15), $\widehat{u}$ is actually a local minimizer of F on B. Now the fact that L satisfies (5.2) directly follows from (4.1) and to check that it also satisfies (5.3) it is enough to recall that $\nabla\widehat{\varphi}$ is Lipschitz and that $\widehat{M}$ is Lipschitz as soon as Ω is $C^{2,1}$. Invoking again Theorem 5.1, we deduce that $\widehat{u} \in W^{1,\infty}_{\rm loc}(B)$. Since $\partial\Omega$ is compact, by a straightforward finite covering argument, we conclude that u is globally Lipschitz on Ω. □

Notice that this proof passing through the results of [15] asks for regularity on f, namely $f \in C^{0,\alpha}$. This was not restrictive in this paper because, for other reasons due to the computation of the divergence of $\widehat{\sigma}$, f should already be supposed Lipschitz continuous. Yet, it is a matter of fact that the sharp assumption for this L^∞ result should be $f \in L^{N+\varepsilon}$, an assumption which does not fit into this (shorter) strategy.

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