SIAM J. MATH. ANAL.
Vol. 37, No. 1, pp. 60–82

MATHEMATICAL ANALYSIS OF A NONLINEAR PARABOLIC EQUATION ARISING IN THE MODELLING OF NON-NEWTONIAN FLOWS*

ERIC CANCÈS[†], ISABELLE CATTO[‡], AND YOUSRA GATI[†]

Abstract. The mathematical properties of a nonlinear parabolic equation arising in the modelling of concentrated suspension flows are investigated. The peculiarity of this equation is that it may degenerate into a hyperbolic equation (in fact, a linear advection equation). Depending on the initial data, at least two situations can be encountered: the equation may have a unique solution in a convenient class, or it may have infinitely many solutions. The present article is the theoretical side of a joint project with rheologists, aiming at better understanding the flows of complex fluids.

Key words. nonlinear parabolic equation, degenerate parabolic equation, viscosity solutions, non-Newtonian flows, complex fluids, concentrated suspensions

AMS subject classifications. 35K50, 35K55, 35K65, 35Q35, 76A05

DOI. 10.1137/S0036141003430044

1. Introduction and description of the model.

Complex fluids are substances that are neither really liquid nor really solid in the classical sense. They include melt polymers, colloids, emulsions, foams, gels, liquid crystals, suspensions, and other materials that form flowable microstructures. Modelling the flow of such fluids is a very intricate problem which is far from being solved up to now. The model we are interested in is an attempt to recover the rheological behavior of the particular type of concentrated suspensions. Examples of such suspensions are numerous and can be found, e.g., in food (pastes), cosmetics (tooth paste), medicine (blood), and building industry (cement). In contrast to some complex fluids such as polymeric liquids for which elaborate rheological models, based on fine mesoscopic physical descriptions, are available, the modelling of concentrated suspensions is still in its infancy.

When simple fluids are sheared, stress and shear rate are linked by a linear relation. The linear response coefficients are well understood and their relation to the microstructure of the fluid are known [7]. On the contrary, complex systems, such as concentrated suspensions of hard or soft spheres, exhibit very nonlinear flow properties far from being understood. These nonlinear properties occur not only at high shear rates, where one expects that linear response theory does not apply, but also at very low shear rates. Let us for instance consider a concentrated suspension of particles. At low concentrations, thermally induced structural relaxations of the particles positions occur. The system behaves as a Newtonian fluid at low shear rates[1]. But, when the concentration is increased, the energetical cost of particles reorganizations

---

*Received by the editors June 19, 2003; accepted for publication (in revised form) November 5, 2004; published electronically August 17, 2005.

http://www.siam.org/journals/sima/37-1/43004.html

[†] CERMICS, Ecole Nationale des Ponts et Chaussées and INRIA, 6 & 8 avenue Blaise Pascal, Cité Descartes, 77455 Marne-la-Vallée Cedex 2, France (cances@cermics.enpc.fr, gati@cermics.enpc.fr).

[‡] CEREMADE, UMR CNRS 7534, Université Paris IX-Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France (catto@ceremade.dauphine.fr).

[1] At higher shear rates, due to hydrodynamics interactions, the relation between the stress and the shear rate may be sublinear, and the system is said to be shear-thinning. We neglect all hydrodynamics interactions in the following description.

is much higher than thermal energy, and structural relaxations are arrested. This implies that, once at rest, the system is not at thermal equilibrium, and exhibits general properties of glasses [1, 11, 13, 14]. This property has a striking consequence on the low shear rate flow of these systems: the stress σ tends to a finite nonzero value σ_c when shear rate goes to zero. This discontinuity of the shear stress *versus* shear rate is an experimental very common feature, but is very poorly understood. One of the most common physical explanations relies on the hypothesis that the system may locally store deformation. It relies on observations of the structures of concentrated dispersions that appear to be frozen in nonequilibrium positions: particular configurations of the dynamically arrested suspension store deformation energy. They do not correspond to configurations of minimal energy, and may thus store finite values of stress and strain. The system may thus be described as an heterogeneous field of stress and strain. Microscopic description of these fields does not exist, and, besides studies which aim at improving phenomenological models (such as the celebrated Herschel–Bulkley model [9]), a few attempts have been made to recover the rheological behavior of complex fluids from elementary physical processes. We consider here the model proposed by Hébraud and Lequeux, in which the system is divided in mesoscopic blocks whose size is large enough so that stress and strain tensors may be defined for each block, but small compared to the characteristic length scale of the stress field. A mesoscopic evolution equation of the stress of each block is then written as:

(i) At low shear, each particle keeps the same neighbors, and a block behaves as an Einstein elastic solid, in which the elasticity arises from interactions between neighboring particles.

(ii) Then, deformation induces local reorganization of the particles, at a given stress threshold σ_c. Above this threshold, the block flows as an Eyring fluid: the configuration reached by shearing the suspension relaxes with a characteristic time T_0 towards a completely relaxed state, where no stress is stored.

(iii) Lastly, coupling between the flow of neighboring blocks must be included. This is taken into account by the introduction of a diffusion term in the evolution equation, where it is assumed that the diffusion coefficient is proportional to the number of reorganizations per unit time.

In the model, each block carries a given shear stress σ (σ is a real number; it is in fact an extra-diagonal term of the stress tensor in convenient coordinates). The evolution of the blocks is described through a probability density $p(t,\sigma)$ which represents the distribution of stress in the assembly of blocks at time t. The equation for the probability density $p(t,\sigma)$ for a block to be under stress σ at time t is written as

$$\begin{cases} \partial_t p = -b(t)\partial_\sigma p + D(p(t))\partial^2_{\sigma\sigma} p - \dfrac{\mathbb{1}_{\mathbb{R}\setminus[-\sigma_c,\sigma_c]}(\sigma)}{T_0} p + \dfrac{D(p(t))}{\alpha}\delta_0(\sigma), \\ t \in (0;T), \quad \sigma \in \mathbb{R}, \\ p \geq 0, \\ p(0,\sigma) = p_0(\sigma), \end{cases} \tag{1.1}$$

where for $f \in L^1(\mathbb{R})$, we denote

$$D(f) = \frac{\alpha}{T_0} \int_{|\sigma| > \sigma_c} f(\sigma) d\sigma. \tag{1.2}$$

The equation satisfied by p in (1.1) is referred to as the SP equation in the following. In this equation, $\mathbb{1}_{\mathbb{R}\setminus[-\sigma_c,\sigma_c]}$ denotes the characteristic function of the open set $\mathbb{R}\setminus[-\sigma_c,\sigma_c]$ and δ_0 the Dirac delta function on $\mathbb{R}$. The three terms arising from the right-hand side of the HL equation model the three physical features described above. When a block is submitted to a shear rate $\dot\gamma(t)$, the stress of this block evolves with a variation rate $b(t) = G_0\dot\gamma(t)$, where G_0 is an elasticity constant. (In this study, the shear rate $\dot\gamma(t)$, and therefore the function $b(t)$, are assumed to be in $L^2_{\mathrm{loc}}(\mathbb{R}^+)$.) When the modulus of the stress overcomes the critical value σ_c, the block becomes unstable and may relax into a state with zero stress after a characteristic relaxation time T_0. This property is expressed by the last two terms in (1.1). This relaxation phenomenon induces a rearrangement of the other blocks and this is finally modelled through the diffusion term $D(p(t))\partial^2_{\sigma\sigma}p$. The diffusion coefficient $D(p(t))$ as given by (1.2) is assumed to be proportional to the density of blocks that rearrange during time T_0, by a proportional parameter α which depends on the microscopic properties of the sample and which is supposed to represent the "mechanical fragility" of the material. This nonlinear diffusion term is introduced to display the importance of collective effects in this kind of samples. For more details on the physical meaning of the model, we refer to the original article by Hébraud and Lequeux [8].

In all that follows, the parameters α, T_0 and σ_c are positive, and the initial data p_0 in (1.1) is a given probability density; that is,

$$p_0 \geq 0, \quad p_0 \in L^1(\mathbb{R}), \quad \int_{\mathbb{R}} p_0 = 1. \tag{1.3}$$

We will be looking for solutions $p = p(t,\sigma)$ in $C^0_t(L^1_\sigma \cap L^2_\sigma)$ such that σp belongs to $L^\infty_t(L^1_\sigma)$ of the nonlinear parabolic partial differential equation (1.1). The subscript σ refers to integration over $\mathbb{R}$ with respect to σ, whereas the subscript t refers to time integration on $[0,T]$ for any $T > 0$. Note that under a mean-field assumption the macroscopic stress in the material is given by

$$\tau(t) = \int_{\mathbb{R}} \sigma p(t,\sigma)d\sigma, \tag{1.4}$$

and therefore the above condition on σp ensures that the average stress is an essentially bounded function of time.

Actually in practice, the shear rate is not uniform in the flow (the shear creates elastic waves in the fluid), and in order to better describe the coupling between the macroscopic flow and the evolution of the microstructure we introduce and study in a second paper [2] a micro-macro model where the shear rate is a function of the velocity of the macroscopic flow. In this model p is also a function of the macroscopic space variables and the average stress defined by (1.4) is inserted into the macroscopic equation governing the velocity of the macroscopic flow (see also section 6 below).

In order to lighten the notation and without loss of generality we assume from now on that $\sigma_c = 1$ and $T_0 = 1$. This amounts to changing the time and stress scales.

The main difficulties one encounters in the mathematical analysis come from the nonlinearity in the diffusion term and even more from the fact that the parabolic equation may degenerate when the viscosity coefficient $D(p)$ vanishes, and this will be shown to appear only when $D(p_0) = 0$. This difficulty is illustrated on a simplified example just below and also in section 5 where we discuss the existence of stationary solutions in the case when the shear rate b is a constant.

Let us first of all look at the following simplified model which already includes the difficulties we are going to face to in the study of (1.1). We consider the equation

$$\begin{cases} \partial_t u = D(u(t))\partial^2_{\sigma\sigma} u, \\ u(0,\sigma) = \frac{1}{2}\mathbb{1}_{]-1,1[}(\sigma), \end{cases} \tag{1.5}$$

where $\mathbb{1}_{]-1,1[}$ is the characteristic function of the interval $]-1,1[$. The initial condition is purposely chosen in such a way that $D(u(t=0)) = 0$. The function $u = \frac{1}{2}\mathbb{1}_{]-1,1[}(\sigma)$ is a stationary solution to this equation and for this solution $D(u(t))$ is identically zero. But it is not the unique solution to (1.5) in $C^0_t(L^2_\sigma) \cap L^\infty_t(L^1_\sigma)$. It is indeed possible to construct a so-called *vanishing viscosity solution* for which $D(u(t)) > 0$ for all $t > 0$, and there are actually infinitely many solutions to this equation. (This statement is obtained as a corollary of Lemma 4.3 in section 4 below.)

As far as (1.1) is concerned, we show that, in the case when $D(p_0) = 0$ and $b \equiv 0$, we may have either a unique or infinitely many solutions, depending on the initial data (see Proposition 4.1 in section 4).

On the other hand, we are able to prove the following existence and uniqueness result in the nondegenerate case when $D(p_0) > 0$.

THEOREM 1.1. *Let the initial data* p_0 *satisfy the conditions*

$$p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad p_0 \geq 0, \int_{\mathbb{R}} p_0 = 1, \text{ and } \int_{\mathbb{R}} |\sigma| p_0 < +\infty, \tag{1.6}$$

and assume that

$$D(p_0) > 0.$$

Then, for every $T > 0$, *there exists a unique solution* p *to* (1.1) *in* $L^\infty_t(L^1_\sigma \cap L^2_\sigma) \cap L^2_t(H^1_\sigma)$. *Moreover,* $p \in L^\infty_{t,\sigma} \cap C^0_t(L^1_\sigma \cap L^2_\sigma)$, $\int_{\mathbb{R}} p(t,\sigma)d\sigma = 1$ *for all* $t > 0$, $D(p) \in C^0_t$ *and for every* $T > 0$ *there exists a positive constant* $\nu(T)$ *such that*

$$\min_{0 \leq t \leq T} D(p(t)) \geq \nu(T).$$

Besides $\sigma p \in L^\infty_t(L^1_\sigma)$ *so that the average stress* $\tau(t)$ *is well-defined by* (1.4) *in* L^∞_t.

The first step towards the existence proof of solutions to (1.1) will consist of the study of so-called vanishing viscosity approximations, which are the unique solutions to the following family of equations:

$$\begin{cases} \partial_t p_\varepsilon = -b(t)\partial_\sigma p_\varepsilon + (D(p_\varepsilon(t)) + \varepsilon)\partial^2_{\sigma\sigma} p_\varepsilon - \mathbb{1}_{\mathbb{R}\setminus[-1,1]} p_\varepsilon + \dfrac{D(p_\varepsilon(t))}{\alpha}\delta_0(\sigma), \\ p_\varepsilon \geq 0, \\ p_\varepsilon(0,\cdot) = p_0. \end{cases} \tag{1.7}$$

(Recall that we have rescaled the time and stress units to get $T_0 = 1$ and $\sigma_c = 1$.) Section 2 below is devoted to the proof of the following proposition.

PROPOSITION 1.2 (existence and uniqueness of vanishing viscosity approximations). *Let* $T > 0$ *be given. We assume that the initial data satisfies the same conditions* (1.6) *as in the statement of the theorem. Then, for every* $T > 0$ *and*

$0 < \varepsilon \le 1$, there exists a unique solution p_ε to (1.7) *in $L^\infty_t(L^1_\sigma \cap L^2_\sigma) \cap L^2_t(H^1_\sigma)$. Moreover, $p_\varepsilon \in L^\infty_{t,\sigma} \cap C^0_t(L^1_\sigma \cap L^2_\sigma)$, $D(p_\varepsilon) \in C^0_t$,*

$$\int_{\mathbb{R}} p_\varepsilon = 1, \tag{1.8}$$

$$0 \le p_\varepsilon \le \|p_0\|_{L^\infty_\sigma} + \sqrt{\frac{\alpha}{\pi}}\sqrt{T}, \tag{1.9}$$

and for every $T > 0$, there exist positive constants $C_1(T, p_0)$, $C_2(T, p_0)$, and $C_3(T, p_0)$ which are independent of ε such that

$$\sup_{0\le t\le T} \int_{\mathbb{R}} |\sigma| p_\varepsilon \le C_1(T, p_0), \tag{1.10}$$

$$\sup_{0\le t\le T} \int_{\mathbb{R}} p_\varepsilon^2 \le C_2(T, p_0), \tag{1.11}$$

and

$$\int_0^T \big(\varepsilon + D(p_\varepsilon)\big) \int_{\mathbb{R}} |\partial_\sigma p_\varepsilon|^2 \le C_3(T, p_0). \tag{1.12}$$

Theorem 1.1 is then proved in section 3 while the degenerate case is investigated in section 4. Lastly, the description of stationary solutions in the constant shear rate case is carried out in section 5.

2. The vanishing viscosity approximation. This section is devoted to the proof of Proposition 1.2. We begin with the following lemma.

LEMMA 2.1 (uniqueness). *Let p_0 satisfy* (1.3). *Then for every $T > 0$ and $0 < \varepsilon$, there exists at most one solution p_ε to* (1.7) *in $L^\infty_t(L^1_\sigma \cap L^2_\sigma) \cap L^2_t(H^1_\sigma)$. Moreover, $p_\varepsilon \in C^0_t(L^2_\sigma)$ (thus, the initial condition makes sense) and*

$$\int_{\mathbb{R}} p_\varepsilon = 1, \tag{2.1}$$

for almost every t in $[0, T]$.

Proof. We begin by proving that every solution to (1.7) in $L^\infty_t(L^1_\sigma \cap L^2_\sigma) \cap L^2_t(H^1_\sigma)$ satisfies (2.1). We fix $R > 1$ and we consider a cut-off C^2 function $\phi_R = \phi_R(\sigma)$ with compact support which is equal to 1 when $0 \le |\sigma| \le R$ and to 0 when $|\sigma| \ge 2R$ and such that

$$|\phi_R'| \le \frac{C}{R}, \tag{2.2}$$

where here and below C denotes a positive constant that is independent of R. Notice that ϕ' is equal to 0 on $]-\infty, -2R]$, on $[-R, R]$ and on $[2R, +\infty[$.

Now, we multiply (1.7) by ϕ_R and integrate over $[0, t] \times \mathbb{R}$ to obtain

$$\begin{aligned}
&\int_{\mathbb{R}} p_\varepsilon(t)\phi_R - \int_{\mathbb{R}} p_0\phi_R \\
&\qquad = -\int_0^t b(s) \int_{\mathbb{R}} \partial_\sigma p_\varepsilon(s)\phi_R - \int_0^t (D(p_\varepsilon(s)) + \varepsilon) \int_{\mathbb{R}} \partial_\sigma p_\varepsilon(s)\phi_R' \\
&\qquad\quad - \int_0^t \int_{|\sigma|>1} p_\varepsilon(s)\phi_R + \frac{1}{\alpha}\int_0^t D(p_\varepsilon(s))\phi_R(0).
\end{aligned}$$

We bound the terms on the right-hand side from above as follows. First, we have

$$\begin{aligned}\left|\int_0^t b(s)\int_{\mathbb{R}}\partial_\sigma p_\varepsilon(s)\phi_R\right| &\le \int_0^t |b(s)|\int_{\mathbb{R}} p_\varepsilon(s)|\phi_R'| \\ &\le \frac{C}{R}\int_0^t |b(s)|\int_{R\le|\sigma|\le 2R} p_\varepsilon(s) \le \frac{C}{R},\end{aligned}$$

thanks to (2.2) and using that $p_\varepsilon \in L^\infty_t(L^1_\sigma)$ and $b \in L^1_t$. Next,

$$\begin{aligned}\int_0^t (D(p_\varepsilon)+\varepsilon)\left|\int_{\mathbb{R}}\partial_\sigma p_\varepsilon \phi'\right| &\le (\varepsilon + \alpha\|p_\varepsilon\|_{L^\infty_t(L^1_\sigma)})\int_0^t \|\partial_\sigma p_\varepsilon\|_{L^2_\sigma}\|\phi_R'\|_{L^2_\sigma} \\ &\le \frac{C\sqrt{t}}{R^{1/2}}\|\partial_\sigma p_\varepsilon\|_{L^2_{t,\sigma}} \le \frac{C}{R^{1/2}},\end{aligned}$$

thanks again to (2.2), the Cauchy–Schwarz inequality and since $\partial_\sigma p_\varepsilon$ is in $L^2_{t,\sigma}$. Finally,

$$\begin{aligned}0 \le \frac{1}{\alpha}\int_0^t D(p_\varepsilon) - \int_0^t\int_{|\sigma|>1} p_\varepsilon\phi_R &= \int_0^t\int_{|\sigma|>1} p_\varepsilon(1-\phi_R) \\ &\le \int_0^t\int_{|\sigma|>R} p_\varepsilon,\end{aligned}$$

and the right-hand side goes to 0 as R goes to infinity since p_ε is in $L^\infty_t(L^1_\sigma)$. All this together yields

$$\int_{\mathbb{R}} p_\varepsilon(t) = \lim_{R\to+\infty}\int_{\mathbb{R}} p_\varepsilon(t)\phi_R = \lim_{R\to+\infty}\int_{\mathbb{R}} p_0\phi_R = \int_{\mathbb{R}} p_0 = 1,$$

for almost every t in $[0,T]$. In particular, this implies that $D(p_\varepsilon) \le \alpha$.

Let us now argue by contradiction by assuming that there exist two solutions p_1 and p_2 to (1.7) corresponding to the same initial data p_0. By subtracting the equations satisfied by p_1 and p_2, respectively, we obtain

$$\tag{2.3}\begin{cases}\partial_t q = -b(t)\partial_\sigma q + D(q)\partial^2_{\sigma\sigma}p_1 + (D(p_2)+\varepsilon)\partial^2_{\sigma\sigma}q \\ \qquad -\mathbb{1}_{\mathbb{R}\setminus[-1,1]}q + \dfrac{D(q)}{\alpha}\delta_0(\sigma), \\ q(0,\sigma) = 0,\end{cases}$$

where $q = p_1 - p_2$. We multiply (2.3) by q and integrate over $\mathbb{R}$ with respect to σ to obtain, after integrations by parts,

$$\tag{2.4}\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} q^2 + (D(p_2)+\varepsilon)\int_{\mathbb{R}}|\partial_\sigma q|^2 + \int_{|\sigma|>1} q^2 \\ &\quad = \frac{D(q)}{\alpha}q(t,0) - D(q)\int_{\mathbb{R}}\partial_\sigma p_1\partial_\sigma q.\end{aligned}$$

We first remark that since $\int_{\mathbb{R}} p_1 = \int_{\mathbb{R}} p_2 = 1$ thanks to (2.1), we get

$$|D(q)| = \alpha\left|\int_{|\sigma|<1} q\right| \le \alpha\sqrt{2}\|q\|_{L^2_\sigma},$$

with the help of the Cauchy–Schwarz inequality. Next, using the Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, we bound the terms on the right-hand side from above in the following way:

$$\begin{aligned}&\left|\frac{D(q)}{\alpha}q(t,0) - D(q)\int_{\mathbb{R}}\partial_\sigma p_1 \partial_\sigma q\right| \\ &\qquad\le \sqrt{2}\|q\|_{L^2_\sigma}\|q\|_{L^\infty_\sigma} + \sqrt{2}\alpha\|q\|_{L^2_\sigma}\int_{\mathbb{R}}|\partial_\sigma p_1 \partial_\sigma q| \\ &\qquad\le \sqrt{2}\|q\|_{L^2_\sigma}\left(\|q\|^2_{L^2_\sigma} + \|\partial_\sigma q\|^2_{L^2_\sigma}\right)^{\frac{1}{2}} + \sqrt{2}\alpha\|q\|_{L^2_\sigma}\|\partial_\sigma p_1\|_{L^2_\sigma}\|\partial_\sigma q\|_{L^2_\sigma} \\ &\qquad\le \frac{1}{\varepsilon}\|q\|^2_{L^2_\sigma} + \frac{\alpha^2}{\varepsilon}\|q\|^2_{L^2_\sigma}\|\partial_\sigma p_1\|^2_{L^2_\sigma} + \frac{\varepsilon}{2}\|q\|^2_{L^2_\sigma} + \varepsilon\|\partial_\sigma q\|^2_{L^2_\sigma}.\end{aligned}$$

Therefore, comparing with (2.4) we deduce that

$$\frac{1}{2}\frac{d}{dt}\|q\|^2_{L^2_\sigma} \le \left(\frac{1}{\varepsilon} + \frac{\alpha^2}{\varepsilon}\|\partial_\sigma p_1\|^2_{L^2_\sigma} + \frac{\varepsilon}{2}\right)\|q\|^2_{L^2_\sigma}.$$

Finally, by applying the Gronwall lemma, we prove that $\|q\|^2_{L^2_\sigma} \le 0$, thus $q = 0$. The uniqueness of the solution follows. ☐

Remark 2.2. The same proof shows that if there exists a solution to (1.1) in $L^\infty_t(L^1_\sigma \cap L^2_\sigma) \cap L^2_t(H^1_\sigma)$ such that $\inf_{0\le t\le T} D(p(t)) > 0$, then it is unique in this space.

We now turn to the existence part in the statement of Proposition 1.2. From now on we fix a positive constant $\varepsilon \le 1$. The proof of Proposition 1.2 will be carried out by the Schauder fixed point theorem. For given positive constants $M(\ge \varepsilon)$ and R, we introduce $\mathcal{D}_{\varepsilon,M}$ and Y_R two closed convex subsets of, respectively, L^2_t and $L^2_{t,\sigma}$ as follows:

$$\begin{aligned}\mathcal{D}_{\varepsilon,M} &= \left\{a \in L^2_t;\ \varepsilon \le a \le M\right\} \\ Y_R &= \left\{p \in L^2_{t,\sigma};\ p \ge 0,\ \sup_{0\le t\le T}\int_{\mathbb{R}}|\sigma| p \le R\right\}.\end{aligned}$$

To simplify notation we denote

$$\begin{cases}\varphi_\eta(x) = \dfrac{1}{\sqrt{2\pi}\eta}\exp\left(-\dfrac{x^2}{2\eta^2}\right) & \text{if } \eta > 0, \\ \varphi_0 = \delta_0.\end{cases}$$

We first prove the following proposition.

PROPOSITION 2.3. *Let $T > 0$ and let $p_0 \in L^2(\mathbb{R})$ such that $p_0 \ge 0$. Then, for every a in $\mathcal{D}_{\varepsilon,M}$ and q in Y_R, there exists a unique solution p in $L^\infty_t(L^2_\sigma) \cap L^2_t(H^1_\sigma)$ to*

$$\begin{cases}\partial_t p(t,\sigma) = -b(t)\partial_\sigma p(t,\sigma) + a(t)\partial^2_{\sigma\sigma}p(t,\sigma) \\ \qquad\qquad -\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(\sigma)p(t,\sigma) + \dfrac{D(q)}{\alpha}\delta_0(\sigma), \\ p(0,\sigma) = p_0(\sigma).\end{cases} \tag{2.5}$$

Moreover, $p \in C^0_t(L^2_\sigma)$, p is nonnegative and

$$p_- \le p \le p_+, \tag{2.6}$$

with

$$p_-(t,\sigma) = e^{-t}\int_{-\infty}^{+\infty} p_0(\sigma')\varphi_{\sqrt{2\int_0^t a}}(\sigma-\sigma'-\chi(t))d\sigma' \tag{2.7}$$

and

$$\begin{aligned} p_+(t,\sigma) = &\int_{-\infty}^{+\infty} p_0(\sigma')\varphi_{\sqrt{2\int_0^t a}}(\sigma-\sigma'-\chi(t))d\sigma' \\ &+\frac{1}{\alpha}\int_0^t D(q(s))\varphi_{\sqrt{2\int_s^t a}}(\sigma-\chi(t)+\chi(s))ds, \end{aligned} \tag{2.8}$$

where $\chi(t) = \int_0^t b(s)ds$. *In addition,*

(i) *If* $p_0 \in L^\infty(\mathbb{R})$, *then* p *is in* $L^\infty_{t,\sigma}$ *and*

$$0 \le p \le \|p_0\|_{L^\infty} + \frac{R\sqrt{T}}{\sqrt{\pi}\sqrt{\varepsilon}}. \tag{2.9}$$

(ii) *If* $\int_\mathbb{R}|\sigma|p_0 < +\infty$ *(thus* $p_0 \in L^1(\mathbb{R})$*), then* $|\sigma|p \in L^\infty_t(L^1_\sigma)$. *More precisely, we have*

$$\begin{aligned} \sup_{0\le t\le T}\int_\mathbb{R}|\sigma|p \le &\int_\mathbb{R}|\sigma|p_0 + \sqrt{T}\|b\|_{L^2(0,T)}\|p_0\|_{L^1} + \frac{2R}{3}T^{3/2}\|b\|_{L^2(0,T)} \\ &+\frac{2}{\sqrt{\pi}}(MT)^{1/2}\|p_0\|_{L^1} + \frac{4R\sqrt{M}}{3\sqrt{\pi}}T^{3/2}. \end{aligned} \tag{2.10}$$

Moreover, $p \in C^0_t(L^1_\sigma)$ *and* $D(p) \in C^0_t$.

Proof. Let us first observe that for every q in Y_R, $D(q) \in L^\infty_t$ since

$$0 \le D(q(t)) \le \alpha\int_{|\sigma|>1}|\sigma|q \le \alpha R, \tag{2.11}$$

for almost every t in $[0,T]$. Therefore the source term $D(q(t))\delta_0(\sigma)$ in (2.5) is in $L^\infty_t(H^{-1}_\sigma)$ and the existence and uniqueness of a solution $p \in C^0_t(L^2_\sigma)\cap L^2_t(H^1_\sigma)$ to the system (2.5) is well known (see, for example, [4]). In particular, the initial condition makes sense. Owing to the fact that the source term is nonnegative, the proof that $p \ge 0$ is also standard (see again [4]).

We now check the pointwise inequality (2.6).

This is ensured by the maximum principle with observing that p_- and p_+ given, respectively, by (2.7) and (2.8), are the unique solutions to the systems

$$\begin{cases} \partial_t p_- = -b\partial_\sigma p_- + a\partial^2_{\sigma\sigma}p_- - p_-, \\ p_-(0,\sigma) = p_0(\sigma), \end{cases}$$

and

$$\begin{cases} \partial_t p_+ = -b\partial_\sigma p_+ + a\partial^2_{\sigma\sigma}p_+ + \dfrac{D(q)}{\alpha}\delta_0(\sigma), \\ p_+(0,\sigma) = p_0(\sigma), \end{cases}$$

respectively. We now turn to the proof of statement (i), and assume that p_0 belongs to $L^\infty(\mathbb{R})$. Then, using the two facts that for every $\nu > 0$, $\int_\mathbb{R} \varphi_\nu = 1$ and $\varphi_\nu \le \frac{1}{\sqrt{2\pi\nu}}$, (2.9) is easily deduced from $p \le p_+$ with the help of (2.11) and since $a \ge \varepsilon$.

Suppose now that $\int_\mathbb{R} |\sigma| p_0 < +\infty$. This together with the assumption $p_0 \in L^2(\mathbb{R})$, guarantees that $p_0 \in L^1(\mathbb{R})$ (see also below). Using (2.6) again, we now have

$$\begin{aligned}
\int_\mathbb{R} |\sigma| p &\le \int_\mathbb{R} |\sigma| p_+ \\
&\le \int_\mathbb{R}\int_\mathbb{R} p_0(\sigma')|\sigma|\varphi_{\sqrt{2\int_0^t a}}(\sigma - \chi(t) - \sigma')\,d\sigma d\sigma' \\
&\quad + \frac{1}{\alpha}\int_0^t D(q(s))\Big(\int_\mathbb{R} |\sigma|\varphi_{\sqrt{2\int_s^t a}}(\sigma - \chi(t) + \chi(s))\,d\sigma\Big)ds \\
&= \int_\mathbb{R}\int_\mathbb{R} p_0(\sigma')|\sigma + \sigma' + \chi(t)|\varphi_{\sqrt{2\int_0^t a}}(\sigma)\,d\sigma d\sigma' \qquad (2.12)\\
&\quad + \frac{1}{\alpha}\int_0^t D(q(s))\Big(\int_\mathbb{R} |\sigma + (\chi(t) - \chi(s))|\varphi_{\sqrt{2\int_s^t a}}(\sigma)\,d\sigma\Big)ds \\
&\le \int_\mathbb{R} |\sigma| p_0(\sigma)\,d\sigma + |\chi(t)|\|p_0\|_{L^1} + \frac{1}{\alpha}\int_0^t |\chi(t) - \chi(s)| D(q(s))\,ds \\
&\quad + \frac{2}{\sqrt{\pi}}\Big(\int_0^t a\Big)^{1/2}\|p_0\|_{L^1} + \frac{2}{\alpha\sqrt{\pi}}\int_0^t D(q(s))\Big(\int_s^t a\Big)^{1/2} ds,
\end{aligned}$$

since $\int_\mathbb{R} |\sigma|\varphi_\nu(\sigma)d\sigma = (2/\pi)^{1/2}\nu$ and $\int_\mathbb{R}\varphi_\nu = 1$. With the help of (2.11), and observing that $|\chi(t) - \chi(s)| \le \sqrt{t-s}\|b\|_{L^2(0,T)}$, we then deduce (2.10).

We now use this bound to check that $p \in C_t^0(L_\sigma^1)$ and $D(p) \in C_t^0$. Indeed, for any t in $[0,T]$, any $A > 1$, and any sequence t_n in $[0,T]$ which converges to t, we have

$$\begin{aligned}
\int_\mathbb{R} \big|p(t_n) - p(t)\big| &= \int_{|\sigma|\le A} \big|p(t_n) - p(t)\big| + \int_{|\sigma|\ge A} \big|p(t_n) - p(t)\big| \\
&\le \sqrt{2A}\Big(\int_\mathbb{R} \big|p(t_n) - p(t)\big|^2\Big)^{1/2} + \frac{1}{A}\int_\mathbb{R} |\sigma|\big(|p(t_n)| + |p(t)|\big) \qquad (2.13)\\
&\le \sqrt{2A}\Big(\int_\mathbb{R} \big|p(t_n) - p(t)\big|^2\Big)^{1/2} + \frac{2}{A}\sup_{0\le t\le T}\int_\mathbb{R} |\sigma||p(t)|.
\end{aligned}$$

For any fixed A, the first term on the right-hand side goes to 0 as n goes to infinity since $p \in C_t^0(L_\sigma^2)$ and then the second term is arbitrarily small as A goes to infinity. The same argument yields the continuity of $D(p(t))$ with respect to t. □

The following proposition aims at checking the required assumptions to apply the Schauder fixed point theorem.

PROPOSITION 2.4. *Let $T_f > 0$ be given. We assume that*

$$p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad p_0 \geq 0, \ \int_{\mathbb{R}} p_0 = 1, \ \text{and} \ \int_{\mathbb{R}} |\sigma| p_0 < +\infty. \tag{2.14}$$

Let $0 < \varepsilon \leq 1$, $R = 1 + \int_{\mathbb{R}} |\sigma| p_0$, and $M = 1 + 2\alpha$. We define

$$T_c = \frac{9}{25}\left[\|b\|_{L^2(0,T_f)} + \frac{2\sqrt{1+2\alpha}}{\sqrt{\pi}}\right]^{-2}. \tag{2.15}$$

Then, for every $T \leq \min(\frac{1}{R}; T_c)$, the function $\mathcal{T} : (a; q) \mapsto (D(p) + \varepsilon; p)$, with p being the solution to the system (2.5), maps $\mathcal{D}_{\varepsilon,M} \times Y_R$ into itself. Moreover, $\mathcal{T}$ is continuous and $\mathcal{T}(\mathcal{D}_{\varepsilon,M} \times Y_R)$ is relatively compact in $L^2(0,T) \times L^2_{t,\sigma}$.

Proof. **Step 1.** *$\mathcal{T}$ is well-defined.* According to Proposition 2.3, p is in $C^0_t(L^1_\sigma)$ and $D(p) \in C^0_t$. We now prove that with our choice for M (which ensures that $\varepsilon + D(p_0) \leq 1 + \alpha \leq M$), $D(p) + \varepsilon \in \mathcal{D}_{\varepsilon,M}$. For this, we again use the inequality $p \leq p_+$, the definition (2.8) of p_+, the rough estimate $\int_{|\sigma|>1} \varphi_\nu \leq \int_{\mathbb{R}} \varphi_\nu = 1$ and (2.11) to obtain

$$\sup_{0\leq t\leq T} D(p(t)) \leq \sup_{0\leq t\leq T} D(p_+(t)) \leq \alpha + \alpha R T \leq 2\alpha,$$

for $T \leq \frac{1}{R}$. It only remains now to check that $\sup_{0\leq t\leq T} \int_{\mathbb{R}} |\sigma| p \leq R$. We thus go back to (2.10) and observe that this condition holds provided

$$T \leq \max\left\{ t > 0; \|b\|_{L^2(0,T_f)} \sqrt{t}\left(1 + \frac{2R}{3}t\right) + \frac{2\sqrt{Mt}}{\sqrt{\pi}} + \frac{4R\sqrt{M}t^{3/2}}{3\sqrt{\pi}} \leq 1 \right\}.$$

Since we have already demanded that $t \leq T \leq \frac{1}{R}$, a sufficient condition is then

$$\sqrt{T}\left[\frac{5}{3}\|b\|_{L^2(0,T_f)} + \frac{10\sqrt{1+2\alpha}}{3\sqrt{\pi}}\right] \leq 1,$$

which reduces to $T \leq T_c$ with T_c given by (2.15).

Our next step will consist of establishing a priori bounds on p in $L^\infty_t(L^2_\sigma) \cap L^2_t(H^1_\sigma)$.

Step 2. *A priori bounds.* If we multiply (2.5) by p and integrate by parts over $\mathbb{R}$ with respect to σ we easily obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} p^2 + a\int_{\mathbb{R}} |\partial_\sigma p|^2 \leq \frac{D(q)}{\alpha} p(t,0).$$

Since from the Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and the bound (2.11) on $D(q)$, we get

$$\begin{aligned}\left|\frac{D(q)}{\alpha} p(t,0)\right| &\leq R\|p\|_{L^\infty_\sigma} \\ &\leq R\big(\|p\|^2_{L^2_\sigma} + \|\partial_\sigma p\|^2_{L^2_\sigma}\big)^{\frac{1}{2}} \\ &\leq \frac{R^2}{2\varepsilon} + \frac{\varepsilon}{2}\|p\|^2_{L^2_\sigma} + \frac{\varepsilon}{2}\|\partial_\sigma p\|^2_{L^2_\sigma},\end{aligned}$$

we may write

$$\frac{1}{2}\frac{d}{dt}\|p\|^2_{L^2_\sigma} + \left(a - \frac{\varepsilon}{2}\right)\|\partial_\sigma p\|^2_{L^2_\sigma} \leq \frac{R^2}{2\varepsilon} + \frac{\varepsilon}{2}\|p\|^2_{L^2_\sigma}. \tag{2.16}$$

We recall that $a \geq \varepsilon$ and we apply the Gronwall lemma to obtain

$$\sup_{0\leq t\leq T} \|p\|^2_{L^2_\sigma} \leq e^{\varepsilon T}\left(\|p_0\|^2_{L^2_\sigma} + \frac{TR^2}{\varepsilon}\right). \tag{2.17}$$

We now return to (2.16) and integrate it over $[0;T]$ to obtain

$$\varepsilon\|\partial_\sigma p\|^2_{L^2_{t,\sigma}} \leq \|p_0\|^2_{L^2_\sigma}(1+\varepsilon T e^{\varepsilon T}) + \frac{TR^2}{\varepsilon}(1+\varepsilon T e^{\varepsilon T}). \tag{2.18}$$

Step 3. *The function $\mathcal{T}$ is continuous.* We consider a sequence $(a_n;q_n)$ in $\mathcal{D}_{\varepsilon,M}\times Y_R$ such that a_n converges to a strongly in L^2_t and q_n converges to q strongly in $L^2_{t,\sigma}$, and we denote $\mathcal{T}(a_n;q_n) = (D(p_n)+\varepsilon;p_n)$. We have to prove that p_n converges strongly to p in $L^2_{t,\sigma}$ and $D(p_n)$ converges to $D(p)$ strongly in L^2_t, with $(D(p)+\varepsilon;p) = \mathcal{T}(a;q)$.

In virtue of (2.17) and (2.18), the sequence p_n is bounded in $L^\infty_t(L^2_\sigma)\cap L^2_t(H^1_\sigma)$. Then, $\partial_\sigma p_n$ is bounded in $L^\infty_t(H^{-1}_\sigma)$ and $\partial^2_{\sigma\sigma}p_n$ is bounded in $L^2_t(H^{-1}_\sigma)$. Since $a_n\partial^2_{\sigma\sigma}p_n$ is bounded in $L^2_t(H^{-1}_\sigma)$, $b\in L^2_t$ and $D(q_n)\delta_0$ is bounded in $L^2_t(H^{-1}_\sigma)$, $\partial_t p_n$ is bounded in $L^2_t(H^{-1}_\sigma)$. This together with the fact that p_n is bounded in $L^2_t(H^1_\sigma)$ implies that, up to a subsequence, p_n converges strongly towards p in $L^2_t(L^2_{\mathrm{loc},\sigma})$ (the convergence being weak in $L^2_t(H^1_\sigma)$) thanks to a well-known compactness result [10]. In particular, p_n converges to p almost everywhere. Thus $p\geq 0$ and by Fatou's lemma, $\int_{\mathbb{R}}|\sigma|p\leq R$ almost everywhere on $[0;T]$. Hence p belongs to Y_R. We are going to show that the convergence is actually strong in $L^2_{t,\sigma}$.

In virtue of (2.9) in Proposition 2.3, we dispose of a uniform a priori bound on p_n in $L^\infty_{t,\sigma}$ (hence also on p). For the strong convergence in $L^2_{t,\sigma}$ we then argue as follows. For any fixed positive real number K, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}|p_n-p|^2 &\leq \int_0^T\int_{|\sigma|\leq K}|p_n-p|^2 + \int_0^T\int_{|\sigma|>K}|p_n-p|^2\\ &\leq \int_0^T\int_{|\sigma|\leq K}|p_n-p|^2 + (\|p_n\|_{L^\infty_{t,\sigma}}+\|p\|_{L^\infty_{t,\sigma}})\frac{2RT}{K},\end{aligned}$$

owing to the fact that p_n and p belong to a bounded subset of $Y_R\cap L^\infty_{t,\sigma}$. We then conclude by letting n, then K, go to infinity.

We now prove that $D(p_n)$ converges to $D(p)$ strongly in L^2_t. We shall actually prove that $D(p_n)$ converges to $D(p)$ strongly in L^1_t and then use the fact that $D(p_n)$ is bounded in L^∞_t, in virtue of (2.11) and because p_n lies in Y_R. Let us fix $K>1$. Then, we have

$$\begin{aligned}\frac{1}{\alpha}\int_0^T|D(p_n)-D(p)| &= \int_0^T\left|\int_{|\sigma|>1}(p_n-p)\right|\\ &\leq \int_0^T\int_{1<|\sigma|<K}|p_n-p| + \frac{1}{K}\int_0^T\int_{|\sigma|>K}|\sigma|(|p_n|+|p|)\\ &\leq \int_0^T\int_{1<|\sigma|<K}|p_n-p| + \frac{2RT}{K},\end{aligned} \tag{2.19}$$

because p and p_n belong to Y_R. Since p_n converges to p strongly in $L^1_t(L^1_{\mathrm{loc},\sigma})$, we conclude that $D(p_n)$ converges to $D(p)$ in L^1_t by letting n, then K, go to infinity in (2.19).

In order to pass to the limit in the equation satisfied by p_n (thereby proving that $(D(p)+\varepsilon; p) = \mathcal{T}(a; q)$), we now observe that the strong convergence of q_n to q in $L^2_{t,\sigma}$, together with the argument in (2.19) above shows that $D(q_n)$ converges to $D(q)$ strongly in L^2_t. It is then easily proved that p is a weak solution to (1.7) and since p is in $L^2_t(H^1_\sigma)$ it is the unique solution to (2.5) corresponding to a and q. In particular, the whole sequence p_n converges and not only a subsequence.

Step 4. *$\mathcal{T}(\mathcal{D}_\varepsilon \times Y_R)$ is relatively compact.* Let $(D(p_n)+\varepsilon; p_n) = \mathcal{T}(a_n; q_n)$ be a sequence in $\mathcal{T}(\mathcal{D}_{\varepsilon,M} \times Y_R)$. We have to prove that we may extract a subsequence which converges strongly in $L^2_t \times L^2_{t,\sigma}$. Exactly as for the proof of the continuity, the a priori estimates (2.17) and (2.18) ensure that the sequence p_n is bounded in $L^\infty_t(L^2_\sigma) \cap L^2_t(H^1_\sigma)$. Since $|\sigma| p_n$ is bounded $L^\infty_t(L^1_\sigma)$, we can mimic the argument in Step 3 above to deduce that up to a subsequence the sequence p_n converges to some p in Y_R strongly in $L^2_{t,\sigma}$ and that $D(p_n)$ converges to $D(p)$ strongly in L^2_t. $\square$

We are now in position to conclude the proof of Proposition 1.2.

Let $T_f > 0$ and $0 < \varepsilon \le 1$ being given. We are going to prove the existence of a unique solution on $[0; T_f]$.

Being given an initial data p_0 which satisfies (1.6), existence of a solution p_ε is ensured from Proposition 2.4 by applying the Schauder fixed point theorem on "short" time interval $[0; T_1]$ with $T_1 = \min(\frac{1}{R_1}, T_c)$ and where $R_1 = 1 + \int_{\mathbb{R}} |\sigma| p_0$. This solution is uniquely defined in virtue of Lemma 2.1 and we know from (2.1) that $\int_{\mathbb{R}} p_\varepsilon(T_1) = 1$. Moreover, from Proposition 2.3 $p_\varepsilon(T_1) \in L^\infty_\sigma$ and by construction $\int_{\mathbb{R}} |\sigma| p_\varepsilon(T_1) \le R_1$. Therefore $p_\varepsilon(T_1)$ satisfies the same conditions (2.14) as p_0. Then, repeating the same argument we may build a solution to (1.7) with initial data $p_\varepsilon(T_1)$ on $[T_1; T_2]$ with $T_2 = \min(\frac{1}{R_2}, T_c)$, where $R_2 = R_1 + 1 = \int_{\mathbb{R}} |\sigma| p_0 + 2$. Thanks to the uniqueness result (Lemma 2.1), if we now glue this solution to p_ε at $t = T_1$ we obtain the unique solution to (1.7) on $[0; T_1 + T_2]$. It is now clearly seen that for any integer $n \ge 1$ we may build a solution to (1.7) on $[0; \sum_{1 \le k \le n} T_k]$ with $T_k = \min((k + \int_{\mathbb{R}} |\sigma| p_0)^{-1}; T_c)$. Since $\sum_{1 \le k \le n} T_k$ obviously goes to $+\infty$ together with n, existence (and uniqueness) of the solution p_ε to (1.7) is obtained on every time interval.

For the proof of (1.9) we argue as for the proof of (2.9) in Proposition 2.3. Defining p_ε^+ as in (2.8) with a replaced by $D(p_\varepsilon) + \varepsilon$ and $D(q)$ by $D(p_\varepsilon)$, we obtain

$$\begin{aligned}
0 \le p_\varepsilon &\le p_\varepsilon^+ \\
&\le \|p_0\|_{L^\infty} + \frac{1}{\alpha\sqrt{\pi}} \int_0^t \frac{D(p_\varepsilon(s))}{2\sqrt{\varepsilon + \int_s^t D(p_\varepsilon)}} ds \\
&\le \|p_0\|_{L^\infty} + \frac{1}{\alpha\sqrt{\pi}} \left[\sqrt{\varepsilon + \int_0^t D(p_\varepsilon)} - \sqrt{\varepsilon} \right] \\
&\le \|p_0\|_{L^\infty} + \frac{1}{\alpha\sqrt{\pi}} \sqrt{\int_0^t D(p_\varepsilon)} \\
&\le \|p_0\|_{L^\infty} + \frac{\sqrt{\alpha}\sqrt{T}}{\sqrt{\pi}}.
\end{aligned}$$

Then

$$\int_{\mathbb{R}} p_\varepsilon^2 \le \|p_\varepsilon\|_{L^\infty_\sigma} \int_{\mathbb{R}} p_\varepsilon,$$

from which (1.11) follows gathering together (2.1) and (1.9) and, with the notation of the proposition,

$$C_2(T, p_0) = \|p_0\|_{L^\infty} + \frac{\sqrt{\alpha}\sqrt{T}}{\sqrt{\pi}}.$$

The proof of (1.10) follows the same lines as the proof of (2.12). Indeed, we again use the pointwise inequality $p_\varepsilon \leq p_\varepsilon^+$ and replace $D(q)$ by $D(p_\varepsilon)(\leq \alpha)$ and a by $D(p_\varepsilon) + \varepsilon(\leq \alpha + 1)$ in (2.12) and use (2.14) to deduce

$$\sup_{0\leq t\leq T} \int_{\mathbb{R}} |\sigma| p_\varepsilon \leq \int_{\mathbb{R}} |\sigma| p_0 + \sqrt{T}\left(\frac{2\sqrt{1+\alpha}}{\sqrt{\pi}} + \|b\|_{L^2(0,T)}\right) + \frac{2}{3}T^{3/2}\left(1 + \frac{2\sqrt{1+\alpha}}{\sqrt{\pi}}\right),$$

whence (1.10) with $C_1(T, p_0)$ being the quantity in the right-hand side of the above inequality.

In order to prove (1.12), we apply p_ε to (1.7) and we integrate by parts over $\mathbb{R}$ with respect to σ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} p_\varepsilon^2 + (D(p_\varepsilon) + \varepsilon)\int_{\mathbb{R}} |\partial_\sigma p_\varepsilon|^2 + \int_{|\sigma|>1} p_\varepsilon^2 = \frac{D(p_\varepsilon)}{\alpha} p_\varepsilon(t, 0). \tag{2.20}$$

We use the L^∞ bound (1.9) to bound the right-hand side and we integrate (2.20) with respect to t over $[0; T]$ to deduce (1.12) with

$$C_3(T, p_0) = \|p_0\|_{L^\infty}\left(\frac{1}{2} + T\right) + \frac{\sqrt{\alpha}}{\sqrt{\pi}} T^{3/2},$$

using $\|p_0\|_{L^2_\sigma}^2 \leq \|p_0\|_{L^\infty} \int_{\mathbb{R}} p_0$.

3. The nondegenerate case: $\mathbf{D(p_0) > 0}$. The main result of this section corresponds to the statement of Theorem 1.1 and fully describes the issue of existence and uniqueness of solutions to the HL equation (1.1) in the nondegenerate case. It is summarized in the following proposition.

PROPOSITION 3.1. *Let* p_0 *satisfy* (1.6). *We assume that* $D(p_0) > 0$. *Then, the HL equation* (1.1) *has a unique solution* p *in* $C_t^0(L_\sigma^2) \cap L_t^2(H_\sigma^1)$ *and* p *is the limit (in* $L^2_{t,\mathrm{loc}}(L^2_\sigma) \cap C^0_{t,\mathrm{loc}}(L^2_\sigma)$*) of* (p_ϵ) *when* ϵ *goes to* 0 *where* p_ε *is the vanishing viscosity solution whose existence and uniqueness is ensured by Proposition* 1.2. *Moreover,* $p \in L^\infty_{t,\sigma} \cap C_t^0(L_\sigma^1)$, $\sigma p \in L_t^\infty(L_\sigma^1)$ *and* $\int_{\mathbb{R}} p = 1$. *Furthermore,* $D(p) \in C_t^0$ *and for every* $T > 0$ *there exists a positive constant* $\nu(T)$ *such that*

$$\min_{0\leq t\leq T} D(p(t)) \geq \nu(T). \tag{3.1}$$

We begin by proving the following lemma.

LEMMA 3.2. *We assume that* p_0 *satisfies* (1.6). *Then, if* $D(p_0) > 0$, $D(p_\varepsilon)(t) > 0$ *for every* $t \in [0, T]$, *with* p_ε *being the unique solution to* (1.7) *provided by Proposition* 1.2 *and, actually, for every* $T > 0$ *there exists a positive constant* $\nu(T)$ *such that*

$$\min_{0\leq t\leq T} D(p_\varepsilon(t)) \geq \nu(T), \tag{3.2}$$

for every $0 < \varepsilon \leq 1$.

Remark 3.3. Note that this bound from below is independent of ε, but it comes out from the proof that it depends on p_0 and on the shear b.

Proof. The proof relies on the bound from below in (2.6) that we integrate over $|\sigma| > 1$ to obtain

$$\begin{aligned} D(p_\varepsilon(t)) &\geq \alpha \int_{|\sigma|>1} p_\varepsilon^- \\ &\geq \alpha e^{-t} \int_{\mathbb{R}} p_0(\sigma') \left(\int_{|\sigma|>1} \varphi_{\sqrt{2\int_0^t (D(p_\varepsilon)+\varepsilon)}} (\sigma - \sigma' - \chi(t)) d\sigma \right) d\sigma'. \end{aligned} \tag{3.3}$$

Let us define $K_\chi = [-1-\chi(t), 1-\chi(t)]$. The function $\sigma \mapsto \varphi_{\sqrt{2\int_0^t (D(p_\varepsilon)+\varepsilon)}}(\sigma - \sigma' - \chi(t))$ is a Gaussian probability density with mean $\sigma' + \chi(t)$ and squared width $2\int_0^t (D(p_\varepsilon) + \varepsilon)$. Therefore, for every $\sigma' \in \mathbb{R} \setminus K_\chi$, we have

$$\int_{|\sigma|>1} \varphi_{\sqrt{2\int_0^t (D(p_\varepsilon)+\varepsilon)}} (\sigma - \sigma' - \chi(t)) d\sigma \geq \frac{1}{2},$$

which implies (3.3) is

$$\geq \frac{\alpha}{2} e^{-T} \int_{\mathbb{R}\setminus K_\chi} p_0 = \frac{\alpha}{2} e^{-T} \int_{|\sigma+\chi(t)|>1} p_0.$$

In the zero shear case ($b \equiv 0$, thus $\chi \equiv 0$) the proof is over and

$$\min_{0\leq t\leq T} D(p(t)) \geq \frac{1}{2} e^{-T} D(p_0).$$

In the general case, a strictly positive bound from below is available as long as the support of p_0 is not contained in K_χ. We thus define

$$t^* = \inf \left\{ t > 0; \int_{|\sigma+\chi(t)|>1} p_0 = 0 \right\}. \tag{3.4}$$

Then $0 < t^*$ (t^* possibly even infinite), the support of p_0 is contained in $[-1-\chi(t^*), 1-\chi(t^*)[$, and for every $T < \frac{t^*}{2}$, (3.2) holds for some positive constant $\nu_1(T)$ defined by

$$\nu_1(T) = \frac{\alpha}{2} e^{-T} \min_{0\leq t\leq T} \int_{|\sigma+\chi(t)|>1} p_0. \tag{3.5}$$

It is worth emphasizing that this quantity is independent of ε. If $t^* = +\infty$, the proof is over and $\nu(T) = \nu_1(T)$ fits. Let us now examine the case when $t^* < +\infty$ and $T \geq \frac{t^*}{2}$.

We go back to (3.3), take t in $[\frac{t^*}{2}; T]$, and denote $x = \int_0^t (D(p_\varepsilon) + \varepsilon)$ for shortness.

Then

$$
\begin{aligned}
D(p_\varepsilon(t)) &\geq \alpha e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{|\sigma|>1} \varphi_{\sqrt{2x}}(\sigma - \sigma' - \chi(t)) d\sigma \right) d\sigma' \\
&= \alpha e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{|\sigma|>1} \frac{e^{-(\sigma-\sigma'-\chi(t))^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma \right) d\sigma' \\
&= \frac{\alpha}{\sqrt{\pi}} e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \Bigg(\int_{-\infty}^{-1+\sigma'+\chi(t)} \frac{e^{-\sigma^2/4x}}{2\sqrt{x}} d\sigma \\
&\qquad\qquad + \int_{1+\sigma'+\chi(t)}^{+\infty} \frac{e^{-\sigma^2/4x}}{2\sqrt{x}} d\sigma \Bigg) d\sigma' \\
&= \frac{\alpha}{\sqrt{\pi}} e^{-T} \int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') \left(\int_{\frac{1+\sigma'+\chi(t)}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt + \int_{\frac{1-\sigma'-\chi(t)}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt \right) d\sigma' \\
&\geq \frac{\alpha}{\sqrt{\pi}} e^{-T} \left(\int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0(\sigma') d\sigma' \right) \Bigg(\int_{\frac{2-\chi(t^*)+\chi(t)}{\sqrt{2t^*\nu_1(t^*/2)}}}^{+\infty} e^{-t^2} dt \\
&\qquad\qquad + \int_{\frac{2+\chi(t^*)-\chi(t)}{\sqrt{2t^*\nu_1(t^*/2)}}}^{+\infty} e^{-t^2} dt \Bigg);
\end{aligned}
$$

hence

$$
D(p_\varepsilon(t)) \geq \frac{\alpha}{\sqrt{\pi}} e^{-T} \min_{t^*/2 \leq t \leq T} \left(\int_{\frac{2-\chi(t^*)+\chi(t)}{\sqrt{2t^*\nu_1(t^*/2)}}}^{+\infty} e^{-t^2} dt + \int_{\frac{2+\chi(t^*)-\chi(t)}{\sqrt{2t^*\nu_1(t^*/2)}}}^{+\infty} e^{-t^2} dt \right), \tag{3.6}
$$

since $\int_{-1-\chi(t^*)}^{1-\chi(t^*)} p_0 = 1$ and $x \geq \int_0^{t^*/2} D(p_\varepsilon) \geq t^* \nu_1(t^*/2)/2$ thanks to (3.5). The proof of Lemma 3.2 then follows by defining

$$
\nu(T) = \min(\nu_1(T); \nu_2(T)),
$$

with $\nu_1(T)$ given by (3.5) and $\nu_2(T)$ being the positive quantity on the right-hand side of (3.6), that is

$$
\nu_2(T) = \frac{\alpha}{\sqrt{\pi}} e^{-T} \min_{t^*/2 \leq t \leq T} \left(\int_{\frac{2-\chi(t^*)+\chi(t)}{\sqrt{2t^*\nu_1(t^*/2)}}}^{+\infty} e^{-t^2} dt + \int_{\frac{2+\chi(t^*)-\chi(t)}{2\sqrt{2t^*\nu_1(t^*/2)}}}^{+\infty} e^{-t^2} dt \right). \qquad \square
$$

Proof of Proposition 3.1. We first go back to the proof of the bound (1.12) on $\partial_\sigma p_\varepsilon$, and more precisely we look at (2.20), and observe that in virtue of (3.1)

$$
\nu(T) \int_0^T \int_{\mathbb{R}} |\partial_\sigma p_\varepsilon|^2 \leq C_3(T, p_0). \tag{3.7}
$$

Now let ε_n denote any sequence in $[0, 1]$ which goes to 0 as n goes to infinity. To shorten the notation we denote by p_n instead of p_{ε_n} the corresponding sequence of solutions to (1.7). With the above bound (3.7) on p_n and (1.11), we know that p_n is bounded in $L^2_t(H^1_\sigma)$ independently of n. Moreover, thanks to (2.1) and (1.9), p_n is bounded in $L^\infty_t(L^1_\sigma \cap L^\infty_\sigma)$ and we also dispose of a uniform bound on $\int_{\mathbb{R}} |\sigma| p_n$ in virtue of (1.10). Therefore arguing exactly as in the proof of Proposition 2.4 (Step 4) where we have proved that the mapping $\mathcal{T}$ is relatively compact in $L^2_t \times L^2_{t,\sigma}$ we show that p_n converges to some p strongly in $L^2_{t,\sigma}$ and $D(p_n)$ converges to $D(p)$ in L^2_t. Therefore, the nonlinear term $D(p_n)\partial^2_{\sigma\sigma}p_n$ converges to $D(p)\partial^2_{\sigma\sigma}p$ strongly in $L^1_t(H^{-2}_\sigma)$ (for instance). Then p is a weak solution to the initial problem (1.1) in $L^2_t(H^1_\sigma) \cap L^\infty_t(L^1_\sigma \cap L^\infty_\sigma)$, $\int_{\mathbb{R}} p = 1$ and $\int_{\mathbb{R}} |\sigma| p < +\infty$. Moreover,

$$\inf_{0 \le t \le T} D(p(t)) \ge \nu(T).$$

This nondegeneracy condition on the viscosity coefficient ensures that there is at most one solution to (1.1) in $L^2_t(H^1_\sigma) \cap L^\infty_t(L^2_\sigma)$ (this follows by an obvious adaptation of the proof of Lemma 2.1 to this case). Therefore the limiting function p is uniquely defined and does not depend on the sequence ε_n. Moreover, the whole sequence p_n converges to this unique limit and not only a subsequence. ☐

As a conclusion of this subsection let us make the following comment which is a by product of Proposition 3.1. If p is a solution to (1.1) in $C^0_t(L^1_\sigma \cap L^2_\sigma)$, then as soon as $D(p(t))$ is positive for some time t it remains so afterwards since the solution can be continued in a unique way starting at time t.

4. The degenerate case: $\mathbf{D(p_0) = 0}$. Throughout this section we assume that p_0 satisfies (1.3) and that $D(p_0) = 0$. Therefore the support of p_0 is included in $[-1; +1]$. Assume that we dispose of a solution to (1.1) in $C^0_t(L^1_\sigma \cap L^2_\sigma)$. We may define $t_* \in \mathbb{R}^+ \cup \{+\infty\}$ by

$$t_* = \max\left\{t > 0; \int_0^t D(p) = 0\right\}. \tag{4.1}$$

According to the comment at the end of the previous section for every $t > t_*$, $D(p(t)) > 0$ while $D(p(t)) = 0$ for all t in $[0; t_*]$. On $[0; t_*[$, the HL equation (1.1) reads

$$\begin{cases} \partial_t p = -b(t)\partial_\sigma p, \\ p \ge 0, \\ p(0, \cdot) = p_0, \\ D(p(t)) = 0. \end{cases}$$

The above system reduces to

$$\begin{cases} p(t,\sigma) = p_0(\sigma - \chi(t)), \\ D(p(t)) = 0 \quad \text{for all } t \text{ in } [0; t_*]. \end{cases} \tag{4.2}$$

The second equation in (4.2) is compatible with the first one as long as

$$\int_{|\sigma + \chi(t)| > 1} p_0 = 0 \quad \text{for all } t \text{ in } [0; t_*].$$

Therefore there exists a maximal time interval $[0; T_c]$ on which the HL equation may reduce to a mere transport equation and this is for an intrinsic time T_c (possibly infinite) defined by

$$T_c = \inf\left\{t > 0; \int_{|\sigma + \chi(t)| > 1} p_0 > 0\right\}. \tag{4.3}$$

Note that T_c is completely determined by the data p_0 and b. If $T_c = +\infty$, the steady state $p(t,\sigma) = p_0(\sigma - \chi(t))$ is a solution of the HL equation for all time. We shall now exhibit circumstances under which it is not the unique solution. For convenience, we restrict ourselves to the case when $b \equiv 0$ (we then have obviously $T_c = +\infty$).

For $p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $p_0 \geq 0$, let us denote by F_{p_0} the function from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by $F_{p_0}(0) = D(p_0)$ and by

$$\text{for all } x > 0, \qquad F_{p_0}(x) = \alpha \int_{|\sigma|>1} \left(\int_{\mathbb{R}} p_0(\sigma')\varphi_{\sqrt{2x}}(\sigma - \sigma')d\sigma' \right) d\sigma.$$

Proposition 4.1. *Let p_0 satisfy* (1.6) *and be such that $D(p_0) = 0$, then*

(i) *If F_{p_0} satisfies*

$$\int_0^1 \frac{dx}{F_{p_0}(x)} = +\infty, \tag{4.4}$$

then $p(t,\sigma) = p_0(\sigma)$ is the unique solution to (1.1) *in $C^0_t(L^2_\sigma)$.*

(ii) *Otherwise,* (1.1) *has an infinite number of solutions in $C^0_t(L^2_\sigma)$. The set of solutions to* (1.1) *is made of the steady state $p(t,\sigma) = p_0(\sigma)$ and of the functions $(q_{t_0})_{t_0 \geq 0}$ defined by*

$$q_{t_0}(t,\sigma) = \left| \begin{array}{ll} p_0(\sigma), & \text{if } t \leq t_0 \\ q(t - t_0, \sigma), & \text{if } t > t_0, \end{array} \right.$$

where q is the unique solution to (1.1) *in $C^0_t(L^2_\sigma)$ such that $D(q) > 0$ on $]0,+\infty[$. Besides,*

$$p_\epsilon \xrightarrow[\epsilon \to 0]{} q \quad \text{strongly in } L^2_{t,\mathrm{loc}}(L^2_\sigma). \tag{4.5}$$

Lemma 4.2. *Let $p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that*

$$p_0 \geq 0, \qquad \int_{\mathbb{R}} p_0 = 1, \qquad D(p_0) = 0.$$

The function F_{p_0} is in $C^0([0,+\infty[) \cap C^\infty(]0,+\infty[)$, and is positive on $]0,+\infty[$. In addition, $F'_{p_0} > 0$ on $]0,+\infty[$.

Proof. It is easy to check that $F_{p_0} \in C^0([0,+\infty[) \cap C^\infty(]0,+\infty[)$, and that $F_{p_0} > 0$ on $]0,+\infty[$. Since $D(p_0) = 0$, the function p_0 is supported in $[-1,1]$. Thus, for any $x > 0$,

$$\begin{aligned} F_{p_0}(x) &= \alpha \int_{|\sigma|>1} \left(\int_{\mathbb{R}} p_0(\sigma')\varphi_{\sqrt{2x}}(\sigma - \sigma')d\sigma' \right) d\sigma \\ &= \alpha \int_{-1}^1 p_0(\sigma') \left(\int_{|\sigma|>1} \frac{e^{-(\sigma-\sigma')^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma \right) d\sigma' \\ &= \alpha \int_{-1}^1 p_0(\sigma') \left(\int_{-\infty}^{-1+\sigma'} \frac{e^{-\sigma^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma + \int_{1+\sigma'}^{+\infty} \frac{e^{-\sigma^2/4x}}{2\sqrt{\pi}\sqrt{x}} d\sigma \right) d\sigma' \\ &= \alpha \frac{1}{\sqrt{\pi}} \int_{-1}^1 p_0(\sigma') \left(\int_{\frac{1+\sigma'}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt + \int_{\frac{1-\sigma'}{2\sqrt{x}}}^{+\infty} e^{-t^2} dt \right) d\sigma'. \end{aligned} \tag{4.6}$$

It follows that for any $x > 0$,

$$F'_{p_0}(x) = \alpha \frac{1}{\sqrt{\pi}} \int_{-1}^{1} p_0(\sigma') \left(\frac{1+\sigma'}{4x^{3/2}} e^{-\frac{(1+\sigma')^2}{4x}} + \frac{1-\sigma'}{4x^{3/2}} e^{-\frac{(1-\sigma')^2}{4x}} \right) d\sigma' > 0. \qquad \square$$

LEMMA 4.3. *Let* $\gamma \geq 0$ *and* $p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *such that*

$$p_0 \geq 0, \qquad \int_{\mathbb{R}} p_0 = 1, \qquad \int_{\mathbb{R}} |\sigma| p_0 < +\infty, \qquad D(p_0) = 0.$$

Let us consider the problem

$$\begin{cases} \partial_t w = D(w(t)) \partial^2_{\sigma\sigma} w - \gamma w, \\ w(0,\sigma) = p_0(\sigma). \end{cases} \tag{4.7}$$

(i) *If* F_{p_0} *satisfies* (4.4), *then* $p(t,\sigma) = p_0(\sigma)$ *is the unique solution to* (4.7) *in* $C^0_t(L^2_\sigma)$.

(ii) *Otherwise,* (4.7) *has an infinite number of solutions in* $C^0_t(L^2_\sigma)$. *The set of solutions to* (4.7) *is made of the steady state* $w(t,\sigma) = p_0(\sigma)$ *and of the functions* $(v_{t_0})_{t_0 \geq 0}$ *defined by*

$$v_{t_0}(t,\sigma) = \left| \begin{array}{ll} p_0(\sigma), & \text{if } t \leq t_0, \\ v(t-t_0,\sigma), & \text{if } t > t_0, \end{array} \right.$$

where v *is the unique solution to* (4.7) *in* $C^0_t(L^2_\sigma)$ *such that* $D(v) > 0$ *on* $]0,+\infty[$.

COROLLARY 4.4. *The initial data* $p_0 = \frac{1}{2}\mathbb{1}_{]-1,1[}$ *fulfills the assumptions of the above lemma and* $\int_0^1 \frac{dx}{F_{p_0}(x)} < +\infty$. *Therefore there are infinitely many solutions to* (1.5) *in the introduction.*

Proof of Corollary 4.4. The only point to be checked is that $\int_0^1 \frac{dx}{F_{p_0}(x)} < +\infty$. With the standard notation $\operatorname{erfc}(z) \equiv \int_z^{+\infty} e^{-t^2} dt$, and by using (4.6) and symmetry considerations, simple calculations yield

$$\begin{aligned} F_{p_0}(x) &= \frac{2\alpha\sqrt{x}}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{x}}} \operatorname{erfc}(\sigma) d\sigma \\ &= \frac{2\alpha}{\sqrt{\pi}} \left[\operatorname{erfc}\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{2}\sqrt{x} e^{-\frac{1}{x}} + \frac{1}{2}\sqrt{x} \right]. \end{aligned}$$

Since $\operatorname{erfc}(z) \sim \frac{1}{2} e^{-z^2}/z$ for z going to $+\infty$, $F_{p_0}(x) \sim \frac{\alpha}{\sqrt{\pi}}\sqrt{x}$ near 0 and the integrability of $1/F_{p_0}$ on $[0;1]$ follows. $\square$

Proof of Lemma 4.3. Let us consider a nonnegative continuous function D on $\mathbb{R}^+$. The unique solution in $C^0_t(L^2_\sigma)$ to the problem

$$\begin{cases} \partial_t w_D = D(t) \partial^2_{\sigma\sigma} w_D - \gamma w_D, \\ w_D(0,\sigma) = p_0(\sigma), \end{cases} \tag{4.8}$$

is given by

$$w_D(t,\sigma) = \left| \begin{array}{ll} e^{-\gamma t} p_0(\sigma), & \text{if } t \leq t^*, \\ e^{-\gamma t} \displaystyle\int_{\mathbb{R}} p_0(\sigma') \varphi_{\sqrt{2\int_0^t D(s)ds}}(\sigma - \sigma') d\sigma', & \text{if } t > t^*, \end{array} \right. \tag{4.9}$$

where $t^* = \inf\{t > 0; \int_0^t D > 0\}$. Any solution to (4.7) thus satisfies $w = w_{D(w)}$ and therefore

$$\begin{aligned} D(w(t)) &= D(w_{D(w)}(t)) \\ &= \alpha \int_{|\sigma|>1} w_{D(w)}(t,\sigma) d\sigma \\ &= \alpha e^{-\gamma t} \int_{|\sigma|>1} \left(\int_{\mathbb{R}} p_0(\sigma') \varphi_{\sqrt{2\int_0^t D(w(s))ds}} (\sigma - \sigma') d\sigma' \right) d\sigma \\ &= e^{-\gamma t} F_{p_0} \left(\int_0^t D(w(s)) ds \right). \end{aligned}$$

It follows that the function $D(w)$ is the solution in $C^0([0,+\infty[)$ to the nonlinear integral equation

$$y(t) = e^{-\gamma t} F_{p_0} \left(\int_0^t y(s) ds \right). \tag{4.10}$$

On the other hand, if $D \in C^0([0,+\infty[)$ is solution to (4.10) it is easy to check that the function w_D defined by (4.9) is solution to (4.8).

If condition (4.4) is fulfilled, (4.10) has a unique solution in $C^0([0,+\infty[)$ (the constant function equal to zero) and the steady state $w(t,\cdot) = p_0$ is therefore the unique solution to (4.7) in $C_t^0(L_\sigma^2)$; otherwise, the set of solutions to (4.10) is made of the steady state $w(t,\cdot) = p_0$ and of the family $(y_{t_0})_{t_0 \geq 0}$ with

$$y_{t_0}(t) = \left| \begin{array}{ll} 0, & \text{if } t \leq t_0, \\ z(t-t_0), & \text{if } t > t_0, \end{array} \right.$$

where the function z is defined on $[0,+\infty[$ by

$$\int_0^{z(t)} \frac{dx}{F(x)} = \begin{cases} \dfrac{1-e^{-\gamma t}}{\gamma}, & \text{if } \gamma > 0, \\ t, & \text{otherwise.} \end{cases}$$

Statement (ii) is obtained by denoting by v the solution to (4.8) associated with the function $z(t)$. □

Proof of Proposition 4.1. The solution p_ϵ to (1.7) satisfies the inequalities

$$p_\epsilon^-(t,\sigma) \leq p_\epsilon(t,\sigma) \leq p_\epsilon^+(t,\sigma) \quad \text{almost everywhere,}$$

where p_ϵ^- and p_ϵ^+ are defined in $C_t^0(L_\sigma^2)$ by

$$\begin{cases} \partial_t p_\epsilon^- = \left(D(p_\epsilon(t)) + \epsilon\right) \partial_{\sigma\sigma}^2 p_\epsilon^- - p_\epsilon^-, \\ p_\epsilon^-(0,\sigma) = p_0(\sigma), \end{cases}$$

and

$$\begin{cases} \partial_t p_\epsilon^+ = (D(p_\epsilon(t)) + \epsilon) \partial_{\sigma\sigma}^2 p_\epsilon^+ + \dfrac{D(p_\epsilon)}{\alpha} \delta_0, \\ p_\epsilon^+(0,\sigma) = p_0(\sigma). \end{cases}$$

Therefore, on the one hand,

$$D(p_\epsilon(t)) \geq D(p_\epsilon^-(t)) = e^{-t} F_{p_0}\left(\int_0^t (D(p_\epsilon)+\epsilon)\right) \tag{4.11}$$

and, on the other hand,

$$\begin{aligned} D(p_\epsilon(t)) &\leq D(p_\epsilon^+(t)) \\ &= F_{p_0}\left(\int_0^t (D(p_\epsilon)+\epsilon)\right) + \int_0^t \frac{D(p_\epsilon)(s)}{\alpha}\left(\int_{|\sigma|>1} \varphi_{\sqrt{2\int_s^t (D(p_\epsilon)+\epsilon)}}\right) ds \\ &\leq F_{p_0}\left(\int_0^t (D(p_\epsilon)+\epsilon)\right) + \frac{1}{\alpha}\int_0^t D(p_\epsilon)(s)ds. \end{aligned}$$

If (4.4) is not fulfilled, using (4.11) and the property that F_{p_0} is strictly increasing on $[0,+\infty[$, we obtain that

$$D(p_\epsilon) \geq z(t),$$

where $z(t)$ is the function defined in the proof of Lemma 4.3. As for any $0 < t_0 \leq T$ there exists $\eta > 0$ such that $z(t) \geq \eta$ on $[t_0, T]$ the same reasoning as in the nondegenerate case leads to the conclusion that (p_ϵ) converges up to an extraction to p in $\mathcal{D}'(]0,+\infty[\times\mathbb{R})$ and in $L^2([t_0,T], L^2(\mathbb{R}))$ for any $0 < t_0 < T < +\infty$, p being a solution to (1.1) in $C^0(]0,+\infty[, L^2_\sigma)$ such that $D(p) > 0$ on $]0,+\infty[$. □

5. Steady states. Throughout this section the shear rate b is assumed to be a given constant and we are looking for solutions in $L^1(\mathbb{R})$ to the following system:

$$\begin{cases} -b\partial_\sigma p + D(p)\partial^2_{\sigma\sigma} p - \mathbb{1}_{\mathbb{R}\setminus[-1,1]} p + \dfrac{D(p)}{\alpha}\delta_0(\sigma) = 0 \quad \text{on } (0;T)\times\mathbb{R}, \\ p \geq 0, \quad \displaystyle\int_{\mathbb{R}} p = 1, \\ D(p) = \alpha \displaystyle\int_{|\sigma|>1} p(\sigma)d\sigma. \end{cases} \tag{5.1}$$

Our main results are summarized in the following proposition.

Proposition 5.1 (existence of steady states).

(i) *If* $b \equiv 0$, *any probability density which is compactly supported in* $[-1;+1]$ *is a solution to* (5.1) *which satisfies* $D(p) = 0$. *If* $\alpha \leq \frac{1}{2}$, *these are the only stationary solutions (and there are infinitely many), whereas when* $\alpha > \frac{1}{2}$ *there exists a unique stationary solution corresponding to a positive value of* D, *which is explicitly given in* (5.2) *and* (5.4) *below. This solution is even and with exponential decay at infinity.*

(ii) *If* $b \not\equiv 0$, *for any* $\alpha > 0$, *there exists a unique stationary solution to* (5.1), *and it corresponds to a positive value for* D, *which is implicitly given in* (5.5) *and* (5.6) *below. This solution has exponential decay at infinity.*

Remark 5.2. The statement in the above proposition was already pointed out by Hébraud and Lequeux [8].

Proof. **The case when b ≡ 0.** We first observe that any nonnegative function p which is normalized in $L^1(\mathbb{R})$ and with support in $[-1;+1]$ is a solution to the system (5.1) since in that case all terms in the equation satisfied by p in (5.1) cancel. We now examine the issue of existence of solutions of (5.1) such that $D(p) > 0$. For simplicity we denote $D = D(p)$. For given constant $D > 0$, it is very easy to calculate explicitly the solutions to (5.1) on each of the three regions $\sigma < -1$, $\sigma \in [-1;+1]$ and $\sigma > 1$. Using compatibility conditions on $\mathbb{R}$ and the fact that p has to be in $L^1(\mathbb{R})$, one obtains

$$p(\sigma) = \begin{cases} \dfrac{\sqrt{D}}{2\alpha} e^{(1+\sigma)/\sqrt{D}}, & \text{if } \sigma \leq -1, \\ \dfrac{1}{2\alpha}\sigma + \dfrac{\sqrt{D}+1}{2\alpha}, & \text{if } -1 \leq \sigma \leq 0, \\ -\dfrac{1}{2\alpha}\sigma + \dfrac{\sqrt{D}+1}{2\alpha}, & \text{if } 0 \leq \sigma \leq 1, \\ \dfrac{\sqrt{D}}{2\alpha} e^{(1-\sigma)/\sqrt{D}}, & \text{if } 1 \leq \sigma. \end{cases} \tag{5.2}$$

The compatibility condition $D = D(p)$ happens to then be automatically satisfied and the normalization constraint $\int_{\mathbb{R}} p = 1$ imposes that D solves

$$D + \sqrt{D} = \alpha - \frac{1}{2}. \tag{5.3}$$

Since $D \geq 0$, we immediately reach a contradiction when $\alpha < \frac{1}{2}$, whereas when $\alpha > \frac{1}{2}$ (5.3) admits a unique positive solution; namely

$$D = -\frac{1}{2} + \frac{\sqrt{4\alpha - 1}}{2}. \tag{5.4}$$

The case when $\mathbf{b \not\equiv 0}$. First of all, we observe that if $D = 0$ all terms in the equation satisfied by p in (5.1) but $b\partial_\sigma p$ vanish. Thus p has to be a nonzero constant which is in contradiction with $p \in L^1(\mathbb{R})$. So necessarily $D > 0$. For given positive constant D, we then solve (5.1) as above and obtain

$$p(\sigma) = \begin{cases} a_1 e^{\beta^+ \sigma}, & \text{if } \sigma \leq -1, \\ a_2 e^{\frac{b}{D}\sigma} + a_2 - \dfrac{D}{b\alpha}, & \text{if } -1 \leq \sigma \leq 0, \\ \left(a_2 - \dfrac{D}{b\alpha}\right) e^{\frac{b}{D}\sigma} + a_2, & \text{if } 0 \leq \sigma \leq 1, \\ a_1 e^{\beta^- \sigma}, & \text{if } 1 \leq \sigma, \end{cases} \tag{5.5}$$

with

$$\beta^\pm = \frac{b}{2D} \pm \frac{1}{2}\sqrt{\frac{b^2 + 4D}{D^2}},$$

$$a_1 = \frac{e^{\frac{1}{2}\sqrt{\frac{b^2}{D^2} + \frac{4}{D}}}}{\alpha\left(\beta^+ e^{b/2D} - \beta^- e^{-b/2D}\right)},$$

and

$$a_2 = \frac{D\beta^+ e^{b/2D}}{\alpha b(\beta^+ e^{b/2D} - \beta^- e^{-b/2D})}.$$

It is tedious but easy to check that this function always fulfills the self-consistency condition $D = D(p)$ and that the normalization condition $\int_{\mathbb{R}} p = 1$ reads

$$\frac{D}{b}\frac{(1+\beta^+) + (\beta^- - 1)e^{-b/D}}{\beta^+ - \beta^- e^{-b/D}} + D = \alpha. \tag{5.6}$$

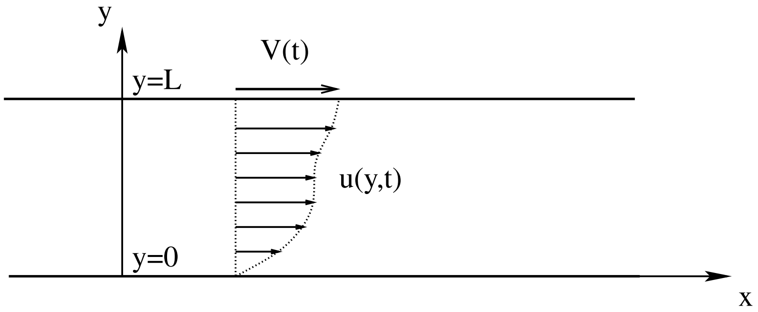

FIG. 1. *Planar Couette flow.*

For any $b > 0$ (the negative values of b are dealt with by replacing σ by $-\sigma$), the left-hand side of (5.6) is a continuous function which goes to $+\infty$ when D goes to infinity and goes to zero when D goes to 0. This already ensures the existence of at least one steady state for any $\alpha > 0$. Moreover, setting $z = b^2/D$ (for example) we may rewrite the left-hand side of (5.6) as

$$f(z) = \frac{b^2}{z} + \frac{2b^2}{z}\left[\frac{1 + \frac{1}{2b}z\coth(z/2b) + \frac{1}{2b}(z^2+4z)^{1/2}}{z + (z^2+4z)^{1/2}\coth(z/2b)}\right].$$

Next we check that the function f is monotone decreasing (thus, the left-hand side of (5.6) is increasing with respect to D), whence the uniqueness result. $\square$

6. Conclusion and future trends. Theorem 1.1 shows that the Hébraud–Lequeux model (1.1) is well-posed when the initial data p_0 is such that $D(p_0) > 0$. On the other hand, Proposition 4.1 claims that the model may have infinitely many solutions for certain p_0 such that $D(p_0) = 0$. This pathological behavior might be a flaw of the model but we think that is not the case. Indeed, our interpretation is that an initial data p_0 such that $D(p_0) = 0$ can be considered as admissible only if it is the long-time limit of a solution $p > 0$ of (1.1) with zero shear rate ($b = 0$); in view of both numerical simulations and heuristic arguments, we suspect that such a p_0 necessarily fulfills condition (4.4). We are currently studying the long-time asymptotics of the HL equation [3] and hope to be able to provide mathematical justifications of this statement in the near future.

To conclude, let us mention that the present article is the theoretical side of a joint project with rheologists aiming at better understanding the flows of complex fluids.

We are currently working on a multiscale model for planar Couette flows of concentrated suspensions (see Figure 1), in which the mesoscopic behavior of the suspension is described by the HL equation. It is indeed experimentally observed that the shear rate $b(t)$ is not homogeneous in space (in particular, the term $-b(t)\partial_\sigma p$ generates elastic waves in the fluid). In order to better describe the coupling between the macroscopic flow and the evolution of the mesostructure, we propose the following multiscale model:

$$
\begin{cases}
\rho\partial_t u(t,y) = \partial_y \tau(t,y) + \mu\partial_{yy}u(t,y), \\
\partial_t p(t,y,\sigma) = -G_0\partial_y u(t,y)\partial_\sigma p(t,y,\sigma) + D(p(t,y))\partial^2_{\sigma\sigma}p(t,y,\sigma) \\
\qquad\qquad - \dfrac{\mathbb{1}_{\mathbb{R}\setminus[-\sigma_c,\sigma_c]}(\sigma)}{T_0}p(t,y,\sigma) + \dfrac{1}{T_0}\left(\displaystyle\int_{|\sigma'|>\sigma_c} p(t,\sigma',y)d\sigma'\right)\delta_0(\sigma), \\
\tau(t,y) = \displaystyle\int_{\mathbb{R}} \sigma p(t,y,\sigma)d\sigma.
\end{cases}
\tag{6.1}
$$

In the above equations, $u(t,y)$ denotes the component along e_x of the velocity field (the flow being laminar and incompressible, the velocity field is of the form $\vec{u} = u(t,y)e_x$ and the pressure does not play any role), ρ is the volumic mass of the fluid and μ some nonnegative viscosity coefficient. This system is complemented by the no-slip boundary conditions

$$
\begin{cases}
u(t,0) = 0 & \text{for almost all } t, \\
u(t,L) = V(t) & \text{for almost all } t.
\end{cases}
$$

The theoretical study of this multiscale model will be the matter of another article. Numerical simulations of (6.1) have already been performed by one of us [6]. The results of these simulations will be compared in the near future with experimental data obtained with a NMR rheometer [12] (this very modern equipment allows measurements of local velocities in opaque fluids).

Acknowledgments. We would like to thank Philippe Coussot for pointing out the Hébraud–Lequeux equation to us. We also warmly thank Pascal Hébraud and Claude Le Bris for helpful discussions.

REFERENCES

[1] J.P. Bouchaud, *Weak ergodicity breaking and aging in disordered systems*, J. Phys. I, 2 (1992), pp. 1705–1713.

[2] E. Cancès, I. Catto, Y. Gati, and C. Le Bris, *On a multiscale model for non-Newtonian flows*, Multiscale Model. Simul., to appear.

[3] E. Cancès and C. Le Bris, preprint, IMA, 2005.

[4] M. Chipot, *Elements of Nonlinear Analysis*, Birkhäuser Verlag, Basel, 2000.

[5] R.S. Fall, J.R. Melrose, and R.C. Ball, *Kinetic theory of jamming in hard-sphere startup flows*, Phys. Rev. E, 55 (1997), pp. 7203–7211.

[6] Y. Gati, *Numerical simulation of micro-macro model of concentrated suspensions*, Internat. J. Numer. Methods Fluids, Special Issue: ICFD Conference on Numerical Methods for Fluid Dynamics, 47 (2005), pp. 1019–1025.

[7] J.-P. Hansen and I.R. McDonald, *Theory of Simple Liquids*, Academic Press, London, 1976.

[8] P. Hébraud and F. Lequeux, *Mode coupling theory for the pasty rheology of soft glassy materials*, Phys. Rev. Lett., 81 (1998), pp. 2934–2937.

[9] R.G. Larson, *The Structure and Rheology of Complex Fluids*, Oxford University Press, London, 1998.

[10] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.

[11] C. Monthus and J.P. Bouchaud, *Models of traps and glass phenomenology*, J. Phys. A, 29 (1996), pp. 3847–3869.

[12] J.S. Raynaud, P. Moucheront, J.C. Baudez, F. Bertrand, J.P. Guilbaud, and Ph. Coussot, *Direct determination by NMR of the thixotropic and yielding behavior of suspensions*, J. Rheol., 46 (2002), pp. 709–732.

[13] P. Sollich, *Rheological constitutive equation for a model of soft glassy materials*, Phys. Rev. E, 58 (1998), pp. 738–759.

[14] P. Sollich, F. Lequeux, P. Hébraud, and M. Cates, *Rheological of soft glassy materials*, Phys. Rev. Lett., 78 (1997), pp. 2020–2023.