

ATTRACTORS FOR A NON-LINEAR PARABOLIC EQUATION MODELLING SUSPENSION FLOWS

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ABSTRACT. In this paper we prove the existence of a global attractor with respect to the weak topology of a suitable Banach space for a parabolic scalar differential equation describing a non-Newtonian flow. More precisely, we study a model proposed by Hébraud and Lequeux for concentrated suspensions.

1. Introduction. Non-Newtonian (or complex) fluids are ubiquitous in nature and industry, appearing for instance in foods, biofluids, personal care products, pharmacology and bioengineering, electronics and optical materials, energy and plastic production, etc. In fact, one could say that Newtonian (or simple) fluids, i.e., those fluids whose stress-tensor is given by the Navier-Stokes ansatz, are rather an exception (if not an idealization), even though they include such a prominent member as water. Attending to their rheologic properties, complex fluids are classified in different categories, including suspensions, colloids, melt polymers, liquid crystals, gels and foams, among others. Needless to say, coping with such a broad diversity of fluids requires physical insight, mathematical sophistication, and a lot of ingenuity.

Non-Newtonian fluids are notoriously difficult to model and to analyze. To begin with, these fluids display very nonlinear flow properties (such as memory effects and discontinuities) that are far from being understood from first principles. As a result one has to resort in general to phenomenological (or macroscopic) descriptions or, in some cases, to mesoscopic models describing the interaction of different types of microstructures (hard or soft spheres, rods, dumb bells, etc.) much larger than the atomic scale. Elaborated mesoscopic models are being successfully used in polymers, liquid crystals and suspensions.

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In this paper we will consider only suspensions and, more specifically, the model proposed by Hébraud and Lequeux [18], in which the system is divided in mesoscopic blocks whose size is large enough so that stress and strain tensors may be defined for each block, but small compared to the characteristic length scale of the stress field. In the Hébraud-Lequeux (HL) model, each block carries a given shear stress $\sigma \in \mathbb{R}$ (in fact, σ is an extra-diagonal term of the stress tensor in convenient coordinates). The evolution of the blocks is described by means of a probability distribution density $p(t, \sigma) \geq 0$ which represents the distribution of stress in the assembly of blocks at time t . The equation satisfied by $p(t, \sigma)$ is

$$\partial_t p = -b(t)\partial_\sigma p + D(p)\partial_{\sigma\sigma}^2 p - \frac{\mathbf{1}_{\mathbb{R}\setminus[-\sigma_c, \sigma_c]}(\sigma)}{T_0} p + \frac{D(p)}{\alpha} \delta_0(\sigma), \quad (1)$$

with the initial condition $p(0, \sigma) = p_0(\sigma)$. Here $\mathbf{1}_{\mathbb{R}\setminus[-\sigma_c, \sigma_c]}$ is the characteristic function of the open interval $\mathbb{R}\setminus[-\sigma_c, \sigma_c]$, δ_0 is the Dirac delta function on \mathbb{R} with support on the origin, and for $f \in L^1(\mathbb{R})$ we denote

$$D(f) = \frac{\alpha}{T_0} \int_{|\sigma| > \sigma_c} f(\sigma) d\sigma.$$

Briefly, the physical interpretation of the parameters appearing in the HL equation (1) is as follows. When a block is submitted to a shear rate $\gamma(t)$, the stress of this block evolves with a variation rate $b(t) = G_0 \dot{\gamma}(t)$, where G_0 is an elasticity constant. The term $b(t)\partial_\sigma p$ models then the blocks behaving as Einstein elastic solids at low shear, their elasticity arising from interactions between neighboring particles. On the other hand, when the modulus of the stress surpasses the critical value σ_c , the block flows as an Eyring fluid: the configuration reached by shearing the suspension relaxes in a characteristic relaxation time T_0 towards a state with zero stress. This relaxation phenomenon induces a rearrangement of the other blocks and this is finally modelled through the diffusion term $D(p(t))\partial_{\sigma\sigma}^2 p$. For more details on the physics of the model, we refer to [18].

The existence and uniqueness of solutions of (1) were studied in [8, Theorem 1.1]. In the present paper we complete this analysis by considering the asymptotic behavior of the solutions and, more specifically, the existence of attractors. In doing so we will suppose that the stress variation rate $b(t)$ is small enough so that the first term on the right hand side of (1) is negligible as compared to the others and, therefore, it may be dropped altogether. In this simplified model, we find that the HL equation has indeed an attractor in the weak topology. Also, we set $\sigma_c = 1$ for simplicity, although the same result can be obtained for an arbitrary $\sigma_c > 0$.

In sum, in this paper we study the asymptotic behavior of the weak solutions of the scalar parabolic equation

$$\partial_t p - D(p(t))\partial_{\sigma\sigma}^2 p + \frac{1}{T_0} \mathbf{1}_{\mathbb{R}\setminus[-1, 1]}(\sigma) p = \frac{D(p(t))}{\alpha} \delta_0(\sigma), \quad (2)$$

where $p = p(\sigma, t)$, $t \geq 0$, $\sigma \in \mathbb{R}$, $D(p(t)) = \frac{\alpha}{T_0} \int_{|\sigma| > 1} p(t, \sigma) d\sigma$, and $\mathbf{1}_{\mathbb{R}\setminus[-1, 1]}$ denotes the characteristic function of the open set $\mathbb{R}\setminus[-1, 1]$. Our aim is to prove the existence of a global attractor of this equation. We note that the theory of global attractors for parabolic equations has been developed intensively in the last twenty years. The first results were obtained in the case where the spacial variable belongs to a bounded domain (see [3], [4], [5], [10], [13], [15], [20], [23], [25], [26], [32], [33], [35], [36]). The problem in an unbounded domain was considered later

on in [6]. In the last year several authors have continued working in this direction (see [1], [14], [16], [17], [28], [29], [30], [31], [37], [38]).

From the mathematical point of view, equation (2) presents several difficulties when studying the asymptotic behavior of solutions.

First, we were not able to prove that the Cauchy problem has a unique solution in a suitable phase space. Hence, we have to work with a multivalued semiflow rather than with a semigroup of operators. This approach has been used before for parabolic equations in [10], [15], [20], [23], [29], [32], [33].

Secondly, due to the terms $D(p(t))$ and $\mathbf{1}_{\mathbb{R} \setminus [-1, 1]}$ on the left-hand side of (2), the dissipative mechanism of the equation is weakened, and we were not able to obtain an absorbing set. Nevertheless, we have proved that the solutions starting at a bounded set of a suitable phase space remain uniformly bounded for positive values of time, and in this way we have obtained the existence of a global (possibly unbounded) attractor with respect to the weak topology of an appropriate Banach space. It is an open question whether this result can be improved by considering the strong topology.

This paper is organized as follows. In Sections 2 and 3 we extend the results on existence of solutions given in [8] and obtain some a priori estimates. In Section 4 we develop a general theory of global attractors for multivalued semiflows in topological spaces. Finally, in Section 5 we prove the main result of this paper, that is, the existence of the global attractor.

2. Setting of the problem and existence of weak solutions. We shall consider the following scalar parabolic equation

$$\begin{cases} \partial_t p - D(p(t)) \partial_{\sigma\sigma}^2 p + \frac{1}{T_0} \mathbf{1}_{\mathbb{R} \setminus [-1, 1]}(\sigma) p = \frac{D(p(t))}{\alpha} \delta_0(\sigma), \\ p \geq 0, \\ p(0, \sigma) = p_0(\sigma), \end{cases} \quad (3)$$

where $p = p(\sigma, t)$, $t \geq 0$, $\sigma \in \mathbb{R}$, $D(p(t)) = \frac{\alpha}{T_0} \int_{|\sigma| > 1} p(t, \sigma) d\sigma$, and $\mathbf{1}_{\mathbb{R} \setminus [-1, 1]}$ denotes the characteristic function of the open set $\mathbb{R} \setminus [-1, 1]$.

Consider an initial condition p_0 satisfying

$$p_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), p_0 \geq 0, \int_{\mathbb{R}} p_0 d\sigma = 1 \text{ and } D(p_0) > 0. \quad (4)$$

Then it is proved in [8, Theorem 1.1] that problem (3) has a unique solution $p(t)$ satisfying the following properties for all $T > 0$:

$$\begin{aligned} p &\in L^\infty(0, T; L^2(\mathbb{R}) \cap L^1(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R})), \\ p &\in L^\infty((0, T) \times \mathbb{R}) \cap C([0, T], L^2(\mathbb{R}) \cap L^1(\mathbb{R})), \\ \int_{\mathbb{R}} p(t, \sigma) d\sigma &= 1, p(t) \geq 0 \text{ for all } t \geq 0, \\ D(p(t)) &\in C([0, T]) \text{ and } \min_{0 \leq t \leq T} D(p(t)) \geq \nu(T) > 0, \\ \sigma p &\in L^\infty(0, T; L^1(\mathbb{R})), \end{aligned}$$

where $\nu(T)$ exists for any $T > 0$. The last property implies that the average stress $\tau(t) = \int_{\mathbb{R}} \sigma p(t, \sigma) d\sigma$ belongs to $L^\infty(0, T)$. From $\delta_0 \in H^{-1}(\mathbb{R})$ and the above properties it follows $\partial_t p \in L^2(0, T; H^{-1}(\mathbb{R}))$.

We can see that $p(t) \in L^\infty(\mathbb{R})$ for a.a. t . Moreover, since p belongs to the space $L^\infty(0, t; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, $p \in C([0, t], L^2(\mathbb{R}))$ and by interpolation the

embedding $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ is continuous, we obtain that $p : [0, t] \rightarrow L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is weakly continuous (i.e. continuous with respect to the weak topology of the space $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$) [34, Lemma 1.4, p.263].

Our aim in this paper is to study the asymptotic behavior of solutions and, in particular, to prove the existence of a global attractor. For this purpose it is necessary to extend this existence result to a more general class of initial conditions.

Let

$$X = \left\{ p \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\sigma| |p| d\sigma < +\infty \right\}.$$

Note that X is a Banach space with the norm $\|p\|_X = \|p\|_{L^2} + \int_{\mathbb{R}} |\sigma| |p| d\sigma$. We denote by X_w the space X endowed with the weak topology. Denote by $\langle \cdot, \cdot \rangle$ pairing between $H^1(\mathbb{R})$ and $H^{-1}(\mathbb{R})$ and by (\cdot, \cdot) the scalar product in $L^2(\mathbb{R})$. Also, the embedding $X \subset L^1(\mathbb{R})$ is continuous.

Put

$$\overline{L}^1(\mathbb{R}) = \left\{ p \in L^1(\mathbb{R}) : \int_{\mathbb{R}} |\sigma| |p| d\sigma < +\infty \right\},$$

which is a Banach space with the norm $\|p\|_{\overline{L}^1} = \int_{\mathbb{R}} (1 + |\sigma|) |p| d\sigma$. From general results concerning the dual space of $L^1(d\mu)$ for general measures μ (see e.g. [11] or [12]) it follows that the dual space $(\overline{L}^1(\mathbb{R}))^*$ can be characterized as

$$(\overline{L}^1(\mathbb{R}))^* = \{ w = (1 + |\sigma|) \xi : \xi \in L^\infty(\mathbb{R}) \},$$

and

$$(p, w)_{\overline{L}^1} = \int_{\mathbb{R}} (1 + |\sigma|) \xi p d\sigma.$$

Nevertheless, for the sake of completeness we give a proof of this fact in the Appendix.

Furthermore, it is easy to see that the norms $\|p\|_X = \|p\|_{L^2} + \int_{\mathbb{R}} (1 + |\sigma|) |p| d\sigma$ and $\|p\|_{\overline{L}^1}$ are equivalent. Clearly, $\|p\|_X \leq \|p\|_{\overline{L}^1}$. Then it is enough to prove that $\|p\|_{\overline{L}^1} \leq C \|p\|_X$. This follows from

$$\|p\|_{\overline{L}^1} = \int_{|\sigma| \leq 1} |p| d\sigma + \int_{|\sigma| > 1} |p| d\sigma \leq \sqrt{2} \|p\|_{L^2} + \int_{\mathbb{R}} |\sigma| |p| d\sigma \leq \sqrt{2} \|p\|_X.$$

Since $X = \overline{L}^1(\mathbb{R}) \cap L^2(\mathbb{R})$, it follows that $X^* = (\overline{L}^1(\mathbb{R}))^* + L^2(\mathbb{R})$.

Let $p(\cdot)$ be the unique solution corresponding to an initial data p_0 satisfying (4). Then the continuous embedding $X \subset L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $p \in L^\infty(0, t; X) \cap C([0, t], L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$ imply that $p : [0, t] \rightarrow X$ is weakly continuous [34, Lemma 1.4, p.263].

Definition 2.1. We say that $p : [0, T] \rightarrow X$ is a weak solution of (3) on $[0, T]$ if we have $p \in L^\infty(0, T; X)$, $D(p)p \in L^2(0, T; H^1(\mathbb{R}))$, $\partial_t p \in L^2(0, T; H^{-1}(\mathbb{R}))$ and

$$\int_0^T \left(\langle \partial_t p, \xi \rangle + \int_{\mathbb{R}} D(p(t)) \partial_\sigma p \partial_\sigma \xi dx + \int_{|\sigma| > 1} \frac{p\xi}{T_0} dx \right) dt = \int_0^T \frac{D(p(t))}{\alpha} \langle \delta_0, \xi \rangle dt, \tag{5}$$

for all $\xi \in L^2(0, T; H^1(\mathbb{R}))$.

We note that $p \in C([0, T], H^{-1}(\mathbb{R}))$ and $p \in L^\infty(0, T; L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$, so that $p : [0, T] \rightarrow L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ is weakly continuous, that is, it is continuous with respect to the weak topology of the space $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (see again [34, Lemma

1.4, p.263]). Then an initial condition $p(0) = p_0 \in X$ makes sense. In particular, $p \in C([0, T], L_w^p(\mathbb{R}))$, $p = 1, 2$, where $L_w^p(\mathbb{R})$ denotes the space $L^p(\mathbb{R})$ endowed with the weak topology.

In fact, we shall prove that $p \in C([0, T], X_w)$ for all weak solutions satisfying $p(t) \geq 0$ and $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$ for all $t \geq 0$. In order to prove this we need the following estimate of the tails of the weak solutions.

In the sequel by x_α or x^α we shall denote a generalized sequence in X_w .

Lemma 2.2. *Let $B \subset X$ be bounded in the norm of the space $\overline{L}^1(\mathbb{R})$. Then for any $\varepsilon > 0$ there exist $T(\varepsilon, B)$, $\overline{k}(\varepsilon)$ such that any weak solution p with $p(0) = p_0 \in B$, $p_0 \geq 0$, and such that $p(t) \geq 0$, $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$ for all $t \geq 0$, satisfies the estimate*

$$\int_{2k \leq |\sigma|} p(t, \sigma) |\sigma| d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \overline{k}. \quad (6)$$

Also, there exists a constant $M(B)$ such that

$$\int_{\mathbb{R}} p(t, \sigma) |\sigma| d\sigma \leq M, \text{ for all } t \geq 0. \quad (7)$$

If $p_0^\alpha \rightarrow p_0$ in X_w , where $p_0^\alpha, p_0 \geq 0$, then for any $\varepsilon > 0$ there exist $\overline{k}(\varepsilon)$, $\Lambda(\varepsilon)$ such that

$$\int_{2k \leq |\sigma|} p_\alpha(t, \sigma) |\sigma| d\sigma \leq \varepsilon \text{ if } t \geq 0, k \geq \overline{k}, \alpha \geq \Lambda, \quad (8)$$

for any weak solution p_α with $p_\alpha(0) = p_0^\alpha$ and such that $p_\alpha(t) \geq 0$, $\int_{\mathbb{R}} p_\alpha(t, \sigma) d\sigma = 1$ for all $t \geq 0$.

Remark 1. We observe that $\overline{k}(\varepsilon)$ does not depend on the set B .

Proof. Let θ be a smooth function such that

$$\theta(s) = \begin{cases} 0, & s \leq 1, \\ 0 \leq \theta(s) \leq 1, & 1 \leq s \leq 2, \\ 1, & s \geq 2. \end{cases} \quad (9)$$

Define $\xi(s) = 1 - \theta(s)$. It is clear that $\xi(s) = 1$ if $s \leq 1$, $0 \leq \xi(s) \leq 1$, if $1 \leq s \leq 2$, and $\xi(s) = 0$ if $s \geq 2$. Also, all the derivatives of the function ξ are uniformly bounded. Let $\rho_k(\sigma) = \sqrt{\theta(\frac{\sigma}{k})}$, $k \geq 1$, $\tau_R(\sigma) = \sqrt{\xi(\frac{\sigma}{R})}$, $R \geq 2k$. We multiply (3) by $\rho_k^2(\sigma) \tau_R^2(\sigma) \sigma$. Since $p, \partial_t p \in L^1(0, T; H^{-1}(\mathbb{R}))$, we have the equality $\langle \partial_t p, \xi \rangle = \frac{d}{dt} \langle p, \xi \rangle$, for all $\xi \in H^1(\mathbb{R})$ [34, p.250]. In particular, this holds for $\xi = \rho_k^2(\sigma) \tau_R^2(\sigma) \sigma \in C_0^\infty(\mathbb{R})$. Hence

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p \sigma d\sigma - \langle D(p(t)) \partial_{\sigma\sigma}^2 p, \rho_k^2(\sigma) \tau_R^2(\sigma) \sigma \rangle \\ & + \frac{1}{T_0} \int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p \sigma d\sigma \\ & = \frac{D(p(t))}{\alpha} \langle \delta_0(\sigma), \tau_R^2(\sigma) \rho_k^2(\sigma) \sigma p \rangle = 0. \end{aligned}$$

Note that

$$\begin{aligned}
& \langle D(p(t)) \partial_{\sigma\sigma}^2 p, \rho_k^2(\sigma) \tau^2(\sigma) \sigma \rangle \\
&= D(p(t)) \int_{\mathbb{R}} p \partial_{\sigma\sigma}^2 (\rho_k^2(\sigma) \tau_R^2(\sigma) \sigma) d\sigma \\
&= D(p(t)) \int_{\mathbb{R}} p \left(\theta'' \left(\frac{\sigma}{k} \right) \xi \left(\frac{\sigma}{R} \right) \frac{\sigma}{k^2} + \xi'' \left(\frac{\sigma}{R} \right) \theta \left(\frac{\sigma}{k} \right) \frac{\sigma}{R^2} \right. \\
&\quad \left. + \theta'(\sigma) \xi'(\sigma) \frac{2\sigma}{kR} + \theta' \left(\frac{\sigma}{k} \right) \xi \left(\frac{\sigma}{R} \right) \frac{2}{k} + \theta \left(\frac{\sigma}{k} \right) \xi' \left(\frac{\sigma}{R} \right) \frac{2}{R} \right) d\sigma.
\end{aligned}$$

Then using that $\theta' \left(\frac{\sigma}{k} \right) = \theta'' \left(\frac{\sigma}{k} \right) = 0$, if $\sigma > 2k$ or $\sigma < k$, and $\xi' \left(\frac{\sigma}{k} \right) = \xi'' \left(\frac{\sigma}{k} \right) = 0$, if $\sigma > 2R \geq 2k$ or $\sigma < R$, $i = 1, 2$, we get

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p \sigma d\sigma + \frac{1}{T_0} \int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p \sigma d\sigma \\
& \leq \frac{C}{k} D(p(t)) \int_{\mathbb{R}} p d\sigma = \frac{C}{k} D(p(t)) \leq \frac{C\alpha}{kT_0}.
\end{aligned}$$

By the Gronwall lemma

$$\int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p \sigma d\sigma \leq e^{-\frac{1}{T_0}t} \int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p_0 \sigma d\sigma + \frac{C\alpha}{kT_0} \int_0^t e^{-\frac{1}{T_0}(t-s)} ds. \quad (10)$$

Now for $p_0 \in B$, a bounded set in $\overline{L^1}(\mathbb{R})$, by $\int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p_0 |\sigma| d\sigma \leq C(B)$, $\rho_k^2(\sigma) \tau_R^2(\sigma) = 1$, if $2k \leq \sigma \leq R$, and $\rho_k^2(\sigma) = 0$ if $\sigma < k$, we obtain

$$\int_{2k \leq \sigma \leq R} p(t, \sigma) \sigma d\sigma \leq e^{-\frac{1}{T_0}t} C(B) + \frac{C\alpha}{k},$$

so that for any $\varepsilon > 0$ there exist $T(\varepsilon, B)$ and $\bar{k}(\varepsilon)$ such that

$$\int_{2k \leq \sigma \leq R} p(t, \sigma) \sigma d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \bar{k},$$

where $R \geq 2k$ is arbitrary. Passing to the limit as $R \rightarrow +\infty$ we have that

$$\int_{2k \leq \sigma} p(t, \sigma) \sigma d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \bar{k}. \quad (11)$$

Also, for $k = 1$, $M(B) = C(B) + C\alpha$ it follows $\int_{2 \leq \sigma} p(t, \sigma) \sigma d\sigma \leq M$, and then

$$\int_{0 \leq \sigma} p(t, \sigma) \sigma d\sigma \leq M + 2, \text{ if } t \geq 0. \quad (12)$$

If $p_0^\alpha \rightarrow p_0$ in X_w , where $p_0^\alpha, p_0 \geq 0$, then for any $\varepsilon > 0$ one can find $k_1(\varepsilon)$, $\Lambda(\varepsilon)$ such that

$$\int_{\mathbb{R}} \rho_{k_1}^2(\sigma) p_0^\alpha (1 + |\sigma|) d\sigma \leq \frac{\varepsilon}{2}, \text{ for all } \alpha \geq \Lambda. \quad (13)$$

Indeed, take $k_1(\varepsilon)$ such that $\int_{\mathbb{R}} \rho_{k_1}^2(\sigma) p_0 (1 + |\sigma|) d\sigma \leq \frac{\varepsilon}{4}$. Since $\rho_{k_1}^2(\sigma) \in L^\infty(\mathbb{R})$, the convergence $p_0^\alpha \rightarrow p_0$ in X_w implies the existence of $\Lambda(\varepsilon) = \overline{\Lambda}(\varepsilon, k_1(\varepsilon))$ such that

$$\left| \int_{\mathbb{R}} \rho_{k_1}^2(\sigma) p_0^\alpha (1 + |\sigma|) d\sigma - \int_{\mathbb{R}} \rho_{k_1}^2(\sigma) p_0 (1 + |\sigma|) d\sigma \right| < \frac{\varepsilon}{4}, \text{ if } \alpha \geq \Lambda.$$

Hence, (13) follows. Then by (10) there exist $\bar{k}(\varepsilon)$, $\Lambda(\varepsilon)$ such that

$$\int_{2k \leq \sigma \leq R} p_\alpha(t, \sigma) \sigma d\sigma \leq \varepsilon \text{ if } t \geq 0, k \geq \bar{k}, \alpha \geq \Lambda,$$

and passing to the limit as $R \rightarrow +\infty$ we get

$$\int_{2k \leq \sigma} p(t, \sigma) \sigma d\sigma \leq \varepsilon \text{ if } t \geq 0, k \geq \bar{k}, \alpha \succeq \Lambda. \quad (14)$$

Let now $\rho_k(\sigma) = \sqrt{\theta\left(\frac{-\sigma}{k}\right)}$, $k \geq 1$, $\tau_R(\sigma) = \sqrt{\xi\left(\frac{-\sigma}{R}\right)}$, $R \geq 2k$. Multiplying (3) by $-\rho_k^2(\sigma) \tau_R^2(\sigma) \sigma$ and arguing as before we have

$$\int_{\mathbb{R}} (-\sigma) \rho_k^2(\sigma) \tau_R^2(\sigma) p d\sigma \leq e^{-\frac{1}{T_0}t} \int_{\mathbb{R}} (-\sigma) \rho_k^2(\sigma) \tau_R^2(\sigma) p_0 d\sigma + \frac{C\alpha}{kT_0} \int_0^t e^{-\frac{1}{T_0}(t-s)} ds.$$

Now for $p_0 \in B$ (a bounded set in $\bar{L}^1(\mathbb{R})$), using $\int_{\mathbb{R}} \rho_k^2(\sigma) \tau_R^2(\sigma) p_0 |\sigma| d\sigma \leq C(B)$, $\rho_k^2(\sigma) \tau_R^2(\sigma) = 1$, if $2k \leq (-\sigma) \leq R$, and $\rho_k^2(\sigma) = 0$ if $\sigma > -k$, we obtain

$$\int_{2k \leq (-\sigma) \leq R} p(t, \sigma) (-\sigma) d\sigma \leq e^{-\frac{1}{T_0}t} C(B) + \frac{C\alpha}{k},$$

so that for any $\varepsilon > 0$ there exist $T(\varepsilon, B)$ and $\bar{k}(\varepsilon)$ such that

$$\int_{2k \leq (-\sigma) \leq R} p(t, \sigma) (-\sigma) d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \bar{k},$$

where $R \geq 2k$ is arbitrary. Passing to the limit as $R \rightarrow +\infty$ we have that

$$\int_{2k \leq (-\sigma)} p(t, \sigma) (-\sigma) d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \bar{k}. \quad (15)$$

Also, for $k = 1$, $M(B) = C(B) + C\alpha$ it follows $\int_{2 \leq (-\sigma)} p(t, \sigma) (-\sigma) d\sigma \leq M$, and then

$$\int_{0 \leq (-\sigma)} p(t, \sigma) (-\sigma) d\sigma \leq M + 2, \text{ if } t \geq 0. \quad (16)$$

If $p_0^\alpha \rightarrow p_0$ in X_w , where $p_0^\alpha, p_0 \geq 0$, then arguing as in the case of inequality (14) we obtain the existence of $\bar{k}(\varepsilon)$, $\Lambda(\varepsilon)$ such that

$$\int_{2k \leq (-\sigma)} p(t, \sigma) \sigma d\sigma \leq \varepsilon \text{ if } t \geq 0, k \geq k, \alpha \succeq \Lambda. \quad (17)$$

Joining (11), (12), (14), (15), (16), (17) we obtain (6)-(8). \square

Remark 2. For the main result of this paper (see Theorem 4.4) it is important that the constant $\bar{k}(\varepsilon)$ in (6) does not depend on the set B .

Lemma 2.3. *Every weak solution p of (3) with initial data $p_0 \in X$, $p_0 \geq 0$, and such that $p(t) \geq 0$, $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$ for all $t \geq 0$, belongs to $C([0, T], X_w)$.*

Proof. We know already that $p \in C([0, T], L_w^2(\mathbb{R}))$. It remains to prove the continuity in the space $\bar{L}_w^1(\mathbb{R})$. Indeed, if $t_n \rightarrow t_0$, then using (8) we have that for any $\xi \in L^\infty(\mathbb{R})$, $\varepsilon > 0$ there exist $K(\varepsilon, \xi) \geq 1$ and $N(\varepsilon, K, \xi)$ such that

$$\begin{aligned} & \left| \int_{\mathbb{R}} (p(t_n) - p(t_0)) (1 + |\sigma|) \xi d\sigma \right| \\ & \leq \left| \int_{|\sigma| \leq K} (p(t_n) - p(t_0)) (1 + |\sigma|) \xi d\sigma \right| \\ & \quad + 2 \|\xi\|_{L^\infty} \int_{|\sigma| > K} |p(t_n)| |\sigma| d\sigma \end{aligned}$$

$$\begin{aligned}
& + 2 \|\xi\|_{L^\infty} \int_{|\sigma| > K} |p(t_0)| |\sigma| d\sigma \\
& \leq 3\varepsilon, \text{ for all } n \geq N.
\end{aligned}$$

□

Let H be the closure in X_w of the set

$$E = \left\{ p \in X : p \in L^\infty(\mathbb{R}), p \geq 0, \int_{\mathbb{R}} p d\sigma = 1 \text{ and } D(p) > 0 \right\}. \quad (18)$$

The set E is convex, which implies that H coincides with the closure of E in X . Then for any $p \in H$ one can find a sequence $p_n \in E$ converging to p strongly in X . Hence $p_n \rightarrow p$ in $L^1(\mathbb{R})$, so that

$$p \geq 0, \int_{\mathbb{R}} p d\sigma = 1, \text{ for all } p \in H. \quad (19)$$

Then every element of H is a probability density.

As we have seen before for every initial condition $p_0 \in E$ there exists a unique globally defined weak solution $p(t)$ such that $p \in C([0, T], L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$, $p \in C([0, T], X_w)$, $p(t) \geq 0$, $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$, $D(p(t)) > 0$, for all $T > 0$ and $t \geq 0$. Also, $p : [0, T] \rightarrow L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is weakly continuous for all $T > 0$, so that $p(t) \in L^\infty(\mathbb{R}) \cap X$ for every $t \geq 0$. Hence, $p(t) \in E$ for any $t \geq 0$.

H is a topological Hausdorff space endowed with the weak topology of X .

We shall prove now some previous estimates.

Lemma 2.4. *Let $p_0 \in E$ be such that $\|p_0\|_{L^2} \leq R$. Then for any $T > 0$ there exists $C(R, T)$ such that the unique weak solution to (3) such that $p(0) = p_0$ satisfies*

$$\begin{aligned}
\|p(t)\|_{L^2} & \leq C(R, T), \text{ for all } t \in [0, T], \\
\|D^{\frac{1}{2}}(p)p\|_{L^2(0, T; H^1)} & \leq C(R, T), \\
\|D(p)p\|_{L^2(0, T; H^1)} & \leq C(R, T), \\
\|\partial_t p\|_{L^2(0, T; H^{-1})} & \leq C(R, T).
\end{aligned} \quad (20)$$

Proof. Fix $T > 0$. We note that $\partial_t p \in L^2(0, T; H^{-1}(\mathbb{R}))$, $p \in L^2(0, T; H^1(\mathbb{R}))$ imply $\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} p^2 d\sigma = \langle \partial_t p, p \rangle$ [34, p.261]. Multiplying (3) by p we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} p^2 d\sigma + D(p(t)) \int_{\mathbb{R}} (\partial_\sigma p)^2 d\sigma \\
& \leq \frac{D(p(t))}{\alpha} p(t, 0) \leq \frac{D(p(t))}{\alpha} \|p(t)\|_{L^\infty(\mathbb{R})} \\
& \leq c \frac{D(p(t))}{\alpha} \left(\int_{\mathbb{R}} (p^2 + (\partial_\sigma p)^2) d\sigma \right)^{\frac{1}{2}} \\
& \leq \frac{D(p(t))}{2} \left(\int_{\mathbb{R}} p^2 d\sigma + \int_{\mathbb{R}} (\partial_\sigma p)^2 d\sigma + \frac{c^2}{\alpha^2} \right),
\end{aligned} \quad (21)$$

where we have used the continuous embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Hence

$$\frac{d}{dt} \int_{\mathbb{R}} p^2 d\sigma + D(p(t)) \int_{\mathbb{R}} (\partial_\sigma p)^2 d\sigma \leq D(p(t)) \left(\int_{\mathbb{R}} p^2 d\sigma + \frac{c^2}{\alpha^2} \right). \quad (22)$$

By the Gronwall lemma and $0 \leq D(p(t)) \leq \frac{\alpha}{T_0}$ we obtain

$$\int_{\mathbb{R}} p^2 d\sigma \leq C(R, T), \text{ for all } t \in [0, T]. \quad (23)$$

Also, note that (22) implies

$$\int_0^T \int_{\mathbb{R}} \left(\partial_\sigma \left((D(p))^{\frac{1}{2}} p \right) \right)^2 d\sigma dt \leq \int_{\mathbb{R}} (p_0)^2 d\sigma + \frac{T\alpha}{T_0} \left(C(R, T) + \frac{c^2}{\alpha^2} \right).$$

Also,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} (\partial_\sigma (D(p)p))^2 d\sigma dt &\leq \int_0^T D(p) \int_{\mathbb{R}} \left(\partial_\sigma \left((D(p))^{\frac{1}{2}} p \right) \right)^2 d\sigma dt \\ &\leq \frac{\alpha}{T_0} \left(\int_{\mathbb{R}} (p_0)^2 d\sigma + \frac{T\alpha}{T_0} \left(C(R, T) + \frac{c^2}{\alpha^2} \right) \right). \end{aligned}$$

From these inequalities (20) follows. \square

Let Z be a Banach space with its dual denoted by Z^* and with pairing between Z and Z^* denoted by $\langle \cdot, \cdot \rangle_{Z^*}$. Let Z_w be the space Z endowed with the weak topology. For $\{u_n\} \subset C([0, T], Z_w)$ the convergence $u_n \rightarrow u$ in $C([0, T], Z_w)$ will mean that for any $v \in Z^*$, $\sup_{t \in [0, T]} |\langle u_n(t) - u(t), v \rangle_{Z^*}| \rightarrow 0$, as $n \rightarrow \infty$.

Lemma 2.5. *Let $T > 0$, $p_0 \in H$, $p_0^\alpha \in E$, $p_0^\alpha \rightarrow p_0$ in H and let $p_\alpha(\cdot)$ be the generalized sequence of weak solutions corresponding to p_0^α . Then there exists a generalized subsequence (denoted again $p_\alpha(\cdot)$) converging to some function $p(\cdot)$ in $C([0, T], H)$. Moreover, p is a weak solution of (3), $p(0) = p_0$, $\int_{\mathbb{R}} p(t) dt = 1$, $p(t) \geq 0$, for all $t \geq 0$, and (20) holds, where R is such that $\|p_0^\alpha\| \leq R$ for all α .*

Proof. From Lemma 2.4 it follows that p_α is bounded in $L^\infty(0, T; L^2(\mathbb{R}))$, $D(p_\alpha)p_\alpha$ is bounded in $L^2(0, T; H^1(\mathbb{R}))$ and $\partial_t p_\alpha$ is bounded in $L^2(0, T; H^{-1}(\mathbb{R}))$. It follows the existence of a generalized subsequence such that

$$p_\alpha \rightarrow p \text{ weakly star in } L^\infty(0, T; L^2(\mathbb{R})), \quad (24)$$

$$D(p_\alpha)p_\alpha \rightarrow \xi \text{ weakly in } L^2(0, T; H^1(\mathbb{R})), \quad (25)$$

$$\partial_t p_\alpha \rightarrow \partial_t p \text{ weakly in } L^2(0, T; H^{-1}(\mathbb{R})). \quad (26)$$

For $k \in \mathbb{N}$, let $B_k = (-k, k)$. Denote by $L_k p$ the restriction of $p(\sigma)$ onto $(-k, k)$. The compact embedding $L^2(B_k) \subset H^{-1}(B_k)$ implies that for any $t \in [0, T]$ the sequence $\{L_k p_n(t)\}$ is precompact in $H^{-1}(B_k)$, the dual space to $H_0^1(B_k)$. Also, by using the previous convergences we obtain that $\partial_t L_k p_\alpha$ is bounded in $L^2(0, T; H^{-1}(B_k))$ and then the family of functions $t \mapsto L_k p_\alpha(t) \in H^{-1}(B_k)$ is equicontinuous on $[0, T]$. By the Ascoli-Arzelà theorem and a standard argument (using the uniqueness of the limit for all converging subsequences) we have that $L_k p_\alpha \rightarrow L_k p$ in $C([0, T], H^{-1}(B_k))$, for all k . Note also that $L_k p \in C([0, T], H^{-1}(B_k)) \cap L^\infty(0, T; L^2(B_k))$ implies that $L_k p$ belongs to the space $C([0, T], L_w^2(B_k))$ [34, p.263]. By a contradiction argument it follows that $L_k p_\alpha \rightarrow L_k p$ in $C([0, T], L_w^2(B_k))$. Indeed, if we assume the opposite, then there would exist $v \in L^2(\mathbb{R})$ such that $\sup_{t \in [0, T]} |(L_k p_\alpha(t) - L_k p(t), v)| \not\rightarrow 0$. Then we could find $t_0, \varepsilon > 0$ and generalized subsequences $L_k p_\alpha(t_\alpha), t_\alpha \rightarrow t_0$, such that

$$|(L_k p_\alpha(t_\alpha) - L_k p(t_\alpha), v)| > \varepsilon, \text{ for all } \alpha.$$

But from $\{L_k p_\alpha(t_\alpha)\}$ one can extract subsequences converging weakly in $L^2(B_k)$ to $L_k p(t_0)$, which is a contradiction. Again, by [34, p.263] $p \in C([0, T], H^{-1}(\mathbb{R}))$ and $p \in L^\infty(0, T; L^2(\mathbb{R}))$ imply that $p \in C([0, T], L^2_w(\mathbb{R}))$. Now, since p_α satisfy (20), (24) implies that $\|p(t)\| \leq C(R, T)$, for a.a. t . Then $p \in C([0, T], L^2_w(\mathbb{R}))$ implies that p satisfies (20). Since $k \in \mathbb{N}$ is arbitrary one can show that $p_\alpha \rightarrow p$ in $C([0, T], L^2_w(\mathbb{R}))$. Indeed, take an arbitrary $v \in L^2(\mathbb{R})$. Then for any $\varepsilon > 0$ there exist $k(\varepsilon, v) \geq 1, \Lambda(\varepsilon, k, v)$, such that

$$\begin{aligned} \sup_{t \in [0, T]} \left| \int_{\mathbb{R}} (p_\alpha(t) - p(t)) v d\sigma \right| &\leq \sup_{t \in [0, T]} \left| \int_{|\sigma| \leq k} (p_\alpha(t) - p(t)) v d\sigma \right| \\ &+ \sup_{t \in [0, T]} (\|p_\alpha(t)\|_{L^2} + \|p(t)\|_{L^2}) \left(\int_{|\sigma| > k} v^2 d\sigma \right)^{\frac{1}{2}} \leq 3\varepsilon, \text{ if } \alpha \succeq \Lambda(\varepsilon, k, v), \end{aligned}$$

where we have used inequality (20) for p_α and p . We note that $p(t)$ belongs to the weak closure of the set E in $L^2(\mathbb{R})$. Since E is convex, it belongs also to the strong closure. Hence, $p(t) \geq 0$ for any $t \geq 0$. We need also to check that $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$. Indeed, we know that $L_k p_\alpha \rightarrow L_k p$ in $C([0, T], L^2_w(B_k))$, so that $\int_{|\sigma| \leq k} p_\alpha(t) d\sigma \rightarrow \int_{|\sigma| \leq k} p(t) d\sigma$ for any k and t . By Lemma 2.2 for an arbitrary $\varepsilon > 0$ one can choose $K(\varepsilon)$ such that

$$1 - \varepsilon \leq \int_{|\sigma| \leq k} p_\alpha(t) d\sigma \leq 1, \forall \alpha, k \geq K.$$

Passing to the limit we obtain

$$1 - \varepsilon \leq \int_{|\sigma| \leq k} p(t) d\sigma \leq 1.$$

Thus $\int_{\mathbb{R}} p(t, \sigma) d\sigma = 1$. It follows from $L_1 p_\alpha \rightarrow L_1 p$ in $C([0, T], L^2_w(B_1))$ that

$$\begin{aligned} &\sup_{t \in [0, T]} \left| \int_{|\sigma| > 1} p_\alpha(\sigma, t) d\sigma - \int_{|\sigma| > 1} p(\sigma, t) d\sigma \right| \\ &= \sup_{t \in [0, T]} \left| \int_{|\sigma| \leq 1} p_\alpha(\sigma, t) d\sigma - \int_{|\sigma| \leq 1} p(\sigma, t) d\sigma \right| \rightarrow 0, \end{aligned}$$

so that $D(p_\alpha) \rightarrow D(p)$ in $C([0, T])$. Since $p_\alpha \rightarrow p$ weakly in $L^2(0, T; L^2(\mathbb{R}))$ and $D(p_\alpha) \rightarrow D(p)$ in $C([0, T])$, we have $D(p_\alpha) p_\alpha \rightarrow D(p) p$ weakly in $L^2(0, T; L^2(\mathbb{R}))$. Then $\xi = D(p) p$. Passing to the limit it follows that p satisfies (5), that is, p is a weak solution. Finally, we prove that $p_\alpha \rightarrow p$ in $C([0, T], \overline{L^1}_w(\mathbb{R}))$. Take an arbitrary $\xi \in L^\infty(\mathbb{R})$. Then by the estimate of the tails given in Lemma 2.2 for any $\varepsilon > 0$ there exist $K(\varepsilon) \geq 1, \Lambda(\varepsilon, K, \xi)$, such that

$$\begin{aligned} \sup_{t \in [0, T]} \left| \int_{\mathbb{R}} (1 + |\sigma|) (p_\alpha(t) - p(t)) \xi d\sigma \right| &\leq \sup_{t \in [0, T]} \left| \int_{|\sigma| \leq K} (1 + |\sigma|) (p_\alpha(t) - p(t)) \xi d\sigma \right| \\ &+ 2 \|\xi\|_{L^\infty} \sup_{t \in [0, T]} \left(\int_{|\sigma| > K} |\sigma| p_\alpha(t) d\sigma + \int_{|\sigma| > K} |\sigma| p(t) d\sigma \right) \leq 3\varepsilon, \text{ if } \alpha \succeq \Lambda(\varepsilon, K, \xi). \end{aligned}$$

Hence, $p_\alpha \rightarrow p$ in $C([0, T], H)$. It is clear that $p(0) = p_0$. □

Remark 3. Since for any $p_0 \in H$ a sequence $p_0^n \in E$ converging to p_0 in H exists, Lemma 2.5 implies that for every $T > 0$, $p_0 \in H$ there exists at least one weak solution, although, generally speaking, it can be non-unique. It is also clear that $p(t) \in H$ for every $t \geq 0$ (and then $p(\cdot)$ satisfies (19)).

Also, Lemma 2.5 implies that for any $p_0 \in H$ there exist $p_0^\alpha \in E$, $p_0^\alpha \rightarrow p_0$ in H , and weak solutions $p_\alpha(\cdot)$, $p(\cdot)$ satisfying $p_\alpha(0) = p_0^\alpha$, $p(0) = p_0$, for any $t \geq 0$, such that $p_\alpha \rightarrow p$ in $C([0, T], H)$.

3. A priori estimates. In this section we shall obtain a uniform (for $t \geq 0$) estimate in the space X of the weak solutions with initial condition $p_0 \in E$. We shall prove that the weak solutions starting at $p_0 \in B \subset E$, a bounded set of X , are uniformly bounded in X for $t \geq 0$.

We need first two technical lemmas.

Lemma 3.1. *If $u \in H^1(\mathbb{R})$ and $\rho \in W^{1,\infty}(\mathbb{R})$, then $u\rho \in H^1(\mathbb{R})$ and $\frac{d}{d\sigma}(u\rho) = \frac{du}{d\sigma}\rho + \frac{d\rho}{d\sigma}u$.*

Proof. First, it is clear that $u\rho \in L^2(\mathbb{R})$. Let $k > 0$ be arbitrary and let $B_k = (-k, k)$. If we restrict the functions u, ρ to the interval B_k , then it is clear that $u, \rho \in H^1(B_k)$. We take sequences $u_n, \rho_n \in C_0^\infty(\mathbb{R})$ such that $u_n \rightarrow u$, $\rho_n \rightarrow \rho$ in $H^1(B_k)$ [7, p.127]. For any $\phi \in C_0^\infty(B_k)$ we have

$$\begin{aligned} - \left\langle \frac{d}{d\sigma}(u_n\rho_n), \phi \right\rangle_{\mathcal{D}'(B_k)} &= \int_{-k}^k u_n\rho_n \frac{d\phi}{d\sigma} d\sigma \\ &= - \int_{-k}^k \frac{d}{d\sigma}(u_n\rho_n) \phi d\sigma = - \int_{-k}^k \left(\frac{du_n}{d\sigma}\rho_n + \frac{d\rho_n}{d\sigma}u_n \right) \phi d\sigma, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}'(B_k)}$ denotes pairing in the sense of the scalar distributions in B_k . It is easy to prove that $\frac{du_n}{d\sigma}\rho_n + \frac{d\rho_n}{d\sigma}u_n$ converges to $\frac{du}{d\sigma}\rho + \frac{d\rho}{d\sigma}u$ in the sense of the scalar distributions in B_k . Passing to the limit we obtain:

$$\frac{d}{d\sigma}(u\rho) = \frac{du}{d\sigma}\rho + \frac{d\rho}{d\sigma}u$$

in the sense of the scalar distributions in B_k . Since k is arbitrary, it is clear that

$$\frac{d}{d\sigma}(u\rho) = \frac{du}{d\sigma}\rho + \frac{d\rho}{d\sigma}u \tag{27}$$

in the sense of the scalar distributions $\mathcal{D}'(\mathbb{R})$. Finally, we need to prove that $u\rho \in H^1(\mathbb{R})$. From (27) and $\rho, \frac{d\rho}{d\sigma} \in L^\infty(\mathbb{R})$, $u, \frac{du}{d\sigma} \in L^2(\mathbb{R})$, we deduce that $\frac{d}{d\sigma}(u\rho) \in L^2(\mathbb{R})$, which concludes the proof. \square

Lemma 3.2. *Let $\rho : \mathbb{R} \rightarrow \mathbb{R}^+$ be such that $\rho \in W^{1,\infty}(\mathbb{R})$. If a function u belongs to $L^2(0, T; H^1(\mathbb{R}))$, and its derivative $\frac{du}{dt}$ belongs to $L^2(0, T; H^{-1}(\mathbb{R}))$, then*

$\left\| \rho^{\frac{1}{2}}u(t) \right\|_{L^2}^2$ is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt} \left\| \rho^{\frac{1}{2}}u \right\|_{L^2}^2 = 2 \left\langle \frac{du}{dt}, \rho u \right\rangle, \text{ for a.a. } t \in (0, T). \tag{28}$$

Proof. By regularizing we obtain a sequence of functions $\{u_m\} \subset C^1([0, T], H^1(\mathbb{R}))$ such that

$$u_m \longrightarrow u \text{ in } L_{loc}^2(0, T; H^1(\mathbb{R})),$$

$$\frac{d}{dt}u_m \longrightarrow \frac{d}{dt}u \text{ in } L^2_{loc}(0, T; H^{-1}(\mathbb{R})),$$

as $m \rightarrow \infty$. Since $\rho \in W^{1, \infty}(\mathbb{R})$, in view of Lemma 3.1 we have that $\rho v(t)$ belongs to $H^1(\mathbb{R})$ for a.a. t and $\frac{\partial}{\partial \sigma}(\rho v(t)) = \rho'v(t) + \rho \frac{\partial}{\partial \sigma}v(t)$, where $v = u_m$ or $v = u$. Hence

$$\rho u_m \longrightarrow \rho u \text{ in } L^2_{loc}(0, T; H^1(\mathbb{R})).$$

Also, it is clear that $\rho^{\frac{1}{2}}$ is measurable and then it belongs to $L^\infty(\mathbb{R})$. It follows that $\rho^{\frac{1}{2}}u_m \in C^1([0, T], L^2(\mathbb{R}))$, $\frac{d}{dt}(\rho^{\frac{1}{2}}u_m) = \rho^{\frac{1}{2}}\frac{du_m}{dt}$, and

$$\frac{d}{dt} \left\| \rho^{\frac{1}{2}}u_m(t) \right\|_{L^2}^2 = 2 \left\langle \frac{d}{dt}u_m(t), \rho u_m(t) \right\rangle = 2 \left\langle \frac{d}{dt}u_m(t), \rho u_m(t) \right\rangle. \quad (29)$$

Passing to the limit in (29) in the distribution sense we have the equality

$$\frac{d}{dt} \left\| \rho^{\frac{1}{2}}u \right\|^2 = 2 \left\langle \frac{du}{dt}, \rho u \right\rangle.$$

Since $\left\langle \frac{du}{dt}, \rho u \right\rangle$ belong to $L^1(0, T)$, (28) holds and $\left\| \rho^{\frac{1}{2}}u(t) \right\|_{L^2}^2$ is absolutely continuous on $[0, T]$. \square

Proposition 1. *For any $B \subset E$ bounded in X there exists $N(B)$ such that*

$$\|p(t)\|_X \leq N,$$

for all $t \geq 0$ and any weak solution such that $p(0) = p_0 \in B$.

Proof. Let θ be as in (9). Define $\rho_\eta(\sigma) = \theta\left(\frac{2\sigma^2}{(1+\eta)^2}\right)$, $\eta > 0$. Multiply (3) by $\rho_\eta(\sigma)p$. From $p \in L^2(0, T; H^1(\mathbb{R}))$ and $\partial_t p \in L^2(0, T; H^{-1}(\mathbb{R}))$, we have that $\frac{d}{dt} \int_{\mathbb{R}} \rho_\eta(\sigma) p^2 d\sigma = 2 \langle \partial_t p, \rho_\eta p \rangle$ (see Lemma 3.2). Then we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho_\eta(\sigma) p^2 d\sigma - \langle D(p(t)) \partial_{\sigma\sigma}^2 p, \rho_\eta(\sigma) p \rangle + \frac{1}{T_0} \int_{\mathbb{R}} \rho_\eta(\sigma) p^2 d\sigma \\ &= \frac{D(p(t))}{\alpha} \langle \delta_0(\sigma), \rho_\eta(\sigma) p \rangle = 0. \end{aligned} \quad (30)$$

Consider now the second term of the last expression. Using Lemma 3.1 we have

$$\begin{aligned} & - \langle D(p(t)) \partial_{\sigma\sigma}^2 p, \rho_\eta(\sigma) p \rangle \\ &= D(p(t)) \left(\int_{\mathbb{R}} \rho_\eta(\sigma) (\partial_\sigma p)^2 d\sigma + \int_{\mathbb{R}} \frac{4\sigma}{(1+\eta)^2} \theta' \left(\frac{2\sigma^2}{(1+\eta)^2} \right) p \partial_\sigma p d\sigma \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{4}{(1+\eta)^2} \int_{\mathbb{R}} \sigma \theta' \left(\frac{2\sigma^2}{(1+\eta)^2} \right) p \partial_\sigma p d\sigma \\ &= - \int_{\mathbb{R}} \frac{4\sigma}{(1+\eta)^2} \theta' \left(\frac{2\sigma^2}{(1+\eta)^2} \right) p \partial_\sigma p d\sigma \\ & \quad - \int_{\mathbb{R}} \frac{16\sigma^2}{(1+\eta)^4} \theta'' \left(\frac{2\sigma^2}{(1+\eta)^2} \right) p^2 d\sigma - \int_{\mathbb{R}} \frac{4}{(1+\eta)^2} \theta' \left(\frac{2\sigma^2}{(1+\eta)^2} \right) p^2 d\sigma. \end{aligned}$$

Hence, using $\theta''\left(\frac{2\sigma^2}{(1+\eta)^2}\right) = 0$ for $|\sigma| \geq 1 + \eta$ we obtain

$$-\langle D(p(t)) \partial_{\sigma\sigma}^2 p, \rho_\eta(\sigma) p \rangle \geq D(p(t)) \left(\int_{\mathbb{R}} \rho_\eta(\sigma) (\partial_\sigma p)^2 d\sigma - \frac{10\beta}{(1+\eta)^2} \int_{\mathbb{R}} p^2 d\sigma \right), \quad (31)$$

where $\beta > 0$ is such that $|\theta'|, |\theta''| \leq \beta$. Using (31) in (30) we have that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho_\eta(\sigma) p^2 d\sigma + D(p(t)) \int_{\mathbb{R}} \rho_\eta(\sigma) (\partial_\sigma p)^2 d\sigma + \frac{1}{T_0} \int_{\mathbb{R}} \rho_\eta(\sigma) p^2 d\sigma \\ & \leq \frac{10\beta}{(1+\eta)^2} D(p(t)) \int_{\mathbb{R}} p^2 d\sigma. \end{aligned}$$

By the Gronwall lemma

$$\begin{aligned} & \int_{|\sigma| \geq 1+\eta} p^2(t, \sigma) d\sigma \quad (32) \\ & \leq \int_{\mathbb{R}} \rho_\eta(\sigma) p^2(t, \sigma) d\sigma \\ & \leq e^{-\frac{2}{T_0}t} \int_{\mathbb{R}} p_0^2(\sigma) d\sigma + \int_0^t e^{-\frac{2}{T_0}(t-s)} \frac{20\beta D(p(s))}{(1+\eta)^2} \int_{\mathbb{R}} p^2(s, \sigma) d\sigma ds. \end{aligned}$$

Define now $\xi_\eta(\sigma) = 1 - \theta\left(\frac{\sigma^2}{(1+\eta)^2}\right)$. Multiply (3) by $\xi_\eta^2(\sigma) p$. Then by Lemma 3.2 we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma - \langle D(p(t)) \partial_{\sigma\sigma}^2 p, \xi_\eta^2(\sigma) p \rangle + \frac{1}{T_0} \int_{|\sigma| \geq 1} \xi_\eta^2(\sigma) p^2 d\sigma \quad (33) \\ & = \frac{D(p(t))}{\alpha} \langle \delta_0(\sigma), \xi_\eta^2(\sigma) p \rangle = \frac{D(p(t))}{\alpha} \xi_\eta^2(0) p(t, 0). \end{aligned}$$

Let $I = [(-1 - \eta)\sqrt{2}, (1 + \eta)\sqrt{2}]$. By Lemma 3.1 $\xi_\eta(\cdot) p(t, \cdot) \in H^1(\mathbb{R})$, so that the restriction of $\xi_\eta(\cdot) p(t, \cdot)$ in I belongs to $H_0^1(I) \subset L^\infty(I)$ for a.a. t . Then $\xi_\eta^2(0) = 1 = \xi_\eta(0)$ and the Poincaré inequality imply the existence of C_I such that

$$\begin{aligned} \frac{D(p(t))}{\alpha} \xi_\eta^2(0) p(t, 0) & = \frac{D(p(t))}{\alpha} \xi_\eta(0) p(t, 0) \\ & \leq \frac{D(p(t))}{\alpha} \|\xi_\eta(\cdot) p(t, \cdot)\|_{L^\infty(I)} \\ & \leq \frac{D(p(t))}{\alpha} C_I \|\partial_\sigma(\xi_\eta(\cdot) p(t, \cdot))\|_{L^2(I)} \\ & \leq \frac{D(p(t))}{2} \int_{\mathbb{R}} (\partial_\sigma(\xi_\eta(\sigma) p))^2 d\sigma + \frac{D(p(t)) C_I^2}{2\alpha^2}. \quad (34) \end{aligned}$$

On the other hand, using again Lemma 3.1, we get

$$\begin{aligned}
& - \langle D(p(t)) \partial_{\sigma\sigma}^2 p, \xi_\eta^2(\sigma) p \rangle \\
& = D(p(t)) \int_{\mathbb{R}} (\partial_\sigma p) \partial_\sigma (\xi_\eta^2(\sigma) p) d\sigma \\
& = D(p(t)) \left(\int_{\mathbb{R}} \xi_\eta'(\sigma) \xi_\eta(\sigma) p (\partial_\sigma p) d\sigma + \int_{\mathbb{R}} \xi_\eta(\sigma) (\partial_\sigma p) \partial_\sigma (\xi_\eta(\sigma) p) d\sigma \right) \\
& = D(p(t)) \left(\int_{\mathbb{R}} \xi_\eta'(\sigma) \xi_\eta(\sigma) p (\partial_\sigma p) d\sigma + \int_{\mathbb{R}} (\partial_\sigma (\xi_\eta(\sigma) p))^2 d\sigma \right. \\
& \quad \left. - \int_{\mathbb{R}} p \xi_\eta'(\sigma) \partial_\sigma (\xi_\eta(\sigma) p) d\sigma \right) \\
& = D(p(t)) \left(\int_{\mathbb{R}} (\partial_\sigma (\xi_\eta(\sigma) p))^2 d\sigma - \int_{\mathbb{R}} p^2 (\xi_\eta'(\sigma))^2 d\sigma \right). \tag{35}
\end{aligned}$$

Using (34)-(35) in (33) we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma + \frac{D(p(t))}{2} \int_{\mathbb{R}} (\partial_\sigma (\xi_\eta(\sigma) p))^2 d\sigma \\
& \leq D(p(t)) \int_{\mathbb{R}} (\xi_\eta'(\sigma))^2 p^2 d\sigma + \frac{D(p(t)) C_I^2}{2\alpha^2}.
\end{aligned}$$

Note that by the Poincaré inequality there exists a positive constant $R_I = R_I(\eta)$ such that

$$\begin{aligned}
& R_I \int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma \\
& = R_I \int_I \xi_\eta^2(\sigma) p^2 d\sigma \leq \int_I (\partial_\sigma (\xi_\eta(\sigma) p))^2 d\sigma \leq \int_{\mathbb{R}} (\partial_\sigma (\xi_\eta(\sigma) p))^2 d\sigma.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma + D(p(t)) R_I \int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma \\
& \leq 2D(p(t)) \int_{\mathbb{R}} (\xi_\eta'(\sigma))^2 p^2 d\sigma + \frac{D(p(t)) C_I^2}{\alpha^2}.
\end{aligned}$$

Multiplying the inequality by $e^{R_I \int_0^r D(p(s)) ds}$ and integrating over $(0, t)$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma & \leq e^{-R_I \int_0^t D(p(s)) ds} \int_{\mathbb{R}} \xi_\eta^2(\sigma) p_0^2 d\sigma \\
& \quad + \int_0^t 2D(p(r)) \int_{\mathbb{R}} (\xi_\eta'(\sigma))^2 p^2 d\sigma e^{-R_I \int_r^t D(p(s)) ds} dr \\
& \quad + \frac{C_I^2}{\alpha^2} \int_0^t D(p(r)) e^{-R_I \int_r^t D(p(s)) ds} dr.
\end{aligned}$$

Hence,

$$\begin{aligned} \int_{|\sigma| \leq 1+\eta} p^2(t, \sigma) d\sigma &\leq \int_{\mathbb{R}} \xi_\eta^2(\sigma) p^2 d\sigma \\ &\leq \left(\int_{\mathbb{R}} p_0^2 d\sigma \right) e^{-R_I \int_0^t D(p(s)) ds} + \frac{C_I^2}{\alpha^2 R_I} \left(1 - e^{-R_I \int_0^t D(p(s)) ds} \right) \\ &\quad + \int_0^t 2D(p(r)) \int_{\mathbb{R}} (\xi'_\eta(\sigma))^2 p^2 d\sigma e^{-R_I \int_r^t D(p(s)) ds} dr. \end{aligned} \quad (36)$$

Joining (32) and (36) we obtain

$$\begin{aligned} &\int_{\mathbb{R}} p^2(t, \sigma) d\sigma \\ &\leq \left(\int_{\mathbb{R}} p_0^2(\sigma) d\sigma \right) \left(e^{-R_I \int_0^t D(p(s)) ds} + e^{-\frac{2t}{T_0}} \right) \\ &\quad + \frac{C_I^2}{\alpha^2 R_I} \left(1 - e^{-R_I \int_0^t D(p(s)) ds} \right) + \int_0^t e^{-\frac{2}{T_0}(t-s)} \frac{20\beta D(p(s))}{(1+\eta)^2} \int_{\mathbb{R}} p^2(s, \sigma) d\sigma ds \\ &\quad + \int_0^t 2D(p(r)) \int_{\mathbb{R}} (\xi'_\eta(\sigma))^2 p^2 d\sigma e^{-R_I \int_r^t D(p(s)) ds} dr. \end{aligned} \quad (37)$$

Let us estimate now the last term in (37). We multiply (3) by $(\xi'_\eta)^2 p$. Then arguing as for (35) and using $\xi'_\eta(\sigma) = 0$, for $|\sigma| \leq 1$, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\xi'_\eta(\sigma))^2 p^2 d\sigma + D(p(t)) \int_{\mathbb{R}} (\partial_\sigma (\xi'_\eta(\sigma) p))^2 d\sigma + \frac{1}{T_0} \int_{\mathbb{R}} (\xi'_\eta(\sigma))^2 p^2 d\sigma \\ &\leq D(p(t)) \int_{\mathbb{R}} (\xi''_\eta(\sigma))^2 p^2 d\sigma. \end{aligned}$$

Therefore, since

$$|\xi'_\eta(\sigma)| \leq \frac{2|\sigma|}{(1+\eta)^2} \left| \theta' \left(\frac{\sigma^2}{(1+\eta)^2} \right) \right| \leq \frac{2\sqrt{2}\beta}{1+\eta},$$

we get

$$\begin{aligned} \int_{\mathbb{R}} (\xi'_\eta(\sigma))^2 p^2(t, \sigma) d\sigma &\leq \frac{8\beta^2}{(1+\eta)^2} e^{-\frac{2}{T_0}t} \int_{\mathbb{R}} p_0^2 dx \\ &\quad + 2 \int_0^t e^{-\frac{2}{T_0}(t-s)} D(p(s)) \int_{\mathbb{R}} (\xi''_\eta(\sigma))^2 p^2 d\sigma ds. \end{aligned} \quad (38)$$

Note that $\theta'' \left(\frac{\sigma^2}{(1+\eta)^2} \right) = 0$, if $\sigma^2 > 2(1+\eta)^2$, implies

$$|\xi''_\eta(\sigma)| \leq \frac{4\sigma^2}{(1+\eta)^4} \left| \theta'' \left(\frac{\sigma^2}{(1+\eta)^2} \right) \right| + \frac{2}{(1+\eta)^2} \left| \theta' \left(\frac{\sigma^2}{(1+\eta)^2} \right) \right| \leq \frac{10\beta}{(1+\eta)^2}. \quad (39)$$

By (37), (38) and (39) we have

$$\begin{aligned}
& \int_{\mathbb{R}} p^2(t, \sigma) d\sigma \tag{40} \\
& \leq \left(\int_{\mathbb{R}} p_0^2(\sigma) d\sigma \right) \left(e^{-R_I \int_0^t D(p(s)) ds} + e^{-\frac{2}{T_0} t} \right) \\
& \quad + \frac{C_I^2}{\alpha^2 R_I} \left(1 - e^{-R_I \int_0^t D(p(s)) ds} \right) + \int_0^t e^{-\frac{2}{T_0}(t-s)} \frac{20\beta D(p(s))}{(1+\eta)^2} \int_{\mathbb{R}} p^2(s, \sigma) d\sigma ds \\
& \quad + \int_0^t D(p(r)) \left(\frac{16\beta^2}{(1+\eta)^2} e^{-\frac{2}{T_0} r} \int_{\mathbb{R}} p_0^2 d\sigma \right) e^{-R_I \int_r^t D(p(s)) ds} dr \\
& \quad + \int_0^t D(p(r)) \left(\frac{400\beta^2}{(1+\eta)^4} \int_0^r e^{-\frac{2}{T_0}(r-s)} D(p(s)) \int_{\mathbb{R}} p^2 d\sigma ds \right) e^{-R_I \int_r^t D(p(s)) ds} dr.
\end{aligned}$$

For further arguments we need to give an explicit expression for the function $R_I(\eta)$. Recall that R_I has to satisfy the inequality $R_I \int_I u^2 d\sigma \leq \int_I (\partial_\sigma u)^2 d\sigma$ for any $u \in H_0^1(I)$. Since $I = [(-1-\eta)\sqrt{2}, \sqrt{2}(1+\eta)]$, for any $u \in H_0^1(I)$ we have

$$\begin{aligned}
|u(\sigma)| & \leq \|\partial_\sigma u\|_{L^1(I)} \leq |I|^{\frac{1}{2}} \|\partial_\sigma u\|_{L^2(I)}, \\
\|u\|_{L^2}^2 & \leq |I|^2 \|\partial_\sigma u\|_{L^2(I)}^2,
\end{aligned}$$

where $|I| = 2\sqrt{2}(1+\eta)$, so that

$$\frac{1}{8(1+\eta)^2} \|u\|_{L^2(I)}^2 \leq \|\partial_\sigma u\|_{L^2(I)}^2$$

and we can take $R_I = \frac{1}{8(1+\eta)^2}$. Thus we can choose η big enough, so that

$$R_I(1+\eta)^4 - 400\beta^2\alpha = \frac{1}{8}(1+\eta)^2 - 400\beta^2\alpha > 0. \tag{41}$$

Suppose that $\overline{K} > 0$ is such that

$$\frac{C_I^2}{\alpha^2 R_I} + \frac{200\beta^2\alpha}{R_I(1+\eta)^4} \overline{K}^2 = \frac{\overline{K}^2}{2},$$

i.e.

$$\overline{K}^2 = \frac{C_I^2 2(1+\eta)^4}{\alpha^2 (R_I(1+\eta)^4 - 400\beta^2\alpha)}, \tag{42}$$

and that $\|p_0\|_{L^2} < K_0$. Then we state that there exists $K > \overline{K}$ such that

$$\|p(t)\|_{L^2} < K, \text{ for all } t \geq 0. \tag{43}$$

Indeed, this is the case if $K > \max\{\sqrt{2}K_0, \overline{K}\}$. If we assume the opposite, then there exists $t_0 > 0$ such that $\|p(t)\|_{L^2} < K$, for $t < t_0$, and $\|p(t_0)\|_{L^2} = K$. From

(40) and $D(f) \leq \frac{\alpha}{T_0}$ we obtain

$$\begin{aligned} & \|p(t)\|_{L^2}^2 \\ & < K_0^2 \left(e^{-R_I \int_0^t D(p(s)) ds} + e^{-\frac{2}{T_0}t} \right) \\ & \quad + \frac{C_I^2}{\alpha^2 R_I} \left(1 - e^{-R_I \int_0^t D(p(s)) ds} \right) + \frac{10\alpha\beta}{(1+\eta)^2} K^2 \left(1 - e^{-\frac{2}{T_0}t} \right) \\ & \quad + \frac{8\alpha\beta^2}{(1+\eta)^2} K_0^2 \left(1 - e^{-\frac{2}{T_0}t} \right) \\ & \quad + \frac{200\beta^2\alpha}{(1+\eta)^4} K^2 \int_0^t D(p(r)) \left(1 - e^{-\frac{2}{T_0}r} \right) e^{-R_I \int_r^t D(s) ds} dr. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^t D(p(r)) \left(1 - e^{-\frac{2}{T_0}r} \right) e^{-R_I \int_r^t D(p(s)) ds} dr & \leq \int_0^t D(p(r)) e^{-R_I \int_r^t D(p(s)) ds} dr \\ & = \frac{1}{R_I} \left(1 - e^{-R_I \int_0^t D(p(s)) ds} \right). \end{aligned}$$

Now we choose η such that $\frac{10\alpha\beta}{(1+\eta)^2} \leq \frac{1}{4}$ and $\frac{8\alpha\beta^2}{(1+\eta)^2} \leq \frac{1}{2}$. Hence, using $K_0^2 < \frac{K^2}{2}$, we get

$$K_0^2 e^{-\frac{2}{T_0}t} + \frac{10\alpha\beta}{(1+\eta)^2} K^2 \left(1 - e^{-\frac{2}{T_0}t} \right) + \frac{8\alpha\beta^2}{(1+\eta)^2} K_0^2 \left(1 - e^{-\frac{2}{T_0}t} \right) < \frac{K^2}{2}.$$

On the other hand

$$K^2 > \overline{K}^2 = \frac{C_I^2 2(1+\eta)^4}{\alpha^2 (R_I(1+\eta)^4 - 400\beta^2\alpha)}$$

implies

$$\frac{C_I^2}{\alpha^2 R_I} + \frac{200\beta^2\alpha}{R_I(1+\eta)^4} K^2 < \frac{K^2}{2}.$$

Thus

$$K_0^2 e^{-R_I \int_0^t D(p(s)) ds} + \left(\frac{C_I^2}{\alpha^2 R_I} + \frac{200\beta^2\alpha}{R_I(1+\eta)^4} K^2 \right) \left(1 - e^{-R_I \int_0^t D(s,y) ds} \right) < \frac{K^2}{2},$$

so that

$$\|p(t)\|_{L^2} < K, \text{ for all } t \leq t_0,$$

which is a contradiction.

It remains to obtain an estimate in the space $\overline{L}^1(\mathbb{R})$. Fix some $\varepsilon > 0$. Using the estimate of the tail (7) we obtain the existence of $M(B)$ such that for any weak solution $p(\cdot)$ starting at B (note that it satisfies $p(t) \geq 0$ and $\int_{\mathbb{R}} p(\sigma) d\sigma = 1$, for all $t \geq 0$), we have

$$\int_{\mathbb{R}} p(t, \sigma) |\sigma| d\sigma \leq \int_{|\sigma| < 1} p(t, \sigma) d\sigma + \int_{|\sigma| \geq 1} p(t, \sigma) |\sigma| d\sigma \leq 1 + M = \tilde{K}, \text{ for all } t \geq 0.$$

Then the result follows for $N = K^{\frac{1}{2}} + \tilde{K}$. \square

Let us obtain now an estimate of the tails in the norm of the space $L^2(\mathbb{R})$.

Lemma 3.3. *Let $B \subset E$ be bounded in the norm of the space $L^2(\mathbb{R})$. Then for any $\varepsilon > 0$ there exist $T(\varepsilon, B)$, $\bar{k}(\varepsilon, B)$ such that any weak solution p with $p(0) = p_0 \in B$ satisfies the estimate*

$$\int_{k \leq |\sigma|} p^2(t, \sigma) d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \bar{k}.$$

If, moreover, B is compact in $L^2(\mathbb{R})$, then there exists $\bar{k}(\varepsilon, B)$ such that

$$\int_{k \leq |\sigma|} p^2(t, \sigma) d\sigma \leq \varepsilon \text{ if } t \geq 0, k \geq \bar{k}.$$

Proof. We put $k = 1 + \eta$ in (32). Then

$$\begin{aligned} & \int_{|\sigma| \geq k} p^2(t, \sigma) d\sigma & (44) \\ & \leq \int_{\mathbb{R}} \rho_\eta(\sigma) p^2 d\sigma \\ & \leq e^{-\frac{2}{T_0}t} \int_{\mathbb{R}} \rho_\eta(\sigma) p_0^2 d\sigma + \frac{20\beta}{k^2} \int_0^t e^{-\frac{2}{T_0}(t-s)} D(p(s)) \int_{\mathbb{R}} p^2(t, \sigma) d\sigma ds. \end{aligned}$$

Now for $p_0 \in B$ (a bounded set in $L^2(\mathbb{R})$), Proposition 1 implies $\|p(t)\|_{L^2} < N(B)$. Then by $0 \leq D(p(t)) \leq \frac{\alpha}{T_0}$ and $\int_{\mathbb{R}} p_0^2 d\sigma \leq K(B)$ we obtain

$$\int_{|\sigma| \geq k} p^2(t, \sigma) d\sigma \leq e^{-\frac{2}{T_0}t} K(B) + \frac{10\beta N^2(B) \alpha}{k^2},$$

so that for any $\varepsilon > 0$ there exist $T(\varepsilon, B)$ and $\bar{k}(\varepsilon, B)$ such that

$$\int_{|\sigma| \geq k} p^2(t, \sigma) d\sigma \leq \varepsilon \text{ if } t \geq T, k \geq \bar{k}.$$

Let now $p_0 \in B$, where B is compact in $L^2(\mathbb{R})$. Then we can obtain the estimate of the tail for an arbitrary $t \geq 0$. Note that as B is compact, for any $\varepsilon > 0$ we can find $\bar{k}_1(\varepsilon, B)$ such that $\int_{\mathbb{R}} \rho_\eta(\sigma) p_0^2 d\sigma \leq \int_{|\sigma| \geq k/\sqrt{2}} p_0^2(t, \sigma) d\sigma \leq \frac{\varepsilon}{2}$ if $k \geq \bar{k}_1$. Then (44) implies that for any $\varepsilon > 0$ there exists $\bar{k}(\varepsilon, B)$ such that

$$\int_{|\sigma| \geq k} p^2(t, \sigma) d\sigma \leq \varepsilon \text{ if } k \geq \bar{k}, \tag{45}$$

for any $t \geq 0$. □

4. Abstract theory of global attractors. We shall develop in this section a general theory of global attractors for multivalued semiflows defined in a topological space, extending similar results obtained before in metric spaces [27]. The results proved here are almost contained in the more general nonautonomous setting given in [21], [24]. However, we write these results for the sake of completeness, and also in order to explain the particular additional properties obtained in the autonomous case. Moreover, the proof in this case is simpler. Also, due to the special difficulties of our problem (3) we need to consider the attraction property of the attractor only with respect to the elements in a given subset of the phase space. Hence, there are slight differences between the theory developed here and the ones in [21], [24].

Let Y be a metric space with metric ρ , $P(Y)$ ($\beta(Y)$, $C(Y)$, $K(Y)$) be the set of all nonempty (nonempty bounded, nonempty closed, nonempty compact) subsets of the space X . Let F be a Hausdorff topological space. We assume that F is a

subset of Y , i.e. $F \subseteq Y$, where the inclusion has to be understood in the sense of sets (not topologically).

Definition 4.1. The multivalued map $G : \mathbb{R}^+ \times F \rightarrow P(F)$ is called a multivalued semi-flow if the next conditions are satisfied:

1. $G(0, \cdot) = I$ is the identity map;
2. $G(t_1 + t_2, x) \subset G(t_1, G(t_2, x))$, $\forall t_1, t_2 \in \mathbb{R}^+, \forall x \in F$,

where $G(t, B) = \bigcup_{x \in B} G(t, x)$, $B \subset F$.

It is called an strict multivalued semi-flow if the converse inclusion $G(t_1, G(t_2, x)) \subset G(t_1 + t_2, x)$ also holds.

Let $E \subset F$ be a subset of F . First, we give the definition of a (Y, F) -global attractor.

Definition 4.2. The set $A \subset F$ is called (Y, F) -attracting for G if for any $B \in \beta(Y)$, $B \subset E$, and an arbitrary neighborhood $N(A)$ of A in F there exists $T = T(N(A), B)$ such that

$$G(t, B) \subset N(A), \forall t \geq T. \quad (46)$$

This property will be denoted by

$$G(t, B) \rightarrow A, t \rightarrow +\infty. \quad (47)$$

Remark 4. If $Y = F$, then condition (47) should be understood as

$$\text{dist}(G(t, B), A) \rightarrow 0, t \rightarrow +\infty, \quad (48)$$

where for $A, C \subset Y$, $\text{dist}(A, C) = \sup_{x \in A} \inf_{y \in C} \rho(x, y)$, that is, for all $\epsilon > 0$ and $B \in \beta(Y)$ there exists $T = T(\epsilon, B)$ such that $G(t, B) \subset O_\epsilon(A)$, $\forall t \geq T$.

Here $O_\epsilon(A) = \{y \in X : \text{dist}(y, A) < \epsilon\}$. For a compact set $A \subset Y$ properties (46) and (48) are equivalent. In the general case we can say only that (46) implies (48).

Definition 4.3. The set \mathcal{K} is called an (Y, F) -global attractor of G if:

1. \mathcal{K} is an (Y, F) -attracting set;
2. \mathcal{K} is negatively semi-invariant, i.e. $\mathcal{K} \subset G(t, \mathcal{K})$, $\forall t \geq 0$.

Further, for $B \subset E$ let us define $\gamma_s(B) = \bigcup_{t \geq s} G(t, B)$ and the ω -limit set $\omega(B) = \bigcap_{s \geq 0} \text{cl}_F(\gamma_s(B))$, where cl_F is the closure in the space F .

It is known [21, p.1971] that $y \in \omega(B)$ if and only if there exist generalized sequences $t_\alpha \rightarrow +\infty$, $\xi_\alpha \in G(t_\alpha, B)$ such that $\xi_\alpha \rightarrow y$ in F .

Now we prove the existence and properties of the ω -limit set for a bounded set B .

Proposition 2. Assume that for any $B \in \beta(Y)$, $B \subset E$, there exists $A(B) \in \beta(Y) \cap K(F)$ (i.e. it is bounded in Y and compact in F) such that

$$G(t, B) \rightarrow A, \text{ as } t \rightarrow +\infty. \quad (49)$$

Assume also that for any $t \geq 0$ the map $x \mapsto G(t, x)$ has closed graph in F .

Then $\omega(B)$ is non-empty, negatively semi-invariant, compact in F , bounded in Y , $\omega(B) \subset A$ and

$$G(t, B) \rightarrow \omega(B), \text{ as } t \rightarrow +\infty. \quad (50)$$

It is the minimal compact set (in F) satisfying (50). Moreover, if F is regular, then it is the minimal closed set (in F) satisfying (50).

Proof. Let us prove that $\omega(B) \neq \emptyset$. If $\omega(B) = \emptyset$, then by the characterization of $\omega(B)$ each net $\xi_\alpha \in G(t_\alpha, B)$, where $t_\alpha \rightarrow \infty$, has no limit points. Then for any $y \in A(B)$ there exists a neighborhood $O(y)$ in F and $T = T(O(y))$ such that $\{\xi_\alpha\} \cap O(y) = \emptyset, \forall t_\alpha \geq T$. The union of these sets $\{O(y) : y \in A(B)\}$ is an open cover of the compact $A(B)$, so that there exists a finite subcover $\{O(y_k) : k = 1, \dots, n\}$. Then for $t_\alpha \geq \max_{1 \leq k \leq n} T(O(y_k))$ one has $\{\xi_\alpha\} \cap (\bigcup_{k=1}^n O(y_k)) = \emptyset$, which is a contradiction with (49). Hence, $\omega(B) \neq \emptyset$. Let us now prove that $\omega(B)$ is negatively semi-invariant. Let $\xi \in \omega(B)$. Then there exists a generalized sequence $\xi_\alpha \in G(t_\alpha, B)$, such that $\xi_\alpha \rightarrow \xi$ in F , as $t_\alpha \rightarrow +\infty$. For $t_\alpha \geq t$ we have $G(t_\alpha, B) \subset G(t, G(t_\alpha - t, B))$ and, therefore, $\xi_\alpha \in G(t, \zeta_\alpha)$, where $\zeta_\alpha \in G(t_\alpha - t, B)$. As we have seen the net ξ_α has a converging subnet. Thus, without loss of generality we can consider that $\xi_\alpha \rightarrow \xi$ and $\zeta_\alpha \rightarrow \zeta$ in F . Since $G(t, \cdot)$ has closed graph, $\xi \in G(t, \omega(B))$. Finally, $\omega(B) \subset G(t, \omega(B)), \forall t \in \mathbb{R}^+$, because $\xi \in \omega(B)$ and $t \in \mathbb{R}^+$ are arbitrary. Let us prove that $\omega(B) \subset A(B)$. Suppose that there exists $y \in \omega(B)$ such that $y \notin A(B)$. Since A is compact, there exist disjoint neighborhoods $O(A(B))$ and $O(y)$ in F . According to the characterization of $\omega(B)$ there exists a net ξ_α converging to y in F and $\xi_\alpha \in G(t_\alpha, B), t_\alpha \rightarrow +\infty$. This is a contradiction with (49). Now, $\omega(B) \subset A(B)$ implies that $\omega(B)$ is bounded in Y . We prove further that $G(t, B) \rightarrow \omega(B)$ as $t \rightarrow +\infty$. Indeed, if $\omega(B)$ does not attract B , then there exists a neighborhood O of $\omega(B)$ and a net $\xi_\alpha \in G(t_\alpha, B), t_\alpha \rightarrow +\infty$, such that $\xi_\alpha \notin O$ for any t_α . In the same way as before we prove that $\{\xi_\alpha\}$ is precompact, which is a contradiction by the characterization of the ω -limit set. By construction $\omega(B)$ is closed. In view of $\omega(B) \subset A(B)$, we obtain that $\omega(B)$ is compact in F . Let F be a regular topological space and Z be a closed set in F such that

$$G(t, B) \rightarrow Z, \text{ as } t \rightarrow +\infty. \tag{51}$$

Let us prove that $\omega(B) \subset Z$. Indeed, if $y \in \omega(B)$ and $y \notin Z$, then according to the regularity of the space F there exist disjoint neighborhoods $O(Z)$ and $O(y)$. Then it is easy to show a contradiction with (51). In the case when F is not regular but Z is compact the proof is the same. \square

Let us prove now the existence of the global attractor.

Theorem 4.4. *Suppose the following conditions:*

1. *for any $B \in \beta(Y), B \subset E$, there exists $A(B) \in \beta(Y) \cap K(F)$ satisfying (49);*
2. *for any $t \geq 0$ the map $x \mapsto G(t, x)$ has closed graph in F .*

Then there exists an (Y, F) -global attractor \mathcal{K} defined by

$$\mathcal{K} = \bigcup_{B \in \beta(Y), B \subset E} \omega(B). \tag{52}$$

Moreover:

1. *If either $F = E$ or F is a regular space, then $\mathcal{K} \subset cl_F Z$ for every attracting set Z , i.e. it is the minimal closed (Y, F) -attracting set.*
2. *If there exists an (Y, F) -attracting set A such that $A \in \beta(Y) \cap K(F)$, then \mathcal{K} is bounded in Y . If, moreover, $E = F$, then \mathcal{K} is compact in F and, moreover, if G is a strict semiflow, then \mathcal{K} is invariant.*

Proof. The fact that \mathcal{K} is a (Y, F) -global attractor is an easy consequence of Proposition 2. Let Z be a closed set in F such that

$$G(t, B) \rightarrow Z, \text{ as } t \rightarrow +\infty,$$

for all $B \in \beta(Y)$, $B \subset E$. By Proposition 2 we have that $\omega(B)$ is negatively semi-invariant and $\omega(B) \in \beta(Y) \cap K(F)$. If $F = E$, then

$$\omega(B) \subset G(t, \omega(B)) \rightarrow Z,$$

so $\omega(B) \subset cl_F Z$, for all $B \in \beta(Y)$ satisfying $B \subset E$. Hence, $\mathcal{K} \subset cl_F Z$. If F is regular, then by Proposition 2 $\omega(B) \subset cl_F Z$, for all $B \in \beta(Y)$ satisfying $B \subset E$, so that $\mathcal{K} \subset cl_F Z$. Assume now that there exists an attracting set A such that $A \in \beta(Y) \cap K(F)$. It follows from Proposition 2 that $\omega(B) \subset A$, for any $B \in \beta(Y)$, $B \subset F$. Then $\mathcal{K} \subset A$. Thus, $\mathcal{K} \in \beta(Y)$. If $E = F$, we shall obtain that $\mathcal{K} \in K(F)$ if we check that it is closed in F . Take an arbitrary net $\{y_\alpha\} \subset \mathcal{K}$ such that $y_\alpha \rightarrow y$. We have to prove that $y \in \mathcal{K}$. Since $\mathcal{K} \subset G(t, \mathcal{K})$, for any $t \geq 0$, we have $y_\alpha \in G(t_\alpha, \mathcal{K})$ for an arbitrary sequence $t_\alpha \rightarrow +\infty$. Then the boundedness of \mathcal{K} in Y and $\mathcal{K} \subset F = E$ imply that $y \in \omega(\mathcal{K}) \subset \mathcal{K}$. Finally, let us prove that $\mathcal{K} \in \beta(Y) \cap K(F)$ is invariant if G is, moreover, an strict semi-flow. For any $t, \tau \in \mathbb{R}^+$, we have

$$G(t, \mathcal{K}) \subset G(t, G(\tau, \mathcal{K})) \subset G(t + \tau, \mathcal{K}).$$

Since $\mathcal{K} \in \beta(Y)$ and $\mathcal{K} \subset F = E$, for any neighborhood $O(\mathcal{K})$ of \mathcal{K} there exists T such that $G(t + \tau, \mathcal{K}) \subset O(\mathcal{K})$, $\forall \tau \geq T$. Since $O(\mathcal{K})$ is arbitrary and \mathcal{K} is compact in F , we have $G(t, \mathcal{K}) \subset cl_F \mathcal{K} = \mathcal{K}$. Hence, $\mathcal{K} = G(t, \mathcal{K})$, $\forall t \in \mathbb{R}^+$. \square

We say that the multivalued map $x \mapsto G(t, x)$ (t is fixed) is upper semicontinuous (w.r.t. F) if for an arbitrary open (in F) set O containing $G(t, x)$ there exists a neighborhood O_x of x in F such that $G(t, O_x) \subset O$.

Theorem 4.5. *Under the conditions 1-2 of the previous theorem, if, moreover, the map $x \mapsto G(t, x)$ is upper semicontinuous (w.r.t. F), has connected values in F and $\mathcal{K} \subset B_1 \subset E$, $B_1 \in \beta(Y)$, where the set B_1 is connected in F , then the global attractor \mathcal{K} is connected in F .*

Proof. Suppose that \mathcal{K} is not connected in F . Then there exist two open sets A_1, A_2 such that $\mathcal{K} \cap A_1 \neq \emptyset$, $\mathcal{K} \cap A_2 \neq \emptyset$, $\mathcal{K} \subset A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Since the map $G(t, \cdot)$ is upper semicontinuous and has connected values, $G(t, B_1)$ is a connected set in F for any $t \geq 0$. Indeed, if $G(t, B_1)$ were not connected, then there would exist open sets U_1 and U_2 in F with $U_1 \cap U_2 = \emptyset$ such that $G(t, B_1) \cap U_i \neq \emptyset$, $i = 1, 2$, and $G(t, B_1) \subset U_1 \cup U_2$. Denote $M_i = \{x \in B_1 : G(t, x) \subset U_i\}$. Since $G(t, \cdot)$ has connected values, $M_1 \cup M_2 = B_1$. Also, $M_1 \cap M_2 = \emptyset$ and $M_i \neq \emptyset$ for $i = 1, 2$. Since $x \mapsto G(t, x)$ is upper semicontinuous, M_i are open sets for $i = 1, 2$ (see [19, p.37] or [2, p.40]), which contradicts the fact that B_1 is a connected set. By $\mathcal{K} \subset G(t, \mathcal{K}) \subset G(t, B_1)$, we have $G(t, B_1) \cap A_1 \neq \emptyset$, $G(t, B_1) \cap A_2 \neq \emptyset$. But $A_1 \cup A_2$ does not cover $G(t, B_1)$ for any $t \geq 0$. Hence there exist $\xi_\alpha \in G(t_\alpha, B_1)$, where $t_\alpha \rightarrow \infty$, such that $\xi_\alpha \notin A_1 \cup A_2$. Arguing as in the proof of Proposition 2 we obtain that the net $\{\xi_\alpha\}$ has a converging subnet and its limit ξ belongs to $\omega(B_1)$ and does not belong to $A_1 \cup A_2$, which is a contradiction. \square

5. The weak global attractor. Let us defined now, for our problem (3), a multivalued semiflow in the Hausdorff topological space H .

We note that Lemma 2.5 implies that for any $T > 0$, $p_0 \in H$ there exist $p_0^\alpha \in E$, $p_0^\alpha \rightarrow p_0$ in H , a generalized sequence of weak solutions $p_\alpha(\cdot)$ and a weak solution $p(\cdot)$ satisfying $p_\alpha(0) = p_0^\alpha$, $p(0) = p_0$, such that $p_\alpha \rightarrow p$ in $C([0, T], H)$.

Let $S(p_0, T)$ be the set of all weak solutions $p(\cdot)$ on $[0, T]$ corresponding to p_0 such that there exist $p_0^\alpha \in E$, $p_0^\alpha \rightarrow p_0$ in H , and a generalized sequence of weak solutions $p_\alpha(\cdot)$ satisfying $p_\alpha(0) = p_0^\alpha$ such that $p_\alpha \rightarrow p$ in $C([0, T], H)$. It is clear that $p(t)$ satisfies (4) for any $t \geq 0$, and also that $S(p_0, T)$ is non-empty for all $p_0 \in H$ and $T > 0$.

Then we define the map $G : \mathbb{R}^+ \times H \rightarrow P(H)$:

$$G(t, p_0) = \{p(t) : p \in S(p_0, T) \text{ for some } T \geq t\}.$$

Lemma 5.1. G is a strict multivalued semiflow.

Proof. Let $y \in G(t + s, x)$. Then $y = p(t + s)$, where $p(\cdot) \in S(x, T)$, $T \geq t + s$. It is clear that $v(\cdot) = p(s + \cdot) \in S(p(s), T - s)$, so that $y = v(t) \in G(t, p(s)) \subset G(t, G(s, x))$. Let now $y \in G(t, G(s, x))$. Then there exist $p(\cdot) \in S(x, T_1)$, $T_1 \geq s$, and $v(\cdot) \in S(p(s), T_2)$, $T_2 \geq t$, such that $y = v(t)$. Define the function

$$z(r) = \begin{cases} p(r), & \text{if } 0 \leq r \leq s, \\ v(r - s), & \text{if } s \leq r \leq s + T_2. \end{cases}$$

It is clear that $z \in S(x, s + T_2)$, so that $y = z(t + s) \in G(t + s, x)$. □

Remark 5. It follows from the proof of Lemma 5.1 that the concatenation of two weak solutions $p(\cdot) \in S(p_0, s)$, $v(\cdot) \in S(p(s), T)$ is a new weak solution which belongs to $S(p_0, s + T)$. In this way we can extend every weak solution $p(\cdot) \in S(p_0, T_p)$ to a global one (i.e. defined for all $t \geq 0$) and such that $p \in S(p_0, T)$ for all $T > 0$. Then an equivalent definition for G is the following:

$$G(t, p_0) = \left\{ \begin{array}{l} p(t) : p(\cdot) \text{ is a globally defined weak} \\ \text{solution such that } p \in S(p_0, T) \text{ for all } T > 0 \end{array} \right\}.$$

Lemma 5.2. The graph of the map $p_0 \mapsto G(t, p_0)$ is closed for any $t \geq 0$.

Proof. Consider a generalized sequence $\xi_\beta \in G(t^*, p_\beta)$ such that $\xi_\beta \rightarrow \xi$, $p_\beta \rightarrow p_0$ in H . We need to prove that $\xi \in G(t^*, p_0)$. Let us consider the set of all open sets O_γ , O_ϕ in H that contain ξ and p_0 , respectively. Let $\Lambda = \{\alpha = (\gamma, \phi) : \gamma \in \Gamma, \phi \in \Phi\}$ and introduce on Λ the following partial order: $(\gamma_1, \phi_1) \succeq (\gamma_2, \phi_2)$ if $O_{\gamma_1} \subseteq O_{\gamma_2}$ and $O_{\phi_1} \subseteq O_{\phi_2}$. Λ is a directed set. For any $\alpha = (\gamma, \phi)$ there exists $\beta(\alpha)$ such that $\xi_{\beta(\alpha)} \in O_\gamma$, $p_{\beta(\alpha)} \in O_\phi$. Then by the definition of $S(p_0, T)$ there exist $p_\alpha \in E \cap O_\phi$ such that $y_\alpha = y^\alpha(t^*) \in O_\gamma$, where $y^\alpha(\cdot)$ is the unique weak solution corresponding to p_α . From Lemma 2.5 we obtain that up to a generalized subsequence $y^\alpha(\cdot)$ converges to a weak solution $y(\cdot)$ in $C([0, T], H)$. Hence, $y(\cdot) \in S(p_0, T)$ and $y(t^*) = \xi \in G(t^*, p_0)$. □

Corollary 1. $G(t^*, p_0)$ is closed in H for any $t^* \geq 0$, $p_0 \in H$.

Lemma 5.3. For any set $B \subset E$, bounded in X , we have that $\gamma_0^+(B) = \cup_{t \geq 0} G(t, B)$ is bounded in X .

Proof. This follows directly from Proposition 1. □

Lemma 5.4. *For any set $B \subset E$, bounded in X , there exists a set A , compact in H and bounded in X , such that*

$$G(t, B) \rightarrow A, \text{ as } t \rightarrow +\infty.$$

Proof. We know by Lemma 5.3 that $\gamma_0^+(B)$ is bounded in X . Then $\gamma_0^+(B)$ is weakly precompact in $L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ is a separable Hilbert space, $L_w^2(\mathbb{R})$ satisfies the first axiom of countability. Then the omega-limit set $\omega_{L_w^2}(B) = \bigcap_{t \geq 0} cl_{L_w^2}(\bigcup_{\tau \geq t} G(t, B))$ can be characterized as

$$\omega_{L_w^2}(B) = \{y : \exists y_n \in G(t_n, B), t_n \rightarrow +\infty, \text{ s.t. } y_n \rightarrow y \text{ in } L_w^2(\mathbb{R})\}.$$

It is clear that $\omega_{L_w^2}(B)$ is non-empty and, as in the proof of Proposition 2, we obtain that $\omega_{L_w^2}(B)$ attracts B in the weak topology of $L^2(\mathbb{R})$. We state that the set $\omega_{L_w^2}(B)$ is a compact set in H and that it is bounded in X . Indeed, first let us check that $\omega_{L_w^2}(B) \subset H$. Let $y \in \omega_{L_w^2}(B)$. Then $y_n \rightarrow y$ in $L_w^2(\mathbb{R})$, where $y_n \in G(t_n, B)$ and $y_n \in H$. We shall prove that $y_n \rightarrow y$ in $\overline{L}^1(\mathbb{R})$. For this we check first that $x_n = (1 + |\sigma|)y_n$ is a weak Cauchy sequence in $L^1(\mathbb{R})$. Take an arbitrary $\xi \in L^\infty(\mathbb{R})$. Then by the estimate of the tails given in Lemma 2.2 for any $\varepsilon > 0$ there exist $K(\varepsilon) \geq 1$, $N(\varepsilon, K, \xi)$, such that

$$\begin{aligned} & \left| \int_{\mathbb{R}} (1 + |\sigma|)(y_n - y_m) \xi d\sigma \right| \\ & \leq \left| \int_{|\sigma| \leq K} (1 + |\sigma|)(y_n - y_m) \xi d\sigma \right| + 2 \|\xi\|_{L^\infty} \left(\int_{|\sigma| > K} |\sigma| y_n d\sigma + \int_{|\sigma| > K} |\sigma| y_m d\sigma \right) \\ & \leq 3\varepsilon, \text{ if } n \geq N(\varepsilon, K, \xi). \end{aligned}$$

Hence, since the space $L^1(\mathbb{R})$ is weakly complete [12, p.290], $x_n \rightarrow x$ weakly in $L^1(\mathbb{R})$. Thus, $y_n \rightarrow u = \frac{x}{1+|\sigma|}$ weakly in $\overline{L}^1(\mathbb{R})$. Note that if we denote by $L_k y_n$ the restriction of y_n into the subset $(-k, k)$, then $L_k y_n \rightarrow L_k y$ weakly in $L^2(-k, k)$, so that $L_k y_n \rightarrow L_k y$ weakly in $\overline{L}^1(-k, k)$. Hence, $L_k y = L_k u$. Since k is arbitrary, it follows that $y = u$. Thus, as H is weakly closed in X , we obtain that $y \in H$. We note that since $\gamma_0^+(B)$ is bounded in X , we have also that $cl_H \gamma_0^+(B)$ is bounded in X . Hence, it follows from $\omega_{L_w^2}(B) \subset cl_H \gamma_0^+(B)$ that $\omega_{L_w^2}(B)$ is bounded in X . Denote now by $L_k^c y_n$ the restriction of y_n into the subset $\Omega_k^c = \mathbb{R} \setminus [-k, k]$. Then using the estimate of the tails for y_n and that $L_k^c y_n \rightarrow L_k^c y$ weakly in $\overline{L}^1(\Omega_k^c)$, we obtain that for any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that

$$\int_{|\sigma| > K} (1 + |\sigma|) y dx \leq \liminf_{n \rightarrow \infty} \int_{|\sigma| > K} (1 + |\sigma|) y_n dx \leq \varepsilon.$$

Thus, the elements of $\omega_{L_w^2}(B)$ satisfy a uniform estimate of the tails in the space $\overline{L}^1(\mathbb{R})$. Further, we note that in an arbitrary Banach space a set A is weakly compact if and only if from any sequence there exists a subsequence converging weakly to some element of A . Take now a sequence $y_n \in \omega_{L_w^2}(B)$. The set $\omega_{L_w^2}(B)$ is bounded in $L^2(\mathbb{R})$, and then up to a subsequence $y_n \rightarrow y$ in $L_w^2(\mathbb{R})$. Finally, for

any $\xi \in L^\infty(\mathbb{R})$ and $\varepsilon > 0$ there exist $K(\varepsilon) \geq 1$, $N(\varepsilon, K, \xi)$, such that

$$\begin{aligned} & \left| \int_{\mathbb{R}} (1 + |\sigma|)(y_n - y) \xi d\sigma \right| \\ & \leq \left| \int_{|\sigma| \leq K} (1 + |\sigma|)(y_n - y) \xi d\sigma \right| + 2 \|\xi\|_{L^\infty} \left(\int_{|\sigma| > K} |\sigma| y_n d\sigma + \int_{|\sigma| > K} |\sigma| y d\sigma \right) \\ & \leq 3\varepsilon, \text{ if } n \geq N(\varepsilon, K, \xi), \end{aligned}$$

so that $y_n \rightarrow y$ in H . Hence, $\omega_{L_w^2}(B)$ is compact in H . Finally, let us prove that $\omega_{L_w^2}(B)$ attracts B in the topology of H . By contradiction, suppose that there exists a neighborhood O of $\omega_{L_w^2}(B)$ in H such that for any $T > 0$ there exist $t \geq T$ and $y \in G(t, B)$ for which $y \notin O$. Then there exists a sequence $y_n \in G(t_n, B)$, $t_n \rightarrow +\infty$, such that $y_n \notin O$ for all n . Since the sequence y_n belongs to a bounded set of $L^2(\mathbb{R})$, passing to a subsequence $y_n \rightarrow y$ in $L_w^2(\mathbb{R})$. Arguing as before we obtain also that $y_n \rightarrow y$ in $\overline{L}_w^1(\mathbb{R})$. Thus, $y_n \rightarrow y$ in H . Since $y \in \omega_{L_w^2}(B)$, we have a contradiction. \square

For the abstract setting developed in Section 4 let $F = H$ and denote by Y the space H with the strong topology of X . We note that in this case every $B \in \beta(Y)$ belongs to F . The set E is given in (18).

Theorem 5.5. *The multivalued semiflow G has a (Y, F) -global attractor defined by*

$$\mathcal{A} = \bigcup_{B \in \beta(Y), B \subset E} \omega_H(B)$$

in the phase space $F = H$, which is bounded in $\overline{L}^1(\mathbb{R})$. Moreover, \mathcal{A} is minimal in the following sense: for any (Y, F) -attracting set Z in H we have $\mathcal{A} \subset cl_{L_w^2} Z$.

Proof. The existence of the (Y, F) -global attractor follows from Lemmas 5.2, 5.4 and Theorem 4.4. Let us prove that \mathcal{A} is bounded in $\overline{L}^1(\mathbb{R})$. Take $y \in \mathcal{A}$. Then $y_n \rightarrow y$ in $L_w^2(\mathbb{R})$, where $t_n \rightarrow +\infty$ and $y_n \in G(t_n, B)$ for some $B \in \beta(Y)$, $B \subset E$. Then using the estimate of the tails given in Lemma 2.2 for y_n and that $L_k^c y_n \rightarrow L_k^c y$ weakly in $\overline{L}^1(\Omega_k^\varepsilon)$, we obtain that for $\varepsilon = 1$ there exists K such that

$$\int_{|\sigma| > K} |\sigma| y dx \leq \varepsilon = 1.$$

Note that K does not depend on B . Thus, the elements of \mathcal{A} satisfy a uniform estimate of the tails in the space $\overline{L}_\sigma^1(\mathbb{R})$. Also, $\int_{\mathbb{R}} y d\sigma = 1$ for all $y \in \mathcal{A}$. Thus

$$\int_{\mathbb{R}} |\sigma| y(\sigma) d\sigma \leq \int_{|\sigma| \leq K} |\sigma| y dx + \int_{|\sigma| > K} |\sigma| y dx \leq K + 1.$$

Hence, \mathcal{A} is bounded in $\overline{L}^1(\mathbb{R})$. Finally, we prove now the minimality. The set \mathcal{A} is defined by

$$\mathcal{A} = \bigcup_{B \in \beta(Y), B \subset E} \omega_H(B).$$

Let Z be an (Y, F) -attracting set in the space H . Then we need to prove that $\omega_H(B) \subset cl_{L_w^2} Z$, for all $B \in \beta(Y)$, $B \subset E$. Take a closed ball B_R in $L^2(\mathbb{R})$ of radius R such that $\gamma_0^+(B) \subset B_R$. Put $Z_R = Z \cap B_R$. Then Z_R is bounded in $L^2(\mathbb{R})$ and attracts B in the space $L_w^2(\mathbb{R})$. Indeed, let O be an arbitrary open set in $L_w^2(\mathbb{R})$ containing Z_R . Set $V = L^2(\mathbb{R}) \setminus B_R$. Since B_R is closed in $L_w^2(\mathbb{R})$, V is

open in $L_w^2(\mathbb{R})$, and then $N = O \cup V$ is an open set in $L_w^2(\mathbb{R})$ containing Z . Hence, there exists T such that $G(t, B) \subset N$, for all $t \geq T$. But note that $\gamma_0^+(B) \subset B_R$, so that $G(t, B) \subset O$, for all $t \geq T$. Hence, Z_R attracts B in the space $L_w^2(\mathbb{R})$. Take the set $\omega_{L_w^2}(B)$ defined in Lemma 5.4. It is clear that

$$\omega_H(B) = \cap_{t \geq 0} cl_H(\cup_{\tau \geq t} G(t, B)) \subset \omega_{L_w^2}(B) = \cap_{t \geq 0} cl_{L_w^2}(\cup_{\tau \geq t} G(t, B)).$$

The space B_R endowed with the induced weak topology of $L^2(\mathbb{R})$ is metrizable. Thus, this space is regular. If there exists $y \in \omega_{L_w^2}(B)$ such that $y \notin cl_{L_w^2} Z_R$, then we can find disjoint neighborhoods (in B_R) $O(y)$ and $O(cl_{L_w^2} Z_R)$. Since Z_R attracts B in $L_w^2(\mathbb{R})$ (and then in B_R), using the characterization of $\omega_{L_w^2}(B)$ given in Lemma 5.4 we obtain that $y \subset cl_{L_w^2} Z_B$, which is a contradiction. Hence, $\omega_H(B) \subset \omega_{L_w^2}(B) \subset cl_{L_w^2} Z_R \subset cl_{L_w^2} Z$. \square

6. Appendix: the dual space of $\bar{L}^1(\mathbb{R})$. We shall prove the characterization of the topological dual space $(\bar{L}^1(\mathbb{R}))^*$.

Lemma 6.1. $(\bar{L}^1(\mathbb{R}))^*$ can be characterized as follows

$$(\bar{L}^1)^* = \{\varphi = (1 + |\sigma|)u : u \in L^\infty(\mathbb{R})\},$$

and

$$\langle \varphi, p \rangle_{\bar{L}^1} = \int_{\mathbb{R}} (1 + |\sigma|) up d\sigma.$$

Proof. Let $\varphi \in (\bar{L}^1(\mathbb{R}))^*$. Then we shall prove the existence of a unique $u \in L^\infty(\mathbb{R})$ such that

$$\langle \varphi, p \rangle = \int_{\mathbb{R}} (1 + |\sigma|) up dx, \text{ for all } p \in \bar{L}^1(\mathbb{R}).$$

Moreover, $\|\varphi\|_{(\bar{L}^1)^*} = \|u\|_{L^\infty}$. Let us take a function $w \in L^2(\mathbb{R})$ such that for any compact set $K \subset \mathbb{R}$, $w(\sigma) \geq \varepsilon_K > 0$. Define $\bar{w}(\sigma) = \frac{w(\sigma)}{1+|\sigma|}$. For any $f \in L^2(\mathbb{R})$ the function $\bar{w}f$ belongs to $\bar{L}^1(\mathbb{R})$ and the functional $L^2(\mathbb{R}) \ni f \mapsto \langle \varphi, \bar{w}f \rangle$ is linear and continuous on $L^2(\mathbb{R})$. Then there exists $v \in L^2(\mathbb{R})$ for which

$$\langle \varphi, \bar{w}f \rangle = \int_{\mathbb{R}} v f d\sigma, \text{ for all } f \in L^2(\mathbb{R}). \tag{53}$$

Put $u(\sigma) = \frac{v(\sigma)}{w(\sigma)}$, which is well defined because $w(\sigma) > 0$. It is clear that $u(\sigma)$ is measurable. Let us prove that $u \in L^\infty(\mathbb{R})$ and $\|u\|_{L^\infty} \leq \|\varphi\|_{(\bar{L}^1)^*}$. By (53) we have

$$\left| \int_{\mathbb{R}} v f d\sigma \right| \leq \int_{\mathbb{R}} |v f| d\sigma \leq \|\varphi\|_{(\bar{L}^1)^*} \|\bar{w}f\|_{(\bar{L}^1)}, \text{ for all } f \in L^2(\mathbb{R}). \tag{54}$$

Put $C > \|\varphi\|_{(\bar{L}^1)^*}$. We shall prove that $A = \{\sigma \in \mathbb{R} : |u(\sigma)| > C\}$ has measure equal to 0. This will imply that $u \in L^\infty(\mathbb{R})$ and $\|u\|_{L^\infty} \leq \|\varphi\|_{(\bar{L}^1)^*}$. Indeed, if A has positive measure, then there exists a measurable subset $B \subset A$ such that $0 < |B| < \infty$. We take in (54) the function

$$f(\sigma) = \begin{cases} 1, & \text{if } \sigma \in B \text{ and } u(\sigma) > 0, \\ -1, & \text{if } \sigma \in B \text{ and } u(\sigma) < 0, \\ 0, & \text{if } \sigma \in \mathbb{R} \setminus B. \end{cases}$$

Then $\int_B |u(\sigma)| w(\sigma) d\sigma \leq \|\varphi\|_{(\overline{L^1})^*} \int_B w(\sigma) d\sigma$, so that

$$C \int_B w(\sigma) d\sigma \leq \|\varphi\|_{(\overline{L^1})^*} \int_B w(\sigma) d\sigma,$$

which is a contradiction, since $\int_B w(\sigma) d\sigma > 0$. It follows that

$$\langle \varphi, \bar{w}f \rangle = \int_{\mathbb{R}} u w f d\sigma.$$

If $g \in C_c(\mathbb{R})$, then $f = \frac{g}{w} \in L^2(\mathbb{R})$. Hence,

$$\langle \varphi, g \rangle = \int_{\mathbb{R}} \frac{w}{\bar{w}} u g d\sigma = \int_{\mathbb{R}} (1 + |\sigma|) u g d\sigma, \forall g \in C_c(\mathbb{R}).$$

We note that $C_c(\mathbb{R})$ is dense in $\overline{L^1}(\mathbb{R})$. Indeed, for any $p \in \overline{L^1}(\mathbb{R})$ put $y = (1 + |\sigma|)p \in L^1(\mathbb{R})$ and take a sequence $y_n \in C_c(\mathbb{R})$ such that $y_n \rightarrow y$ in $L^1(\mathbb{R})$. Then $p_n = \frac{y_n}{1 + |\sigma|} \in \overline{L^1}(\mathbb{R})$ and $p_n \rightarrow p$ in $\overline{L^1}(\mathbb{R})$. Then

$$\langle \varphi, p \rangle = \int_{\mathbb{R}} (1 + |\sigma|) u p dx, \text{ for all } p \in \overline{L^1}(\mathbb{R}).$$

Finally,

$$|\langle \varphi, p \rangle| \leq \|u\|_{L^\infty} \|p\|_{\overline{L^1}}, \text{ for all } p \in \overline{L^1}(\mathbb{R}),$$

so that $\|\varphi\|_{(\overline{L^1})^*} \leq \|u\|_{L^\infty}$. Hence, $\|\varphi\|_{(\overline{L^1})^*} = \|u\|_{L^\infty}$. It is easy to see that u is unique. \square

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