Mumford-Shah Level-Set Ideas for Problems in Medical Imaging

Paris Dauphine 2005

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General Mumford-Shah like functionals

We consider the problem of determining an unknown function $v : D \to \mathbb{R}^N$ with might be singular (discontinuous) across an unknown singularity set $\Gamma \subset D$ from given data $y_d \in Y$.

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Suppose for the moment that the singularity set Γ is known. Then the function $v|_{D\setminus\Gamma}$ is an element in a Hilbert space $X(\Gamma)$ which might be dependent on Γ.

Note: $X(\Gamma)$ is a space of functions with domain of definition given by $D \setminus \Gamma$. The norm on $X(\Gamma)$ does not measure what happens on/across Γ.

The relation between v and the given (exact) data is supposed to be of the form

$$y_d = Kv \tag{1}$$

where $K = K(\Gamma)$ with $K : X(\Gamma) \to Y$ is a continuous (possibly nonlinear) operator.

Simultaneous determination of functional and geometric data

The singularity set Γ and the function $v \in X(\Gamma)$ are to be simultaneously determined as solution to

$$\min_{\substack{\Gamma \in \mathcal{G} \\ v \in X(\Gamma)}} J_{\mathrm{MS}}(v, \Gamma)$$

$$J_{\mathrm{MS}}(v, \Gamma) = \frac{1}{2}\|Kv - y_d\|_Y^2 + \frac{\nu}{2}\|v\|_{X(\Gamma)}^2 + \mu \int_\Gamma 1 \, dS$$

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- Data fit
- Regularization without penalization on/across Γ
- Control of Γ

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Step 2: Consider the shape optimization problem

$$\min_{\Gamma \in G} J_{\mathrm{MS}}(v(\Gamma), \Gamma).$$

Calculate a descent direction F using techniques from shape sensitivity analysis.

Step 3: Update the geometric variable Γ in the chosen descent direction using a level-set formulation for the propagation of the geometric variable. Solve

$$u_t + F|\nabla u| = 0$$

for an appropriate time-step. Here $\Gamma = \{u = 0\}$.

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For linear K:

$$K^*(Kv(\Gamma) - y_d) + \nu v(\Gamma) = 0$$

with $v(\Gamma) \in X(\Gamma)$.

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Since $v(\Gamma)$ satisfies the Euler-Lagrange equations for J_{MS} w.r.t. v, the first term in (2) vanishes.

Descent directions and the choice of an appropriate metric

Shape differential and steepest descent direction

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Suppose $F \in Z$, (Z ... space of perturbations). The shape differential $\delta J(\Gamma) \in Z'$ of J at Ω is an element in Z' satisfying

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A steepest descent direction F_{sd} is an element in Z satisfying

$$dJ(\Gamma; F_{\mathrm{sd}}) = \min_{\substack{F \in Z \\ \|F\|_Z \le 1}} dJ(\Gamma; F).$$

(F_{sd} ... contravariant repr. of the shape derivative)

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Restrictive condition on the choice of Z: $\delta \in Z'$.

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To find the steepest descent direction w.r.t. the H_0^1-norm, we have to solve the constrained optimization problem

$$\min_{\substack{F \in H_0^1(\Gamma) \\ \|F\|_{H_0^1(\Gamma)} \leq 1}} \langle G, F|_\Gamma \rangle_{H^{-1}(\Gamma), H_0^1(\Gamma)}$$

Introducing the Lagrange functional

$$\mathcal{L}(F,\lambda) = \langle G, F_\Gamma \rangle + \lambda\Big(\int_\Gamma \left(|\nabla_\Gamma F|^2 + |F|^2\right) dS - 1 \Big)$$

we get the optimality condition

$$F = -\frac{1}{2\lambda}(-\Delta_\Gamma + \mathrm{id})^{-1} G \qquad (3)$$

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The 'gradient direction' $-G$ is preconditioned by an inverse elliptic operator. The multiplier λ is chosen such that the H^1_0-norm of F is one.

A piecewise constant Mumford-Shah approach for x-ray tomography (with Ronny Ramlau)

Objectives:

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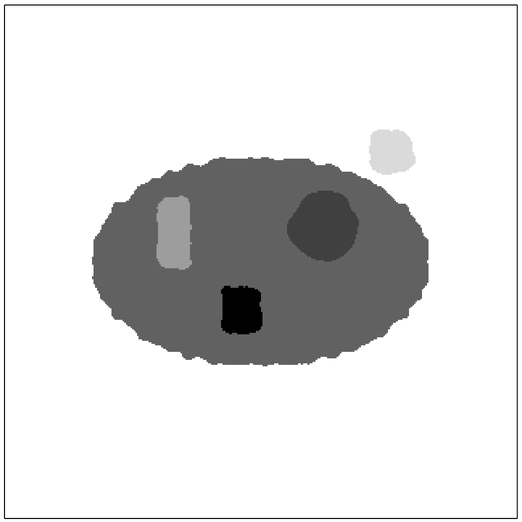

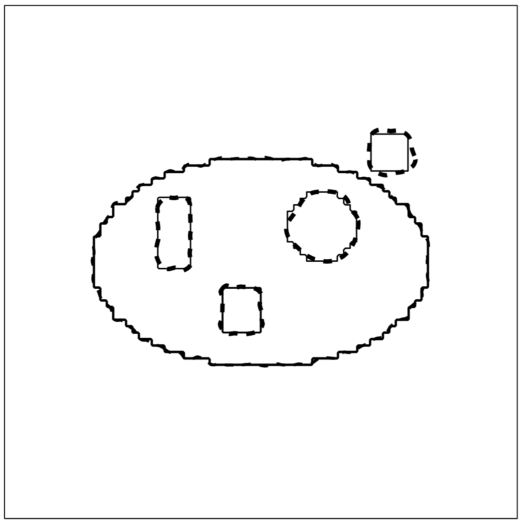

Simultaneously reconstruct the density distribution and the boundaries of regions with approximately homogeneous densities from Radon-transform data.

- Segment directly from the data, estimate objects
- Reduce ill-posedness by reducing the dimension of the unknown — inversion of noisy data
- Comparable in quality and speed to established methods

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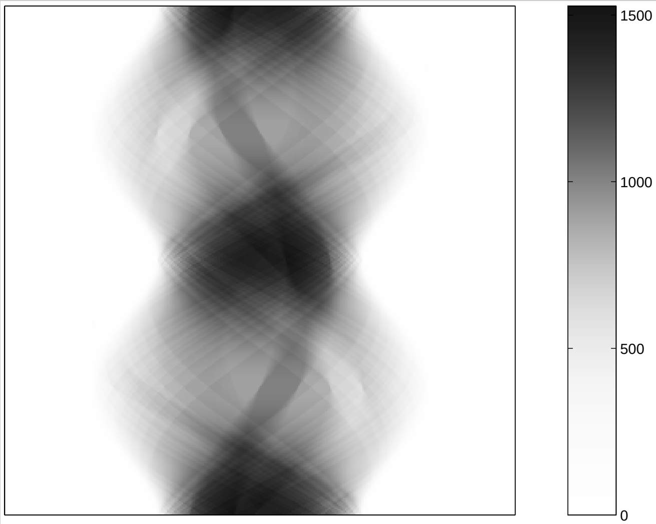

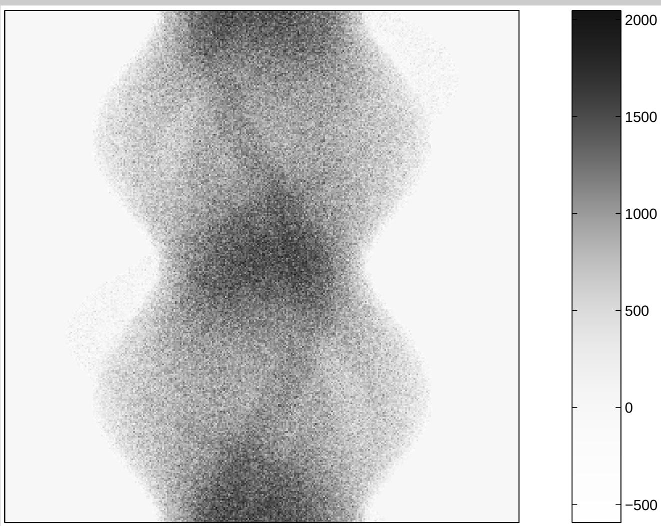

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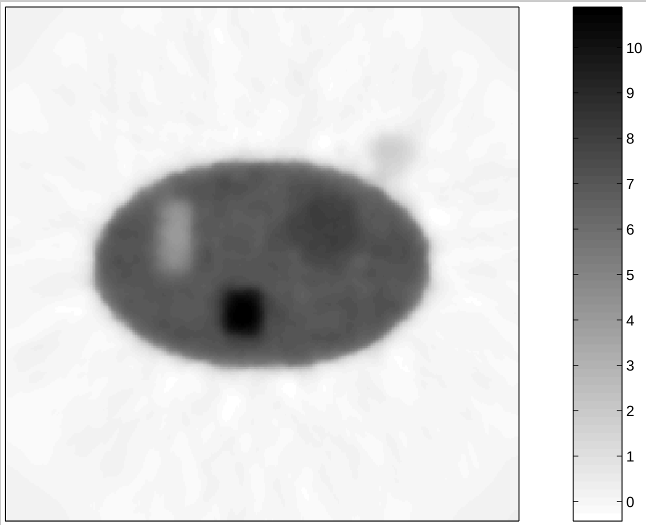

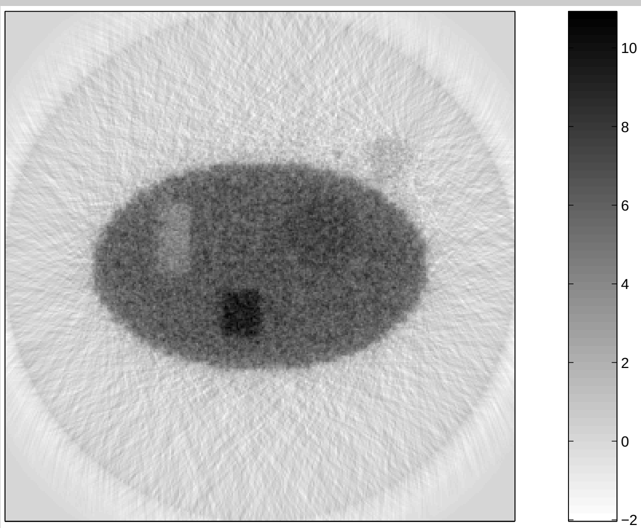

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The functional and the resulting geometric flow

We fix the space of admissible densities as:

$$X(\Gamma) = PC(D \setminus \Gamma) = \Big\{ \sum_{i=1}^{n(\Gamma)} f_i \, \chi_{\Omega_i^\Gamma} \, : \, f_i \in \mathbb{R} \Big\} \subset L^2(D) \tag{4}$$

Ω_i^Γ ... connected component of $D \setminus \Gamma$.

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The "Radon Mumford-Shah" functional is given as

$$J(f, \Gamma) = \|Rf - g_d\|^2_{L^2(\mathbb{R}\times S^1)} + \alpha|\Gamma|. \tag{5}$$

Solving the Euler-Lagrange System (Step 1):

$$\partial_f J(f(\Gamma), \Gamma)\, h = \langle Rf(\Gamma) - g_d, Rh \rangle_{L^2(\mathbb{R}\times S^1)} = 0 \quad \forall h. \qquad (6)$$

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This is equivalent to

$$A\mathbf{f}(\Gamma) = \mathbf{g} \tag{7}$$

where $A = (a_{ij})$ and $\mathbf{g} = (g_i)^t$.

$$a_{ij} = -2\int_{\mathbf{x}\in\partial\Omega_i^\Gamma}\int_{\mathbf{y}\in\partial\Omega_j^\Gamma} |\mathbf{y}-\mathbf{x}|\,\langle \mathbf{n}_i(\mathbf{x}), \mathbf{n}_j(\mathbf{x})\rangle\, dS(\mathbf{y})\, dS(\mathbf{x}). \tag{8}$$

$$g_i = \int_{\Omega_i^\Gamma} R^* g_d\, d\mathbf{x} = \int_{\mathbf{x}\in\Omega_i^\Gamma}\int_{\omega\in S^1} g(\omega\cdot\mathbf{x},\omega)\, d\omega\, d\mathbf{x}. \tag{9}$$

The shape derivative (Step 2):

$$dJ(\Gamma;F) =$$

$$4\sum_k \int_{\mathbf{x}\in\Gamma_k} \sum_{p\in d(k)} s_p f_p \left(\sum_l \sum_{q\in d(l)} s_q f_q \int_{\mathbf{y}\in\Gamma_l} \left\langle \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}, \mathbf{n}^k(\mathbf{y}) \right\rangle dS(\mathbf{y}) \right) \cdot F(\mathbf{x})\, dS(\mathbf{x})$$

$$-\,2\sum_k \int_{\mathbf{x}\in\Gamma_k} \left(\sum_{p\in d(k)} s_p f_p \int_{\omega\in S^1} g_d(\omega\cdot\mathbf{x},\omega)\, d\omega \right) F(\mathbf{x})\, dS(\mathbf{x})$$

$$+\sum_k \int_{\mathbf{x}\in\Gamma_k} \kappa(\mathbf{x})\, F(\mathbf{x})\, dS(\mathbf{x}).$$

f_p...densitiy values, $s_p = \pm 1$, κ...curvature.

Numerical observations

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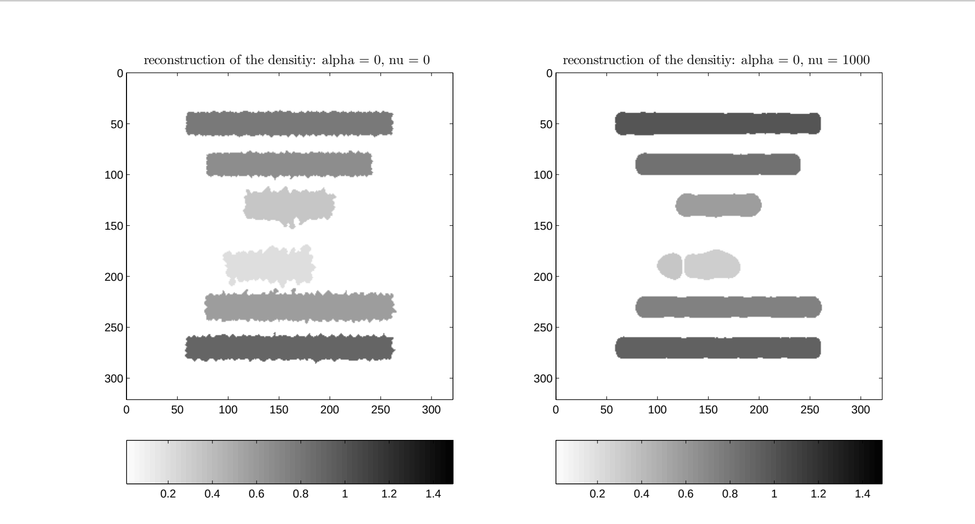

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	$\nu = 10^2$	$\nu = 10^1$	$\nu = 10^0$	$\nu = 10^{-1}$	$\nu = 10^{-2}$	$\nu = 10^{-3}$	$\nu = 0$
$\alpha = 10^1$	74	52	50	–	62	68	–
$\alpha = 10^2$	105	79	134	526	1077	–	1243

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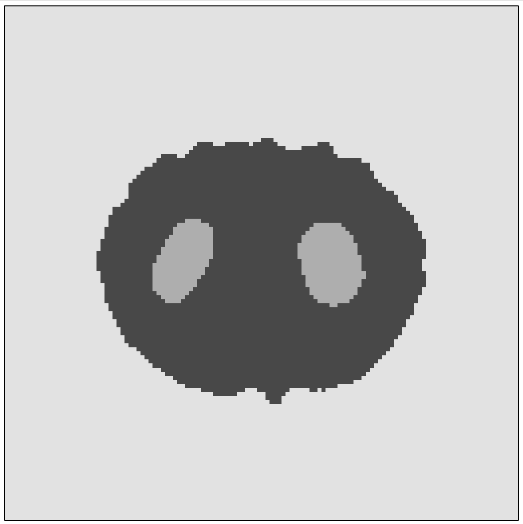

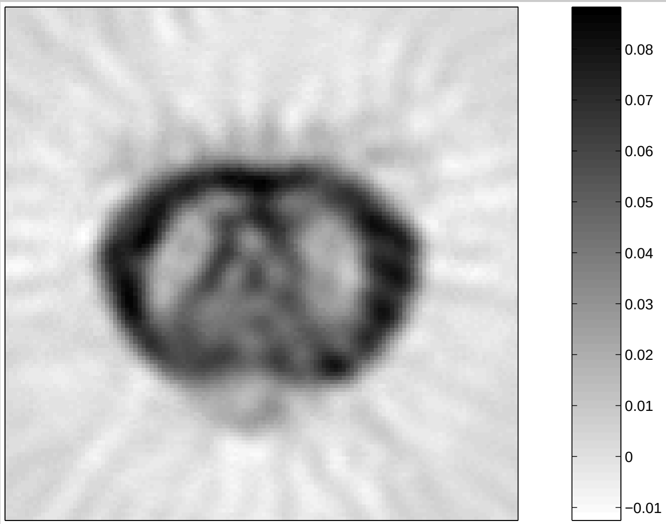

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A Mumford-Shah like approach for simultaneous registration and segmentation of multimodal data sets. (with Marc Droske)

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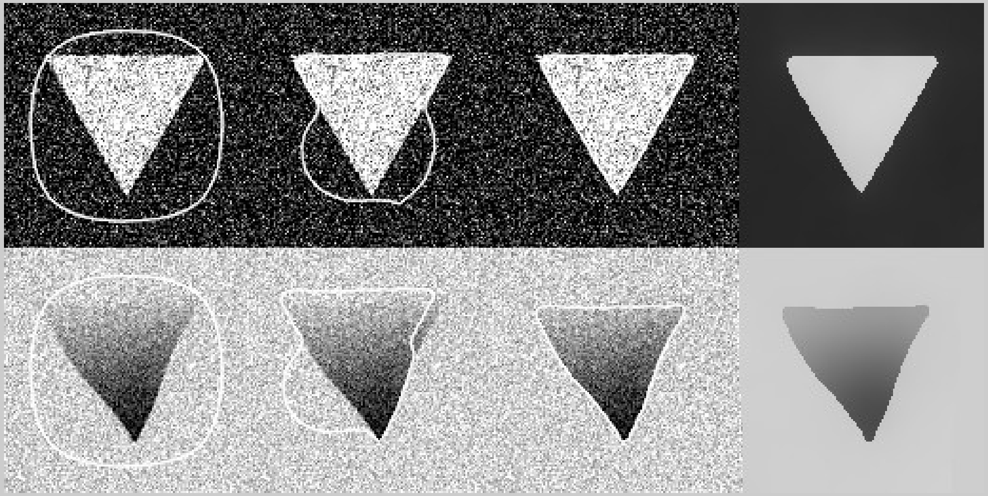

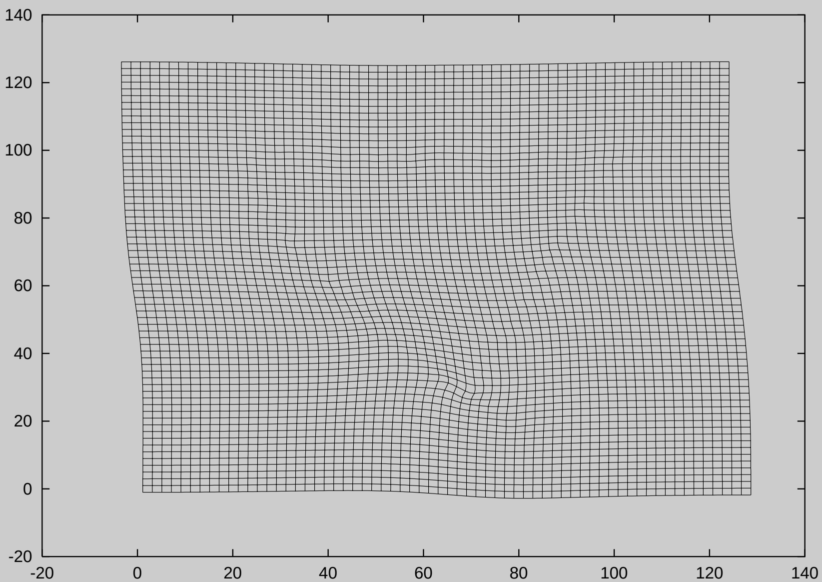

Simultaneously segment two images and match similar structures onto each other.

- Work with noisy multimodal images
- Transfer features which are strong in one image onto the other, where the feature is weak
- Work with real data-sets

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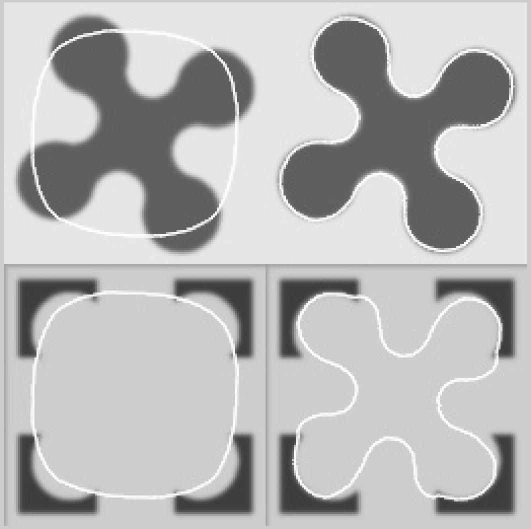

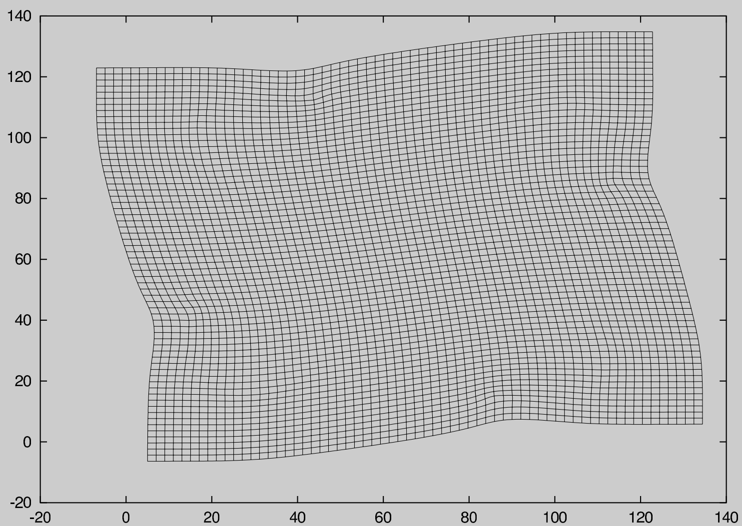

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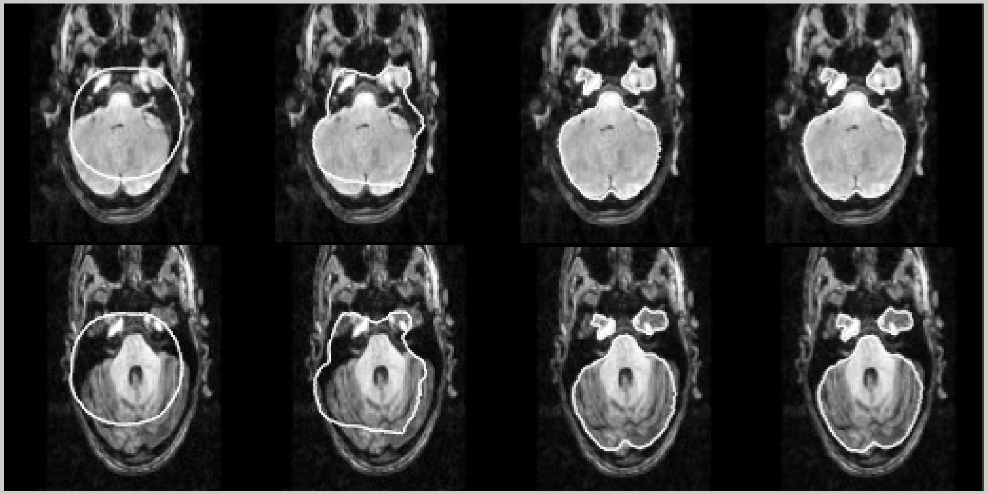

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The functional:

$$E_{\mathrm{MS}}(\Gamma,\Phi,R,T)=\frac{1}{2}\int_D|R-R_0|^2\,\mathrm{d}\mathbf{x}+\frac{\mu}{2}\int_{D\setminus\Gamma}|\nabla R|^2\,\mathrm{d}\mathbf{x}$$
$$+\frac{1}{2}\int_D|T-T_0|^2\,\mathrm{d}\mathbf{x}+\frac{\mu}{2}\int_{D\setminus\Gamma^\Phi}|\nabla T|^2\,\mathrm{d}\mathbf{x}+\alpha\mathcal{H}^{N-1}(\Gamma)+\nu E_{\mathrm{reg}}(\Phi)$$

$R,T\ldots$ reference and template images,

$\Phi\ldots$ transformation between the images,

$\Gamma\ldots$ edge-set in the reference image,

$\Gamma^\Phi=\Phi(\Gamma)\ldots$ edge-set in the template image,

$E_{\mathrm{reg}}\ldots$ regularization term penalizing deviation from a rigid body motion.

Algorithm:

- Minimize w.r.t. T and R for fixed Γ, Φ. Consider the reduced functional

$$\hat{E}(\Gamma, \Phi) = E(\Gamma, \Phi, R(\Gamma), T(\Gamma, \Phi)).$$

- Calculate descent directions with respect to Γ and Φ. Use composite finite elements and multi-grid solvers.

- Update Γ and ϕ. Use the level-set equation for the update of Γ.

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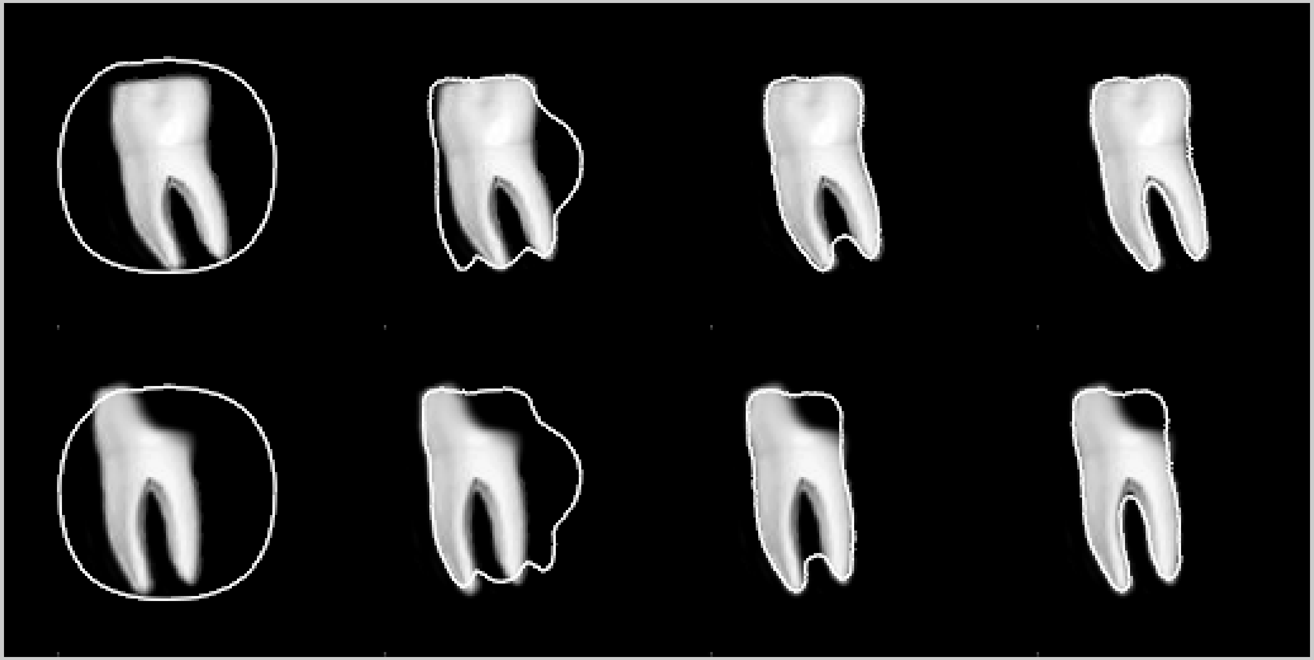

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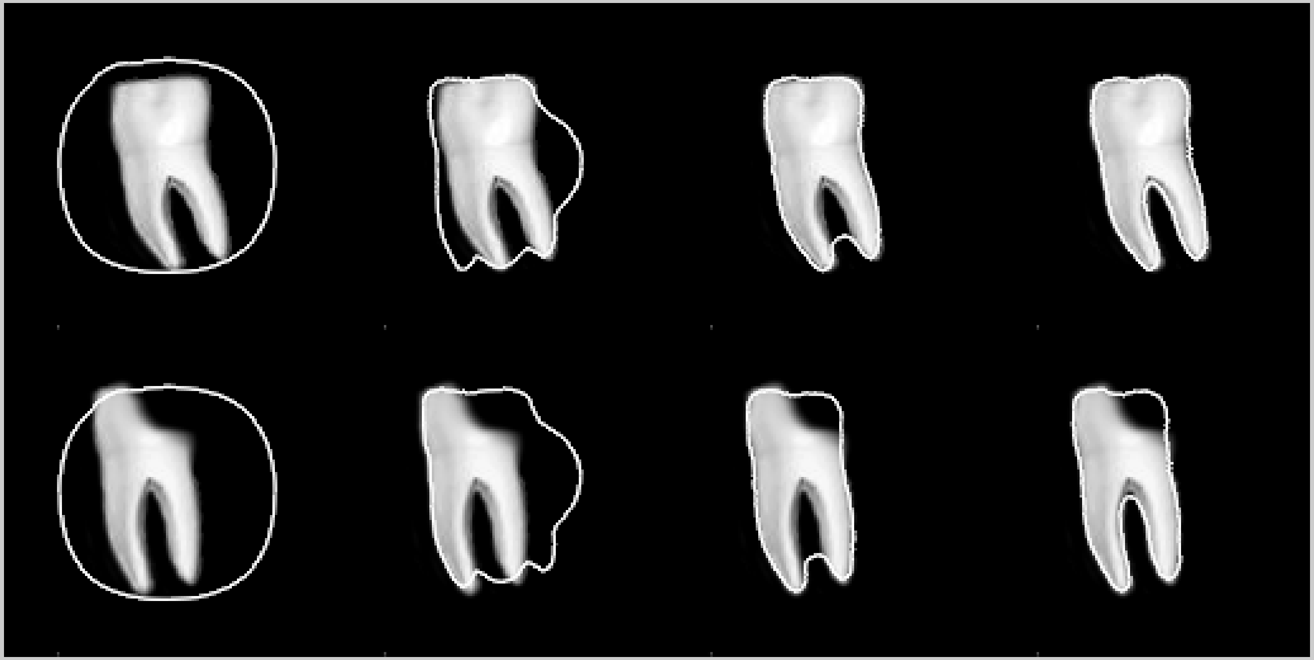

Further applications

- Optical flow estimation
- Inversion of SPECT data
- Occlusions
- etc.