

The Group Marching Method: An $\mathcal{O}(N)$ Level Set Eikonal Solver

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Abstract

The level set method is a numerical technique to compute advancing fronts. It has been adapted for the computation of the first-arrival traveltime (FATT) and implemented with the narrow band technique, called the *fast marching method* (FMM). The FMM requires sorting the solution at each step of the narrow band. For the binary tree sorting, its computation cost becomes $\mathcal{O}(N \log_2 N)$. In this article we develop an $\mathcal{O}(N)$ level set algorithm called the *group marching method* (GMM). The GMM advances a group of grid points at a time, rather than sorting the solution in the narrow band to march forward a single grid point. The GMM improves efficiency, while maintaining the same accuracy as the FMM. Accuracy and efficiency of the GMM are compared with those of the second-order method ENO-DNO-PS.

1. Introduction

A propagating interface such as the first-arrival traveltime can develop corners and discontinuities as it advances. The level set method is a numerical technique to compute advancing fronts and has been applied to a wide range of important physical problems; see e.g. [17, 18] and references therein. Even though its solution shows first-order accuracy, the level set method has been widely used mostly due to its built-in stability.

It has been adapted for the computation of the first-arrival traveltime (FATT) and implemented with the narrow band technique, called the *fast marching method* (FMM) in which the narrow band points form a neighborhood of advancing wavefronts [11, 18, 19]. The FMM requires sorting the solution at each step of the narrow band; it costs $\mathcal{O}(N \log_2 N)$, where N is the number of grid points. The main object of the

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article is to develop an $\mathcal{O}(N)$ level set algorithm called the *group marching method* (GMM). The new method is based on the narrow band approach as in the FMM. However, the GMM is incorporating a correction-by-iteration strategy to advance a group of grid points at a time. After selecting a group of grid points appropriately, the GMM advances the group in two iterations, for the cost of slightly larger than one iteration.

There have been various numerical techniques for FATTs: ray-tracing methods, finite difference (FD) schemes, and algorithms based on the Fermat’s principle [15]. For a comprehensive treatment of ray theory, see [1, 2, 6]. For FD schemes, see [3, 4, 7, 8, 10, 20, 21] for standard expanding-box methods and [12, 13] for the level set method.

2. Finite difference scheme

The eikonal equation in an isotropic medium is given by

$$|\nabla\tau(\mathbf{x}_s, \mathbf{x})|^2 = \frac{1}{v^2(\mathbf{x})}, \quad (2.1)$$

where $\tau(\mathbf{x}_s, \mathbf{x})$ is the travelttime of the acoustic wave from the source \mathbf{x}_s to the location \mathbf{x} and $v(\mathbf{x})$ denotes the velocity of propagating wavefront at \mathbf{x} . For down-going wavefronts (advancing in the $(z+)$ -direction), for example, the equation can be rewritten in the following evolutionary form

$$\tau_z = \sqrt{s^2 - \tau_x^2 - \tau_y^2}, \quad (2.2)$$

where $s(= 1/v)$ is called the slowness.

Let $\tau_{i,j}^k = \tau(\mathbf{x}_s, \mathbf{x}_{i,j}^k)$, where $\mathbf{x}_{i,j}^k = (x_i, y_j, z_k)$, a grid point. Define the forward (+) and backward (−) difference operators for τ_x at the point $\mathbf{x}_{i,j}^k$:

$$D_x^\pm \tau_{i,j}^k = \pm \frac{\tau_{i\pm 1,j}^k - \tau_{i,j}^k}{\Delta x}.$$

Define the *upwind* FD scheme for τ_x incorporating the first-arrivals:

$$\widehat{D}_x \tau_{i,j}^k = \text{mod_max}(\max(D_x^- \tau_{i,j}^k, 0), \min(D_x^+ \tau_{i,j}^k, 0)),$$

where “mod_max” returns the larger value in modulus. Note that

$$|\widehat{D}_x \tau_{i,j}^k| = \max(D_x^- \tau_{i,j}^k, -D_x^+ \tau_{i,j}^k, 0),$$

which can be utilized in practical implementation.

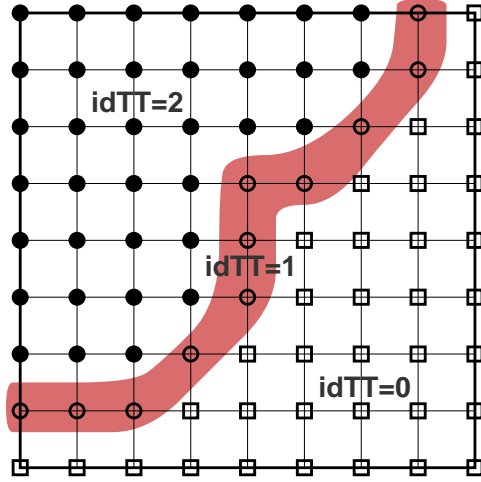


Figure 1: The group marching method (GMM) illustrated in 2D. The closed circles indicate the points already computed, the open circles form a neighborhood of a wavefront, and the open boxes correspond to the downwind points to be computed at the current stage or later.

After introducing the analogues for τ_y and τ_z , i.e., $\widehat{D}_y \tau_{i,j}^k$ and $\widehat{D}_z \tau_{i,j}^k$, we formulate the upwind FD scheme for (2.1):

$$(\widehat{D}_x \tau_{i,j}^k)^2 + (\widehat{D}_y \tau_{i,j}^k)^2 + (\widehat{D}_z \tau_{i,j}^k)^2 = (s_{i,j}^k)^2. \quad (2.3)$$

For down-going wavefronts as in (2.2), the scheme can be explicitly rewritten as

$$\tau_{i,j}^{k+1} = \tau_{i,j}^k + \Delta z \cdot H[\tau]_{i,j}^k, \quad (2.4)$$

where

$$H[\tau]_{i,j}^k = \sqrt{(s_{i,j}^k)^2 - (\widehat{D}_x \tau_{i,j}^k)^2 - (\widehat{D}_y \tau_{i,j}^k)^2}.$$

3. The group marching method

Here we introduce a new narrow band level set algorithm, called *group marching method* (GMM) whose total computation cost is $\mathcal{O}(N)$.

Consider a neighborhood Γ of a wavefront; see the shaded area in Figure 1. In the current stage of the GMM, we will select a group of points G out of Γ , recompute the traveltimes at neighboring points of G that are not completed, register the neighboring points as members of Γ if they are not, and finally tag “completed” for the points in G . The group of points should be carefully chosen in such a way that the computed solution does not violate causality, since in our algorithm the traveltimes are not sorted from the smallest to the largest. The main objective in this section is to develop a way to select such a group of points out of the narrow band Γ .

We consider the case $h = \Delta x = \Delta y = \Delta z$, for a simple presentation. Define

$$s_{\Gamma, \min} = \min\{s_{ij}^k : \mathbf{x}_{ij}^k \in \Gamma\}.$$

Then, it is easy to show the following [9]: *Given two nearby points $\mathbf{x}_{i_1, j_1}^{k_1}$ and $\mathbf{x}_{i_2, j_2}^{k_2}$, if their travelttime difference is less than*

$$\delta\tau \equiv \frac{1}{\sqrt{3}} \cdot h \cdot s_{\Gamma, \min}, \quad (3.1)$$

the angle between the line segment $\overline{\mathbf{x}_{i_1, j_1}^{k_1} \mathbf{x}_{i_2, j_2}^{k_2}}$ and the wavefront normal is larger than 45 degrees, i.e., rather perpendicular than parallel.

Now, we are ready to choose the group G to be *completed* at a time. Define $\tau_{\Gamma, \min} = \min\{\tau_{ij}^k : \mathbf{x}_{ij}^k \in \Gamma\}$ and select G as follows:

$$G = \{\mathbf{x}_{ij}^k \in \Gamma : \tau(\mathbf{x}_{ij}^k) \leq \tau_{\Gamma, \min} + \delta\tau\}. \quad (3.2)$$

Let $\mathbf{x}_{i_1, j_1}^{k_1}$ and $\mathbf{x}_{i_2, j_2}^{k_2}$ be two points in G . When they are not adjacent, it is clear that their traveltimes do not affect each other in the update procedure. If they are adjacent, one can barely affect the other, since the wavefront normal is nearer to perpendicular, rather than parallel, to the line segment formed by the points. However, whether the two points are adjacent or not, their neighboring downwind points can be affected by both of two points. It can be the case in particular for intersecting wavefronts. To avoid any possible instability, we update all the neighboring points of the group G twice – one in an order and the other in the opposite order. The double computation fixes instability. To see it, imagine that we try to update the neighboring points of G one more time. We can readily see that none of the neighboring points changes its travelttime during the extra updates.

We summarize the above arguments as in the following algorithm, called the *group marching method* (GMM).

- Initialization:

- (I1) Assign a large number for traveltimes TT, e.g., $\text{TT}(\cdot, \cdot, \cdot) \equiv 1.0e5$;
- (I2) Set zero for the travelttime index idTT, i.e., $\text{idTT}(\cdot, \cdot, \cdot) \equiv 0$;
- (I3) Set $\text{delTAU} = \frac{1}{\sqrt{3}} \cdot \min(\Delta x, \Delta y, \Delta z) \cdot \min_{i, j, k} s_{i, j}^k$;
- (I4) On the box of $(2 \times 2 \times 2)$ cells having the source at its center,
 - assign the analytic value of TT on the box;
 - set $\text{idTT}(\cdot, \cdot, \cdot) = 2$, at the source;
 - set $\text{idTT}(\cdot, \cdot, \cdot) \equiv 1$, on the surface of the box; save those point indices to the interface indicator array $\text{GAMMA}(\cdot, \cdot, \cdot)$;
 - set TM to be the minimum of TT on the surface of the box;

- Marching Forward:

- (M1) Set $\mathbf{TM} = \mathbf{TM} + \mathbf{delTAU}$;
- (M2) For each (i, j, k) in \mathbf{GAMMA} , in the reverse order, if $(\mathbf{TT}(i, j, k) \leq \mathbf{TM})$, recompute traveltimes of neighboring points (ℓ, m, n) where $\mathbf{idTT} \leq 1$;
- (M3) For each (i, j, k) in \mathbf{GAMMA} , in the forward order, if $(\mathbf{TT}(i, j, k) \leq \mathbf{TM})$,
 - (a) recompute traveltimes of neighboring points (ℓ, m, n) where $\mathbf{idTT} \leq 1$;
 - (b) if $\mathbf{idTT} = 0$ at a neighboring point (ℓ, m, n) , set $\mathbf{idTT}(\ell, m, n) = 1$ and save (ℓ, m, n) into \mathbf{GAMMA} ;
 - (c) remove the index (i, j, k) out of \mathbf{GAMMA} ; set $\mathbf{idTT}(i, j, k) = 2$;
- (M4) If $\mathbf{GAMMA} \neq \emptyset$, go to (M1);

Apparently, the reverse computation (M2) is cheaper to carry out than (M3); the double-computation (M2)-(M3) does not increase the computation cost twice. It can be made cheaper as follows. A step of the GMM advances a group of points including not only the global minimum (of the narrow band) but also all local minima that are less than or equal to \mathbf{TM} . It is not difficult to see if a point in the group is a local minimum of the narrow band. Since it is not necessary to compute twice at local minima, one can modify the algorithm to skip the double-computation there. Let the point (i, j, k) in (M2) be recognized as a local minimum. Then, one can update \mathbf{GAMMA} and \mathbf{idTT} as in (b) and (c) of (M3); the point (i, j, k) would be skipped by (M3), since it is already out of \mathbf{GAMMA} . More than half the points in a group seems a local minimum, in practice. The double-computation increases the computation cost, not twice, but *slightly*.

The GMM is in fact an iterative update procedure, converging in two iterations. One may want to select G with a larger $\delta\tau$. In the case, the number of iterations must become larger. Rouy and Tourin [14] has chosen all the grid points as one group and carried out iterations up to convergence. The GMM can be viewed as an intermediate algorithm between the FMM [16, 18] ($\delta\tau = 0$) and the purely iterative algorithm of Rouy and Tourin [14] ($\delta\tau = \infty$).

If one chooses $\delta\tau$ in (3.1)-(3.2) smaller, the order of traveltime computation becomes *essentially* the same as that of the FMM. The solutions of the GMM have been observed identical for smaller $\delta\tau$ for all velocity models tested, which means that the GMM has already the same accuracy as the FMM for the given $\delta\tau$ in (3.1).

When the slowness is constant, two group marchings advance the wavefront to a completely different outer surface. Such a feature can make the GMM more efficient than point-wise methods such as the FMM.

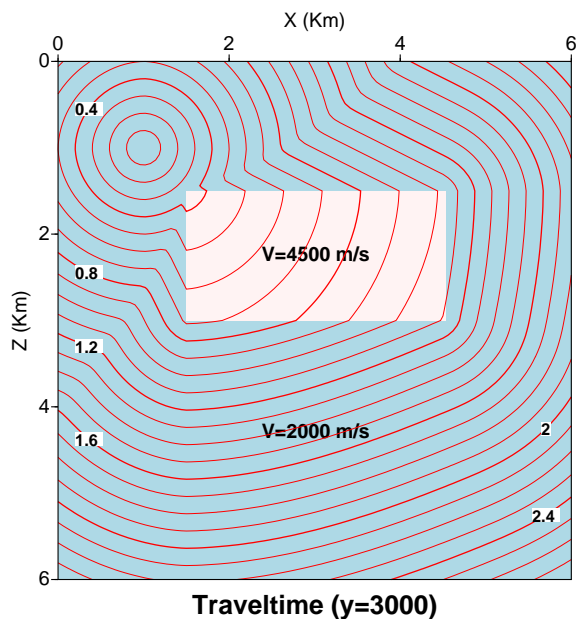


Figure 2: The computed traveltime is superposed on the cross section $\{y = 3000\}$ of the velocity $v = v_1$.

4. Numerical experiments

The GMM presented in the previous section is implemented in three dimensions for the first-arrival traveltimes of (2.1). Set the domain $\Omega = (0, 6000 \text{ m})^3$. We consider two different velocity models: for $\mathbf{x} \in \Omega$,

$$v_1(\mathbf{x}) = \begin{cases} 4500 \text{ m/s}, & \text{if } \mathbf{x} \in [1500, 4500]^3, \\ 2000 \text{ m/s}, & \text{else,} \end{cases}$$

$$v_2(\mathbf{x}) = 1000 + 0.2x + 0.2y + 0.5z \text{ m/s}.$$

Point sources are imposed inside or on the surface of the domain. The domain is partitioned into 100 grid points in each direction: $h = \Delta x = \Delta y = \Delta z = 6000/100 = 60 \text{ m}$. The traveltimes in the linear velocity can be computed analytically; the numerical error is measured as

$$E(\tau^h) = \|\tau^h - \tau_{\text{analytic}}\|_{\infty},$$

where τ^h denotes the computed traveltime with a grid size h . The main/driver routines are written in C++ and the core computation routines are in F77. The computation is carried out on a Gateway Solo, a 266 MHz laptop having 128M memory and a Linux operating system.

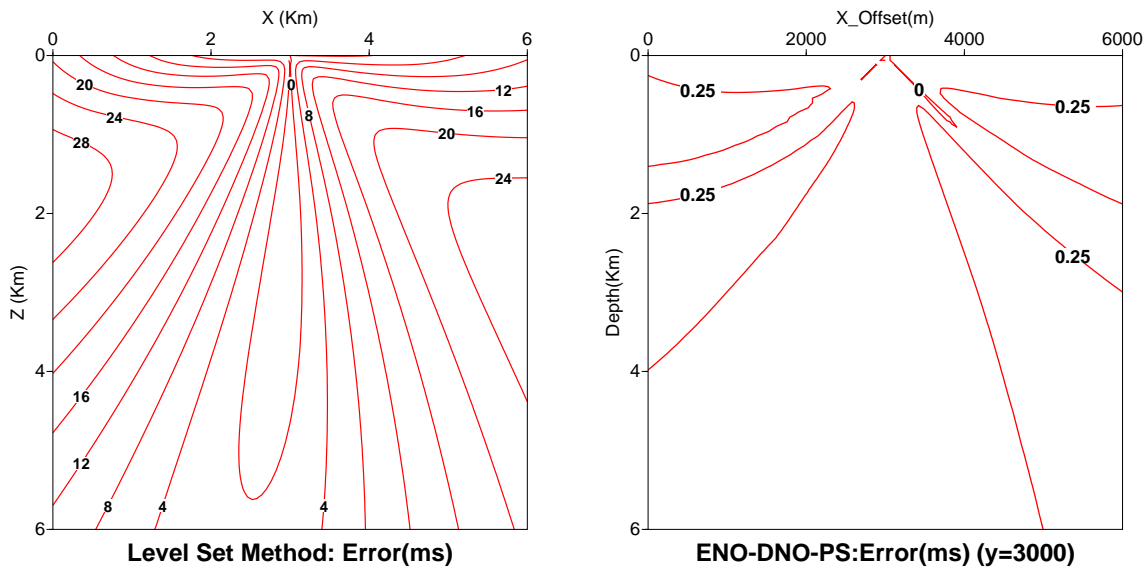


Figure 3: The error $\tau^h - \tau_{\text{analytic}}$ (in millisecond) for the GMM (left) and the ENO-DNO-PS (right).

In Fig. 2, we depict the computed traveltimes superposed on the cross section $\{y = 3000\}$ of the velocity model $v = v_1$. The source is located at $(x, y, z) = (1000, 3000, 1000)$. The GMM takes 39.03 seconds to solve the problem of one million unknowns. One can see from the picture that shocks are developed during the advancement of the wavefronts due to the headwave.

In Fig. 3, we compare accuracy and efficiency between the first-order level set method (GMM) and the second-order scheme ENO-DNO-PS [7, 10]. Set $v = v_2$ and the point source is located at $\mathbf{x}_s = (3000, 3000, 0)$. The GMM takes 38.48 seconds resulting in the maximum error of 59 milliseconds, while the ENO-DNO-PS takes 8.60 seconds and its maximum error becomes 0.58 milliseconds. The second-order scheme is 100 times more accurate and even 4.5 times faster than the first-order method, for the same size problem!

5. Discussion and Conclusions

An *optimal* level set method called the *group marching method* (GMM) has been developed. The GMM is implemented and tested for various models to have the same accuracy as the fast marching method (FMM). We believe that the GMM is a few times faster than the FMM, since the GMM costs $\mathcal{O}(N)$, while the FMM costs $\mathcal{O}(N \log_2 N)$. At least it will be true for large problems [5]. The GMM, an improved version of the FMM, is compared with a second-order scheme so-called the

ENO-DNO-PS [10]. The GMM still takes a few times more CPU time than the ENO-DNO-PS for the same size problem of a linear velocity; the first-order scheme shows 100 times larger error than the second-order algorithm. For efficiency and accuracy of ENO-DNO-PS in real velocity models, see [10].

We may investigate the high computation cost of the level set methods as follows. Both the FMM and the GMM access the data (such as the solution, the narrow band points, and the binary tree (FMM)) in an almost random manner, rather than a systematic way along the array indices. As a consequence, the algorithms can be more expensive than expected.

The level set methods are hard to incorporate high-order FD schemes. Sethian [18] suggested one-sided high-order schemes to be considered whenever the traveltimes at the corresponding points are available. Here one should be noticed that one-sided high-order FD schemes can produce a less accurate solution than the first-order scheme, in particular, in heterogeneous media. (That is indicated in most elementary numerical analysis textbooks.) Even if the medium is constant, the one-sided second-order scheme easily produces under-estimated solutions, as one can see from [18]. There is no way to correct under-estimated traveltimes, while over-estimation can be fixed through a minimizing process utilizing more of available values.

Overall, the level set methods are yet to be improved in both accuracy and efficiency to be *reasonably utilized in seismic applications*.

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