

# Snakes: sur la convexité de la fonctionnelle d'énergie

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## Abstract

### Résumé

*Un contours actif ou snake est une courbe régulière qui se déforme pour représenter le contour d'un objet présent dans une image. Le déplacement d'un snake au cours du temps est régi par la minimisation d'une fonctionnelle, en résolvant une équation d'évolution qui, à la limite, donne une solution du problème de minimisation. La fonctionnelle n'étant pas convexe, il n'est pas sûr que le minimum soit obtenu de cette façon, et la solution dépend de l'initialisation de la courbe. Si les algorithmes convergent généralement, il n'y a quasiment pas de preuve mathématique pour la convergence de l'algorithme vers une solution. On propose donc un algorithme assurant la convergence du snake vers le minimum de la fonctionnelle en un certain sens, en s'inspirant de l'algorithme de non convexité graduée (G.N.C.) pour l'énergie de Blake et Zisserman.*

*Mots clés : Contours Actifs, Modèles déformables, carte de distance, minimisation d'énergie.*

## 1 Introduction

The use of energy-minimizing curves, known as “snakes”, to extract features of interest in image has been introduced by Kass, Witkin and Terzopoulos [1], based on the use of deformable models [2]. Further improvements of this model were successively done by many authors [3, 4, 5, 6] and many other contributors. Recently, a different approach was introduced which defines the curve as a level set of a surface [7, 8]. One difficulty with minimizing the snake energy is that the potential term is not convex, while the smoothing term is quadratic. The same problem arises in minimizing the energy of Blake & Zisserman [9], fitting a deformable membrane to an image allowing discontinuities. There, the smoothing term is bounded, in order to introduce discontinuities. This makes the energy non convex because of this smoothing term, while the data term is quadratic. To obtain the minimization of this non convex energy, they modify the smoothing term to get a convex energy and then use the Graduated Non Convexity algorithm (G.N.C.). We use the same idea but apply it to the potential term.

*An active contour or snake is a smooth curve that deforms to represent the boundary of an object in an image. The deformation of a snake is defined by the minimization of an energy, solving the evolution equation that gives, as steady state, a solution of the minimization problem. Since the functional is not convex, it is not sure that the minimum can be obtained this way, and the evolution depends also of the curve initialization. If the algorithms generally converge, there is no mathematical proof of convergence of algorithms to a solution. We propose an algorithm that ensures the convergence of the snake to a minimum of the functional in some way, based on the Graduated Non Convexity algorithm (G.N.C.) of Blake et Zisserman's energy.*

*keywords : Active contours, deformable models, distance map, energy minimization.*

## 2 Motivation

### 2.1 Active Contour Models

The contour model, as introduced in [1], is a curve  $v(s) = (x(s), y(s))$  that minimizes an energy of the following form:

$$E(v) = \int_{\Omega} w_1 \|v'(s)\|^2 + w_2 \|v''(s)\|^2 + P(v(s)) ds$$

where  $P$  is the potential associated to the external forces.

If  $v$  is a local minimum for  $E$ , it satisfies the associated Euler-Lagrange equation:

$$\begin{cases} -(w_1 v')' + (w_2 v'')'' = -\nabla P(v) = F(v) \\ v(0), v'(0), v(1) \text{ and } v'(1) \text{ given.} \end{cases} \quad (1)$$

As explained in [3, 4], the energy is not convex, because of the potential term and there may be many local minima of  $E$ . To find a good contour in a given area, we suppose we have a rough estimate of the curve. We impose the condition to be “close” to this initial data by solving the

associated evolution equation :

$$\begin{cases} \frac{\partial v}{\partial t} - (w_1 v')' + (w_2 v'')'' = F(v), \\ v(0, s) = v_0(s), \\ v(t, 0) = v_0(0), \quad v(t, 1) = v_0(1), \\ v'(t, 0) = v'_0(0), \quad v'(t, 1) = v'_0(1). \end{cases} \quad (2)$$

A solution to the static problem (1) is achieved when the solution  $v(t)$  stabilizes.

## 2.2 Attraction Potential

As introduced and used in [3, 10, 4], the potential is derived from a set  $\mathcal{S}$  of already extracted **edge points**. The difficulty is then to solve the problems of segmentation and reconstruction from this unstructured set of points (see [4, 11]). The potential is a function of the distance  $d$  (computed from [12, 13]) to the closest edge point. For example, a gaussian function models weak strings that break when too long :

$$P(v) = -e^{-\sigma d(v)^2}$$

Remark that for the potential defined by  $P(v) = g(d(v))$ , the force becomes  $F(v) = -\nabla P(v) = -g'(d(v))\nabla d(v)$ . We can see that  $\|\nabla d(v)\| = 1$  outside the skeleton of  $\mathcal{S}$ . In our case, the force becomes

$$F(v) = -\nabla P(v) = -2\sigma d(v)e^{-\sigma d(v)^2}\nabla d(v),$$

So a good choice of  $\sigma$  permits to control the norm of the attraction force.

## 2.3 Local and global minimum

With the classic approaches, the solution depends on the initialization, and the curve may be trapped in a local minima. Thus, the starting curve has to be close enough to the solution. In previous work, various approaches were used to avoid being too much sensitive to the initialization and avoid local minima (balloon in [3], homogeneity of the inside region [14], geodesic approach [15]). In this paper, we study a way to apply the idea of the G.N.C. (Graduated Non Convexity) like Blake & Zisserman [9] for the discrete approach of the Mumford& Shah energy [16]. The main idea is that one term is "very" convex while the other lacks convexity. So, the non convex term is slightly modified in a way such that the sum of the two terms  $E_c$  is convex. Then, after finding the minimum for  $E_c$ , the minimum is found for a sequence of problems for which a progressive variation of the energy is made until the original problem is solved.

While in [9], the regularization term is non convex and the potential term is quadratic, in the snake energy, it is the contrary. The regularization term is quadratic and the potential is not convex. The idea is then to modify the potential using the same kind of function, but with a larger smoothing to make the energy convex. Our work can give some justification to the algorithm used in [17].

## 3 Convexity of the discrete energy

The convexity of the energy was also studied in [18], but with a different kind of potential to detect thick curves. A different approach to make convex our minimization problem using auxiliary variables is introduced in [11] giving interpretation of many two-step algorithms.

We now present our approach which is explained in more details in [19].

### 3.1 Discretization of the Active contour

We first formulate the discretization of the equation by finite differences. Representing the curve by a set of  $N$  nodes  $v_i = (x_i, y_i) \in \mathbb{R}^2$ ,

$$V = (x_0, \dots, x_N, y_0, \dots, y_N) \in \mathbb{R}^{2(N+1)},$$

A snake is obtained as the minimum of  $E_d$  defined from  $\mathbb{R}^{2(N+1)}$  to  $\mathbb{R}$  :

$$E_d(V) = E_r(V) + E_p(V)$$

Where  $E_r(V)$  is :

$$\frac{Nw_1}{2} \sum_{i=1}^N \|v_i - v_{i-1}\|^2 + \frac{N^3w_2}{2} \sum_{i=1}^{N-1} \|-2v_i + v_{i-1} + v_{i+1}\|^2,$$

$$E_p(V) = \frac{1}{N} \sum_{i=1}^N P(v_i).$$

A minimum  $V$  of  $E_d$  must verify the Euler-Lagrange condition.

If we define

$$B_1 = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & \ddots & 2 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 6 & -4 & 1 & & 0 \\ -4 & \ddots & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & 1 \\ & & \ddots & \ddots & -4 \\ 0 & & & 1 & -4 & 6 \end{pmatrix}$$

We show that

$$\text{for } i = 1, 2 \quad A_i = \begin{pmatrix} B_i & 0 \\ 0 & B_i \end{pmatrix} \quad H_P = \begin{pmatrix} \Sigma & \Gamma \\ \Gamma & \Psi \end{pmatrix}$$

$$A = Nw_1 A_1 + N^3 w_2 A_2$$

$$F = \frac{-1}{N} \left( \frac{\partial P}{\partial x}(v_0), \dots, \frac{\partial P}{\partial x}(v_N), \frac{\partial P}{\partial y}(v_0), \dots, \frac{\partial P}{\partial y}(v_N) \right),$$

then the Euler–Lagrange condition (1) becomes :

$$AV = F(V)$$

After formulating the evolution problem using finite differences with time step  $\tau$  we obtain a system of the form

$$(Id + \tau A)v^t = (v^{t-1} + \tau F(v^{t-1})), \quad (3)$$

where  $Id$  denotes the identity matrix. Thus, we obtain a linear system and we have to solve a pentadiagonal banded symmetric positive system. The solution is usually obtained using a  $LU$  decomposition of  $(Id + \tau A)$ . As shown in [10], the use of an integration on a segment between two nodes for computation of the force term enables to use much less nodes, in a way similar to the finite elements [4].

### 3.2 Condition for Convexity

To study the local convexity of the energy, we need to calculate its Hessian matrix:

If we set

$$Q_{k,i} = \begin{cases} 1 & \text{if } k = i \\ -1 & \text{if } k = i + 1 \\ 0 & \text{else} \end{cases}$$

$$R_{k,i} = \begin{cases} 2 & \text{if } k = i \\ -1 & \text{if } k = i + 1 \text{ or } k = i - 1 \\ 0 & \text{else} \end{cases}$$

We have  $A_1 = Q^t Q$ , and  $A_2 = R^t R$  and the Hessian matrix  $H$  of the energy functional  $E$  is

$$H = Q^t Q + R^t R + H(P) = H_{reg} + H_P,$$

where  $H_P$  is the Hessian matrix of the potential. The energy  $E$  is convex if and only if its Hessian matrix  $H$  is positive. It is simple to show that

**Lemma 3.1** *If  $A$  and  $B$  are two real symmetric matrices, the smallest eigenvalue of  $A + B$  is larger than or equal to the sum of the smallest eigenvalues of  $A$  and  $B$ .*

where  $\Sigma, \Gamma, \Psi$  are diagonal matrices.

$Q^t Q, R^t R$  and  $H(P)$  are real and symmetric with respectively smallest eigenvalues  $\lambda_1, \lambda_2, \lambda_3$ .

So, a sufficient condition for  $H$  to be positive, is:

$$\lambda_1 + \lambda_2 + \lambda_3 > 0. \quad (4)$$

Thus, we need to calculate  $\lambda_1, \lambda_2, \lambda_3$ . And then, we want to use (4) to find a condition under which the potential is convex.

$A_1$  and  $A_2$  are Toeplitz matrices. Using [20], we can calculate their smallest eigenvalues  $\lambda_1^N$  and  $\lambda_2^N$ :

$N^2 \lambda_1^N$  and  $N^4 \lambda_2^N$  converge fairly fast to values close respectively to  $\pi^2$  and  $\pi^4$  and therefore, in practice we can simplify the calculus by setting  $N^2 \lambda_1 \approx \pi^2$  and  $N^4 \lambda_2 \approx \pi^4$ . This also corresponds to the continuous case. Then, we have:

**Theorem 1** *Let  $\lambda_3$  be the smallest eigenvalue of the Hessian matrix of the potential,*

*if  $\lambda_3 > -2w_1\pi^2 - 2w_2\pi^4$ , then  $E$  is convex.*

We now want to calculate  $\lambda_3$ . Let  $c_i$  be the curvature at the closest edge point to  $v_i : p(v_i)$ , we show that, if  $\lambda_3^k$  is the  $k^{\text{th}}$  eigenvalue of  $H_P$ ,

$$\min_k \{\lambda_3^k\} \geq \min_i \left\{ \frac{2\sigma d_i e^{-\sigma d_i^2}}{d_i - \frac{1}{c_i}}, 2\sigma e^{-\sigma d_i^2} (1 - 2\sigma d_i^2) \right\}$$

If we suppose that the largest curvature  $c = \max_i \{c_i\}$ , satisfies:

$$c < S = e^{\frac{7}{4}} \sqrt{\omega_1 \pi^2 + \omega_2 \pi^4},$$

then we have:

**Theorem 2**

*If  $\sigma < \frac{1}{2}(\omega_1 \pi^2 + \omega_2 \pi^4)$ , then  $E$  is convex.*

In fact, in a way similar to [18], this only means that the energy is convex in a neighborhood of the minimum. The set of nodes have to be far enough from the skeleton of  $\mathcal{S}$ , this is usually the case almost immediately since skeleton points are maxima of distance and thus unstable for the minimization. This ensures convergence to the minimum when the curve is close enough to

the solution. This is a result that justifies the fact that many algorithms converge to an equilibrium. In the numerous works on active contour models, (except in [18, 7]), there is almost no justification showing such a property.

#### 4 Varying Scale Algorithm.

The energy  $E$  is approximated by a new functional which is now convex in the neighborhood of the solution and hence descent on  $E$  must converge to the minimum.

We consider  $E$  as a function of  $\sigma$ :  $E = E(\sigma)$ . Therefore, the original discrete energy  $E := E(\sigma^0)$  is not convex because  $\sigma^0$  is “too large”. Let  $E := E(\sigma^1)$ , where  $\sigma^1 < \frac{1}{2}(w_1\pi^2 + w_2\pi^4)$ , be the convex approximation of the the discrete energy.

Then, the strategy is to use a whole sequence of functionals  $E(\sigma^{(p)})$  for  $0 \leq p \leq K$ . These are chosen such that  $E(\sigma^{(0)}) = E(\sigma^1)$ , the convex approximation to  $E$ , and  $E(\sigma^{(K)}) := E(\sigma^0)$ , the original energy. In between,  $\sigma^{(p)}$  increases, between  $\sigma^1$  and  $\sigma^0$ .

The algorithm analog to G.N.C consists in the successive optimization of the whole sequence  $E(\sigma^{(p)})$ , one after the other, using the result of one optimization as the starting point for the next.

- i. choose  $K$ , the number of iterations. The smoothing discretization step will be  $\frac{|\sigma^0 - \sigma^1|}{K}$  and  $\sigma^{(0)} := \sigma^1$ . This defines  $\sigma^{(p)}$ , for all  $p = 0, \dots, K$ .
- ii. choose  $V^0$ , the initial curve
- iii. for all  $p = 0, \dots, K$ , starting with  $V^p$ , we minimize  $E(\sigma^{(p)})$  using (2), we obtain  $V^{p+1}$ .
- iv.  $V^{K+1}$  is the solution for the minimization of  $E(\sigma^0)$

#### 5 Conclusion

We showed a way to modify the attraction potential in order to make the energy locally convex. This gives a mathematical justification to the fact that the algorithm converges to a minimum of the energy. To minimize an energy of this kind, a varying scale algorithm is presented. It begins with the convexity hypothesis satisfied and then makes the scale vary from this initial scale to the original scale of the problem.

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