Theory and Applications of Classical and Quantum Kinetic Theory

Short time behavior for Rayleigh problem

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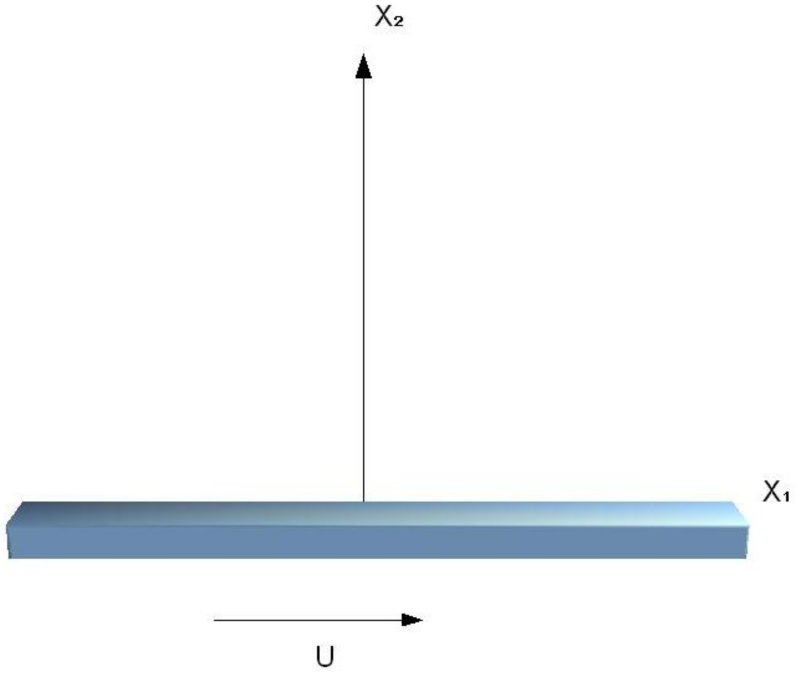
1 Introduction

1.1 Rayleigh Problem and motivation

What is Rayleigh problem?

Consider a flow in the upper half space bounded by a rigid plate coinciding with the X_1 axis.
At $t = 0$ the plate suddenly moves in the tangential direction at constant velocity U.
Find the development of the fluid motion in the region $X_2 > 0$.



Rayleigh considered that the fluid is viscous and imcompressible and that its motion is describable by the Navier-Stokes equation subject to the non-slip boundary condition.
Assume that the flow is uniform in X_1 and X_3.

$$\frac{\partial v_1}{\partial t} = \nu \frac{\partial^2 v_1}{\partial X_2^2},$$

where ν is kinematic viscosity and the boundary conditions and the initial condition are

$$\begin{aligned} v_1 &= U, \quad X_2 = 0, \quad t > 0, \\ v_1 &\to 0, \quad X_2 \to \infty, \quad t > 0, \\ v_1 &= 0, \quad X_2 > 0, \quad t = 0. \end{aligned}$$

The solution is well-known

$$v_1 = \frac{U}{2} Erfc(\frac{X_2}{2\sqrt{\nu t}}) = \frac{2U}{\sqrt{\pi}} \int_{\frac{X_2}{2\sqrt{\nu t}}}^{\infty} \exp(-r^2) dr.$$

Motivation:

Sone, Y. (1964), Kinetic Theory Analysis of Linearized Rayleigh Problem, J. Phys. Soc. Jpn 19.

Prof. Y. Sone discussed this Rayleigh problem by using the linearized BKW model equation. He shows

- For short time the solution represents a perturbation to the free molecular flow.
- For large time the solution has essential difference from the classical slip flow near the boundary.

In this talk, we concentrate our attention on short time behavior. Prof. Y. Sone gives the result in his paper:

For $\frac{x_2}{\sqrt{2RT_0 t}} \ll 1$,

$$\frac{v_1(x_2,t)}{U} \cong \frac{1}{2} - \frac{1}{\sqrt{2RT_0}\sqrt{\pi}}\frac{x_2}{t} + \lambda t\left[\frac{1}{8} + \frac{1}{2}\frac{1}{\sqrt{2RT_0}\sqrt{\pi}}\frac{x_2}{t}\cdot\left\{\log\frac{x_2}{\sqrt{2RT_0}t} + 1 + \frac{\gamma}{2} - \sqrt{2}\log\left(\sqrt{2}+1\right)\right\}\right] .$$

T_0 is the initail temperature of the system (or the wall temperature at any instant).

R is the specific gas constant.

λ is a constant (collision frequency) related to the mean free path.

$\gamma = 0.577\cdots$ is the Euler's constant.

In particular, on the plate

$$\frac{v_1(0,t)}{U} \cong \frac{1}{2} + \frac{\lambda t}{8}.$$

1.2 Boltzmann equation and BKW model equation

Let m be the mass of a molecule, $\xi = (\xi_1, \xi_2, \xi_3)$ the velocity of a molecule and $f = f(X_2, t, \xi)$ the velocity distribution function of the gas molecules. Then the Boltzmann equation with hard sphere model for the present problem may be written as

$$\frac{\partial f}{\partial t} + \xi_2 \frac{\partial f}{\partial X_2} = \frac{1}{m} \mathcal{J}(f, f),$$

where $\mathcal{J}(f, f)$ is the collision integral defined by

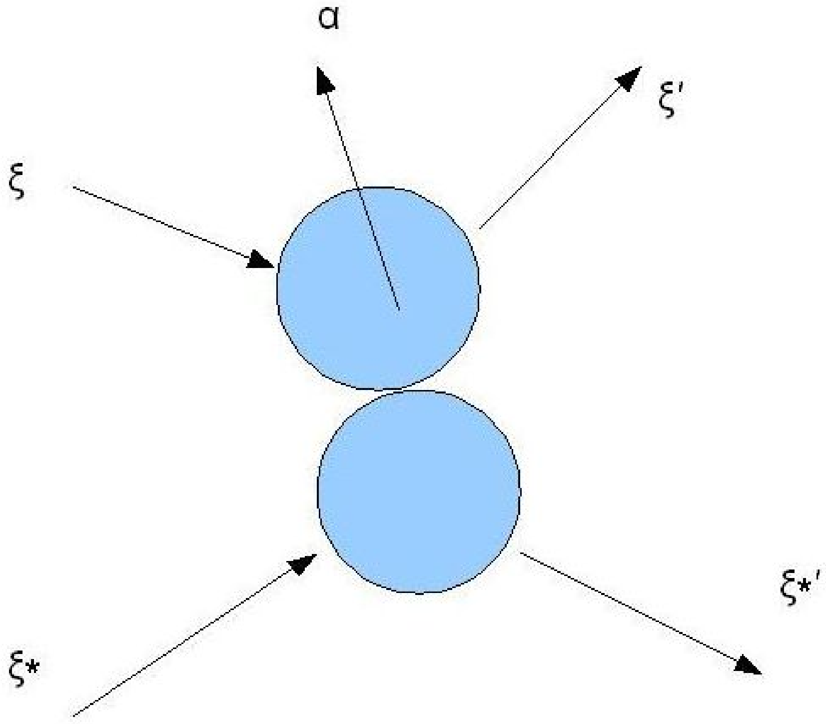
$$\begin{aligned} \mathcal{J}(f, f) &= \int\int (f' f'_* - f f_*) B(V, \alpha) d\Omega(\alpha) d\xi_* \\ &= \int\int f' f'_* B(V, \alpha) d\Omega(\alpha) d\xi_* - \int\int f f_* B(V, \alpha) d\Omega(\alpha) d\xi_*, \end{aligned}$$

$$f = f(X_2, t, \xi), \quad f_* = f(X_2, t, \xi_*), \quad f' = f(X_2, t, \xi'), \quad f'_* = f(X_2, t, \xi'_*),$$

$$V = \xi_* - \xi, \quad \xi' = \xi + (V \cdot \alpha)\, \alpha, \quad \xi'_* = \xi_* - (V \cdot \alpha)\alpha, \quad B(V, \alpha) = \frac{d^2 |V \cdot \alpha|}{2} \quad \text{(for hard sphere).}$$

d : the diameter of molecule.

α : a unit vector which expresses the variation of the direction of the molecular velocity owing to a molecular collsion.



The BKW(*Boltzmann-Krook-Welander*) model equation may be written as

$$\frac{\partial f}{\partial t}+\xi_2\frac{\partial f}{\partial X_2}=A_c\rho\left(f_e-f\right),$$

where A_c is a constant and f_e is the local Maxwellian defined by

$$f_e=\frac{\rho}{\left(2\pi RT\right)^{3/2}}\exp\left[-\frac{|\xi-v|^2}{2RT}\right].$$

Macroscopic variables :

The density ρ, fluid velocity $v=(v_1,v_2,v_3)$, and temperature T are given by

$$\rho(X_2,t)=\int f(X_2,t,\xi)d\xi,$$

$$v_i(X_2,t)=\frac{1}{\rho}\int \xi_i f(X_2,t,\xi)d\xi,\quad i=1,2,3,$$

$$T(X_2,t)=\frac{1}{3R\rho}\int |\xi-v|^2 f(X_2,t,\xi)d\xi.$$

The initial distribution is an equilibrium state

$$f = f_0 = \frac{\rho_0}{(2\pi R T_0)^{3/2}} \exp\left[-\frac{|\xi|^2}{2RT_0}\right] \text{ at } t = 0.$$

At the plate, the velocity and temperature are kept constant at U and $T_w = T_0$. Also assume the interaction of the gas with the boundary is described by diffuse reflection condition:

$$f_w(\xi, t) = \frac{\sigma_w(t)}{(2\pi R T_w)^{3/2}} \exp\left[-\frac{(\xi_1 - U)^2 + \xi_2^2 + \xi_3^2}{2RT_w}\right], \quad X_2 = 0, \xi_2 > 0,$$

where

$$\sigma_w(t) = -\left(\frac{2\pi}{RT_w}\right)^{1/2} \int_{\xi_2 < 0} \xi_2 f(0, t, \xi) d\xi.$$

At inifinity, we assume

$$f \to f_0 \text{ as } X_2 \to \infty.$$

1.3 Dimensionless

For a physical interpretation, we consider the following scale:

$$\begin{cases} \widehat{t} = \frac{t}{t_0} & x_i = \frac{X_i}{L} & \zeta_i = \frac{\xi_i}{(2RT_0)^{1/2}} & \widehat{f} = \frac{f}{\rho_0(2RT_0)^{-3/2}} \\ \widehat{\rho} = \frac{\rho}{\rho_0} & \widehat{v}_i = \frac{v_i}{(2RT_0)^{1/2}} & \widehat{T} = \frac{T}{T_0} & \\ u_{w1} = \frac{U}{(2RT_0)^{1/2}} & \widehat{v}_{w2} = 0 & \widehat{v}_{w3} = 0 & \widehat{T}_w = \frac{T_w}{T_0} = 1 \end{cases}$$

- reference length $L = \frac{\sqrt{\pi}}{2} l_0$ where l_0 is the mean free path for the initial global Maxwellian.
- reference time $t_0 = \frac{\sqrt{\pi}}{2} \frac{l_0}{(2RT_0)^{1/2}}$ is relative to mean free time for the initial global Maxwellian.
- reference magnitude of velocity $(2RT_0)^{1/2}$ is relative to the sound speed in the initial state.
- Define $E = \pi^{-3/2} \exp\left[-|\zeta|^2\right]$ which satisfies $f_0 = \frac{\rho_0}{(2RT_0)^{3/2}} E$.
- Note that $\frac{x_2}{\widehat{t}} = \frac{X_2/L}{t/t_0} = \frac{X_2/t}{(2RT_0)^{1/2}}$.

Thus we consider the initial-boundary value problem for the Boltzmann equation:

$$\begin{cases} \frac{\partial \widehat{f}}{\partial \widehat{t}} + \zeta_2 \frac{\partial \widehat{f}}{\partial x_2} = \widehat{\mathcal{J}}(\widehat{f}, \widehat{f}),\ x_2 > 0,\ \widehat{t} > 0, \\ \widehat{f} = E,\ x_2 > 0,\ \widehat{t} = 0, \\ \widehat{f} = \frac{\widehat{\sigma}_w(\widehat{t})}{\pi^{3/2}} e^{-(\zeta_1 - u_{w1})^2 - \zeta_2^2 - \zeta_3^2},\ x_2 = 0,\ \zeta_2 > 0,\ \widehat{t} > 0 \\ \widehat{\sigma}_w(\widehat{t}) = -2\sqrt{\pi} \int_{\zeta_2 < 0} \zeta_2 \widehat{f}(0, \widehat{t}, \zeta) d\zeta, \\ \widehat{f} \to E \ \text{ as } x_2 \to \infty. \end{cases} \tag{1}$$

For the BKW model equation, the first equation of (1) is replaced by

$$\frac{\partial \widehat{f}}{\partial \widehat{t}} + \zeta_2 \frac{\partial \widehat{f}}{\partial x_2} = \widehat{\rho} \left(\widehat{f}_e - \widehat{f} \right),$$

where the nondimensional form of local Maxwellian is given by

$$\widehat{f}_e = \frac{\widehat{\rho}}{\left(\pi \widehat{T}\right)^{3/2}} \exp\left[-\frac{|\zeta - \widehat{v}|^2}{\widehat{T}} \right].$$

The nondimensional form of density $\widehat{\rho}$, fluid velocity $\widehat{v} = (\widehat{v}_1, \widehat{v}_2, \widehat{v}_3)$, and temperature $\widehat{T}$ are given by

$$\widehat{\rho} = \int \widehat{f} d\zeta,$$

$$\widehat{\rho}\widehat{v}_i = \int \zeta_i \widehat{f} d\zeta, \ i = 1, 2, 3,$$

$$\frac{3}{2}\widehat{\rho}\widehat{T} = \int |\zeta - \widehat{v}|^2 \widehat{f} d\zeta.$$

1.4 Perturbation around the Maxwellian

Consider a pertubation of (1) around the initial Maxwellian E by $\widehat{f} = E + F\phi$ and thereby the initial-boundary value problem

$$\begin{cases} \frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} = \mathcal{L}\phi + \mathcal{J}(\phi, \phi),\ x_2 > 0,\ \widehat{t} > 0, \\ E + F\phi = \frac{\widehat{\sigma}_w(\widehat{t})}{\pi^{3/2}} e^{-(\zeta_1 - u_{w1})^2 - \zeta_2^2 - \zeta_3^2}\text{, } x_2 = 0,\ \zeta_2 > 0,\ \widehat{t} > 0, \\ \widehat{\sigma}_w(\widehat{t}) = 1 - 2\sqrt{\pi} \int_{\zeta_2 < 0} \zeta_2 F\phi(0, \widehat{t}, \zeta) d\zeta, \\ \phi = 0\text{, } \widehat{t} = 0,\ x_2 > 0, \\ \phi \to 0 \text{ as } x_2 \to \infty. \end{cases} \tag{2}$$

where

$$F = \sqrt{E},$$

$$\mathcal{L}\phi = \int\int F_* \left(F'_* \phi' + F' \phi'_* - F_* \phi - F \phi_* \right) \widehat{B} d\Omega(\alpha) d\zeta_*,$$

$$\textsf{J}(\phi, \phi) = \int\int F_* \left(\phi'_* \phi' - \phi_* \phi \right) \widehat{B} d\Omega(\alpha) d\zeta_*.$$

For the BKW model equation, the first equation of (2) is replaced by

$$\frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} = (1+\omega)\left(\phi_e - \phi\right),$$

where

$$F\phi_e = \frac{1+\omega}{\pi^{3/2}\left(1+\tau\right)^{3/2}} \exp\left[-\frac{|\zeta - u|^2}{1+\tau}\right] - E.$$

The perturbation variables of density ω, fluid velocity $u = \left(u_1, u_2, u_3\right)$, and temperature τ are given by

$$\omega = \int \phi F d\zeta,$$

$$(1+\omega)u_i = \int \zeta_i \phi F d\zeta,\ \ i = 1, 2, 3,$$

$$\frac{3}{2}(1+\omega)\tau = \int \left(|\zeta|^2 - \frac{3}{2}\right) \phi F d\zeta - (1+\omega)|u|^2.$$

1.5 Linearized Boltzmann equation and linearized BKW model equation

When the velocity of the plate U is assumed to be much less than the velocity of sound in the gas (i.e. $u_{w1} = \frac{U}{2RT_0} << 1$) so that the equation as well as the boundary condtion may be linearized, we neglect all higher order terms and consider the linearized Boltzmann equation and BKW model equation:

$$\begin{cases} \frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} = \mathcal{L}\phi, \ x_2 > 0, \ \widehat{t} > 0, \\ \phi = \check{\sigma}_w(\widehat{t})F + 2\zeta_1 u_{w1} F, \, x_2 = 0, \ \zeta_2 > 0, \ \widehat{t} > 0, \\ \check{\sigma}_w(\widehat{t}) = -2\sqrt{\pi} \int_{\zeta_2<0} \zeta_2 F \phi(0, \widehat{t}, \zeta) d\zeta, \\ \phi = 0, \, \widehat{t} = 0, \ x_2 > 0, \\ \phi \to 0 \ \text{ as } x_2 \to \infty. \end{cases} \tag{3}$$

$$\begin{cases} \frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} = -\phi + \omega F + 2(u \cdot \zeta) F + \tau \left(|\zeta|^2 - \frac{3}{2} \right) F, \ x_2 > 0, \ \widehat{t} > 0, \\ \phi = 0, \, \widehat{t} = 0, \ x_2 > 0, \\ \phi \to 0 \ \text{ as } x_2 \to \infty, \\ \phi = \check{\sigma}_w(\widehat{t})F + 2u_{w1}\zeta_1 F, \, x_2 = 0, \ \zeta_2 > 0, \ \widehat{t} > 0, \\ \check{\sigma}_w(\widehat{t}) = -2\sqrt{\pi} \int_{\zeta_2<0} \zeta_2 F \phi(0, \widehat{t}, \zeta_i) d\zeta. \end{cases} \tag{4}$$

The linearized version of the perturbed density ω, the fluid velocity $u = (u_1, u_2, u_3)$, and the temperature τ are given by

$$\begin{cases} \omega = \int \phi F d\zeta, \\ u_i = \int \zeta_i \phi F d\zeta, \ i = 1,\ 2,\ 3, \\ \frac{3}{2}\tau = \int (|\zeta|^2 - \frac{3}{2}) \phi F d\zeta. \end{cases} \tag{5}$$

The linearized collision operator is of the form

$$\mathcal{L}(\phi) = K(\phi) - \nu(\zeta)\phi$$

which K is an integral operator with kernal

$$k(\zeta, \zeta_*) = 2^{-3/2}\pi^{-1} \left\{ 2|\zeta_* - \zeta|^{-1} \exp\left[-\frac{1}{4} \frac{\left(|\zeta_*|^2 - |\zeta|^2\right)^2}{|\zeta_* - \zeta|^2} - \frac{1}{4}|\zeta_* - \zeta|^2 \right] \right.$$
$$\left. -|\zeta_* - \zeta| \exp\left[-\frac{1}{2}\left(|\zeta_*|^2 + |\zeta|^2\right) \right] \right\}$$

and $\nu(\zeta)$ is a miltiplicative operator

$$\nu(\zeta) = 2^{-3/2} \left[\exp(-|\zeta|^2) + \left(2|\zeta| + \frac{1}{|\zeta|} \right) \int_0^{|\zeta|} \exp\left(-r^2\right) dr \right].$$

2 Main Results

Theorem 1 (Linearized BKW model equation) There exists a positive solution $u_1^*(\cdot,\widehat{t}) \in L^\infty(\mathbb{R}^+)$ *for* $\widehat{t} > 0$ *which satisfies the integral equation .*

$$u_1(x_2,\widehat{t}) = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^{\widehat{t}} u_1(y,s)\frac{1}{\widehat{t}-s}\exp\left[-\left(\frac{x_2-y}{\widehat{t}-s}\right)^2 + s - \widehat{t}\right] dsdy + \frac{u_{w1}}{\sqrt{\pi}}\int_{\frac{x_2}{\widehat{t}}}^\infty \exp\left(-r^2 - \frac{x_2}{r}\right) dr,$$

and

$$\omega(x_2,\widehat{t}) = u_2(x_2,\widehat{t}) = u_3(x_2,\widehat{t}) = \tau(x_2,\widehat{t}) = 0. \tag{6}$$

Moreover,

$$u_1^*(x_2,\widehat{t}) \le \frac{u_{w1}}{2}e^{-\left(\frac{x_2}{\widehat{t}}\right)^2}\left\{1+\widehat{t}+O(\widehat{t}^2)\right\} \textit{ for } \widehat{t} << 1, \tag{7}$$

$$u_1^*(x_2,\widehat{t}) \ge \frac{u_{w1}}{4e^2}e^{-\left(\frac{x_2}{\widehat{t}}\right)^2} + \frac{u_{w1}}{8e^3}e^{-\left(\frac{x_2}{\widehat{t}}\right)^2}\widehat{t} \textit{ for } 0 < x_2 < \widehat{t} << 1, \tag{8}$$

On the plate,

$$\frac{u_{w1}}{2} + \frac{u_{w1}}{4}\frac{e^{-1}}{2^{3/2}}\frac{\widehat{t}}{2} \le u_1^*(0,\widehat{t}) \le \frac{u_{w1}}{2} + \frac{u_{w1}}{2}\widehat{t} + O(\widehat{t}^2) \textit{ for } \widehat{t} << 1. \tag{9}$$

Theorem 2 (Linearized Boltzmann equation) There exists a unique solution $\phi(x_2,\widehat{t},\cdot)\in L^\infty_\zeta$ *of equation (*3*) which satisfies*

$$\left\|\phi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta}\equiv\sup_{\zeta\in\mathbb{R}^3}|\phi(x_2,\widehat{t},\zeta)|\leq u_{w1}\sqrt{2}\pi^{-3/4}e^{-1/4}e^{-\nu_0x_2+C_1\widehat{t}},\ x_2\geq 0,\ \widehat{t}\geq 0, \tag{10}$$

for some constant $C_1>0$. *Furthermore, the flow velocity* u_1 *in* x_1*-direction satisfies*

$$u_1(x_2,\widehat{t})\leq\frac{u_{w1}}{\sqrt{\pi}}e^{-\nu_0x_2}J_0\left(\nu_0x_2\right)+u_{w1}\frac{4\sqrt{2}C_1}{\sqrt{\pi}e^{1/4}}e^{-\nu_0x_2}\widehat{t}+u_{w1}O(\widehat{t}^2)\ \textit{for}\ \widehat{t}<<1, \tag{11}$$

$$u_1(x_2,\widehat{t})\geq Cu_{w1}e^{-2\nu_1x_2-2\left(\frac{x_2}{\widehat{t}}\right)^2}-u_{w1}\frac{4\sqrt{2}C_1}{\sqrt{\pi}e^{1/4}}e^{-\nu_0x_2}\widehat{t}-u_{w1}O(\widehat{t}^2)\ \textit{for}\ \widehat{t}<<1, \tag{12}$$

and

$$\omega(x_2,\widehat{t})=u_2(x_2,\widehat{t})=u_3(x_2,\widehat{t})=\tau(x_2,\widehat{t})=0. \tag{13}$$

Here $J_0(x)$ *is defined as:* $J_0(x)\equiv\int_0^\infty e^{-r^2-\frac{x}{r}}dr$.
In particular, on the plate the flow velocity satisfies

$$|u_1(0,\widehat{t})-\frac{u_{w1}}{2}|=u_{w1}\frac{2\sqrt{2}C_1}{\sqrt{\pi}e^{1/4}}\widehat{t}+u_{w1}O(\widehat{t}^2)\ \textit{as}\ \widehat{t}<<1. \tag{14}$$

Remark 1 (12) shows that near the plate and for short time, $0 < x_2 < \widehat{t} << 1$

$$u_1(x_2, \widehat{t}) > \varepsilon u_{w1} > 0 \text{ for some } \varepsilon > 0,$$

and (14) shows at the plate for short time $\left(0 < \widehat{t} << 1\right)$

$$u_1(0, \widehat{t}) > \frac{u_{w1}}{4} > 0 \ .$$

Theorem 3 (Full Boltzmann equation) For each $\Gamma > 0$ *, there exsits* $\delta > 0$ *such that whenever* $u_{w1} \in (0,\delta)$ *the solution of (2) exists for* $0 < \widehat{t} \leq \Gamma$ *and satisfies*

$$\left\|\phi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \leq 2u_{w1}C_2 e^{-\nu_0 x_2 + 2(C_3+1)C_1\widehat{t}},\ \ x_2 \geq 0,\ 0 < \widehat{t} \leq \Gamma, \tag{15}$$

for some constants C_1, C_2, $C_3 > 0$ *and* $0 < \widehat{t} \leq \Gamma$*. Futhermore*

$$\widehat{\rho}(x_2,\widehat{t})u_1(x_2,\widehat{t}) \leq \frac{u_{w1}}{\sqrt{\pi}} e^{-\nu_0 x_2} J_0\left(\nu_0 x_2\right) + u_{w1} 2C_1C_2 e^{-\nu_0 x_2} 4\pi^{1/4}\widehat{t} + O(u_{w1}^3 + u_{w1}^2\widehat{t} + u_{w1}\widehat{t}^2), \tag{16}$$

$$\widehat{\rho}(x_2,\widehat{t})u_1(x_2,\widehat{t}) \geq C u_{w1} e^{-2\nu_1 x_2 - 2\left(\frac{x_2}{\widehat{t}}\right)^2} - u_{w1} 2C_1C_2 e^{-\nu_0 x_2} 4\pi^{1/4}\widehat{t} - O(u_{w1}^3 + u_{w1}^2\widehat{t} + u_{w1}\widehat{t}^2), \tag{17}$$

where $C > 0$ *is a constant.*
In particular, at the plate

$$|\widehat{\rho}(0,\widehat{t})u_1(0,\widehat{t}) - \frac{u_{w1}}{2}| = u_{w1} 4\pi^{1/4} C_1C_2\widehat{t} + O(u_{w1}\widehat{t}^2 + u_{w1}^2\widehat{t}). \tag{18}$$

Here $\widehat{\rho} = 1 + \omega$ *and* $\omega = O(u_{w1}^2)$.

3 Compare with the free molecular flow

In our resultes, we condiser short time $\widehat{t} = \frac{t}{t_0} << 1.$
That is, time t is much less than the mean free time t_0.
In this case, collision term is not important. Thus we consider the free transport equation

$$\begin{cases} \frac{\partial \widehat{f}}{\partial \widehat{t}} + \zeta_2 \frac{\partial \widehat{f}}{\partial x_2} = 0,\ x_2 > 0,\ \widehat{t} > 0, \\ \widehat{f} = E,\ \widehat{t} = 0,\ x_2 > 0, \\ \widehat{f}_w = \frac{\widehat{\sigma}_w}{\pi^{3/2}} e^{-(\zeta_1 - u_{w1})^2 - \zeta_2^2 - \zeta_3^2},\ x_2 = 0,\ \zeta_2 > 0,\ \widehat{t} > 0, \\ \widehat{\sigma}_w(\widehat{t}) = -2\sqrt{\pi} \int_{\zeta_2 < 0} \zeta_2 \widehat{f}(0, \zeta_i, \widehat{t}) d\zeta, \\ \widehat{f} \to E \text{ as } x_2 \to \infty. \end{cases}$$

By direct computations one can obtain the solution $\widehat{f}$ and the flow velocity in x_1 direction

$$\begin{cases} \widehat{f}(x_2, \widehat{t}, \zeta) = E \text{ for } \zeta_2 < \frac{x_2}{\widehat{t}}, \\ \widehat{f}(x_2, \widehat{t}, \zeta) = \frac{1}{\pi^{3/2}} e^{-(\zeta_1 - u_{w1})^2 - \zeta_2^2 - \zeta_3^2} \text{ for } \zeta_2 > \frac{x_2}{\widehat{t}}, \end{cases}$$

Thus we obtain the macroscopic variables

$$\widehat{v}_1(x_2,\widehat{t}) = \frac{u_{w1}}{2} Erfc\left(\frac{x_2}{\widehat{t}}\right) = \frac{u_{w1}}{\sqrt{\pi}} \int_{\frac{x_2}{\widehat{t}}}^{\infty} \exp(-\zeta_2^2) d\zeta_2, \tag{19}$$

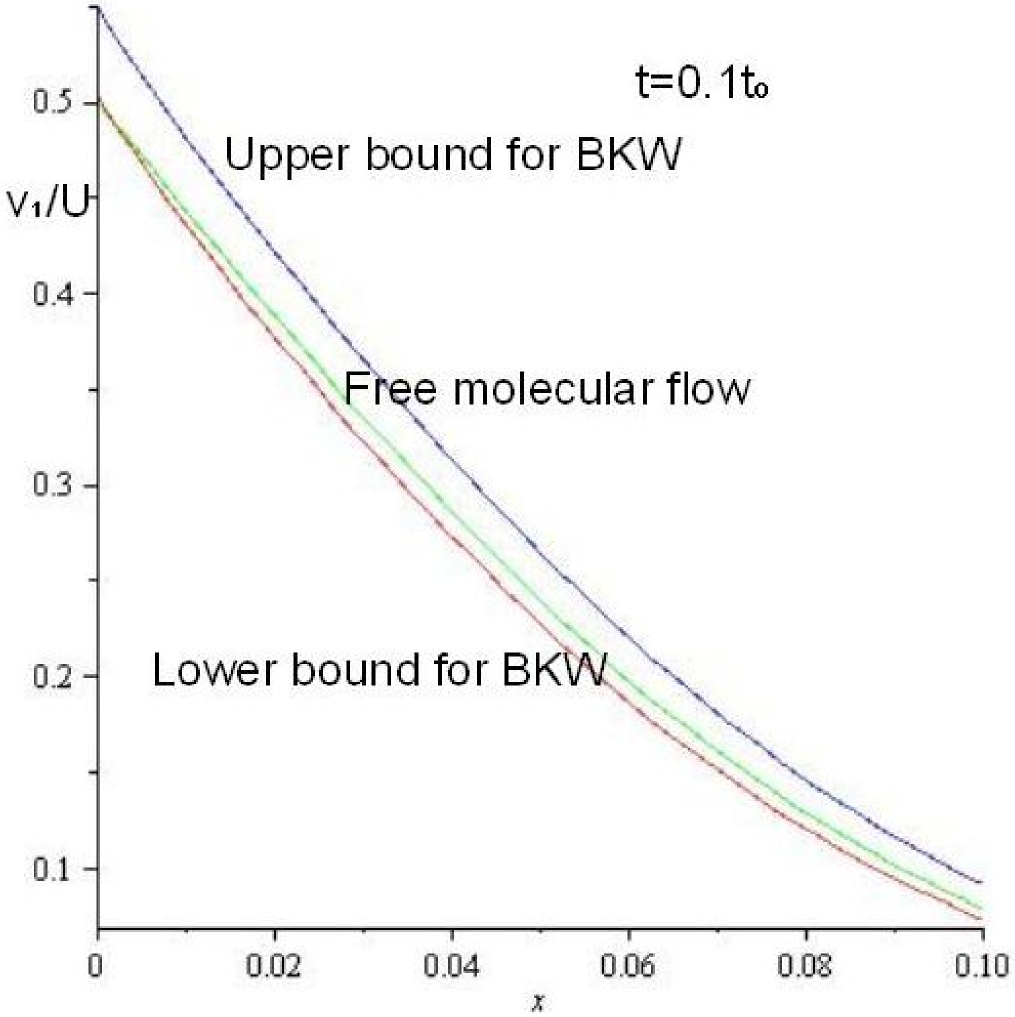
$$\widehat{v}_2(x_2,\widehat{t}) = 0$$

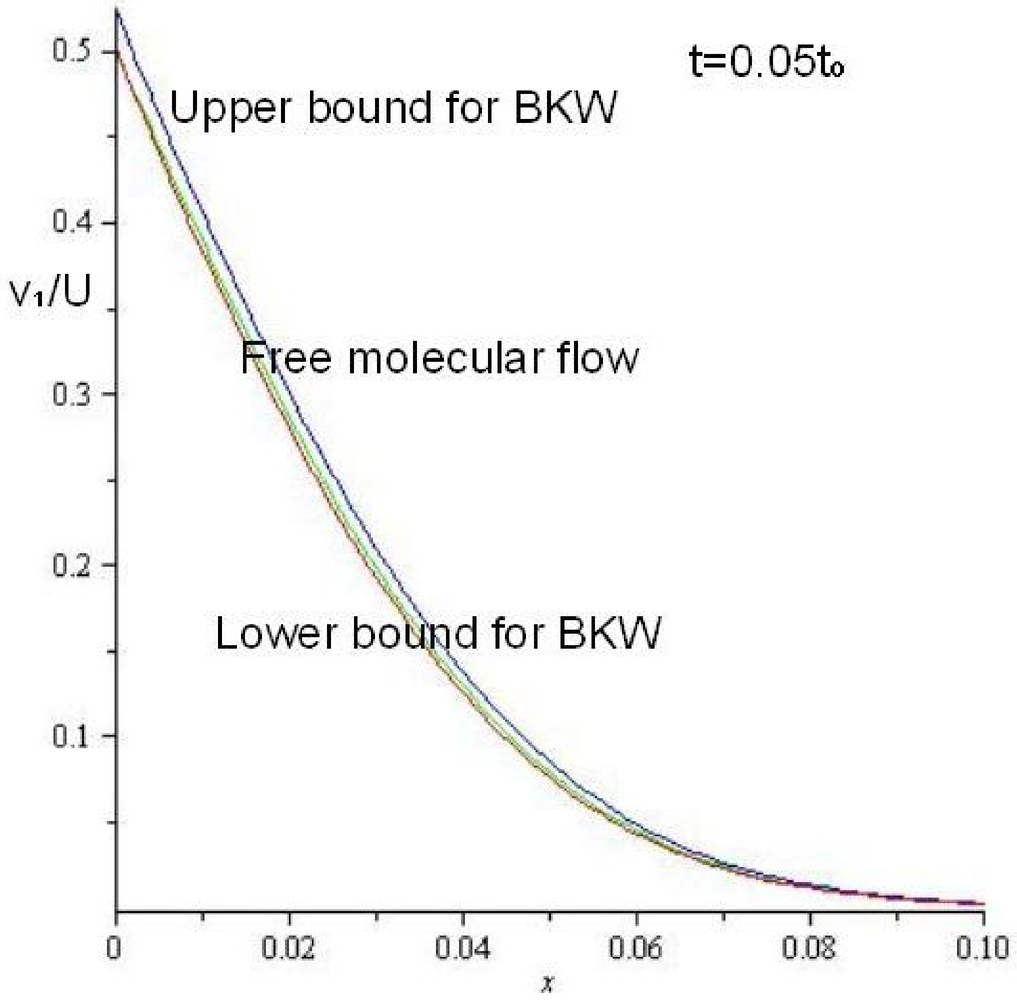
$$\widehat{v}_3(x_2,\widehat{t}) = 0,$$

$$\widehat{\rho}(x_2,\widehat{t}) = 1, \text{ or } \omega(x_2,\widehat{t}) = 0,$$

$$\frac{3}{2}\widehat{T} = \frac{3}{2} + \widehat{v}_1\left(u_{w1} - \widehat{v}_1\right), \text{ or } \tau(x_2,\widehat{t}) = \widehat{v}_1\left(u_{w1} - \widehat{v}_1\right) = O(u_{w1}^2),$$

In particular, in the original scaling $v_1(0,t) = \frac{U}{2}$ on the plate.


t=0.1t₀
Upper bound for BKW
Free molecular flow
Lower bound for BKW
V₁/U
0.5
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.10
x


t=0.05t₀
Upper bound for BKW
Free molecular flow
Lower bound for BKW
v₁/U
0.5
0.4
0.3
0.2
0.1
0
0.02
0.04
0.06
0.08
0.10
x

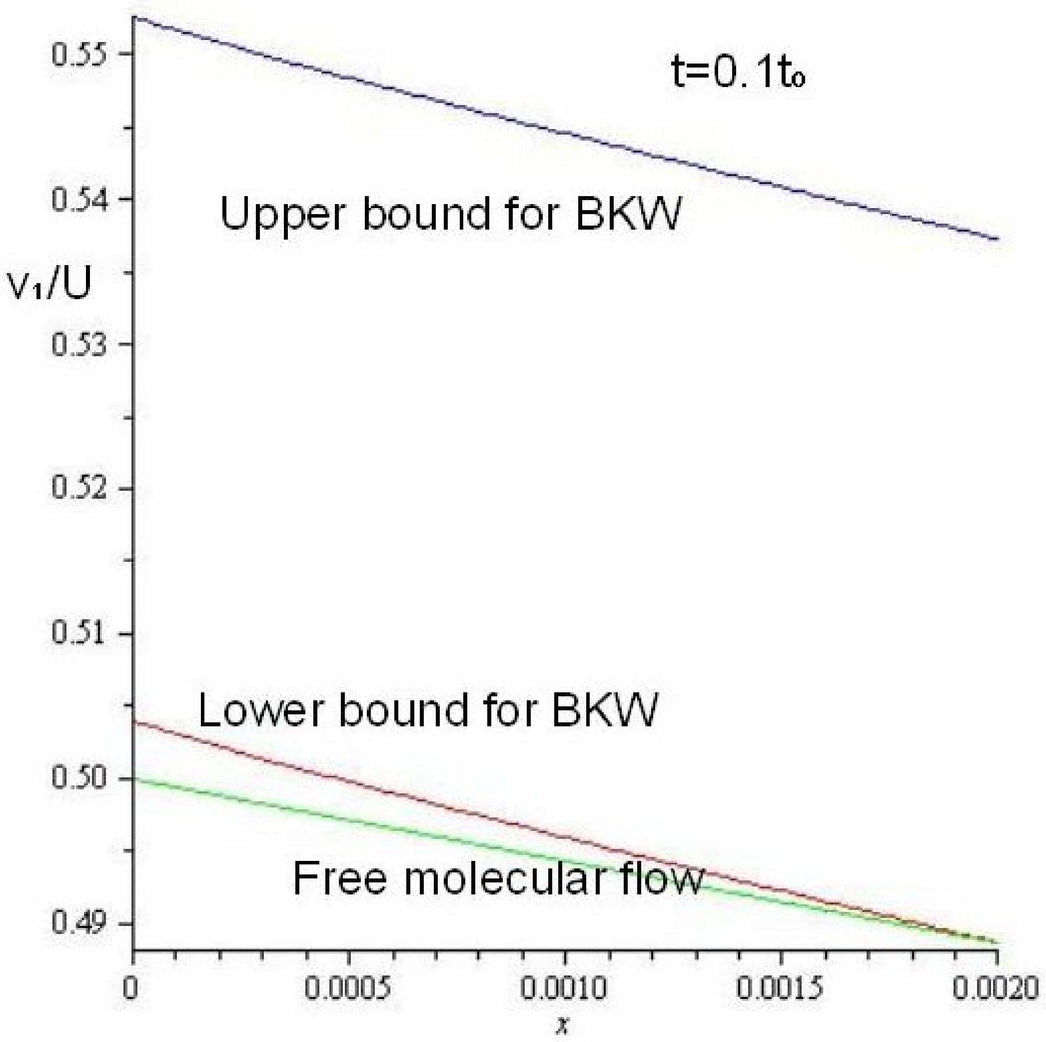
In the case of the linearized BKW model equation, we have, more precisely,

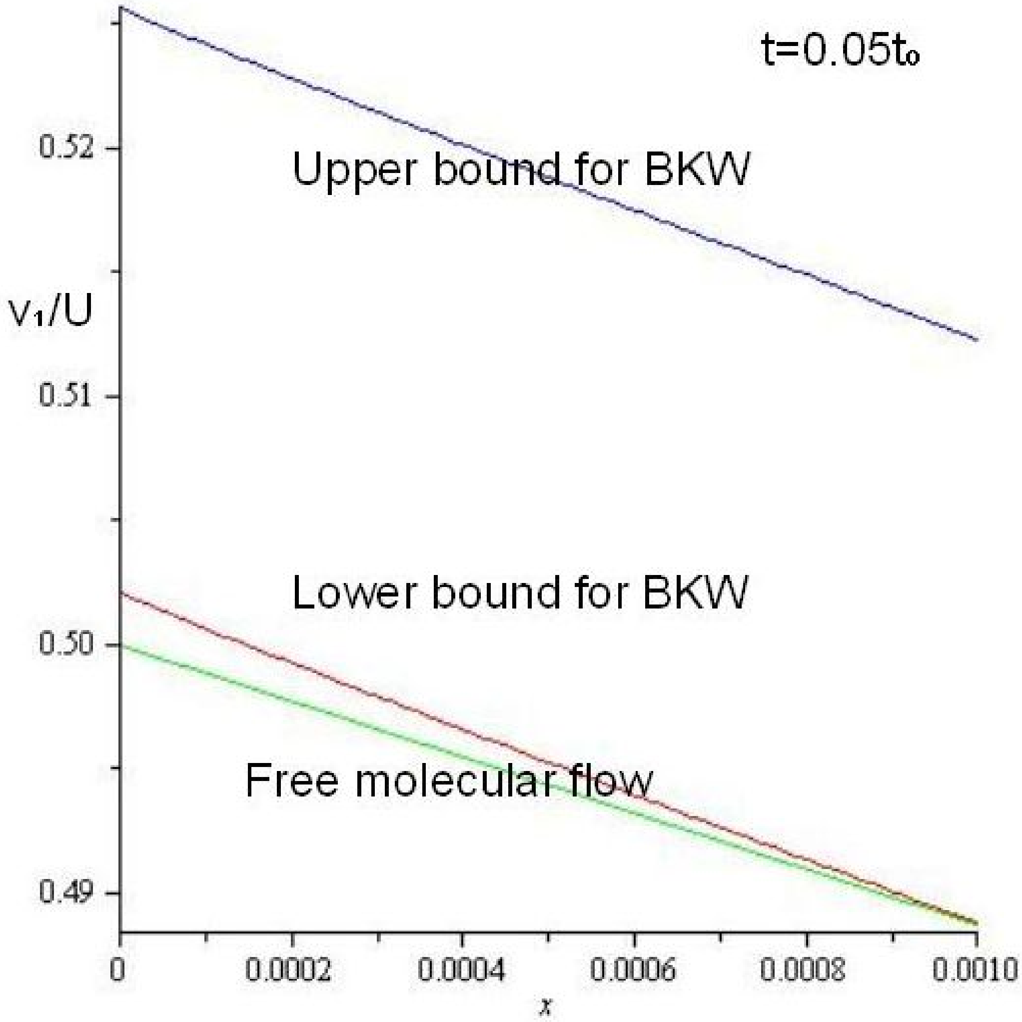
$$u_1^*(x_2,\widehat{t}) \leq \frac{u_{w1}}{\sqrt{\pi}} \int_{\frac{x_2}{\widehat{t}}}^{\infty} \exp\left(-r^2 - \frac{x_2}{r}\right) dr + \frac{u_{w1}}{2} e^{-\left(\frac{x_2}{\widehat{t}}\right)^2} \left[e^{\widehat{t}} - 1\right]$$

and

$$u_1^*(x_2,\widehat{t}) \geq \frac{u_{w1}}{\sqrt{\pi}} \int_{\frac{x_2}{\widehat{t}}}^{\infty} \exp\left(-r^2 - \frac{x_2}{r}\right) dr + u_{w1} \frac{\widehat{t}}{16\sqrt{2}} e^{-\widehat{t}} e^{-3\left(\frac{x_2}{\widehat{t}}\right)^2}. \tag{20}$$

Comparing (19) with (20) and (9) in the region $\frac{x_2}{\widehat{t}} \ll 1$ or $\frac{X_2}{t} \ll (2RT_0)^{1/2}$ (the original scale).


t=0.1t₀
Upper bound for BKW
Lower bound for BKW
Free molecular flow
V₁/U
0.55
0.54
0.53
0.52
0.51
0.50
0.49
0
0.0005
0.0010
0.0015
0.0020
x


t=0.05t₀
Upper bound for BKW
Lower bound for BKW
Free molecular flow
V₁/U
0.52
0.51
0.50
0.49
0
0.0002
0.0004
0.0006
0.0008
0.0010
x

Conclusions:

- The flow velocity is a small perturabtion of the order of $\widehat{t}$ to the free molecular flow for short time.
- Due to the collision effect, the flow is more accelerated than the free molecular flow on and near the plate.

Remark 2 The conclusions above agree with Professor Sone's results.

Remark 3 We choose a microscopic scale ($L \sim$ mean free path ; $t_0 \sim$ mean frre time) and hence the particlelike waves dominate the initail layer. But these scales are not adequate for studying the long time behavior.

4 Sketch of Proof

Recall the linearized collision integral $\mathcal{L} = K - \nu$.

Lemma 1 There exists a constant $C_1 > 0$ *such that* $\int |k(\zeta, \zeta_*)| d\zeta_* \leq C_1$ *for all* $\zeta \in \mathbb{R}^3$.

Lemma 2 There exists constants $\nu_0, \nu_1 > 0$ *such that* $\nu_0(1 + |\zeta|) \leq \nu(\zeta) \leq \nu_1(1 + |\zeta|)$ *for all* $\zeta \in \mathbb{R}^3$.

Lemma 3 If ϕ *solves* (3) *or* (4)*, then* ϕ *is an odd function in* ζ_1.

Proof. We set $\phi = \phi_o + \phi_e$. ϕ_o is odd in ζ_1 and ϕ_e is even in ζ_1. For system (3), from the isotropy property of $\mathcal{L}$ and linearity, we have

$$\begin{cases} \frac{\partial \phi_e}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi_e}{\partial x_2} = \mathcal{L}(\phi_e), \ x_2 > 0, \ \widehat{t} > 0, \\ \phi_e = 0, \, x_2 > 0, \ \widehat{t} = 0, \\ \phi_e = -2\sqrt{\pi} \int_{\zeta_2 < 0} \zeta_2 F \phi_e(0, \widehat{t}, \zeta) d\zeta F, \, x_2 = 0, \ \zeta_2 > 0, \ \widehat{t} > 0. \\ \phi_e \to 0 \text{ as } x_2 \to \infty. \end{cases}$$

Then by uniqueness of the linearized Boltzmann equation, $\phi_e = 0$ and so $\phi = \phi_o$, odd in ζ_1.
Similar arguments apples for the linearized BKW model. ■

Theorem 1 (Linearized BKW model equation)

1. By Lemma 3, we have

$$\check{\sigma}_w(\widehat{t}) = \omega(x_2,\widehat{t}) = u_2(x_2,\widehat{t}) = u_3(x_2,\widehat{t}) = \tau(x_2,\widehat{t}) = 0$$

because (5) and the odd property of ϕ. This proves (6). Then we consider equation

$$\begin{cases} \frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} = -\phi + 2u_1\zeta_1 F, \ x_2 > 0, \ \widehat{t} > 0, \\ \phi = 0, \ x_2 > 0, \ \widehat{t} = 0, \\ \phi = 2u_{w1}\zeta_1 F, \ x_2 = 0, \ \zeta_2 > 0, \ \widehat{t} > 0, \\ \phi \to 0 \ \text{ as } x_2 \to \infty. \end{cases} \tag{21}$$

and derive the integral form of ϕ :

$$\begin{cases} \phi\left(x_2,\widehat{t},\zeta\right) = F\int_0^{\widehat{t}} e^{s-\widehat{t}} 2\zeta_1 u_1(x_2 - \zeta_2(\widehat{t}-s), s)ds \ \text{ for } \zeta_2 < \frac{x_2}{\widehat{t}}, \\ \phi\left(x_2,\widehat{t},\zeta\right) = 2u_{w1}\zeta_1 F e^{\frac{-x_2}{\zeta_2}} + F\int_{\widehat{t}-\frac{x_2}{\zeta_2}}^{\widehat{t}} e^{s-\widehat{t}} 2\zeta_1 u_1(x_2 - \zeta_2(\widehat{t}-s), s)ds \text{ for } \zeta_2 > \frac{x_2}{\widehat{t}}. \end{cases} \tag{22}$$

2. Derive the integral equation for $u_1(x_2, \widehat{t})$ by (22) and (5)

$$u_1(x_2, \widehat{t}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^{\widehat{t}} u_1(y, s) \frac{1}{\widehat{t} - s} \exp\left[-\left(\frac{x_2 - y}{\widehat{t} - s} \right)^2 + s - \widehat{t} \right] dsdy + \frac{u_{w1}}{\sqrt{\pi}} \int_{\frac{x_2}{\widehat{t}}}^\infty \exp\left(-r^2 - \frac{x_2}{r} \right) dr$$

and then use iteration method to find the solution.

3. Observe that

$$\frac{u_{w1}}{4} \exp\left(-\widehat{t}\right) \exp\left[-2 \left(\frac{x_2}{\widehat{t}} \right)^2 \right] \leq \frac{u_{w1}}{\sqrt{\pi}} \int_{\frac{x_2}{\widehat{t}}}^\infty \exp\left(-r^2 - \frac{x_2}{r} \right) dr \leq \frac{u_{w1}}{2} \exp\left[-\left(\frac{x_2}{\widehat{t}} \right)^2 \right]$$

and plug this into iteration to find the upper bound and lower bound for $u_1(x_2, \widehat{t})$

Theorem 2 (Linearized Boltzmann equation)

1. From Lemma 3 we have

$$\check{\sigma}_w(\widehat{t}) = \omega(x_2, \widehat{t}) = u_2(x_2, \widehat{t}) = u_3(x_2, \widehat{t}) = \tau(x_2, \widehat{t}) = 0.$$

This proves (13) and the diffuse reflection condition is simplified. Thus we consider

$$\begin{cases} \frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} = \mathcal{L}\phi, \\ \phi = 2\zeta_1 u_{w1} F,\ x_2 = 0,\ \zeta_2 > 0, \\ \phi = 0,\ \widehat{t} = 0, \\ \phi \to 0 \text{ as } x_2 \to \infty. \end{cases} \tag{23}$$

2. Rewrite the equation as

$$\frac{\partial \phi}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi}{\partial x_2} + \nu(\zeta)\phi = K\phi.$$

Then use the Picard iteration. That is, define a sequence $\{\phi^n\}$ which satisfies

$$\begin{cases} \frac{\partial \phi^0}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi^0}{\partial x_2} + \nu\phi^0 = 0, \\ \phi^0 = 0,\ \widehat{t} = 0, \\ \phi^0 = 2u_{w1}\zeta_1 F,\ x_2 = 0,\ \zeta_2 > 0, \end{cases}$$

$$\begin{cases} \frac{\partial \phi^{n+1}}{\partial \widehat{t}} + \zeta_2 \frac{\partial \phi^{n+1}}{\partial x_2} + \nu \phi^{n+1} = K\phi^n, \\ \phi^{n+1} = 0, \ \widehat{t} = 0, \\ \phi^{n+1} = 0, \ x_2 = 0, \ \zeta_2 > 0. \end{cases}$$

Then $\sum \phi^n$ converges to the solution ϕ in L^∞_ζ for $x_2, \widehat{t} > 0$ and we get a pointwise estimate for ϕ

$$\left\| \phi \left(x_2, \widehat{t}, \cdot \right) \right\|_{L^\infty_\zeta} \leq u_{w1} C_1 e^{-\nu_0 x_2} e^{C_2 \widehat{t}} \tag{24}$$

and the integral representation for ϕ is

$$\begin{cases} \phi\left(x_2, \widehat{t}, \zeta\right) = \int_0^{\widehat{t}} e^{\nu(s-\widehat{t})} K(\phi)(x_2 - \zeta_2(\widehat{t} - s), s) ds \text{ for } \zeta_2 < \frac{x_2}{\widehat{t}}, \\ \phi\left(x_2, \widehat{t}, \zeta\right) = 2u_{w1}\zeta_1 F e^{\frac{-\nu x_2}{\zeta_2}} + \int_{\widehat{t}-\frac{x_2}{\zeta_2}}^{\widehat{t}} e^{\nu(s-\widehat{t})} K(\phi)(x_2 - \zeta_2(\widehat{t} - s), s) ds \text{ for } \zeta_2 > \frac{x_2}{\widehat{t}}. \end{cases} \tag{25}$$

3. To estimate the flow velocity , we multiply (25) by ζ_1 and integrate over ζ to yield

$$u_1(x_2, \widehat{t}) = \int \int \int_{\frac{x_2}{\widehat{t}}}^{\infty} 2u_{w1} \zeta_1^2 E e^{-\frac{\nu x_2}{\zeta_2}} d\zeta + \int_0^{\widehat{t}} \int_{-\infty}^{\frac{x_2}{\widehat{t}-s}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\nu(s-\widehat{t})} K(\phi)(x_2 - \zeta_2(\widehat{t} - s), s) \zeta_1 F d\zeta_1 d\zeta_3 d\zeta_2 ds \tag{26}$$

(a) First we find an upper bound and a lower bound of the above first integral from the boundary effect. This can be done by Lemma 2 and direct computations.

(b) We then estimate the second integral of (26) by using (24)

Theorem 3 (Full Boltzmann equation)

1. To solve the full Boltzmann equation, we consider the iteration

$$\begin{cases} \frac{\partial\phi^{n+1}}{\partial\widehat{t}}+\zeta_2\frac{\partial\phi^{n+1}}{\partial x_2}=\mathcal{L}\phi^{n+1}+\mathcal{J}(\phi^n,\phi^n), \\ \phi^{n+1}=0,\ \widehat{t}=0, \\ \phi^{n+1}=\frac{1-2\sqrt{\pi}\int_{\zeta_2<0}\zeta_2 F\phi^{n+1}(0,\widehat{t},\zeta)d\zeta}{\pi^{3/2}F}e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2}-F,\ x_2=0,\ \zeta_2>0. \end{cases}$$

(a) Take $\phi^0=0$ and so we start with this problem

$$\begin{cases} \frac{\partial\phi}{\partial\widehat{t}}+\zeta_2\frac{\partial\phi}{\partial x_2}=\mathcal{L}\phi, \\ \phi=0,\ \widehat{t}=0, \\ \phi=\frac{1-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\phi(0,\widehat{t},\zeta_*)d\zeta_*}{\pi^{3/2}F}e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2}-F,\ x_2=0,\ \zeta_2>0. \end{cases} \tag{27}$$

Lemma 4 When $0<u_{w1}<1$ *there exists a solution of (27) satisfying*

$$\left\|\phi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta}\le C_2u_{w1}e^{-\nu_0x_2+(C_3+1)C_1\widehat{t}}\ \text{for } x_2>0,\ \widehat{t}>0. \tag{28}$$

To prove Lemma 4 we need the following estimates and use similar argument in the case of the linearized Boltzmann equation.

$$
\begin{aligned}
A_1 &= \frac{e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2}}{\pi^{3/2}F} - F,\\
A_2 &= \frac{-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\phi(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*}{\pi^{3/2}F}e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2}\\
&= -2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\phi(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*\pi^{-3/4}e^{-(\zeta_1-u_{w1})^2+\frac{\zeta_1^2}{2}-\frac{\zeta_2^2}{2}-\frac{\zeta_3^2}{2}}\\
&\equiv A_{21}A_{22},
\end{aligned}
$$

$$
\begin{aligned}
A_{21} &= \int_{\zeta_{2*}<0}\zeta_{2*}F_*\phi(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*,\\
A_{22} &= -2\sqrt{\pi}\pi^{-3/4}e^{-(\zeta_1-u_{w1})^2+\frac{\zeta_1^2}{2}-\frac{\zeta_2^2}{2}-\frac{\zeta_3^2}{2}}.
\end{aligned}
$$

Lemma 5 There exists a $C_2 > 0$ *such that* $|A_1| \leq C_2 u_{w1}$ *for* $0 < u_{w1} < 1$*.*

Lemma 6 For $u_{w1} << 1$*,* $A_{22} = O(1)$ *and there exists* $C_3 > 0$ *such that*

$$
|\frac{-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*d\zeta_*}{\pi^{3/2}F}|e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2} \leq C_3 \text{ for } u_{w1} < 1.
$$

(b) Next we consider the following problem with ψ a given function

$$\begin{cases} \frac{\partial\phi}{\partial\widehat{t}} + \zeta_2 \frac{\partial\phi}{\partial x_2} = \mathcal{L}\phi + \mathcal{J}(\psi,\psi), \\ \phi = \frac{1 - 2\sqrt{\pi}\int_{\zeta_{2*}<0} \zeta_{2*} F_* \phi(0,\widehat{t},\zeta_*) d\zeta_*}{\pi^{3/2}F} e^{-(\zeta_1 - u_{w1})^2 - \zeta_2^2 - \zeta_3^2} - F,\ x_2 = 0,\ \zeta_2 > 0, \\ \phi = 0 \ \ \text{, } \widehat{t} = 0, \\ \phi \to 0 \text{ as } x_2 \to \infty. \end{cases} \tag{29}$$

Lemma 7 Let $\left\|\psi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \leq D_1 u_{w1} e^{-\nu_0 x_2 + D_2 \widehat{t}}$ *with* $D_2 > 2(C_3+1)C_1 - \nu_0$. *For each time interval* $0 < \widehat{t} < \Gamma$ *there exists a* $0 < \delta = \delta(\Gamma) << 1$ *such that for* $u_{w1} \in (0,\delta)$ *the solution of (29) exists and satisfies*

$$\left\|\phi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \leq C_2 u_{w1} e^{-\nu_0 x_2 + (C_3+1)C_1 \widehat{t}} + \frac{D_1}{2} u_{w1} e^{-\nu_0 x_2 + D_2 \widehat{t}}.$$

To prove this lemma we need the following two lemmas:
We define the function $\widetilde{\psi}$

$$\begin{cases} \widetilde{\psi}(x_2,\widehat{t},\zeta) = \int_0^{\widehat{t}} e^{\nu(s-\widehat{t})}\mathcal{J}(\psi,\psi)(x_2-\zeta_2(\widehat{t}-s),s,\zeta)ds \text{ if } \zeta_2 < \frac{x_2}{\widehat{t}}, \\ \widetilde{\psi}(x_2,\widehat{t},\zeta) = \widetilde{\psi}_w(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_i)e^{\frac{-\nu x_2}{\zeta_2}} + \int_{\widehat{t}-\frac{x_2}{\zeta_2}}^{\widehat{t}} e^{\nu(s-\widehat{t})}\mathcal{J}(\psi,\psi)(x_2-\zeta_2(\widehat{t}-s),s,\zeta)ds \text{ if } \zeta_2 > \frac{x_2}{\widehat{t}}, \end{cases}$$

where

$$\begin{cases} \widetilde{\psi}_w(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_i) = \frac{1-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\widetilde{\psi}(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*}{\pi^{3/2}F}e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2} - F,\ \zeta_2>0, \\ \widetilde{\psi}(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*) = \int_0^{\widehat{t}-\frac{x_2}{\zeta_2}} e^{\nu_*(s-\widehat{t}+\frac{x_2}{\zeta_2})}\mathcal{J}(\psi,\psi)(-\zeta_2(\widehat{t}-\frac{x_2}{\zeta_2}-s),s,\zeta_*)ds. \end{cases}$$

Lemma 8 Let $D_1, D_2 > 0$ *and for any* ψ *with* $\left\|\psi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \le D_1 u_{w1} e^{-\nu_0 x_2 + D_2\widehat{t}}$.
For each $\Gamma > 0$ *there exists a* $\delta = \delta(\Gamma) > 0$ *such that the function* $\widetilde{\psi}$ *satisfies*

$$\left\|\widetilde{\psi}(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \le C_2 u_{w1} e^{-\nu_0 x_2} + \frac{D_1}{4} u_{w1} e^{-\nu_0 x_2 + D_2\widehat{t}}$$

whenever $u_{w1} \in (0,\delta)$ *and* $0 < \widehat{t} \le \Gamma$.

Next for a given function φ we define a function $\widetilde{\varphi}$

$$\begin{cases} \widetilde{\varphi}(x_2,\widehat{t},\zeta) = \int_0^{\widehat{t}} e^{\nu(s-\widehat{t})} K\varphi(x_2-\zeta_2(\widehat{t}-s),s,\zeta)ds \text{ for } \zeta_2 < \frac{x_2}{\widehat{t}} \text{ ,} \\ \widetilde{\varphi}(x_2,\widehat{t},\zeta) = \widetilde{\varphi}_w(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_i)e^{\frac{-\nu x_2}{\zeta_2}} + \int_{\widehat{t}-\frac{x_2}{\zeta_2}}^{\widehat{t}} e^{\nu(s-\widehat{t})} K\varphi(x_2-\zeta_2(\widehat{t}-s),s,\zeta)ds \text{ for } \zeta_2 > \frac{x_2}{\widehat{t}} \text{ ,} \end{cases}$$

where

$$\begin{cases} \widetilde{\varphi}_w(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_i) = \frac{-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\widetilde{\varphi}(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*}{\pi^{3/2}F} e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2}, \ \zeta_2 > 0, \\ \widetilde{\varphi}(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*) = \int_0^{\widehat{t}-\frac{x_2}{\zeta_2}} e^{\nu_*(s-\widehat{t}+\frac{x_2}{\zeta_2})} K\widetilde{\varphi}\left(-\zeta_2(\widehat{t}-\frac{x_2}{\zeta_2}-s),s,\zeta_*\right)ds. \end{cases}$$

Lemma 9 If $\left\|\varphi(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \le D_1 u_{w1} e^{-\nu_0 x_2 + D_2\widehat{t}}$ *then*

$$\left\|\widetilde{\varphi}(x_2,\widehat{t},\cdot)\right\|_{L^\infty_\zeta} \le \frac{D_1}{2} u_{w1} e^{-\nu_0 x_2 + D_2\widehat{t}} \text{ for } D_2 > 2(C_3+1)C_1 - \nu_0.$$

(c) Reture to the iteration. Therefore, from Lemma 4 we have

$$\left\|\phi^1\left(x_2,\widehat{t},\cdot\right)\right\|_{L^\infty_\zeta}\le C_2u_{w1}e^{-\nu_0x_2+(C_3+1)C_1\widehat{t}}.$$

By Lemma 7, taking $D_1=C_2$, $D_2=2(C_3+1)C_1$, we have

$$\left\|\phi^2\left(x_2,\widehat{t},\cdot\right)\right\|_{L^\infty_\zeta}\le\frac{3}{2}C_2u_{w1}e^{-\nu_0x_2+2(C_3+1)C_1\widehat{t}}.$$

Apply Lemma 7 repeatedly we obtain the convergence of $\{\phi^n\}$ to the solution ϕ satisfying

$$\left\|\phi\left(x_2,\widehat{t},\cdot\right)\right\|_{L^\infty_\zeta}\le 2C_2u_{w1}e^{-\nu_0x_2+2(C_3+1)C_1\widehat{t}}. \tag{30}$$

2. The integral representation for ϕ is

$$\begin{cases} \phi(x_2,\widehat{t},\zeta) = \int_0^{\widehat{t}} e^{\nu(s-\widehat{t})}\left[K\phi(x_2-\zeta_2(\widehat{t}-s),s,\zeta)+\mathcal{J}(\phi,\phi)(x_2-\zeta_2(\widehat{t}-s),s,\zeta)\right]ds \text{ for } \zeta_2<\frac{x_2}{\widehat{t}}, \\ \phi(x_2,\widehat{t},\zeta) = \phi_w(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_i)e^{\frac{-\nu x_2}{\zeta_2}} + \\ \int_{\widehat{t}-\frac{x_2}{\zeta_2}}^{\widehat{t}} e^{\nu(s-\widehat{t})}\left[K\phi(x_2-\zeta_2(\widehat{t}-s),s,\zeta)+\mathcal{J}(\phi,\phi)(x_2-\zeta_2(\widehat{t}-s),s,\zeta)\right]ds \text{ for } \zeta_2>\frac{x_2}{\widehat{t}}, \end{cases} \tag{31}$$

$$\begin{cases} \phi_w(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_i) = \frac{1-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\phi(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*}{\pi^{3/2}F}e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2} - F, \\ \phi(0,\widehat{t}-\frac{x_2}{\zeta_2},\zeta_*) = \int_0^{\widehat{t}-\frac{x_2}{\zeta_2}} e^{\nu_*(s-\widehat{t}+\frac{x_2}{\zeta_2})}. \\ \left[K\phi(x_2-\zeta_2(\widehat{t}-s),s,\zeta_*)+\mathcal{J}(\phi,\phi)(x_2-\zeta_2(\widehat{t}-s),s,\zeta_*)\right]ds. \end{cases} \tag{32}$$

Multiply (31) by ζ_1 and integrate with respect to ζ , we have

$$\begin{aligned} &\rho(x_2,\widehat{t})u_1(x_2,t) \\ &= \int_0^t \int_{-\infty}^{\frac{x_2}{t-s}} \int\int \zeta_1 F e^{\nu(s-t)}\left[K\phi(x_2-\zeta_2(\widehat{t}-s),s,\zeta)+\mathcal{J}(\phi,\phi)(x_2-\zeta_2(\widehat{t}-s),s,\zeta)\right]d\zeta_1 d\zeta_3 d\zeta_2 ds \\ &+ \int_{\zeta_2>\frac{x_2}{t}} \zeta_1\left(1-2\sqrt{\pi}\int_{\zeta_{2*}<0}\zeta_{2*}F_*\phi(0,t-\frac{x_2}{\zeta_2},\zeta_*)d\zeta_*\right)\cdot\pi^{-3/2}e^{-(\zeta_1-u_{w1})^2-\zeta_2^2-\zeta_3^2} - Ee^{\frac{-\nu x_2}{\zeta_2}}d\zeta \\ &\equiv I_1+I_2+B_1+B_2. \end{aligned} \tag{33}$$

3. Expand $e^{-(\zeta_1 - u_{w1})^2}$ in u_{w1}:

$$e^{-(\zeta_1 - u_{w1})^2} = e^{-\zeta_1^2}\left[1 + 2\zeta_1 u_{w1} + (2\zeta_1^2 - 1)u_{w1}^2 + O(u_{w1}^3)\right],$$

whence

$$\begin{aligned} B_1 &= \int_{\zeta_2 > \frac{x_2}{t}} \zeta_1 \left\{\pi^{-3/2} e^{-(\zeta_1 - u_{w1})^2 - \zeta_2^2 - \zeta_3^2} - E\right\} e^{\frac{-\nu x_2}{\zeta_2}} d\zeta \\ &= \int_{\zeta_2 > \frac{x_2}{t}} \zeta_1 E\left[2\zeta_1 u_{w1} + (2\zeta_1^2 - 1)u_{w1}^2 + O(u_{w1}^3)\right] e^{\frac{-\nu x_2}{\zeta_2}} d\zeta \\ &= \int_{\zeta_2 > \frac{x_2}{\widehat{t}}} 2\zeta_1^2 u_{w1} E e^{\frac{-\nu x_2}{\zeta_2}} d\zeta + O(u_{w1}^3). \end{aligned}$$

(a) The integral in B_1 is the same as the first integral of (26) and then follow the result in the case of the linearized Boltzmann equation we have

$$C u_{w1} e^{-2\nu_1 x_2 - 2\left(\frac{x_2}{\widehat{t}}\right)^2} - O(u_{w1}^3) \le B_1 \le \frac{u_{w1}}{\sqrt{\pi}} e^{-\nu_0 x_2} J_0\left(\nu_0 x_2\right) + O(u_{w1}^3)$$

for some $C > 0$ and $\widehat{t} << 1$.

(b) We then estimate the other integrals of (33) by using (30).

Thank you for your attention !