Evaporation problems in gravitation

Landau and Fokker-Planck equations on a bounded velocity domain

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Outline

Vlasov-Landau-Poisson system with evaporation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla \phi \cdot \nabla_v f = Q_L(f) \\ \Delta \phi = \int f \, \mathrm{d}v, \qquad \phi(x) \xrightarrow[|x| \to +\infty]{} 0, \\ f_{|t=0} = f_0 \geq 0, \\ f(t,x,v) = 0 \text{ if } e := |v|^2/2 + \phi(x,t) \geq 0, \end{cases}$$

- $f(t,x,v)$ is the density distribution
- Q_L is the Landau collision kernel:

$$Q_L(f) := \nabla_v \cdot \int |v - v_*|^\gamma \Pi(v - v_*)(f_* \nabla_v f - f \nabla_v f_*) \mathrm{d}v_*,$$

with $\gamma \in [-3, 1[$, $\Pi(z)$ the orthogonal projection on $(z\mathbb{R})^\perp$.

Landau equation with evaporation

Instead of the previous system, we will study a simplified model where

- f is homogeneous,
- the escape velocity is a constant $R > 0$.

$$\begin{cases} \partial_t f = Q_L(f), \\ f_{|t=0} = f_0 \geq 0, \\ f(t,v) = 0 \text{ if } |v| \geq R, \end{cases}$$

A Fokker-Planck equation with evaporation

Besides we assume that:

- f is spherically symmetric,
- the interaction potential is Maxwellian, i.e. $\gamma = 0$.

With these hypotheses the system becomes:

$$\begin{cases} \partial_t f = Q_{FP}(f) := \nabla \cdot (E_f \nabla f + 3 v M_f f), \\ f_{|t=0} = f_0 \geq 0, \\ f(t,v) = 0 \text{ if } |v| \geq R, \end{cases}$$

- E_f is the energy: $E_f := \int_{|v| \leq R} f v^2 \mathrm{d}v$,
- M_f is the mass: $M_f := \int_{|v| \leq R} f \mathrm{d}v$.

Bounded domain versus unbounded domain

In the unbounded case ($v \in \mathbb{R}^3$)

- E_f and M_f and the momentum $\displaystyle\int fv \,\mathrm{d}v$ are conserved,
- the entropy $\displaystyle\int f \log f$ is nonincreasing,
- f is in $L^\infty(L \log L)$.

In the bounded case ($|v| \leq R$)

- E_f and M_f are nonincreasing
- The momentum is not conserved and the entropy may increase.
- f is in $L^\infty([0, T], L^2)$ for any $T > 0$.

Conservation law

Using Stokes formula we have

$$\frac{\mathrm{d}}{\mathrm{d}t} M_f(t) = E_f(t) \int_{|v|=R} \nabla f(t,v) \cdot n \mathrm{d}\sigma = -4\pi R^2 E_f(t) |\partial_r f(t,R)|,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_f(t) = E_f(t) \int_{|v|=R} v^2 \nabla f(t,v) \cdot n \mathrm{d}\sigma = -4\pi R^4 E_f(t) |\partial_r f(t,R)|.$$

We have the conservation law

$$R^2 M_f - E_f = R^2 M_0 - E_0,$$

which implies that

$$M_f \geq M_0 + \frac{E_0}{R^2} > 0.$$

The energy does not vanish in finite time

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{|v|\le R} f^2(t,.)\mathrm{d}v &= -2E_f(t)\int_{|v|\le R}|\nabla f(t,.)|^2\mathrm{d}v\\ &\quad -3M_f(t)\int_{|v|\le R} v\cdot\nabla(f^2)(t,.)\\ &\le 9M_f(0)\int_{|v|\le R} f^2(t,.).\end{aligned}$$

Thus, for any $T>0$, $f\in L^\infty([0,T],L^2)$. Furthermore we prove that

$$E_f(t)\ge C\|f\|_{L^2}^{-\frac{4}{3}}M_f^{\frac{7}{3}}.$$

Then $E_f\ge e_T>0$. This prove the uniform ellipticity.

The energy tends to 0 when time goes to infinity

Lemma

$$\lim_{t \to +\infty} E_f(t) = 0.$$

Idea of the proof

If you suppose $\lim_{t \to +\infty} E_f(t) =: E_\infty > 0$, *then* $\lim_{t \to +\infty} M_f(t) =: M_\infty > 0$, *and the equation is equivalent when time goes to* $+\infty$ *to*

$$\partial_t f = \nabla \cdot (E_\infty \nabla f + 3 v M_\infty f).$$

We should have $f \underset{+\infty}{\to} 0$, *that is impossible because* $\inf M_f > 0$.

Stationary state

We have proved that

- $E_f \underset{+\infty}{\longrightarrow} 0$,
- $M_f \underset{+\infty}{\longrightarrow} M_\infty := M_0 - \dfrac{E_0}{R^2} > 0$.

Consequently we have

$$f \rightharpoonup M_\infty \delta_0.$$

Asymptotic behaviour of energy (1/3)

We have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_f = -4\pi R^4 E_f |\partial_r f(t,R)|.$$

Thus

$$E_f(t) = E_0 e^{-4\pi R^4 \int_0^t |\partial_r f(s,R)| \mathrm{d}s}.$$

Thanks to a supersolution that vanishes on ∂B we prove that

$$\partial_r f(t,R) \xrightarrow[t\to+\infty]{} 0.$$

This is sufficient to prove that the decrease of E_f is not exponential.

Asymptotic behaviour of energy (2/3)

More precisely, we consider a Maxwellian distribution

$$\mathcal{M}(t,v) := \frac{\alpha}{\beta^{\frac{3}{2}}(t)} e^{-\frac{v^2}{\beta(t)}} + C,$$

with well-chosen constants α and C. It is supersolution (i.e. $0 \leq f \leq \mathcal{M}$) iff

$$\beta' = 4E_f - 6M_f\beta.$$

From this ODE we have

$$\beta \underset{+\infty}{\sim} \frac{2E_f}{3M_\infty}.$$

Asymptotic behaviour of energy (3/3)

From that we deduce an asymptotic upper bound for $\partial_r f(., R)$ in function of E_f which gives a differential inequality on E_f:

$$-\frac{E_f'}{E_f^2} e^{\frac{c_1}{E_f}} \leq c_2,$$

for some positive constants c_1 and c_2. This gives a logarithmic asymptotic lower bound for energy. We find that

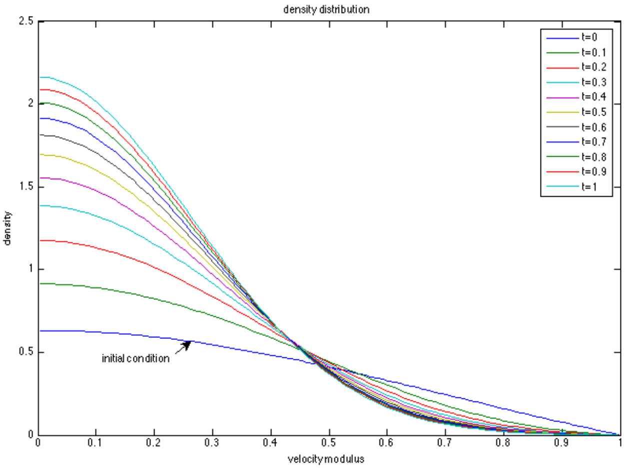
$$E_f(t) \underset{t \to +\infty}{\sim} \frac{3 M_\infty R^2}{2 \log t}.$$

We used a conservative numerical scheme in the sense that

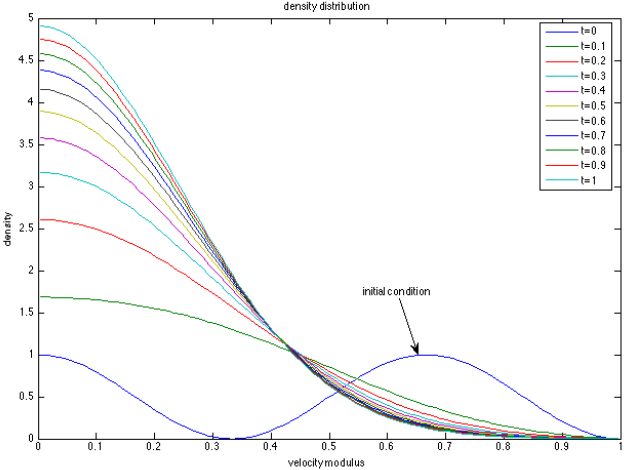
- the discrete mass and energy are nonincreasing,
- the conservation law is respected,

Thank's to these conservative properties, for fixed time and velocity steps, the scheme converges, when time goes to infinity, to a discrete mass of Dirac. A basic non conservative scheme tends to 0.

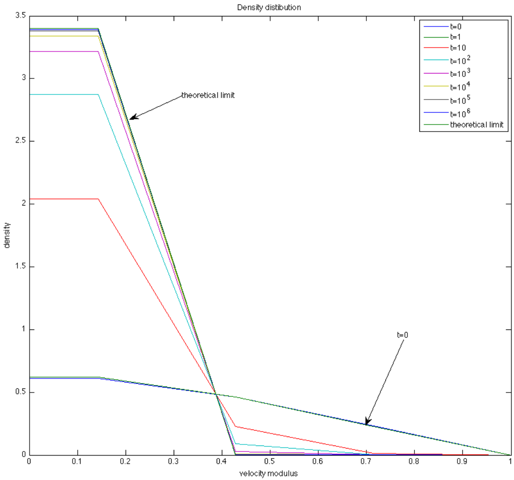
Influence of initial profile (1/2)



Influence of initial profile (2/2)



Convergence to a discrete Dirac



Work in progress, perspectives

- The study of the non spherically symmetric case with $\gamma \geq 0$
- non homogeneous case.