

# Free energies, nonlinear flows and functional inequalities

Jean Dolbeault

<http://www.ceremade.dauphine.fr/~dolbeault>

<http://www.sciencesmaths-paris.fr/>

Ceremade, Université Paris-Dauphine

December 7, 2011

Oberwolfach workshop  
*Variational Methods for Evolution*  
(2011, December 4-9)

# A – Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

# Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in  $\mathbb{R}^d$ ,  $d \geq 3$ ,

$$\|u\|_{L^{2^*}(\mathbb{R}^d)}^2 \leq S_d \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \quad \forall u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \quad (1)$$

and the Hardy-Littlewood-Sobolev inequality

$$S_d \|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \geq \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx \quad \forall v \in L^{\frac{2d}{d+2}}(\mathbb{R}^d) \quad (2)$$

are **dual** of each other. Here  $S_d$  is the Aubin-Talenti constant and  $2^* = \frac{2d}{d-2}$ . Can we recover this using a nonlinear flow approach? Can we improve it?

Keller-Segel model: another motivation [Carrillo, Carlen and Loss] and [Blanchet, Carlen and Carrillo]

# Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0, \quad x \in \mathbb{R}^d \quad (3)$$

If we define  $H(t) := H_d[v(t, \cdot)]$ , with

$$H_d[v] := \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \|v\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2$$

then we observe that

$$\frac{1}{2} H' = - \int_{\mathbb{R}^d} v^{m+1} \, dx + S_d \left( \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}} \, dx \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}} \, dx$$

where  $v = v(t, \cdot)$  is a solution of (3). With the choice  $m = \frac{d-2}{d+2}$ , we find that  $m + 1 = \frac{2d}{d+2}$

# A first statement

## Proposition

[J.D.] Assume that  $d \geq 3$  and  $m = \frac{d-2}{d+2}$ . If  $v$  is a solution of (3) with nonnegative initial datum in  $L^{2d/(d+2)}(\mathbb{R}^d)$ , then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[ \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \|v\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 \right] \\ = \left( \int_{\mathbb{R}^d} v^{m+1} \, dx \right)^{\frac{2}{d}} \left[ S_d \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \|u\|_{L^{2^*}(\mathbb{R}^d)}^2 \right] \geq 0 \end{aligned}$$

The HLS inequality amounts to  $H \leq 0$  and appears as a consequence of Sobolev, that is  $H' \geq 0$  if we show that  $\limsup_{t>0} H(t) = 0$

Notice that  $u = v^m$  is an optimal function for (1) if  $v$  is optimal for (2)

# Improved Sobolev inequality



By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when  $d \geq 5$  for integrability reasons

## Theorem

[J.D.] Assume that  $d \geq 5$  and let  $q = \frac{d+2}{d-2}$ . There exists a positive constant  $C \leq (1 + \frac{2}{d}) (1 - e^{-d/2}) S_d$  such that

$$S_d \|w^q\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} w^q (-\Delta)^{-1} w^q dx \leq C \|w\|_{L^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left[ \|\nabla w\|_{L^2(\mathbb{R}^d)}^2 - S_d \|w\|_{L^{2^*}(\mathbb{R}^d)}^2 \right]$$

for any  $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

## Solutions with *separation of variables*

Consider the solution of  $\frac{\partial v}{\partial t} = \Delta v^m$  vanishing at  $t = T$ :

$$\bar{v}_T(t, x) = c (T - t)^\alpha (F(x))^{\frac{d+2}{d-2}}$$

where  $F$  is the Aubin-Talenti solution of

$$-\Delta F = d(d-2) F^{(d+2)/(d-2)}$$

Let  $\|v\|_* := \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{d+2} |v(x)|$

### Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]  
 For any solution  $v$  with initial datum  $v_0 \in L^{2d/(d+2)}(\mathbb{R}^d)$ ,  $v_0 > 0$ , there exists  $T > 0$ ,  $\lambda > 0$  and  $x_0 \in \mathbb{R}^d$  such that

$$\lim_{t \rightarrow T_-} (T - t)^{-\frac{1}{1-m}} \|v(t, \cdot) / \bar{v}(t, \cdot) - 1\|_* = 0$$

with  $\bar{v}(t, x) = \lambda^{(d+2)/2} \bar{v}_T(t, (x - x_0)/\lambda)$

## Improved inequality: proof (1/2)

$J(t) := \int_{\mathbb{R}^d} v(t, x)^{m+1} dx$  satisfies

$$J' = -(m+1) \|\nabla v^m\|_{L^2(\mathbb{R}^d)}^2 \leq -\frac{m+1}{S_d} J^{1-\frac{2}{d}}$$

If  $d \geq 5$ , then we also have

$$J'' = 2m(m+1) \int_{\mathbb{R}^d} v^{m-1} (\Delta v^m)^2 dx \geq 0$$

Notice that

$$\frac{J'}{J} \leq -\frac{m+1}{S_d} J^{-\frac{2}{d}} \leq -\kappa \quad \text{with} \quad \kappa T = \frac{2d}{d+2} \frac{T}{S_d} \left( \int_{\mathbb{R}^d} v_0^{m+1} dx \right)^{-\frac{2}{d}} \leq \frac{d}{2}$$

## Improved inequality: proof (2/2)

By the **Cauchy-Schwarz inequality**, we have

$$\begin{aligned} \frac{J'^2}{(m+1)^2} &= \|\nabla v^m\|_{L^2(\mathbb{R}^d)}^4 = \left( \int_{\mathbb{R}^d} v^{(m-1)/2} \Delta v^m \cdot v^{(m+1)/2} dx \right)^2 \\ &\leq \int_{\mathbb{R}^d} v^{m-1} (\Delta v^m)^2 dx \int_{\mathbb{R}^d} v^{m+1} dx = \text{Cst } J'' J \end{aligned}$$

so that  $Q(t) := \|\nabla v^m(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 \left( \int_{\mathbb{R}^d} v^{m+1}(t, x) dx \right)^{-(d-2)/d}$  is **monotone decreasing**, and

$$H' = 2J(S_d Q - 1), \quad H'' = \frac{J'}{J} H' + 2JS_d Q' \leq \frac{J'}{J} H' \leq 0$$

$$H'' \leq -\kappa H' \quad \text{with} \quad \kappa = \frac{2d}{d+2} \frac{1}{S_d} \left( \int_{\mathbb{R}^d} v_0^{m+1} dx \right)^{-2/d}$$

By writing that  $-H(0) = H(T) - H(0) \leq H'(0) (1 - e^{-\kappa T})/\kappa$  and using the estimate  $\kappa T \leq d/2$ , the proof is completed □

# Fast diffusion equations

- 1 entropy methods
- 2 linearization of the entropy
- 3 improved Gagliardo-Nirenberg inequalities

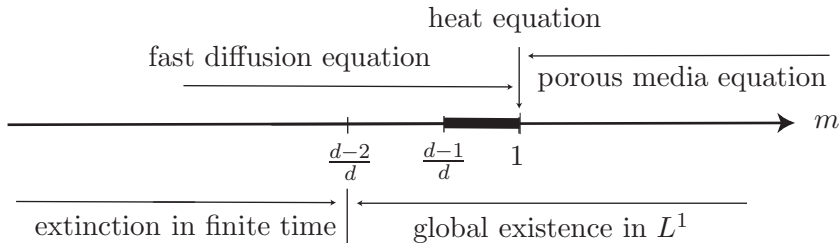
# B1 – Fast diffusion equations: entropy methods

# Existence, classical results

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d, \quad t > 0$$

Self-similar (Barenblatt) function:  $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$  as  $t \rightarrow +\infty$

[Friedmann, Kamin, 1980]  $\|u(t, \cdot) - \mathcal{U}(t, \cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter  $m$

# Time-dependent rescaling, Free energy

- Time-dependent rescaling: Take  $u(\tau, y) = R^{-d}(\tau) v(t, y/R(\tau))$  where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}, \quad R(0) = 1, \quad t = \log R$$

- The function  $v$  solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x v), \quad v|_{\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional:

**Generalized entropy** or **Free energy**

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left( \frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) dx - \mathcal{F}_0$$

Entropy production is measured by the **Generalized Fisher information**

$$\frac{d}{dt} \mathcal{F}[v] = -\mathcal{I}[v], \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

# Relative entropy and entropy production

- Stationary solution: choose  $C$  such that  $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \frac{1-m}{2m} |x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix  $\mathcal{F}_0$  so that  $\mathcal{F}[v_\infty] = 0$

- Entropy – entropy production inequality

## Theorem

$d \geq 3$ ,  $m \in [\frac{d-1}{d}, +\infty)$ ,  $m > \frac{1}{2}$ ,  $m \neq 1$

$$\mathcal{I}[v] \geq 2\mathcal{F}[v]$$

## Corollary

A solution  $v$  with initial data  $u_0 \in L^1_+(\mathbb{R}^d)$  such that  $|x|^2 u_0 \in L^1(\mathbb{R}^d)$ ,  $u_0^m \in L^1(\mathbb{R}^d)$  satisfies  $\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[u_0] e^{-2t}$

# An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left( \frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} \mathcal{I}[v]$$

Rewrite it with  $p = \frac{1}{2m-1}$ ,  $v = w^{2p}$ ,  $v^m = w^{p+1}$  as

$$\frac{1}{2} \left( \frac{2m}{2m-1} \right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left( \frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \geq 0$$

- for some  $\gamma$ ,  $K = K_0 \left( \int_{\mathbb{R}^d} v dx = \int_{\mathbb{R}^d} w^{2p} dx \right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$  is optimal

## Theorem

[Del Pino, J.D.] *With  $1 < p \leq \frac{d}{d-2}$  (fast diffusion case) and  $d \geq 3$*

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq A \|\nabla w\|_{L^2(\mathbb{R}^d)}^\theta \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

$$A = \left( \frac{y(p-1)^2}{2\pi d} \right)^{\frac{\theta}{2}} \left( \frac{2y-d}{2y} \right)^{\frac{1}{2p}} \left( \frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})} \right)^{\frac{\theta}{d}}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1}$$

## ... a proof by the Bakry-Emery method

Consider the generalized Fisher information

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v |Z|^2 dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt} \mathcal{I}[v(t, \cdot)] + 2 \mathcal{I}[v(t, \cdot)] = -2 \int_{\mathbb{R}^d} u^m [|\nabla Z|^2 - (1-m)(\nabla \cdot Z)^2] dx$$

- the Fisher information decays exponentially:

$$\mathcal{I}[v(t, \cdot)] \leq \mathcal{I}[u_0] e^{-2t}$$

- $\lim_{t \rightarrow \infty} \mathcal{I}[v(t, \cdot)] = 0$  and  $\lim_{t \rightarrow \infty} \mathcal{F}[v(t, \cdot)] = 0$
- $\frac{d}{dt} (\mathcal{I}[v(t, \cdot)] - 2 \mathcal{F}[v(t, \cdot)]) \leq 0$  means  $\mathcal{I}[v] \geq 2 \mathcal{F}[v]$

[Otto], [Carrillo, Toscani], [Jüngel, Markowich, Toscani], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

# B2 – Fast diffusion equations: sharp asymptotic rates by linearization of the entropy

# Entropy methods and linearization: sharp rates

*Generalized Barenblatt profiles:*  $V_D(x) := (D + |x|^2)^{\frac{1}{m-1}}$

Assumptions on the initial datum  $v_0$

**(H1)**  $V_{D_0} \leq v_0 \leq V_{D_1}$  for some  $D_0 > D_1 > 0$

**(H2)** if  $d \geq 3$  and  $m \leq m_*$ ,  $(v_0 - V_D) \in L^1$  for some  $D \in [D_1, D_0]$

## Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if  $m < 1$  and  $m \neq m_* := \frac{d-4}{d-2}$ , the entropy decays according to*

$$\mathcal{E}[v(t, \cdot)] \leq C e^{-2(1-m)\Lambda_{\alpha,d} t} \quad \forall t \geq 0$$

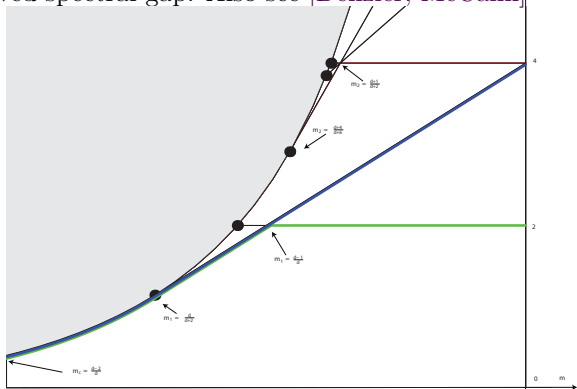
where  $\Lambda_{\alpha,d} > 0$  is the best constant in the Hardy-Poincaré inequality

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_{\alpha} \quad \forall f \in H^1(d\mu_{\alpha})$$

with  $\alpha := 1/(m-1) < 0$ ,  $d\mu_{\alpha} := h_{\alpha} dx$ ,  $h_{\alpha}(x) := (1 + |x|^2)^{\alpha}$

## Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that  $m \in (m_1, 1)$ ,  $d \geq 3$ . Under Assumption (H1), if  $v$  is a solution of the fast diffusion equation with initial datum  $v_0$  such that  $\int_{\mathbb{R}^d} x v_0 dx = 0$ , then the asymptotic convergence holds with an improved rate corresponding to the improved spectral gap. Also see [Denzler, McCann]



## Higher order matching asymptotics

[J.D., G. Toscani] For some  $m \in (m_c, 1)$  with  $m_c := (d-2)/d$ , we consider on  $\mathbb{R}^d$  the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot (u \nabla u^{m-1}) = 0$$

**Without choosing  $R$** , we may define the function  $v$  such that

$$u(\tau, y + x_0) = R^{-d} v(t, x), \quad R = R(\tau), \quad t = \frac{1}{2} \log R, \quad x = \frac{y}{R}$$

Then  $v$  has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[ v \left( \sigma^{\frac{d}{2}(m-m_c)} \nabla v^{m-1} - 2x \right) \right] = 0 \quad t > 0, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on  $R$ )

$$2 \sigma^{-\frac{d}{2}(m-m_c)} = R^{1-d(1-m)} \frac{dR}{d\tau}$$

## Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left( C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}^d \quad (4)$$

Note that  $\sigma$  is a function of  $t$ : as long as  $\frac{d\sigma}{dt} \neq 0$ , the Barenblatt profile  $B_\sigma$  is *not* a solution (it plays the role of a **local Gibbs state**) but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} [v^m - B_\sigma^m - m B_\sigma^{m-1} (v - B_\sigma)] dx$$

The time derivative of this relative entropy is

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t, \cdot)] = \underbrace{\frac{d\sigma}{dt} \left( \frac{d}{d\sigma} \mathcal{F}_\sigma[v] \right) \Big|_{\sigma=\sigma(t)}}_{\text{choose it = 0}} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left( v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t} dx$$

$$\iff \text{Minimize } \mathcal{F}_\sigma[v] \text{ w.r.t. } \sigma \iff \int_{\mathbb{R}^d} |x|^2 B_\sigma dx = \int_{\mathbb{R}^d} |x|^2 v dx$$

# The entropy / entropy production estimate

Using the new change of variables, we know that

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t, \cdot)] = -\frac{m \sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} v \left| \nabla \left[ v^{m-1} - B_{\sigma(t)}^{m-1} \right] \right|^2 dx$$

Let  $w := v/B_{\sigma}$  and observe that the relative entropy can be written as

$$\mathcal{F}_{\sigma}[v] = \frac{m}{1-m} \int_{\mathbb{R}^d} \left[ w - 1 - \frac{1}{m} (w^m - 1) \right] B_{\sigma}^m dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_{\sigma}[v] := \int_{\mathbb{R}^d} \left| \frac{1}{m-1} \nabla \left[ (w^{m-1} - 1) B_{\sigma}^{m-1} \right] \right|^2 B_{\sigma} w dx$$

so that 
$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t, \cdot)] = -m(1-m) \sigma(t) \mathcal{I}_{\sigma(t)}[v(t, \cdot)] \quad \forall t > 0$$

*When linearizing, one more mode is killed and  $\sigma(t)$  scales out*

# Improved rates of convergence



## Theorem (J.D., G. Toscani)

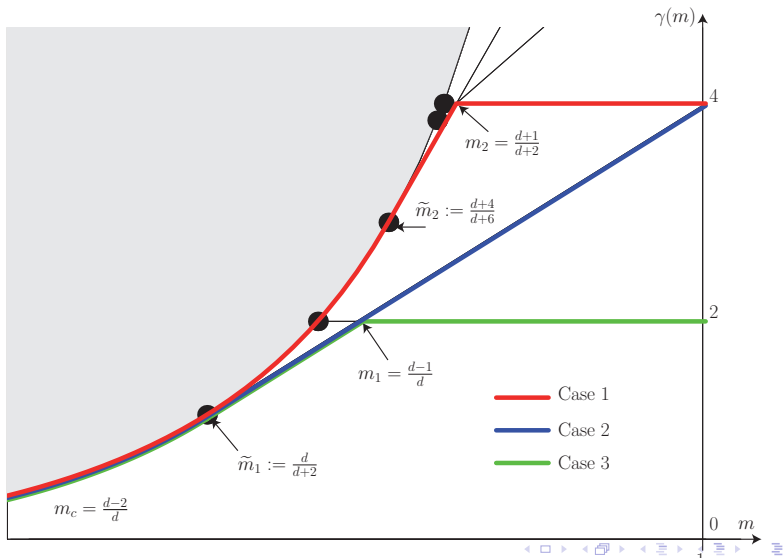
Let  $m \in (\tilde{m}_1, 1)$ ,  $d \geq 2$ ,  $v_0 \in L^1_+(\mathbb{R}^d)$  such that  $v_0^m, |y|^2 v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t, \cdot)] \leq C e^{-2\Lambda(m)t} \quad \forall t \geq 0$$

where

$$\Lambda(m) = \begin{cases} \frac{((d-2)m - (d-4))^2}{4(1-m)} & \text{if } m \in (\tilde{m}_1, \tilde{m}_2] \\ 4(d+2)m - 4d & \text{if } m \in [\tilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

# Spectral gaps and best constants



# B3 – Gagliardo-Nirenberg and Sobolev inequalities : improvements

[J.D., G. Toscani]

## Best matching Barenblatt profiles

(Repeating) Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[ u \left( \sigma \frac{d}{2}(m-m_c) \nabla u^{m-1} - 2x \right) \right] = 0 \quad t > 0, \quad x \in \mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 u(x, t) dx, \quad K_M := \int_{\mathbb{R}^d} |x|^2 B_1(x) dx$$

where

$$B_\lambda(x) := \lambda^{-\frac{d}{2}} \left( C_M + \frac{1}{\lambda} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[ u^m - B_\lambda^m - m B_\lambda^{m-1} (u - B_\lambda) \right] dx$$

## Three ingredients for *global improvements*

- 1  $\inf_{\lambda>0} \mathcal{F}_\lambda[u(x, t)] = \mathcal{F}_{\sigma(t)}[u(x, t)]$  so that

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[u(x, t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot, t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u] := \lambda^{\frac{d}{2}(m-m_c)} \frac{m}{1-m} \int_{\mathbb{R}^d} u |\nabla u^{m-1} - \nabla B_\lambda^{m-1}|^2 dx$$

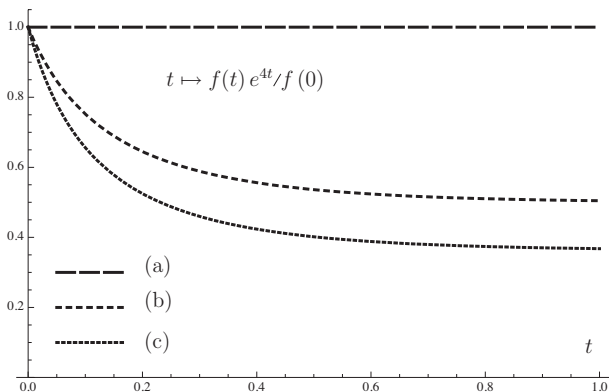
- 2 In the *Bakry-Emery method*, there is **an additional (good) term**

$$4 \left[ 1 + 2 C_{m,d} \frac{\mathcal{F}_{\sigma(t)}[u(\cdot, t)]}{M^\gamma \sigma_0^{\frac{d}{2}(1-m)}} \right] \frac{d}{dt} (\mathcal{F}_{\sigma(t)}[u(\cdot, t)]) \geq \frac{d}{dt} (\mathcal{J}_{\sigma(t)}[u(\cdot, t)])$$

- 3 The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u] \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m dx} C_M^2 \|u - B_\sigma\|_{L^1(\mathbb{R}^d)}^2$$

# Improved decay for the relative entropy



**Figure:** Upper bounds on the decay of the relative entropy:  $t \mapsto f(t) e^{4t} / f(0)$ .

(a): estimate given by the entropy-entropy production method

(b): exact solution of a simplified equation.

(c): numerical solution (found by a shooting method)

## An improved Sobolev inequality: the setting

Sobolev's inequality on  $\mathbb{R}^d$ ,  $d \geq 3$  can be written as

$$\int_{\mathbb{R}^d} |\nabla f|^2 dx - S_d \left( \int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} dx \right)^{\frac{d-2}{d}} \geq 0 \quad \forall f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

and optimal functions take the form

$$f_{M,y,\sigma}(x) = \frac{1}{\sigma^{\frac{d}{2}} \left( C_M + \frac{|x-y|^2}{\sigma} \right)^{d-2}} \quad \forall x \in \mathbb{R}^d$$

where  $C_M$  is uniquely determined in terms of  $M$  by the condition that  $\int_{\mathbb{R}^d} f_{M,y,\sigma}^{\frac{2d}{d-2}} dx = M$  and  $(M, y, \sigma) \in \mathcal{M}_d := (0, \infty) \times \mathbb{R}^d \times (0, \infty)$ . Define the manifold of the optimal functions as

$$\mathfrak{M}_d := \{ f_{M,y,\sigma} : (M, y, \sigma) \in \mathcal{M}_d \}$$

and consider the *relative entropy* functional

$$\mathcal{R}[f] := \inf_{g \in \mathfrak{M}_d} \int_{\mathbb{R}^d} \left[ g^{-\frac{2}{d-2}} \left( |f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \right) - \frac{d}{d-1} \left( |f|^{\frac{d-1}{d-2}} - g^{\frac{d-1}{d-2}} \right) \right] dx$$

# An improved Sobolev inequality: the result (1/2)

## Theorem

[J.D., G. Toscani] Let  $d \geq 3$ . For any  $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ , we have

$$\int_{\mathbb{R}^d} |\nabla f|^2 dx - S_d \left( \int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} dx \right)^{\frac{d-2}{d}} \geq \frac{C_d \mathcal{R}[f]^2}{\| |x|^2 f^{\frac{2d}{d-2}} \|_{L^1(\mathbb{R}^d)}}$$

The functional  $\mathcal{R}[f]$  is a measure of the distance of  $f$  to  $\mathfrak{M}_d$  and because of the **Csiszár-Kullback inequality**, we get

$$\frac{\mathcal{R}[f]}{\| |x|^2 f^{\frac{2d}{d-2}} \|_{L^1(\mathbb{R}^d)}^{1/2}} \geq \frac{C_{CK}}{\| f \|_{L^{2^*}(\mathbb{R}^d)}^{\frac{3d+2}{d-2}}} \inf_{g \in \mathfrak{M}_d} \| |f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \|_{L^1(\mathbb{R}^d)}^2$$

with explicit expressions for  $C_d$  and  $C_{CK}$

## An improved Sobolev inequality: the result (2/2)

### Corollary

[J.D., G. Toscani] *Let  $d \geq 3$ . For any  $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ , we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2 dx - S_d \left( \int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} dx \right)^{\frac{d-2}{d}} \\ \geq \frac{\mathfrak{C}_d}{\|f\|_{L^{2^*}(\mathbb{R}^d)}^2} \inf_{g \in \mathfrak{M}_d} \| |f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \|_{L^1(\mathbb{R}^d)}^4$$

- The expression of  $\mathfrak{C}_d$  is also explicit
- This solves an old open question of [Brezis, Lieb (1985)] with (partial) answers given in [Bianchi-Egnell (1990)] and [Cianchi, Fusco, Maggi, Pratelli (2009)]
- A similar result holds for Gagliardo-Nirenberg inequalities with  $p \in (1, \frac{d}{d-2})$

Thank you for your attention !