

Scientific Project

Maximal operators, Fourth Order Equations and Quantum Drift-Diffusion

Introduction

This proposal is aimed at pursuing and amplifying a fruitful collaboration between the main researchers, students and post-doctoral trainees from France and Chile. The main researchers already constitute a mature team with an ample history of collaboration, developed during the years, and including the co-guidance of Ph. D. students and post-doctoral training.

Our research plan is organized along three new basic lines of research: Liouville-type theorems for non-linear maximal operators, fourth order problems and quantum drift-diffusion equations. These topics are entirely new with respect to former collaborations, except possibly for the first. However, still in this case there is an important change in the point of view regarding the general class of maximal operators, rather than only Pucci's operators. New techniques are required to treat them.

Liouville-type theorems for non-linear maximal operators

One remarkable result in elliptic partial differential equations is the Liouville-type theorem for the Laplace operator proved by Gidas and Spruck. This theorem stands at the heart of the theory of elliptic partial differential equations, with deep consequences in the existence theory, connecting critical Sobolev embeddings, *a priori* estimates and non-existence results. Since its proof in 1981, generalizations in various directions have been obtained, both for more general operators and systems. During the last five years, a series of advances have re-invigorated the investigations on these type of results, making this topic again a very active area of research. See for example, recent results for fully nonlinear operators, Pucci's extremal operators, conformally invariant operators and degenerate p-laplacian like operators in the work by Birindelli and Demengel [7], Cutri and Leoni [11], Capuzzo Dolcetta and Cutri [8], Felmer and Quaas [24], Li and Zhang [33] and Li and Li [31].

In this proposal we plan to consider various questions concerning Liouville type theorems and related existence results. One major problem has to do with the existence and characterization of the critical exponent for nonlinear autonomous extremal operators, in the general sense of Pucci. Given numbers $0 < \lambda \leq \Lambda$ and \mathcal{A} a set of

symmetric matrices satisfying

$$\lambda \leq A \leq \Lambda \quad \forall A \in \mathcal{A},$$

that is stable under orthogonal transformations, we can define, for a symmetric matrix D , the operator

$$\mathcal{M}_{\mathcal{A}}(D) = \sup_{A \in \mathcal{A}} \text{tr}(AD).$$

If u is a C^2 function, then we define a second order differential operator replacing D by D^2u . This differential operator is uniformly elliptic, homogeneous, autonomous and, in general, it does not have divergence form. They are Pucci's operators in the general sense.

Our project aims at understanding the critical exponents for this type of operators, that is, the structure of positive solutions to the equation

$$\mathcal{M}_{\mathcal{A}}(D^2u) + u^p = 0, \quad \text{in } \Omega \subset \mathbb{R}^N,$$

in terms of the parameter p , for a given $N \geq 3$. Associated to $\mathcal{M}_{\mathcal{A}}$ there are two extreme exponents

$$p_- = \frac{N_- + 2}{N_- - 2} \quad \text{and} \quad p_+ = \frac{N_+ + 2}{N_+ - 2},$$

where $N_- = \frac{\lambda}{\Lambda}(N - 1) + 1$ and $N_+ = \frac{\Lambda}{\lambda}(N - 1) + 1$. One conjecture is that for the operator $\mathcal{M}_{\mathcal{A}}$ there is a critical exponent (in the sense of existence/non-existence threshold value) and that it belongs to the interval $[p_-, p_+]$, see the work by Esteban, Felmer and Quaas [23]. This general conjecture seems out of reach for the moment. However we have some preliminary steps in mind.

Problem 1. We plan to study the radial case, still a formidable problem. We remark that the case of regular Pucci's operators was treated by Felmer and Quaas [24], where non-trivial analysis was made. In this case we still can use some reminiscence of Pohozaev's identity and energy/integration by parts. In the general case these tools will simply not work.

Problem 2. A second problem is related to the exploration of the critical exponent for the Pucci's operator without assuming radial symmetry and its connection with existence. See some recent results by Esteban, Felmer and Quaas [23]. The idea here is to treat some special domains and some particular operators in order to obtain valuable information about the way to go from the radial case to the non-radial case.

Problem 3. A third problem we will consider has to do with the extension to regular Pucci's operators of a recent result about bubbling near criticality by del Pino, Dolbeault and Musso in [19, 18]. The analysis in this case is somehow analogous, but there are various technical difficulties that need to be addressed. In particular the associated dynamical systems are not smooth.

As already mentioned, all these problems deal with nonlinear non-divergence form operators, where many of the usual techniques for dealing with the Laplacian are no longer available, such as integration by parts, Pohozaev identities and the Kelvin transform. This makes the problem interesting and challenging.

Fourth Order problems

During the past two years, variational principles associated to nonlinear fourth order equations have been studied. These equations arise in many problems in Mathematical Physics. Fourth order operators share with Pucci's operators difficulties due to the lack of the Maximum Principle and new types of loss of compactness.

In the framework of the ECOS project # C02E06, we have studied the following variational principle,

$$\int_{\mathbb{R}} (u'')^2 dx - \int_{\mathbb{R}} u'' u^2 dx \geq I \int_{\mathbb{R}} u^4 dx$$

for all $u \in L^4(\mathbb{R})$, such that $u'' \in L^2(\mathbb{R})$. This problem has oscillating minimizers in the line, see the work by Benguria, Catto, Dolbeault and Monneau [4]. Such an inequality is relevant in the study of Lieb-Thirring inequalities for fourth order linear operators in the line, see the work by Hoppe, Laptev and Ostensson [28].

Presently, Benguria, Catto and Dolbeault are studying a Gagliardo Nirenberg type inequality involving second derivatives (which are connected to fourth order nonlinear equations). This inequality also arises in connection with Lieb-Thirring inequalities. The goal is to find the best constants $C(p)$ for the inequality,

$$\left(\int_{\mathbb{R}} (\phi')^{2p} dx \right)^{1/p} \leq C(p) \left(\int_{\mathbb{R}} (\phi'')^2 dx \right)^{\theta} \left(\int_{\mathbb{R}} \phi^2 dx \right)^{1-\theta}$$

with $p \in [1, +\infty]$ (when $p = +\infty$ the left-hand side is simply the L^∞ norm), and $\theta = (1/4)(3 - (1/p))$. If $p = 1$, then $\theta = 1/2$, and the best constant is $C(1) = 1$. The optimizers are periodic functions (of any period or amplitude). The conjecture is that the same feature holds for any $p > 1$. Our goal is to catalog examples of such oscillating minimizers and if possible to characterize which fourth order operators exhibit such a qualitative behavior, which has already been observed in many other contexts.

A characterization of the speed of propagation of fronts for the Extended Fisher-Kolmogorov equation has been obtained in collaboration with M. Cristina Depassier. The equation is of the form:

$$u_t = u_{xxxx} + f(u),$$

on the line, see Benguria and Depassier [6]. This is again a problem that involves a fourth order operator. In the context of this problem, we would like to obtain a variational characterization for the speed of propagation of monotonic fronts, in the spirit of our variational characterization for the propagation of fronts of the regular KPP problem, see the work by Benguria and Depassier [5]. We will come back to nonlinear drift-diffusion equations and parabolic operator of higher order at the end of the next section.

Quantum drift-diffusion equations

The third topic of this project deserves some explanations on the mathematical and physical motivations. The collaboration has already started within the framework of

the ECOS project # C02E08, but fully deserves to be continued in this new proposal for reasons which will appear below.

The 6th Hilbert problem is devoted to an axiomatic justification of physical laws especially in mechanics. The purpose is for instance to obtain the equations of fluid mechanics starting from Liouville equations for systems of particles. This program has been divided into several steps: reduction from a description by statistical mechanics to a kinetic description of matter like Boltzmann's equation (Lanford, [30] and more recently Cercignani, Illner and Pulvirenti, [10]) and derivation of fluid (see the recent contributions of Bardos, Golse, Levermore [3, 2] and Saint-Raymond [27]) or diffusive limits (Poupaud and Goudon [12]). All these studies have been done in the context of classical mechanics.

There are cases where the underlying microscopical system can be described only by quantum mechanics. This occurs in the study of VLSI semi-conductor devices. Up to now, empirical models have been able to model most of the experimental situations, but for the development of new types of devices at a smaller scale, quantum effects cannot be neglected any more. The reduction from systems of interacting particles to a kinetic type description through a BBGKY hierarchy is not under control, and models with collisions are not well established. One of the reasons is that physics is by far more complex than for rarefied gases: Various types of particles (electrons, holes) have eventually long range interactions. These particles also interact with the crystal (phonons) and impurities, etc. The program of the 6th Hilbert problem may look hopeless at this stage. However, what is needed for practical purposes is only a hydrodynamical or diffusive description of the electrons. By analogy with classical mechanics (see for instance Levermore's derivation of hydrodynamical systems), a method based on moments and entropy closure has recently been proposed by Degond and Ringhofer [14]. On the diffusive time scale, it provides nonlocal diffusive models which have been formally derived by Degond, Méhats and Ringhofer [13]. A semi-discretized (in time) version of this model has been worked out by Gallego and Méhats [25] last year, with a major difficulty: Because of the lack of control on the null set, it has proved impossible to obtain the solutions of the continuous model by letting the time step tend to zero.

In an ongoing collaboration, Dolbeault, Mayorga and Méhats [21] are now trying to understand the mathematical justification of the quantum diffusive model with a new approach involving two ingredients:

- The diffusive equation is understood as the gradient flow of an appropriate "entropy" (free energy) with respect to the Wasserstein distance. Such an approach turns out to be an extremely powerful way of thinking for nonlinear diffusion equations. Results obtained by Del Pino and Dolbeault for (doubly) nonlinear diffusion equations [16, 15, 17] have been reformulated with these tools by Otto, McCann, Gangbo, Carlen, Agueh [1, 9, 26, 34] after an initial work on the heat equation by Jordan, Kinderlehrer and Otto [34].
- The estimates of Lieb-Thirring type which have been established by Dolbeault, Felmer, Loss and Patrel [20] are expected to provide the appropriate *a priori*

estimates to work within the physical space on which the entropy is simply finite.

This project offers very interesting perspectives:

Problem 1. A lot of freedom on the underlying statistical mechanics can be taken into account. Not only maxwellian statistics but actually any type of distribution of the occupation numbers (like Fermi-Dirac statistics), are allowed, under a convexity assumption. Such an approach is the quantum counterpart of a result which has been obtained recently in classical mechanics by Dolbeault, Ölz, Markowich and Schmeiser [22]. The method associates a nonlinear diffusion equation to each statistical distribution (Gibbs distribution in the classical framework). The first objective is therefore to provide a systematic approach of quantum drift-diffusion equations for general statistics (modelization) and to prove existence results simultaneously.

Problem 2. Justify the formal derivation by Degond and Ringhofer of quantum drift-diffusion equations and hydrodynamical models.

Problem 3. *A priori* estimates should also provide the right setting to understand in which limit quantum corrections to classical diffusive models can be established and provide justifications for instance to the classical quantum drift-diffusion model which involves the fourth order operator

$$\nabla \cdot \left[n \nabla \left(\frac{\Delta \sqrt{n}}{\sqrt{n}} \right) \right]$$

(the term $\Delta \sqrt{n}/\sqrt{n}$ is the so-called Bohm potential). We plan to justify formal expansions involving such fourth order terms.

This research program is connected with two other international projects:

- *Equations de diffusions nonlinéaires et problèmes d'interaction fluide-structures* – Projet d'Action Intégrée France-Espagne - Contrat Picasso no. 07222TG – <http://www.ceremade.dauphine.fr/~dolbeaul/Picasso/>
- *Modèles diffusifs pour les systèmes quantiques et les équations de Wigner* – Projet d'Action Intégrée France-Allemagne - Contrat Procope no. 09608ZL – <http://www.ceremade.dauphine.fr/~dolbeaul/Procope/>

The first one deals with nonlinear diffusion equations using entropy methods, with an emphasis on fourth order parabolic equations. The second one has more specifically to do with quantum drift-diffusion equations.

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