

# ENTROPY METHODS IN NONLINEAR EVOLUTION EQUATIONS

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Our scientific project can be divided into two main topics:

## 1. ENTROPY METHODS

We want to understand how far one can push entropy methods in the study of nonlinear diffusion equations. For instance, we would like to obtain results in some of the following directions:

- (1) *Generalize Beckner type entropies to nonlinear diffusion equations.* The generalized Poincaré inequalities have been introduced by W. Beckner in order to interpolate between Poincaré and logarithmic Sobolev inequalities. They enjoy a remarkable number of properties, like tensorization. From the point of view of evolution PDEs, they are closely linked to linear equations of Fokker-Planck type. Recently people have started the study of nonlinear problems like porous medium or thin film equations using families of inequalities similar to the ones introduced by W. Beckner, see [5, 6], in one space dimension. This raises a large number of questions: what are the corresponding inequalities in higher dimensions ? Which rates of decay can be established ? How do the nonlinear estimates interplay with their linearized versions, which should be some weighted Poincaré inequalities ? Can the asymptotic regime be controlled in terms of the linear estimates ?
- (2) *Gradient flows distances.* In [12], it was proved that the Fokker-Planck equation is the gradient flow of the entropy with respect to the Wasserstein distance. In [10], it has also been proved that the heat equation can be seen as the gradient flow of the entropies which appear in Beckner's inequalities with respect to an appropriate distance, which interpolates between the Wasserstein and  $H^{-1}$  distances. Apart from the case of the standard generalized entropy for the porous media, see [13], not much has been done for general nonlinear diffusions and this is a direction of research that we would like to investigate.
- (3) *Functional inequalities.* Porous media and Gagliardo-Nirenberg-Sobolev inequalities are closely related, as it has been shown in [8]. Such a result is also connected to mass transportation techniques, see for instance [1, 7]. However, in terms of general inequalities of Gagliardo-Nirenberg-Sobolev, the class in which such a connection has been established is rather small and we would like to understand if one can go beyond or if there is a fundamental obstruction.
- (4) *Perturbation results for nonlinear diffusion equations* has apparently never been studied. The goal would be to generalize Holley-Stroock estimates, which are well known in the case of linear diffusions. This would have important consequences also for nonlinear problems, for instance in case of mean field coupling.
- (5) *Lévy processes.* The justification of Lévy processes from microscopical equations, for instance in Fourier variables, is still to be done. An alternative approach would be to obtain it from a diffusive scaling limit of equations written at the kinetic level, in the spirit of [9], but for collision kernels, which would be by far more delicate to handle.

## 2. MEAN FIELD PARABOLIC EQUATIONS

The second topic of this project is more applied. The goal is to understand qualitative properties of the solutions to parabolic equations coupled with a Poisson potential in dimension two and three. In this direction, we can for instance list the following questions:

- (1) *Keller-Segel model.* The Keller-Segel model has been a subject of investigation of both teams in the recent years, see [2, 3, 4]. Many questions are however still open, like convergence rates in self-similar variables. Some Fourier analysis could eventually provide partial answers. The case corresponding to a critical normalized mass equal to  $8\pi$  has recently been studied in [2, 3] for radial solutions, but is still to be understood in the non radially symmetric case. The underlying functional inequalities which allow to control the free energy of the system are either of logarithmic Hardy-Littlewood-Sobolev or of Moser-Trudinger type and should definitely be related to McOwen's inequality. How the free energy production term controls the free energy has still to be studied, but raises difficult issues. For instance, it is not even known if the entropy production term is a convex functional. In the subcritical case, we also plan to study the uniqueness of the branch of the stationary solutions in bounded open sets using methods based on the Rellich-Pohozaev identity, in the spirit of [11, 14].

A better understanding of the simplest form of the Keller-Segel model will provide guidelines for the study of more realistic models which are widely used in mathematical biology, although the modeling in this area of research goes far beyond the scope of the present project.

- (2) *Porous media equation coupled to the gravitational Poisson equation.* Such an equation has many interesting features, ranging from good thermodynamical entropies, called *Tsallis entropies*, to interesting features of the stationary solutions, like compact support properties. One of the main challenges is to prove stability properties for the nonlinear evolution problem. To give a classification of the various types of stationary solutions which can be achieved as critical points of entropy functionals is also of interest in frameworks which are relevant from the point of view of physics. A typical example is the case of systems in solid rotation.

## REFERENCES

- [1] M. AGUEH, N. GHOUSSEUB, AND X. KANG, *Geometric inequalities via a general comparison principle for interacting gases*, *Geom. Funct. Anal.*, 14 (2004), pp. 215–244.
- [2] P. BILER, G. KARCH, P. LAURENÇOT, AND T. NADZIEJA, *The  $8\pi$ -problem for radially symmetric solutions of a chemotaxis model in a disc*, *Topol. Methods Nonlinear Anal.*, (2006).
- [3] ———, *The  $8\pi$ -problem for radially symmetric solutions of a chemotaxis model in the plane*, *Mathematical Methods in the Applied Sciences*, (2006).
- [4] A. BLANCHET, J. DOLBEAULT, AND B. PERTHAME, *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, *Electron. J. Differential Equations*, (2006).
- [5] M. J. CÁCERES, J. A. CARRILLO, AND G. TOSCANI, *Long-time behavior for a nonlinear fourth-order parabolic equation*, *Trans. Amer. Math. Soc.*, 357 (2005), pp. 1161–1175 (electronic).
- [6] J. A. CARRILLO, J. DOLBEAULT, I. GENTIL, AND A. JÜNGEL, *Entropy-energy inequalities and improved convergence rates for nonlinear parabolic equations*, *Discrete and continuous dynamical systems B*, (2006).
- [7] D. CORDERO-ERAUSQUIN, B. NAZARET, AND C. VILLANI, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, *Adv. Math.*, 182 (2004), pp. 307–332.
- [8] M. DEL PINO AND J. DOLBEAULT, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, *J. Math. Pures Appl.* (9), 81 (2002), pp. 847–875.
- [9] J. DOLBEAULT, P. MARKOWICH, D. OELZ, AND C. SCHMEISER, *Nonlinear diffusions as limit of kinetic equations with relaxation collision kernels*, *Archive for Rational Mechanics and Analysis*, (2006).
- [10] J. DOLBEAULT, B. NAZARET, AND G. SAVARÉ, *Heat equations and gradient flows*. In preparation, 2006.
- [11] J. DOLBEAULT AND R. STANCZY, *Uniqueness results by Rellich-Pohožaev type estimates*. Work in progress.
- [12] R. JORDAN, D. KINDERLEHRER, AND F. OTTO, *The variational formulation of the Fokker-Planck equation*, *SIAM J. Math. Anal.*, 29 (1998), pp. 1–17 (electronic).
- [13] F. OTTO, *The geometry of dissipative evolution equations: the porous medium equation*, *Comm. Partial Differential Equations*, 26 (2001), pp. 101–174.
- [14] R. SCHAFF, *Uniqueness for semilinear elliptic problems: supercritical growth and domain geometry*, *Adv. Differential Equations*, 5 (2000), pp. 1201–1220.