Convex Sobolev inequalities and spectral gap

Inégalités de Sobolev convexes et trou spectral

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Abstract

This note is devoted to the proof of convex Sobolev (or generalized Poincaré) inequalities which interpolate between spectral gap (or Poincaré) inequalities and logarithmic Sobolev inequalities. We extend to the whole family of convex Sobolev inequalities results which have recently been obtained by Cattiaux [11] and Carlen and Loss [10] for logarithmic Sobolev inequalities. Under local conditions on the density of the measure with respect to a reference measure, we prove that spectral gap inequalities imply all convex Sobolev inequalities with constants which are uniformly bounded in the limit approaching the logarithmic Sobolev inequalities. We recover the case of the logarithmic Sobolev inequalities as a special case.

Résumé Cette note est consacrée à la preuve d'inégalités de Sobolev convexes (ou inégalités de Poincaré généralisées) qui interpolent entre des inégalités de trou spectral (ou de Poincaré) et des inégalités de Sobolev logarithmiques. Nous étendons à la famille des inégalités de Sobolev convexes toute entière des résultats qui ont été obtenus récemment par Cattiaux [11] et Carlen et Loss [10] pour des inégalités de Sobolev logarithmiques. Sous des conditions locales sur la densité de la mesure par rapport à une mesure de référence, nous démontrons que les inégalités de trou spectral entraînent toutes les inégalités de Sobolev convexes avec des constantes qui sont bornées uniformément dans la limite qui approche les inégalités de Sobolev logarithmiques. Nous retrouvons le cas des inégalités de Sobolev logarithmiques comme un cas particulier.

Key words : generalized Poincaré inequalities ; convex Sobolev inequalities ; Poincaré inequalities ; spectral gap inequalities ; logarithmic Sobolev inequalities ; interpolation ; perturbation ; entropy – entropy production method

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Soit μ une mesure de probabilité sur $\mathbb{R}^d$. On dira que μ admet une *inegalité de Sobolev logarithmique (tendue)* s'il existe une constante $C_1(\mu)$ telle que

$$\int u^2 \, \log \Big(\frac{u^2}{\|u\|^2_{L^2(\mu)}} \Big) \, d\mu \leq C_1(\mu) \, \|\nabla u\|^2_{L^2(\mu)} \quad \forall \, u \in H^1(\mu) \ , \tag{1}$$

et une *inégalité de Poincaré*, ou encore de *trou spectral*, s'il existe une constante $C_2(\mu)$ telle que

$$\int |u - \bar{u}|^2 \, d\mu \leq C_2(\mu) \, \|\nabla u\|^2_{L^2(\mu)} \quad \text{avec} \quad \bar{u} = \int u \, d\mu \ , \quad \forall \, u \in H^1(\mu) \ . \tag{2}$$

Si l'inégalité (1) est vérifiée, alors (2) est aussi vraie avec $C_2(\mu) \leq \frac{1}{2} \, C_1(\mu)$. La réciproque est fausse en général (considérer $d\mu(x) = C \, \exp(-|x|^\alpha)$ avec $\alpha \in [1, 2)$, voir [9,12] et [15,6,5] pour plus de détails). Cependant, Cattiaux dans [11] puis Carlen et Loss dans [10] ont donné des conditions nécessaires sur μ pour que (1) se déduise de (2). Le but de cette note est d'améliorer certains de ces résultats en considérant une famille d'inégalités qui interpole entre (1) et (2). A la suite de Beckner [8], pour $p \in (1, 2]$, on dira que μ vérifie une *inégalité de Poincaré généralisée* s'il existe une constante positive finie $C_p(\mu)$ telle que

$$\frac{1}{p-1} \left[\int |u|^2 \, d\mu - \Big(\int |u|^{2/p} \, d\mu \Big)^p \right] \leq C_p(\mu) \int |\nabla u|^2 \, d\mu \quad \forall \, u \in H^1(\mu) \ . \tag{3}$$

Le cas $p = 2$ correspond à (2) et dans la limite $p \to 1$, on retrouve (1) si $\liminf_{p \to 1} C_p(\mu)$ est finie. On peut montrer que

$$\frac{2}{p} \, C_2(\mu) \leq C_p(\mu) \quad \forall \, p \in [1, 2] \quad \text{et} \quad C_p(\mu) \leq \frac{1}{p-1} \, C_2(\mu) \quad \forall \, p \in (1, 2] \tag{4}$$

(voir [1,7] pour plus de détails). On en déduit que si $C_1(\mu)$ est finie, alors $C_p(\mu)$ est aussi finie pour tout $p \in (1, 2]$. Par contre, il n'est pas possible d'en déduire une estimation de $C_1(\mu)$ sachant que $C_2(\mu)$ est finie. Dans la suite, $C_p(\mu)$ désignera pour tout $p \in [1, 2]$ la valeur optimale de la constante.

Gross a montré dans [13] que (1) est vérifiée pour les mesures gaussiennes :

$$\mu(x) = \nu_\sigma(x) := (2\pi\sigma^2)^{-d/2} \, e^{-\frac{|x|^2}{2\sigma^2}} \ ,$$

et, en utilisant les polynômes d'Hermite, Beckner a établi dans [8] que (3) est aussi vérifiée lorsque $\mu = \nu_\sigma$, pour tout $p \in (1, 2)$, avec

$$C_p(\nu_\sigma) = \frac{2}{p} \, \sigma^2 \ .$$

La méthode d'entropie – production d'entropie de Bakry et Emery [4] permet de montrer (3) dans le cas de mesures du type $\mu = e^{-V}$ lorsque V est strictement convexe, voir [2]. Pour cette raison, les inégalités (3) sont aussi appelées *inégalités de Sobolev convexes.* On montre ainsi que

$$C_p(e^{-V}) \ \leq \ \frac{2}{p} \ \left[\inf_{\xi \in S^{d-1}, \, x \in \mathbb{R}^d} \ (\, D^2 V(x) \, \xi , \, \xi \,) \right]^{-1} \ .$$

Cela conduit naturellement à rechercher des conditions suffisantes sur V pour borner $C_1(\mu)$ en fonction de $C_2(\mu)$, ou, en d'autres termes, pour que l'inégalité de Poincaré entraîne l'inégalité de Sobolev logarithmique. Nous allons nous intéresser à des estimations de $C_p(\mu)$ pour tout $p \in (1, 2)$, et, en prenant la limite $p \to 1$, retrouver et améliorer les résultats obtenus pour $p = 1$ par Cattiaux dans [11] puis par Carlen et Loss dans [10]. Notre principal résultat est un résultat de perturbation pour les inégalités de Sobolev convexes (3) qui diffère toutefois de la méthode classique de Holley-Stroock [14,2,1].

Théorème 1 *Soit $p \in [1,2)$ et $p' = p/(p-1)$. Si μ et ν sont deux mesures de probabilités de densités respectives e^{-V} et e^{-W} par rapport à la mesure de Lebesgue telles que $C_p(\nu)$ et $C_2(\mu)$ soient finies et si $Z := \frac{1}{2}(V - W)$ est une fonction de $L^{p'}(d\nu)$ telle que*

$$\inf_{x\in\mathbb{R}^d} \big(|\nabla Z|^2 - \Delta Z + \nabla Z \cdot \nabla W\big) > -\infty \,,$$

alors

$$C_p(\mu) \le \mathcal{C}_p := \frac{2}{p}\, C_2(\mu) + \Big(\frac{2}{p} - 1\Big) \, \Big[C_p(\nu) + C_2(\mu)\, \big(2\, \|Z\|_{L^{p'}(\mu)} - m\, C_p(\nu)\big)_+\Big] \,.$$

Par passage à la limite $p \to 1$, on obtient un résultat pour les inégalités de Sobolev logarithmiques (1).

Corollaire 2 *Avec les mêmes notations que ci-dessus, si les hypothèses du Théorème 1 sont vérifiées uniformément dans la limite $p \to 1$, alors μ vérifie l'inégalité de Sobolev logarithmique (1) avec $C_1(\mu) \le \liminf_{p\to 1} \mathcal{C}_p$.*

La preuve du Théorème 1 consiste comme dans [10] pour $p = 1$ à établir d'abord une inégalité restreinte :

$$\frac{\int |v|^2\, d\mu - \Big(\int |v|^{2/p}\, d\mu\Big)^p}{(p-1)\int |\nabla v|^2\, d\mu} \le \mathcal{C}_p^* \quad \forall\, v \in H^1(\mu) \quad \text{tel que} \quad \bar{v} = 0 \,.$$

Ensuite on en déduit le cas général grâce au

Lemme 3 *Soit $q \in [1,2]$. Pour toute fonction $u \in L^1 \cap L^q(\mu)$, si $\bar{u} := \int u\, d\mu$, alors*

$$\Big(\int |u|^q\, d\mu\Big)^{2/q} \ge |\bar{u}|^2 + (q-1)\Big(\int |u - \bar{u}|^q\, d\mu\Big)^{2/q} \,.$$

1. Introduction and main result

Consider a probability measure μ on $\mathbb{R}^d$. We say that there is a *(tight) logarithmic Sobolev inequality* associated to μ if there exists a finite constant $C_1(\mu)$ such that

$$\int u^2 \log\Big(\frac{u^2}{\|u\|^2_{L^2(\mu)}}\Big)\, d\mu \le C_1(\mu)\, \|\nabla u\|^2_{L^2(\mu)} \quad \forall\, u \in H^1(\mu) \,, \tag{1}$$

and a *Poincaré inequality* associated to μ if there exists a finite constant $C_2(\mu)$ such that

$$\int |u - \bar{u}|^2\, d\mu \le C_2(\mu)\, \|\nabla u\|^2_{L^2(\mu)} \quad \text{with} \quad \bar{u} = \int u\, d\mu \,, \quad \forall\, u \in H^1(\mu) \,. \tag{2}$$

This inequality is often called the *spectral gap inequality* for the following reason. Consider in $H^1(\mu)$ the Rayleigh quotient $\|\nabla u\|^2_{L^2(\mu)}/\|u\|^2_{L^2(\mu)}$. The lowest critical value, zero, corresponds to constant functions, and the optimal value for $C_2(\mu)^{-1}$ is therefore associated with the second critical value: $u - \bar{u}$ is the projection on the orthogonal of the constants with respect to the $L^2(\mu)$ norm. It is well known that if (1) holds, then (2) is also true with

$$C_2(\mu) \le \frac{1}{2}\, C_1(\mu)\ .$$

This is easily checked by writing $u = 1+\varepsilon\, v$, with $\bar v = 0$, and by letting $\varepsilon \to 0$. The reverse implication is a much harder question, and not true in general. With no additional assumption, we may have $C_1(\mu) = +\infty$ and $C_2(\mu) < \infty$. An example of such a situation is given by $\mu(x) = \exp(-|x|^\alpha)$ in $\mathbb{R}^d$ with $\alpha \in [1,2)$, see, *e.g.,* [9,12] and [15,6,5] for more details. Cattiaux in [11], and then Carlen and Loss in [10] with more elementary tools, gave sufficient conditions on μ under which (1) is a consequence of (2). The goal of this note is to revisit some of these results by considering a family of inequalities which interpolate between (1) and (2).

According to Beckner in [8], we shall say that, for some $p \in (1,2]$, there is a *generalized Poincaré inequality* associated to μ if there exists a positive constant $C_p(\mu)$ such that

$$\frac{1}{p-1}\left[\int |u|^2\, d\mu - \Big(\int |u|^{2/p}\, d\mu\Big)^p\right] \le C_p(\mu) \int |\nabla u|^2\, d\mu \quad \forall\, u \in H^1(\mu)\ . \tag{3}$$

Throughout this paper, we will assume that for any $p \in [1,2]$, $C_p(\mu)$ is the optimal constant. We will not consider "defective" logarithmic Sobolev inequality (see, *e.g.,* [11,16]) and will omit the word "tight" whenever we mention Inequality (1). The limit case $p = 2$ corresponds to (2), at least for nonnegative solutions. However, in the general case, (2) looks different of (3) in the limit case $p = 2$. We indeed get

$$\int |u|^2\, d\mu - \Big(\int |u|\, d\mu\Big)^2 \le C_2(\mu) \int |\nabla u|^2\, d\mu \quad \forall\, u \in H^1(\mu)$$

in that case, which is equivalent to (2) only for nonnegative functions. However, if the inequality holds for a function $u - a = v \ge 0$, a straightforward computation shows that

$$\int |u-\bar u|^2\, d\mu = \int |(u-a)-(\bar u - a)|^2\, d\mu = \int |v-\bar v|^2\, d\mu \le C_2(\mu)\int |\nabla v|^2\, d\mu = C_2(\mu)\int |\nabla u|^2\, d\mu\,,$$

so that (2) holds for any $u \in H^1(\mu)$ such that $u_- \in L^\infty(\mu)$. By density, we extend it to any $u \in H^1(\mu)$: (3) with $p = 2$ is therefore equivalent to (2). On the other hand, by taking the limit $p \to 1$ in (3), we find $C_1(\mu) \le \liminf_{p\to 1} C_p(\mu)$, which proves (1) if the right hand side is finite. By considering again $u = 1 + \varepsilon\, v$, with $\bar v = 0$, in the limit $\varepsilon \to 0$, we get:

$$C_p(\mu) \ge \frac{2}{p}\, C_2(\mu) \quad \forall\, p \in [1,2]\ .$$

By Hölder's inequality, $(\int u\, d\mu)^2 \le (\int |u|^{2/p}\, d\mu)^p$ for any $p \in [1,2]$. As a consequence, for any $p \in (1,2]$,

$$C_p(\mu) = \sup_{u \in H^1(\mu)} \frac{\int |u|^2\, d\mu - \Big(\int |u|^{2/p}\, d\mu\Big)^p}{(p-1)\int |\nabla u|^2\, d\mu} \le \frac{1}{p-1} \sup_{u\in H^1(\mu)} \frac{\int |u|^2\, d\mu - \Big(\int u\, d\mu\Big)^2}{\int |\nabla u|^2\, d\mu} = \frac{C_2(\mu)}{p-1}\ .$$

We refer to [1,7] for more details. Summarizing, we know that

$$\frac{2}{p}\, C_2(\mu) \le C_p(\mu) \quad \forall\, p \in [1,2] \quad \text{and} \quad C_p(\mu) \le \frac{1}{p-1}\, C_2(\mu) \quad \forall\, p \in (1,2]\ . \tag{4}$$

It follows that if $C_1(\mu) < \infty$, then for all $p \in (1,2]$, $C_p(\mu) < \infty$. However, at this stage, it is clear that we have no estimate on $C_1(\mu)$ if we only know that $C_p(\mu)$ is finite for some $p \in (1,2]$.

Inequality (1) has been established by Gross in [13] in the case of Gaussian measures:

$$\mu(x)=\nu_\sigma(x):=(2\pi\sigma^2)^{-d/2}\,e^{-\frac{|x|^2}{2\sigma^2}}\;,$$

and using Hermite polynomials Beckner in [8] proved that (3) holds with

$$C_p(\nu_\sigma)=\frac{2}{p}\,\sigma^2\,.$$

An alternative method based on the entropy – entropy production method of Bakry and Emery [4] has been adapted in [2] to prove (3) in more general situations which for instance cover the case of measures $\mu=e^{-V}$ for some strictly convex function V. For this reason, the family of inequalities (3) has been called *convex Sobolev inequalities.* The entropy – entropy production method gives an upper bound on the best constant:

$$C_p(e^{-V})\;\le\;\frac{2}{p}\,\left[\inf_{\xi\in S^{d-1},\,x\in\mathbb{R}^d}\left(\,D^2V(x)\,\xi,\,\xi\,\right)\right]^{-1}\;=:\;\frac{2}{p\,\lambda_1}\,.$$

This shows that at least in some circumstances, the bounds in (4) are not optimal. As already mentioned, Cattiaux in [11] and then Carlen and Loss in [10] gave sufficient conditions on V to bound $C_1(\mu)$ in terms of $C_2(\mu)$, or, in other words, to deduce logarithmic Sobolev inequalities from spectral gap inequalities. Our purpose is to extend these results to $C_p(\mu)$ for any $p\in(1,2)$, and recover and improve their results by deriving uniform estimates in the limit $p\to1$.

Note for completeness that improvements of several types have been obtained, for instance by considering L^∞ perturbations of V based on Holley-Stroock type estimates [14,2,1]. This allows to relax the strict convexity condition on V. One can also refine the entropy – entropy production method [1], thus giving for instance the improved inequality

$$\left(\frac{p}{p-1}\right)^2\left[\int|u|^2\,d\mu-\left(\int|u|^{2/p}\,d\mu\right)^{2(p-1)}\left(\int|u|^2\,d\mu\right)^{\frac{2}{p}-1}\right]\le\frac{4}{\lambda_1}\int|\nabla u|^2\,d\mu\quad\forall\,u\in H^1(\mu)\,.$$

Although it differs in nature from the Holley and Stroock perturbation lemma, our main result can be seen as a perturbation result as well. It applies to convex Sobolev inequalities (3).

Theorem 1 *Let $p\in[1,2)$. Let μ and ν be two probability measures with respective densities e^{-V} and e^{-W} relatively to Lebesgue's measure such that, for some $p\in(1,2]$, $C_p(\nu)$ and $C_2(\mu)$ are finite. Assume that*

$$Z:=\frac{1}{2}(V-W)\in L^{p'}(d\nu)\quad\text{and}\quad m:=\inf_{x\in\mathbb{R}^d}\delta(x)>-\infty\,,$$

where $\delta:=|\nabla Z|^2-\Delta Z+\nabla Z\cdot\nabla W$, and define $\mathcal{C}_p^*:=C_p(\nu)+C_2(\mu)\,\big(2\,\|Z\|_{L^{p'}(\mu)}-m\,C_p(\nu)\big)_+$. Then we have

$$C_p(\mu)\le\mathcal{C}_p:=\frac{2}{p}\,C_2(\mu)+\Big(\frac{2}{p}-1\Big)\,\mathcal{C}_p^*\,.$$

We denote by $p'=p/(p-1)\in[2,\infty]$ the Hölder conjugate of $p\in[1,2]$. By relative density, we simply mean, *e.g.*, $d\mu(x)=e^{-V(x)}\,dx$. With these notations, $\mu=e^{-2Z}\,\nu$. Taking the limit $p\to1$, we obtain a result analoguous to Theorem 1 for the logarithmic Sobolev inequalities (1).

Corollary 2 *With the above notations, if the assumptions of Theorem 1 hold uniformly in the limit $p\to1$ and if $\liminf_{p\to1}\mathcal{C}_p$ is finite, then the logarithmic Sobolev inequality (1) associated to μ holds with $C_1(\mu)\le\liminf_{p\to1}\mathcal{C}_p$.*

2. Proof of the main result

As in [10], we first prove Theorem 1 in the *restricted case* which corresponds to $\bar{u}=0$ and then extend it to the *unrestricted case.*

Lemma 3 *Under the assumptions of Theorem 1,*

$$\sup_{v\in H^1(\mu),\,\bar{v}=0} \frac{\int |v|^2\,d\mu - \left(\int |v|^{2/p}\,d\mu\right)^p}{(p-1)\int |\nabla v|^2\,d\mu} \le \mathcal{C}_p^*\;.$$

Proof. Define

$$\mathcal{A}(t) := \|\nabla v\|^2_{L^2(\mu)} - \frac{t}{(p-1)\,C_p(\nu)}\left[\int |v|^2\,d\mu - \left(\int |v|^{2/p}\,d\mu\right)^p\right]\;.$$

Proving that for some $t>0$, $\mathcal{A}(t)\ge 0$ for any v in $H^1(\mu)$ with $\bar{v}=0$ is equivalent to the result of Theorem 1 in the case $\bar{u}=0$, *i.e.* $u=v$, the so-called *restricted case* in [10]. Let us write

$$\mathcal{A}(t) = \text{(I)} + \text{(II)} + \text{(III)}$$

with

$$\begin{aligned}
\text{(I)} &= (1-t)\int |\nabla v|^2\,d\mu\;,\\
\text{(II)} &= t\int |\nabla v|^2\,d\mu\;,\\
\text{(III)} &= \frac{-t}{(p-1)\,C_p(\nu)}\left[\int |v|^2\,d\mu - \left(\int |v|^{2/p}\,d\mu\right)^p\right]\;.
\end{aligned}$$

Let $v = g\,e^Z$:

$$\int |v|^2\,d\mu = \int |g|^2\,d\nu \quad\text{and}\quad \int |\nabla v|^2\,d\mu = \int |\nabla g|^2\,d\nu + \int \delta\,|g|^2\,d\nu\;.$$

Using the spectral gap assumption on μ, we get

$$\text{(I)} \ge \frac{1-t}{C_2(\mu)}\int |v|^2\,d\mu = \frac{1-t}{C_2(\mu)}\int |g|^2\,d\nu\;.$$

Using the fact that (3) holds for ν and the above expression of $\int |\nabla v|^2\,d\mu$, we obtain

$$\text{(II)} \ge \frac{t}{(p-1)\,C_p(\nu)}\left(\int |g|^2\,d\nu - \left(\int |g|^{2/p}\,d\nu\right)^p\right) + t\int \delta\,|g|^2\,d\nu\,.$$

As for the last term, we can write it as

$$\text{(III)} = \frac{t}{(p-1)\,C_p(\nu)}\left(\left(\int |v|^{2/p}\,d\mu\right)^p - \int |g|^2\,d\nu\right)\;.$$

Collecting these estimates, we have

$$\mathcal{A}(t) \ge \int \left(\frac{(1-t)}{C_2(\mu)} + t\,\delta\right)|g|^2\,d\mu + \frac{\mathcal{B}\,t}{(p-1)\,C_p(\nu)}\;,\quad \mathcal{B} := \left(\int |v|^{2/p}\,d\mu\right)^p - \left(\int |g|^{2/p}\,d\nu\right)^p.$$

Let $d\pi := \frac{|g|^{2/p}}{\int |g|^{2/p}\,d\nu} d\nu$. By Jensen's inequality applied to the convex function $t \mapsto e^{-t}$, we get

$$\frac{\int |v|^{2/p}\,d\mu}{\int |g|^{2/p}\,d\nu} = \frac{\int |g|^{2/p} e^{-2(1-\frac{1}{p})Z}\,d\nu}{\int |g|^{2/p}\,d\nu} = \int e^{-2(1-\frac{1}{p})Z}\,d\pi \geq \exp\Big[-2\Big(1-\frac{1}{p}\Big)\int Z\,d\pi\Big]\,.$$

Using the lower estimate $e^{-t} \geq 1-t$, we infer that

$$\left(\frac{\int |v|^{2/p}\,d\mu}{\int |g|^{2/p}\,d\nu}\right)^p = e^{-2(p-1)\int Z\,d\pi} \geq 1 - 2(p-1)\int Z\,d\pi\;,$$

$$\Big(\int |v|^{2/p}\,d\mu\Big)^p - \Big(\int |g|^{2/p}\,d\nu\Big)^p \geq -2(p-1)\int Z\,|g|^{2/p}\,d\nu \Big(\int |g|^{2/p}\,d\nu\Big)^{p-1}\,.$$

By Hölder's inequality, we have

$$\Big(\int |g|^{2/p}\,d\nu\Big)^{p-1} \leq \Big(\int |g|^2\,d\nu\Big)^{1-1/p} \quad \text{and} \quad \int Z\,|g|^{2/p}\,d\nu \leq \Big(\int |g|^2\,d\nu\Big)^{1/p}\,\|Z\|_{L^{p'}(\nu)}\,,$$

$$\mathcal{B} \geq -2\,(p-1)\,\|Z\|_{L^{p'}(\mu)}\int |g|^2\,d\nu\;.$$

Altogether, we get

$$\mathcal{A}(t) \geq \left[\frac{1-t}{C_2(\mu)} + t\left(m - \frac{2\,\|Z\|_{L^{p'}(\mu)}}{C_p(\nu)}\right)\right]\int |g|^2\,d\nu\;.$$

This proves that $\mathcal{A}(t) \geq 0$ for any $t \in (0, t^*]$ with

$$t^* := 1 \text{ if } m - \frac{1}{C_2(\mu)} - \frac{2\,\|Z\|_{L^{p'}(\mu)}}{C_p(\nu)} \geq 0\;,\quad t^* := \left[1 + \frac{C_2(\mu)}{C_p(\nu)}\Big(2\,\|Z\|_{L^{p'}(\mu)} - m\,C_p(\nu)\Big)\right]^{-1} \text{otherwise}\,.$$

This ends the proof with $\mathcal{C}_p = C_p(\nu)/t^*$. □

The general case $\bar{u} \neq 0$ in Theorem 1 is a consequence of the following estimate, which is the counterpart for $p < 2$ of a Lemma given in [3] for $p > 2$ (see Remark 2 below).

Lemma 4 *Let $q \in [1,2]$. For any function $u \in L^1 \cap L^q(\mu)$, if $\bar{u} := \int u\,d\mu$, then*

$$\Big(\int |u|^q\,d\mu\Big)^{2/q} \geq |\bar{u}|^2 + (q-1)\Big(\int |u-\bar{u}|^q\,d\mu\Big)^{2/q}\,.$$

Proof. Let $v := u - \bar{u}$, $\phi(t) := \big(\int |\bar{u} + t\,v|^q\,d\mu\big)^{2/q}$. We may notice that $\phi(0) = |\bar{u}|^2$, $\phi'(0) = 0$, $\phi(1) = (\int |u|^q\,d\mu)^{2/q}$ and

$$\frac{1}{2}\,\phi''(t) = (2-q)\Big(\int |w|^q\,d\mu\Big)^{\frac{2}{q}-2}\Big(\int |w|^{q-2}\,w\,v\,d\mu\Big)^2 + (q-1)\Big(\int |w|^q\,d\mu\Big)^{\frac{2}{q}-1}\int |w|^{q-2} v^2\,d\mu$$

with $w := \bar{u} + t\,v$. The first term of the right hand side is nonnegative. As for the second one, we may use Hölder's inequality:

$$\Big(\int |v|^q\,d\mu\Big)^{\frac{2}{q}} = \Big(\int |w|^{\frac{q}{2}(2-q)}\cdot|v|^q\,|w|^{\frac{q}{2}(q-2)}\,d\mu\Big)^{\frac{2}{q}} \leq \Big(\int |w|^q\,d\mu\Big)^{\frac{2}{q}-1}\cdot\int |w|^{q-2}\,|v|^2\,d\mu\,.$$

Thus we get: $\frac{1}{2}\,\phi''(t) \geq (q-1)\,(\int |v|^q\,d\mu)^{2/q}$, which proves that $\phi(1) \geq \phi(0) + (q-1)\,(\int |v|^q\,d\mu)^{2/q}$ and completes the proof. □

Proof of Theorem 1. Let $v := u - \bar{u}$ and apply Lemma 4 with $q = \frac{2}{p} \in [1,2)$. Since $\int |u|^2\,d\mu - |\bar{u}|^2 = \int |u-\bar{u}|^2\,d\mu = \int |v|^2\,d\mu$, we can write

$$\begin{aligned}
\int |u|^2\,d\mu - \Big(\int |u|^{2/p}\,d\mu\Big)^p &\le \int |u|^2\,d\mu - |\bar{u}|^2 - \Big(\frac{2}{p}-1\Big)\Big(\int |u-\bar{u}|^{\frac{2}{p}}\,d\mu\Big)^p \\
&= \int |v|^2\,d\mu - \Big(\frac{2}{p}-1\Big)\Big(\int |v|^{\frac{2}{p}}\,d\mu\Big)^p \\
&= 2\,\frac{p-1}{p}\int |v|^2\,d\mu + \frac{2-p}{p}\left[\int |v|^2\,d\mu - \Big(\int |v|^{2/p}\,d\mu\Big)^p\right].
\end{aligned}$$

We can then apply (2) and Lemma 3, and the result holds with $\mathcal{C}_p = \frac{2}{p}\,C_2(\mu) + (\frac{2}{p}-1)\,\mathcal{C}_p^*$. □

Remark 1 – To deduce the unrestricted inequality from the restricted inequality, Carlen and Loss in [10] use the following inequality:

$$\int |u|^q\,d\mu \le |\bar{u}|^q + \frac{1}{2}\,q\,(q-1)\,\|u\|_{L^q(\mu)}^{q-2}\,\|v\|_{L^q(\mu)}^2 \quad \forall u \in L^q(\mu)\,,\quad v = u - \bar{u} \quad \forall\, q \in [2,\infty)\,.$$

The proof is essentially the same as for Lemma 4. We can also write a similar result for $q \le 2$:

$$\int |u|^q\,d\mu \ge |\bar{u}|^q + \frac{1}{2}\,q\,(q-1)\,\|u\|_{L^q(\mu)}^{q-2}\,\|v\|_{L^q(\mu)}^2 \quad \forall u \in L^q(\mu)\,,\quad v = u - \bar{u} \quad \forall\, q \in (1,2]\,.$$

Remark 2 – In the case $q > 2$, according to [3], the following result holds:

$$\Big(\int |u|^q\,d\mu\Big)^{2/q} \le |\bar{u}|^2 + (q-1)\Big(\int |u-\bar{u}|^q\,d\mu\Big)^{2/q}.$$

Remark 3 – For evident reasons, $\mathcal{C}_p^* \le \mathcal{C}_p$: to the restricted case corresponds an improved inequality, stated in Lemma 3. On the other hand, from (4) and Theorem 1, we obtain

$$0 \le \Big(\frac{2}{p}-1\Big)C_2(\mu) \le C_p(\mu) - C_2(\mu) \le \mathcal{C}_p - C_2(\mu) \le \frac{2-p}{p}\,\big(C_2(\mu) - \mathcal{C}_p^*\big)\,.$$

This means that for any $p \in (1,2)$, under the assumptions of Theorem 1,

$$\mathcal{C}_p^* \le C_2(\mu) \le \mathcal{C}_p\,.$$

3. Application to the euclidean space

To compare our results with those of [10], we can state a result for generalized Poincaré inequalities corresponding to Gaussian weights, *i.e.* $\nu = \nu_\sigma$, and recover in the limit $p \to 1$ the logarithmic Sobolev inequality. We can optimize the choice of $W(x) = |x|^2/(2\sigma^2)$ and cover, for instance, all harmonic potentials, which was not the case in [10]. This freedom in the choice of the parameter σ corresponds to the scaling invariance in the Euclidean space, which is however not so easy to write in the case of generalized Poincaré inequalities. In the case where $\nu = \nu_\sigma$, it is known that inequality (3) holds. Theorem 1 becomes

Corollary 5 *Let $\nu = e^{-V}$ a probabality measure. If there exists $\sigma \in (0,\infty)$ such that*

$$V - \frac{|x|^2}{2\sigma^2} \in L^{p'}(d\nu_\sigma) \quad \textit{and} \quad \inf_{x\in\mathbb{R}^d}\left(|\nabla V|^2 - 2\,\Delta V - \frac{|x|^2}{\sigma^4}\right) > -\infty\,,$$

then Inequality (1) holds.

References

[1] A. Arnold and J. Dolbeault, *Refined convex Sobolev inequalities*, tech. rep., Ceremade, preprint no. 0431, 2004.

[2] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter, *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations*, Comm. Partial Differential Equations, 26 (2001), pp. 43–100.

[3] D. Bakry, *L'hypercontractivité et son utilisation en théorie des semigroupes*, in Lectures on probability theory (Saint-Flour, 1992), vol. 1581 of Lecture Notes in Math., Springer, Berlin, 1994, pp. 1–114.

[4] D. Bakry and M. Émery, *Hypercontractivité de semi-groupes de diffusion*, C. R. Acad. Sci. Paris Sér. I Math., 299 (1984), pp. 775–778.

[5] F. Barthe, P. Cattiaux, and C. Roberto, *Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry*, tech. rep., HAL Preprint no. 00002203, 2004.

[6] F. Barthe and C. Roberto, *Sobolev inequalities for probability measures on the real line*, Studia Math., 159 (2003), pp. 481–497. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday (Polish).

[7] J.-P. Bartier, J. Dolbeault, R. Illner, and M. Kowalczyk, *A qualitative study of linear drift-diffusion equations with time-dependent or vanishing coefficients.* 2005.

[8] W. Beckner, *A generalized Poincaré inequality for Gaussian measures*, Proc. Amer. Math. Soc., 105 (1989), pp. 397–400.

[9] S. G. Bobkov and F. Götze, *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal., 163 (1999), pp. 1–28.

[10] E. Carlen and M. Loss, *Logarithmic Sobolev inequalities and spectral gaps*, in Recent advances in the theory and applications of mass transport, vol. 353 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2004, pp. 53–60.

[11] P. Cattiaux, *Hypercontractivity for perturbed diffusion semi-groups*, Preprint, (2004).

[12] I. Gentil, A. Guillin, and L. Miclo, *Modified logarithmic Sobolev inequalities and transportation inequalities*, tech. rep., Ceremade, preprint no. 04-29, 2004.

[13] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math., 97 (1975), pp. 1061–1083.

[14] R. Holley and D. Stroock, *Logarithmic Sobolev inequalities and stochastic Ising models*, J. Statist. Phys., 46 (1987), pp. 1159–1194.

[15] R. Latała and K. Oleszkiewicz, *Between Sobolev and Poincaré*, in Geometric aspects of functional analysis, vol. 1745 of Lecture Notes in Math., Springer, Berlin, 2000, pp. 147–168.

[16] M. Ledoux, *The geometry of Markov diffusion generators*, Ann. Fac. Sci. Toulouse Math. (6), 9 (2000), pp. 305–366. Probability theory.