Free energies, nonlinear flows and functional inequalities

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(joint work with Giuseppe Toscani)

Consider on $\mathbb{R}^d$, $d \geq 3$, the fast diffusion equation

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left(v \, \nabla v^{m-1} \right) = 0 \tag{1}$$

for some $m \in [m_1, 1)$ with $m_1 := (d-1)/d$. Assume that the initial data is a given nonnegative function u_0 in $L^1(\mathbb{R}^d)$ such that $u_0^m \in L^1(\mathbb{R}^d)$ and $|x|^2 \, u_0 \in L^1(\mathbb{R}^d)$. Large time asymptotics of the solution are governed by Barenblatt self-similar profiles, which can be studied either by comparison methods as in [7] or using time-dependent rescalings and free energy functionals. This second approach goes as follows.

Define the function u such that

$$v(\tau, y + x_0) = R^{-d} \, u(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \log R \;, \quad x = \frac{y}{R} \tag{2}$$

where v is a solution of (1) with initial datum u_0. A simple computation shows that u has to be a solution of

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[u \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla u^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d \;, \tag{3}$$

with initial datum u_0 if we assume that R is chosen such that $R(0) = 1$. The diffusion coefficient σ given by

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{1-d\,(1-m)} \, \frac{dR}{d\tau} \;.$$

A standard choice is to choose $\sigma = 1$ (which amounts to do a self-similar change of variables) and study the convergence of u as $t \to \infty$ towards the stationary solution B_1, where

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\sigma} \, |x|^2 \right)^{\frac{1}{m-1}} \quad \forall \, x \in \mathbb{R}^d$$

and the constant C_M is chosen so that $\int_{\mathbb{R}^d} B_1 \, dx = M := \int_{\mathbb{R}^d} u_0 \, dx$. Consider the *free energy* and *relative Fisher information* functionals respectively defined by

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\sigma^m - m \, B_\sigma^{m-1} \, (u - B_\sigma) \right] \, dx$$

$$\text{and} \quad \mathcal{J}_\sigma[u] := \sigma^{\frac{d}{2}(m-m_c)} \, \frac{m}{1-m} \int_{\mathbb{R}^d} u \, \left| \nabla u^{m-1} - \nabla B_\sigma^{m-1} \right|^2 \, dx \;.$$

It has been established in [4] that

$$\mathcal{F}_\sigma[u] \leq \frac{1}{4} \, \mathcal{J}_\sigma[u] \;, \tag{4}$$

which amounts to an interpolation inequality of Gagliardo-Nirenberg type. As a special case for $m = m_1$, with $u = |f|^{2^*}$, it is equivalent to Sobolev's inequality

$$\|\nabla f\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2 \geq 0 \quad \forall \, f \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{5}$$

where $2^* = \frac{2d}{d-2}$, S_d is the optimal constant of T. Aubin and G. Talenti and $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is the completion with respect to the norm $\|\cdot\|$ defined by $\|f\|^2 = \|\nabla f\|_2^2 + \|f\|_{2d/(d-2)}^2$ of the set of smooth functions with compact support. Since

$$\frac{d}{dt}\mathcal{F}_\sigma[u(t,\cdot)] = -\mathcal{J}_\sigma[u(t,\cdot)] \tag{6}$$

if u is a solution of (3), we find that $\mathcal{F}_\sigma[u(t,\cdot)] \le \mathcal{F}_\sigma[u_0]\, e^{-4t}$ for any $t \ge 0$, which proves the convergence of $u(t,\cdot)$ to B_1 in various norms if $\sigma = 1$. Such results have a nice interpretation in terms of gradient flows as was observed first in [8].

Rates of convergence are related to the following Hardy-Poincaré inequality. For any $\alpha \in (-\infty, \alpha_*) \cup (\alpha_*, 0)$, there is a positive constant $\Lambda_{\alpha,d}$ such that

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in L^2\,(d\mu_{\alpha-1}) \tag{7}$$

under the additional condition $\int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$. Here $\mu_\alpha(x) := (1+|x|^2)^\alpha$, has to be applied with $\alpha = 1/(m-1) < 0$ and $\alpha_* := -(d-2)/2$. The proof relies on the observation that $\mathcal{F}_\sigma[u(t,\cdot)]$ is asymptotically equivalent to $\int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1}$ if $u(t,\cdot) = B_1\,(1 + f\, B_1^{1-m})$ while $\mathcal{F}_\sigma[u(t,\cdot)]$ can be controlled by $\int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$. The method covers a range which is actually not restricted to $m \in [m_1, 1)$: see [1] for details. Moreover, according to [2], for any $m \in (m_1, 1)$ there exists two constants $\mathsf{C} > 0$ and $\Lambda > 4$ such that

$$\mathcal{F}_\sigma[u(t,\cdot)] \le \mathsf{C}\, e^{-\Lambda t} \quad \forall\, t \ge 0 \tag{8}$$

if $x_0 = \frac{1}{M}\int_{\mathbb{R}^d} x\, u_0(x)\; dx$. The spectral gap given by (7) gives exactly $\Lambda = 4$ but the associated eigenspace is generated by the translations of B_1 and is discarded by the above choice of x_0. At this point no improvement is achieved in the Sobolev case, that is for $m = m_1$. The next eigenspace is generated by the dilations of B_1. It can also be discarded as it has been shown in [5], thus showing that Λ can be taken strictly larger than 4 even for $m = m_1$, but for the solution of a different equation. Namely, we shall consider the case where σ is now time-dependent and chosen in order to minimize $\sigma \mapsto \mathcal{F}_\sigma[u(t,\cdot)]$. As measured by the relative entropy, by doing so we are choosing the *best matching Barenblatt profile* $B_{\sigma(t)}$ among all possible ones. An easy computation shows that this amounts to fix $\sigma = \sigma(t)$ such that $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\; dx = \int_{\mathbb{R}^d} |x|^2\, u(t,x)\; dx$, thus making (3) non-local, and (2) non explicit. Three main ingredients have now to be taken into account:

(i) Estimates (6) and (4) are unchanged in the new choice of $\sigma(t)$,
(ii) When applying the Bakry-Emery method, $\frac{d}{dt}\mathcal{J}_\sigma[u(t,\cdot)]$ involves an additional term which has the right sign because $\frac{d\sigma}{dt}$ can be related to $\mathcal{F}_\sigma[u(t,\cdot)]$,
(iii) Using $\|u\|_1 = \|B_\sigma\|_1$ and $\int_{\mathbb{R}^d} |x|^2\, u\; dx = \int_{\mathbb{R}^d} |x|^2\, B_\sigma\; dx$, one can prove a Csiszár-Kullback type inequality according to which we have

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \ge \frac{m}{8\int_{\mathbb{R}^d} B_1^m\; dx}\left(C_M\,\|u - B_\sigma\|_1 + \frac{1}{\sigma}\int_{\mathbb{R}^d} |x|^2\,|u - B_\sigma|\; dx\right)^2.$$

With $u=|f|^{2^*}$, we obtain an improvement of Sobolev's inequality (5), which gives an answer to the question of H. Brezis and E. Lieb in [3, Question (c), p. 75].

Theorem 1 ([6]). *Let $d\ge 3$. There is some explicit constant $\mathfrak{C}_d$ such that*

$$\|\nabla f\|_2^2-\mathsf{S}_d\,\|f\|_{2^*}^2\ge\frac{\mathfrak{C}_d}{\|f\|_{2^*}^{2\,\frac{3\,d+2}{d-2}}}\inf_{g\in\mathfrak{M}_d}\left\||f|^{\frac{2d}{d-2}}-g^{\frac{2d}{d-2}}\right\|_1^4\quad\forall\,f\in\mathcal{D}^{1,2}(\mathbb{R}^d)\;.$$

Here $\mathfrak{M}_d$ is the manifold of optimal functions for (5).

Estimate (8) is optimal in the large t regime, but the price we pay for it is that the constant C is not explicitly known in terms of the initial datum. On the other hand, the result of Theorem 1 provides an improvement of the decay rate of $\mathcal{F}_\sigma[u(t,\cdot)]$ when it is large, that is for small values of t. This raises the open question of giving sharp estimates of the decay of $\mathcal{F}_\sigma[u(t,\cdot)]$ at any time $t\ge 0$.

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