FUNCTIONAL INEQUALITIES, ASYMPTOTICS AND DYNAMICS OF FRONTS

1. Participants:

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2. Topics:

Line 1. *Transversal instability of planar traveling fronts in reaction-diffusion models* (B. Perthame, M. Kowalczyk)
Line 2. *Brownian ratchets: asymptotic regimes, rates of convergence and efficiency* (J. Dolbeault, M. Kowalczyk, and A. Blanchet)
Line 3. *Sharp functional inequalities and nonlinear flows* (J. Dolbeault, M. del Pino, and J. Campos (currently doctorate student of del Pino and Dolbeault, postdoc after december 2011)).

3. Description of the project

3.1. **Transversal instability of planar traveling fronts in reaction-diffusion models.** Instabilities of some simple states often lead to remarkable effects that can be observed, in nature or in experiments, as complicated patterns. This is a major line of research that attracts the attention of physicists, geophysicists and biologists, with many applications in engineering, in fluid mechanics, chemistry or biological modeling of cell colonies growth, to name a few.

Among the many scenarios of instability, the so-called *transversal instabilities* are remarkable both because they are frequent and might be among the most surprising effects that the dimensionality of the system has on its behavior. How is it that a stable (in all possible sense) monotonic, one dimensional traveling wave becomes unstable when considered as a planar wave in dimension $N \geq 2$? Such effects are certainly impossible for scalar equations like Fisher/KPP/monostable or Allen-Cahn/bistable equations. On the other hand they appear in systems of two or more components which are *not* cooperative, *i.e.,* when no monotonicity comparison principle is available. For instance, in a class of FitzHugh-Nagumo (or bistable) systems, transversal instabilities of radial, or periodic flat fronts have been studied in [21, 11, 20]. The system we put forward here (see below) seems rather different.

Our goal here is to study transversal instability as it arises in the usual scenario of the formation of dentritic patterns in cell colonies [19, 17] or in combustion theory. More precisely our goal is to understand the mechanism of appearance of transversal instabilities for a very simple model system

$$\begin{cases} -c\,u_y - u_{yy} = h(u)\,v\;, \\ -c\,v_y - \Lambda v_{yy} = -h(u)\,v\;, \\ \lim_{y\to-\infty}(u,v) = (1,0)\;, \quad \lim_{y\to+\infty}(u,v) = (0,1)\;. \end{cases}$$

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In combustion theory Λ is the inverse of the Lewis number. It is usually considered as large. This is also the case of wild forest fires [1]. For models of bacterial growth, the unknown v represents a molecular nutrient and it usually diffuses much faster than the cells themselves. The relevant regime is then again Λ large; see [17]. This is the regime already proved to be pathological in [9] and this is precisely the regime which we will study here. The paper [17] coined the name *diffusive Fisher* equation for this system and we adopt this terminology. As for the nonlinearity two different cases are proposed in both combustion and biology literature depending on h

$$h \in C^1(0,1), \quad h(0) = 0 \ , \quad h(1) = 1, \quad h'(u) > 0 \quad \text{for} \ \ 0 < u \leq 1 \ . \tag{1}$$

$$h(0) = 0 \quad \text{for} \ \ 0 \leq u \leq \theta < 1 \ , \quad h(\theta^+) = h^+ \geq 0 \ , \quad h(1) = 1 \ , \quad h'(u) > 0 \quad \text{for} \ \ \theta < u \leq 1 \ . \tag{2}$$

In (1) we may take for instance $h(u) = u^p$, $p = 1, 2$.

Does instability occurs for this system or are some other terms needed ? It is well known in the combustion/biophysics literature that the degeneracy of the function $h(\cdot)$ plays a role. Can one characterize the precise mathematical situation where this can be proved ? When transversal instabilities occur it is also certainly necessary that stiffness occurs (i.e the linearized problem has eigenvalues at various orders of the parameter Λ are present), but can one make this statement rigorous ?

3.2. **Brownian ratchets: asymptotic regimes, rates of convergence and efficiency.** In [7, 6] we studied the long time behavior of the following one-dimensional model for a *Brownian ratchet:*

$$[u_x + (\psi\, u)_x]_x = u_t \ , \quad (x,t) \in \mathbb{R} \times (0, \infty) \ , \quad u(x,0) = u_0(x) \geq 0 \ , \quad \int_{\mathbb{R}} u_0 = 1 \ , \tag{3}$$

where $\psi = \psi(x - \omega t)$, and $\psi(\cdot)$ is a periodic, smooth function. Among other things we showed in [7] that as $t \to \infty$ the density function u converges to a Gaussian, modulated by some periodic oscillations, and at the same time traveling with an effective speed $c = c(\omega)$, where ω is the speed of the traveling potential. The effective diffusion of *traveling ratchet* is $\kappa = \kappa(\omega)$. In general we have $c(\omega) < \omega$ and $\kappa(\omega) \neq 1$ but the dependence of the effective speed and the effective diffusion on ω is quite complicated. In [6] we have studied this issue using numerical and formal considerations. We have developed a notion of the *efficiency* of the traveling ratchet, intuitively understood as the ability of the ratchet to move the assembly of Brownian particles in one direction.

The approach used in [7] relies on finding a suitable approximation of the asymptotic profile, and then on using some functional inequalities together with the relative entropy method to control the error of the approximation. In particular, we needed to *homogenize* a certain class of generalized Poincaré inequalities and a limiting logarithmic Sobolev inequality. In some case we were able to find optimal constants for these inequalities. However in the case of the logarithmic Sobolev inequality it is an open problem whether, in the homogenized limit, the optimal constant is given by the effective diffusion constant, as one can reasonably conjecture. We will come back to this question in Section 3.3.

Another issue we propose to study is the problem of optimization of the traveling potential to maximize the efficiency defined in [6]. Our goal is two-fold: to find a procedure, perhaps numerical to determine the most efficient ratchet, and to show rigorously that there is a maximal theoretical efficiency, at least for a reasonable class potentials.

In numerical simulations of the traveling ratchet two time scales are in general observed. At fist no significant, unidirectional motion of the center of mass is seen, and periodic modulations develop over the initial data. In this initial stage localized influence of the potential dominates over the transport of mass over long distances (*i.e.* across the wells of the potential). After this first, rather short stage, transport becomes dominating, as we have described in [7]. Still, the question of capturing the first stage remains. Can one use relative entropies or scalings, to describe the initial dynamics of some smooth, suitably spread data ? How to measure the initial time scale theoretically ? Numerically, how can we built a time-dependent basis which gives an accurate picture of the ratchet in all time scales regimes ?

Understanding the issues we have proposed so far would be important in studying the effect of periodic, spatially inhomogeneous, and time evolving energy landscapes in other problems. For instance, if we replace the standard diffusion term by a non-linear term of fast diffusion / porous medium type, can we still identify the asymptotic long time behavior profile ? Is it possible to determine the effective speed of the center of mass ? Eventually, one would like to involve also nonlinear effects in form of some reaction term or a convection term as in fluid modeling.

Although (3) is an extremely simple model, for which there exists a large literature in physics and biophysics (see a partial review in [6]), not much is known from a mathematical point of view. The problem has an interest by itself in view of modeling in physics and biology, but it is also the occasion to develop a toolbox for a large class of problems (see for instance [12]) that should easily adapt to higher dimensions. Just to mention one example, characterizing the optimal constant of the logarithmic Sobolev inequality in the asymptotic, homogenized regime corresponding to a tilted, periodic channel subject to gravity, would exactly as in the one dimensional case provide a detailed description of the diffusion of particles in a fluid confined to such a channel.

3.3. **Sharp functional inequalities and nonlinear flows.** Functional inequalities like the logarithmic Sobolev inequality in homogenized settings have already been considered above. The conjecture is that the optimal constant is determined by an appropriately linearized problem, that is a Poincaré inequality. Such a situation seems rather generic for a large class of inequalities and can be observed in various cases. Consider for instance the interpolation inequality introduced by W. Beckner in [3]:

$$\frac{q-2}{N}\int_{\mathbb{S}^N}|\nabla u|^2\,d\mu+\int_{\mathbb{S}^N}|u|^2\,d\mu\geq\left(\int_{\mathbb{S}^N}|u|^q\,d\mu\right)^{2/q}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^N,d\mu)\,,\tag{4}$$

for any $q\in(2,2^*]$ with $2^*=2\,N/(N-2)$ if $N\geq 3$ and for any $q\in(2,\infty)$ if $N=2$. Here $d\mu$ is the uniform probability measure on the N-dimensional sphere, that is the measure induced by Lebesgue's measure on $\mathbb{S}^N\subset\mathbb{R}^{N+1}$, up to a normalization factor such that $\mu(\mathbb{S}^N)=1$. Remarkably, the constant in the right hand side of (4) is 1, so that extremal functions are the constants; the stereographic projection then shows that the best constant in Sobolev's inequality on $\mathbb{R}^N$ is $\frac{1}{4}\,N\,(N-2)\,|\mathbb{S}^N|^{1-2/2^*}$ ($N\geq 3$) and constant functions are tranformed into the Aubin-Talenti functions. The optimal constant in (4) is the factor $(q-2)/N$; the $q-2$ term is easily understood in the linearization procedure, while N is nothing else than the first non-zero eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^N$. Several proofs are available in the literature, none of them being elementary. The method of W. Beckner in [3] is based on Legendre's duality, on the Funk-Hecke formula and on the expression of the best constants for the Hardy-Littlewood-Sobolev inequality on $\mathbb{S}^N$ found by E.H. Lieb in [18]. Alternatively, D. Bakry's approach in [2] is based on semi-group estimates. A third proof can be derived from [4] using the Bochner-Lichnerowicz-Weitzenböck identity and the method of B. Gidas and J. Spruck in [16].

Our purpose is to clarify in which cases the best constant in functional inequalities of interpolation is determined by a linear eigenvalue problem, when either the geometry is non trivial or metric issues are replaced by weighted functional settings (we actually have in mind probability densities).
• In the first direction, we can for instance consider the Gagliardo-Nirenberg-Sobolev inequality on the cylinder $\mathcal{C}:=\mathbb{R}\times\mathbb{S}^{N-1}$, namely

$$\|v\|^2_{\mathrm{L}^p(\mathcal{C})}\leq\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)\left(\,\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\right)^{\theta}\,\|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}\quad\forall\,v\in\mathrm{H}^1(\mathcal{C})$$

which is related to Caffarelli-Kohn-Nirenberg inequalities on the euclidean space $\mathbb{R}^N$. An important open question is to determine whether optimal functions of $(s,\omega)\in\mathcal{C}$ do not depend on the angular variable $\omega \in \mathbb{S}^{N-1}$, or if there is *symmetry breaking.* The value of Λ for which the first eigenvalue of the operator obtained by linearization around functions depending only on s becomes negative is certainly a lower bound, in terms ofΛ , for symmetry breaking: see [13], but whether this bound is optimal or not is an open question which has attracted a significant research effort: see [15]. On $\mathcal{C}$, the Ricci tensor degenerates, but the right setting should include a weight along the axis of the cylinder arising from s-dependent extremal functions.

• A second example is the homogenized logarithmic Sobolev inequality (with small parameter $\varepsilon > 0$) introduced in [7], which appears as an endpoint ($p = 1$) of a family of inequalities depending on a parameter $p \in [1, 2]$. For any $p > 1$, the limit of the best constant as $\varepsilon \to 0$ has been identified, but limits $p \to 1$ and $\varepsilon \to 0$ cannot be interverted and the case $p = 1$ is not understood yet. The strategy is to establish that the best constant as $\varepsilon \to 0$ is given by the linearized problem, which is the same as for $p > 1$. A consequence would then be the characterization of the optimal convergence rate as $t \to \infty$ of the solutions of the Brownian ratchet model (3) towards their asymptotic profiles in an appropriate two-scale convergence topology, and a complete characterization of the *efficiency.*

Optimal constants in functional inequalities can be used to determine optimal asymptotic rates in nonlinear flows, like in [8]. Connection between the inequalities and spectral properties of an operator obtained by an appropriate linearization of *relative entropy functionals* and *relative Fisher informations* are expected to provide new results in various problems, for instance in the case of the Keller-Segel model: J. Campos and J. Dolbeault are working on this issue in the spirit of [5]. A last and very promising field of investigation which has recently appeared is the opposite point of view, which consists in building nonlinear equations to study the energy landscape and to relate various functional inequalities: see [10, 14] for two recent contributions in this direction. This topic is more prospective but will certainly be a subject of interaction between the chilean and the french teams.

4. Main objectives

Line 1. Find the mechanism of emergence of nontrivial patterns as a result of transversal instabilities.
Line 2. Analyze in details qualitative behavior of the Brownian ratchet.
Line 3. Understand the role of linearization in optimal functional inequalities of interpolation.

5. Methodology

The methodology is common to the three lines of research. We will use the methods of nonlinear PDEs, techniques proper to functional analysis and calculus of variations to attack the problems presented. We will rely also on formal asymptotic expansions and numerical simulations when appropriate.

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