NONLINEAR ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS (NAPDE)

A PROPOSAL FOR MATHAMSUD

1. Introduction

Participants to this proposal have a long record of joint research projects, mostly in the framework of individual projects of international cooperation or small group projects such as ECOS (3 of them are currently being executed). Some of the investigators have been involved in organization of larger networks or events (Alfa, CIMPA summer school, conferences). The list of joint publications is 59 long testifying to a success story of the previous undertakings. Training of young researchers through exchanges and insertion in joint research projects has been an important characteristic of past collaborations. The participants feel that all these factors constitute a very strong basis to meet challenges of a large scale collaborative project which is expected to have a broad impact within the mathematical community in France and Latin America.

The project has two centers of gravity: one is the nonlinear analysis group at the Ceremade, `http://www.ceremade.dauphine.fr/`, Université Paris-Dauphine, and the other is the Center for Analysis of Partial Differential Equations, `http://www.capde.cl/`, whose members come from leading mathematics departments in Chile (CMM UMI CNRS nr. 2807, Universidad de Chile, Pontificia Universidad Católica, Universidad Técnica Santa María, Universidad de Santiago). The scientists of these two groups have a record of past collaborations that extends for more than ten years. Nearly all others have participated in joint projects with members of the aforementioned groups. One of our aims is to broaden the existing links by incorporation of Argentinian and Brazilean teams, comprising IMPA (UMI CNRS nr. 2924).

The proposed lines of research are new and come from currently emerging areas in the respective fields of investigations. Our hope is that the NAPDE project will reinforce existing collaborations, put its members in a leadership position at the international level and open new opportunities for young researchers of the participating countries.

A summary of scientific topics. The scientific project is organized around three main axes:

(1) Nonlinear analysis: fully nonlinear elliptic problems and functional inequalities
(2) Concentration phenomena, isoperimetric problems and Riemanian manifolds
(3) Applications in mathematical biology: ratchet models, molecular motors and adaptive evolution models

Participants. The topics are mentioned between parentheses.

France.

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Chile.

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2. Scientific project

2.1. **Nonlinear analysis: fully nonlinear elliptic problems and functional inequalities.** The theory of *fully nonlinear elliptic equations* and the viscosity approach to their analysis is now a fundamental area of investigation in PDEs, because of its profound theoretical implications and its wide range of applications. Starting from linear operators in non-divergence form, the theory has already moved to include a large class of elliptic operators, including Bellman and Issacs operators in stochastic game theory. Fully nonlinear equations naturally appear also in geometric analysis including the Monge-Ampere and the range of k-Hessian operators as well as in many applications to modeling and analysis in economics and financial mathematics, population ecology, transport theory and homogenization, *etc.*

One important problem in the area is the understanding of the role of dimension like numbers in critical exponents, fundamental solutions and removable singularities where the Chilean team has made advances [20, 21, 22]. We plan to continue doing research in this direction, where many questions remain open. Another direction is Liouville type theorems in the general case, in connection with regularity theory and Pucci's conjecture, [26]. Based on the new results for the existence of first eigenvalues [23, 24] and on multiplicity of solutions [27] for the Dirichlet problem for fully nonlinear elliptic equations, P. Felmer, A. Quaas and B. Sirakov will address the resonance analysis of the (two) first eigenvalues and its relation with existence theory and exact multiplicity for non-proper equations and systems in bounded domains.

Similar questions for a class of non-uniformly (including some integral) fully nonlinear operators, starting with Alexandrov-Bakelman-Pucci and Harnack inequalities, will then be addressed, based on results for nonlocal diffusion operators recently obtained by L. Caffarelli et al. in [7, 25], and the expertise in the study of nonlocal dispersals of J. Dávila and S. Martínez. Collaborations in these topics will include M. Esteban and B. Sirakov.

Concerning *functional inequalities,* two main directions of fields of applications have to be mentioned. The first one is the study of large time asymptotics of evolution equations, typically diffusive or hypo-coercive kinetic equations, and its connection with functional inequalities of logarithmic Sobolev type or hyper-contractivity properties of semi-groups. French and Chilean participants have played a key role at the beginning of the applications to nonlinear diffusions, [9, 8, 11, 10, 13], but new applications of *relative entropy methods* have been recently discovered, [5, 6] (see Section 2.3 for applications in mathematical biology). The second direction of applications is *mathematical physics,* in view of the general theory of functional analysis and Sobolev spaces, with applications in quantum mechanics and the stability of large systems. Let us mentioned a few questions in this area. French participants have been involved in the study of various versions of Hardy's inequality, [4, 15, 16, 14, 17]. A pending question is for instance to write logarithmic Sobolev inequalities with critical weights, that is weights of the order of $|x|^{-2}$. This question is currently being studied and preliminary results have been achieved, [12]. Another question is connected with the very elegant paper [3], which establishes optimal constants in the case of a H^1_0-norm. By using Pohozaev's estimates and variational methods, the result should extend to a large family of inequalities corresponding to cases where the energy is not quadratic anymore, [1]. Finally, another important question is the study of Lieb-Thirring inequalities and their optimal constants. A relation between these inequalities and the minimization of a certain functional depending on a potential V was discussed in [19]. This approach has lead to the study of a nonlinear functional of the same kind in [18] for $V = 1/|x|$, which needs to be generalized. Another interesting question is the study of Lieb-Thirring-type inequalities for fourth-order operators, in the spirit of the preliminary work [2].

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2.2. **Concentration phenomena, isoperimetric problems and Riemanian manifolds.** Topics addressed here stand at the crossing point between analysis and nonlinear analysis on one side, and Riemannian and differential geometry on the other side.

The role of concentration phenomena in isoperimetric type problems. Very much related to the isoperimetric problem is the problem of finding extremal domains on a Riemanian manifold. This amounts to finding a sub-domain of the manifold which maximizes the first eigenvalue of the Laplacian with Dirichlet boundary condition, among domains with fixed volume V, see [12, 16, 18] and references therein. In general, this problem is very hard to solve and only few results are known (the extremal domains have the property that the first eigenfunction of the Laplacian with Dirichlet boundary condition also has constant Neumann data on the boundary). It is known that there are strong links between the study of the isoperimetric profile and the study of the first eigenvalue of the Laplacian. The expertise of L. Hauswirth, F. Pacard, M. Del Pino, P. Felmer, M. Kowalczyk and M. Musso in the study of the concentration phenomena should be useful to handle the extremal domain problem, where the knowledge is not as advanced yet.

We will start by considering the case where the prescribed volume V is small or close to the volume of the compact manifold, in order to to get precise asymptotic estimates on this extremal domain and on the asymptotic value of the first eigenvalue. In each situation, we expect to be able to describe concentration phenomena where critical points of the scalar curvature play a crucial role. In [14], F. Pacard and P. Sicbaldi have already obtained some example of extremal domains when the manifold has no boundary using nondegenerate critical points of the scalar curvature function. Tools from geometric measure theory should be adapted and new techniques have yet to be developed to prove that their result characterizes all solutions, or to extend it to the case where the ambient manifold has a boundary.

Of interest would be determination of the extremal domains in some manifolds with simple structure like flat torii. Some important progress have been made in the framework of the isoperimetric problem by Hauswirth and his collaborators and we expect to obtain similar results in the case of extremal domains.

The study of extremal domains for the first eigenvalue of the Laplacian raises some interesting questions concerning the domains for which elliptic problems with overdetermined boundary data can be solved. For example, it is known since the work of J. Serrin, [17], that smooth bounded domains in Euclidean space for which the Laplace equation with right hand side constant, zero Dirichlet boundary data and constant Neumann data can be solved, are round balls. It is an open problem to study noncompact domains where this overdetermined problem is solvable. We suspect that the boundaries of these domains share many properties with constant mean curvature surfaces in Euclidean space, [15]. We intend to work on various related questions.

Concentration along submanifolds. The concentration phenomena is related to singularly perturbed problems and has been a subject of broad study in the last 30 years. The concentration at points phenomena are by now very well understood and applications arise in many different fields. It allows for instance to prove the existence of branches of solutions to some highly nonlinear problems, blowing up at finitely many points. On the other hand, when concentration takes place along higher dimensional sets, that is for instance the energy density of a sequence of solutions concentrates along a submanifold, minimal submanifolds (and in some cases constant mean curvature surfaces) seem to play a decisive role as the set of concentration. In some cases, the situation becomes even more complicated due to the existence of a resonance phenomena (related to what is known in physics as Rayleigh's instability). This is a fairly recent area of research where most phenomena and techniques remain to be discovered. From the "constructive method" perspective and particularly, from a methodological point of view, methods developed so far can and should be greatly simplified in order to address such complex issues.

The Allen-Cahn problem which models two-phase mixtures for which every hypersurface with constant mean curvature gives rise to a concentration set, as shown by M. Kowalczyk, F. Pacard and their collaborators, see [8, 13]. Related phenomena have then been described by A. Malchiodi and M. Montenegro [11] and A. Malchiodi and F. Mahmoudi, a former student of F. Pacard, [9] in certain semilinear Neumann problems, and by M. Del Pino, M. Kowalczyk and J. Wei [4] for nonlinear Schrödinger equations. The current studies focus on the existence of multilayer solutions in the framework of semilinear elliptic problems which arise in the standing-wave nonlinear Schrödinger equation as well as in the Allen-Cahn models. Some important new results have been obtained by M. Del Pino, M. Kowalczyk, F. Pacard and J. Wei [2] (also [10]). While giving a much better understanding of the possible configuration of multilayer that appear in these models, the results that have been obtained so far are only preliminary for a new, open and completely unexplored field of investigation concerning solutions of some semilinear elliptic equations in the entire space. These solutions can be understood as fundamental building blocks to construct and possibly classify solutions with multilayer concentration sets for large class of problems.

The solutions constructed by M. Del Pino, M. Kowalczyk, F. Pacard and J. Wei [3] for the nonlinear Schrödinger equation, have their counterpart for the Allen-Cahn equation. This new family of solutions of the Allen-Cahn equation in the entire space (also see [7]) is of interest form the point of view of the study of the corresponding moduli spaces and in particular in determining its dimension. We should emphasize that this approach to the analysis of the structure of the solution set is motivated by the analogy with some problems in the theory of CMC surfaces in geometry but it is entirely new in the context of PDEs.

Recently, M. Del Pino, M. Musso and F. Pacard [5] have realized that there is a very interesting link between solutions to some semilinear elliptic equation in exterior domains and critical points of the length functional for curves defined in the exterior domain (this is a classical obstacle problem). So far the construction works when the curve is embedded in the boundary of the domain. It is expected that this result extends to lower codimensional objects such as submanifolds which are critical points of the k-volume functional in the exterior domain. It is also expected that the embedding of the submanifold in the boundary is not necessary. In other words, the construction should work for any critical point of the k-volume functional in the exterior domain.

Eigenfunctions on a Riemannian manifold. We are interested in the existence of exceptional sequences of eigenfunctions on a Riemannian manifold, namely sequences of eigenfunctions, with corresponding eigenvalues tending to infinity and whose density of energy concentrates along a submanifold of lower dimension (typically one should think about sequences of eigenfunctions on the 2-sphere whose density of energy concentrates along a great circle). Bourgain [1] has shown that exceptional sequences can be produced by perturbation of the metric on the flat torus (see also [6]). We have strong evidence that, when symmetries are not present, some techniques developed for nonlinear problems are precisely the right tool for this problem. Even though the problem is linear, perturbations are nonlinear due to the geometric properties of the concentration set. This line of research is in the scope of interest of F. Pacard, L. Hauswirth, P. Sicbaldi on the French side and M. Del Pino, M. Kowalczyk, M. Musso on the Chilean side.

Singularities in non-linear elliptic equations and minimal surfaces. The construction of minimal surfaces with conic singularities can be achieved by finding suitable solutions to a nonlinear PDE involving the mean curvature operator and singular nonlinearities. The singularities of the surface then correspond to the points where the solution vanishes. This problem is analogous to finding singular solutions to supercritical semilinear PDEs, an area in which several of us have made contributions. We aim at finding new examples of singular solutions with point and higher dimensional singular sets, and solutions in manifolds with compact boundaries. To prescribe higher dimensional singular sets is difficult in supercritical problems but seems achievable in some situations. We are also interested in the converse. Given a reasonable weak solution representing a singular minimal surface, we want to determine the optimal regularity and understand its singular set. This topic stems out of the program pursued J. D. Dávila, M. Montenegro and L. Dupaigne, O. Goubet and A. Ponce.

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2.3. **Applications in mathematical biology: ratchet models, molecular motors and adaptive evolution models.** We first plan to study mathematical models of molecular motors. Apart from their evident interest in biology, such motors raise interesting questions from the physics point of view. Gradients of temperature or electrostatic potentials are insufficient to explain how they operate. This is why one has to take into account the geometry of these motors at molecular level and observe the changes of their conformation. However, a crucial observation is that motion takes place in a highly viscous, overdamped and noisy environment, [1, 17]. Hence, at least in a certain class of motors, one has to consider *Brownian motors* or *ratchets*, which give raise to very interesting mathematically PDE models. Brownian ratchet models have been studied first in a series of papers in the bounded case, see for instance [4, 5, 6, 11, 12, 16, 15]. Two recent contributions have clarified the notion of transport in unbounded domains. The first one is based on a Hamilton-Jacobi reformulation of the problem [19]. The second one uses functional inequalities and tools of homogenization theory to determine observable quantities such as the effective diffusion and asymptotic drift, [2, 3], which allow to quantify the coherence of the transport. The resulting efficiency criterion suggest a method to optimize the design of molecular motors, which will be studied by M. Kowalczyk, J. Dolbeault and A. Blanchet. Other related questions, like the comparison of various notions of efficiency for *tilted ratchet models*, or of the various notions of diffusivity, will also be tackled. For the moment, only one-dimensional problems have been studied, and the domain geometry effects should play a role in higher dimensions.

In population dynamics the collective movement of a species is due to the random motion of individuals and to the interactions between individuals. An example of this is self- and cross-diffusion [20], which is a mechanism that can explain coexistence of species. We propose a new approach to explain density dependent diffusions in reaction diffusion equations based on the Wasserstein distance hybrid variational method, [13]. This problem in a natural way combines the experience of M. Kowalczyk and D. Kinderleherer on the Wasserstein distance based variational approach, some recent results on generalized entropies and gradient flows [8, 7], and the approach developed by S. Martinez for cross- and self-diffusion systems.

In adaptive evolution theory, diffusion or non-local dispersion is used to model mutation and non-local interaction accounts for the competition between individuals with different traits. The study of concentrating solutions when the mutation rate is small is an important problem, since it represents the appearance of dominant traits [18]. It has also been observed that these models exhibit Turing instabilities, and one may conjecture the existence of many interesting steady states. Our general objectives are: to study the behavior of stationary solutions as different parameters vary, to construct solutions exhibiting spatial patterns, and to study concentration in models where a non-local operator represents the mutation. Of special interest is the phenomenon of splitting, a process that could be a model of emergence of new species. The expertise J. Dávila and S. Martinez in the theory of non-local problems will be complemented with that of P. Felmer and A. Quaas in the theory of viscosity solutions and M. Kowalczyk in the theory pattern formation. Recently, B. Perthame et al. have applied numerical methods to validate the Hamilton-Jacobi technique for the tracking of such Dirac concentrations [10]. Such an approach looks extremely promising. It also raises the question of the modeling in presence of additional variables, like traits and position in space. J. Coville will be involved in this research topic. Recently J. Zubelli has been working on the inverse problem for a size structured model and we also expect interactions in this topic.

The Keller-Segel system is a model of a directed, collective movement of bacteria (chemotaxis), [14], where the drift force is due to a gradient of a chemical substance, the chemo-attractant. It is actively studied for its connections with angio-genesis in cancer tumors. We aim at describing conditions on the initial data or certain parameters of the model that produce blow-up, and do determine the blow-up profile. Some methods of adaptive evolution theory could apply, as well as generalized entropy based approaches, [9], will be studied. We will concentrate on two- three-dimensional cases, and extensions, for example, to problems with non-local diffusion. In this line of investigation, which shares many properties with models of gravitation in stellar dynamics, M. Del Pino and I. Guerra will interact with J. Dolbeault and B. Perthame.

Scientific computing. The concentration phenomena arising in the parabolic problems issued from adaptive evolution models will also be studied from the numerical point of view. Presently only one dimensional simulations have been performed which are costly because a fine mesh is needed to resolve the Dirac structures. In higher dimensions, it is a challenge to design new algorithms that will avoid such fine grids. The main research direction towards such algorithm is to possibly use the limiting constrained Hamilton-Jacobi equation that describes the concentration phenomena. This should also apply to gravitational models.

The project BANG has developed several finite elements codes in order to compute parabolic systems exhibiting instabilities or singularittes common in biological systems as the Keller-Segel system for chemotaxis, more realistic extended versions that include several Michaelis-Menten reactions or the Mimura system for dentritic bacterial colony growth. This activity is going to be extended with the team in Chili and efficiency of the different algorithmic methods will be compared.

For Brownian ratchets, the various notions of efficiency, based on thermodynamics, energy balance or designed to measure specifically properties of coherence during transport, will also be studied numerically.

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3. Objectives of the NAPDE project

3.1. **Publications.** The goal of the project is to produce research papers on the topics of the scientific program, that will be submitted to international scientific journals of high level. Exchanges of researchers will be organized between teams participating to the project and financed through the NAPDE project. This is however part of a larger research effort which involves the organization of specialized workshops on the themes of the project and training of young researchers.

3.2. **Workshops.** The NAPDE project will be associated the following events, which are already scheduled.

(1) Workshop on Mathematical Methods and Modeling of Biophysical Phenomena (IMPA, 15-23/3/2009)
(2) Workshop on Elliptic and Parabolic Problems in Geometric Analysis (Capde, 1/2010)
(3) Workshop on Fully Nonlinear Partial Differential Equations (Capde, 1/2011)
(4) Workshop on Solitons, Travelling Fronts and Nonlinear Diffusions (Capde, 1/2012)

Provided an adequate funding can be found, we will also organize a summer school.

3.3. **Training of young researchers.** Young researchers (PhD students and post-docs) will be encouraged to participate in the above mentioned workshops. These events should provide up-to-date informations to the participants on rapidly evolving topics. Otherwise, young researchers will be involved in joint research projects; visits of confirmed researchers and grants for short-term visits will help them to acquire new techniques and expose them to the scientific competition at the best possible level.

4. A brief record of past collaborations

Past joint projects and initiatives. Participants have been involved in the past in several joint projects. First of all, there has been a long tradition of joint ECOS projects between mostly mathematicians of the Ceremade, Université Paris-Dauphine and the DIM at Universidad de Chile, and also the department of Physics at the Catholic University of Chile. We can quote for instance the contracts ECOS C02E06 (Benguria – Séré), C02E08 (Dolbeault – Felmer) and C05E09 (Dolbeault – Felmer). See `http://www.ceremade.dauphine.fr/~dolbeaul/ECOS/` for more details. We should also mention the collaborations between mathematicians of the DIM and, for instance, a PDE group of the Université d'Evry (C05E05, Del Pino – Pacard) or of the Université d'Amiens Jules Verne (C05E04, Davila – Dupaigne).

A second level of collaboration has been achieved through various medium- to long-term visits (J. Dolbeault, 4 months in 2001 and 2 months in 2007), invitations as Professeur invité (R. Benguria, M. Kowalczyk, M. Del Pino) and exchanges of doctoral and post-doctoral students (C. Cid, A. Quaas, C. Flores, I. Flores, J. Mayorga, M. Lewin, A. Blanchet, J. Coville).

Last but not least, several events have been organized with a strong participation of many of the participants, including a summer school in Temuco which goes back as far as 1998. Argentinian, Brazilean, Chilean and French teams have already been part of an Alfa network (funded by the European community and coordinated by J. Zubelli), `http://w3.impa.br/∼alfa/` which has been very successful in the training of young researchers.

It would probably be a tedious enterprise to comment the contents of all papers co-published by participants (see a partial list below) for a very simple reason. What characterizes this collaboration is not only that it has been very productive, but also that it has been able to constantly open new areas of research and new topics, some of them with a considerable impact at international level.

A list of joint papers. This list is made of references of joint papers involving participants to the proposal of at least two different countries. Many other papers have been written within teams or involve two participants of the same country and are not listed here.

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