

# CONCENTRATION FOR AN ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY

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ABSTRACT. We are interested in non-trivial solutions of the equation

$$-\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p, \quad u \geq 0 \quad \text{in } \Omega$$

with  $u = 0$  on  $\partial\Omega$ , where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$  is a bounded domain with smooth boundary,  $0 < \beta < 1$ ,  $1 \leq p < \frac{N+2}{N-2}$  if  $N \geq 3$  ( $p \geq 1$  if  $N = 2$ ) and  $\lambda > 0$ . If  $p > 1$  we prove existence of nontrivial solutions for every  $\lambda > 0$ . As  $\lambda \rightarrow +\infty$  we find that the *least energy* solutions concentrate around a point that maximizes the distance to the boundary. We also study the behavior as  $\lambda \rightarrow 0$ . When  $p = 1$  we have similar results, extending previous works for radial solutions in a ball.

## 1. INTRODUCTION

We are interested in non-trivial solutions of the equation

$$(1.1) \quad \begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$  is a bounded domain with smooth boundary,  $0 < \beta < 1$ ,  $1 \leq p < \frac{N+2}{N-2}$  if  $N \geq 3$  ( $p \geq 1$  if  $N = 2$ ) and  $\lambda > 0$ . By a solution of (1.1) we mean a function  $u \in H_0^1(\Omega)$ ,  $u \geq 0$  such that  $\int_{[u>0]} u^{-\beta} < \infty$  and

$$\int_{\Omega} \nabla u \nabla \varphi + \int_{[u>0]} u^{-\beta} \varphi = \lambda \int_{\Omega} u^p \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

We use the notation  $[u > 0] = \{x \in \Omega : u(x) > 0\}$ .

### 1.1. The case $1 < p < \frac{N+2}{N-2}$ ( $p > 1$ if $N = 2$ ).

The radial problem

$$(1.2) \quad \begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p, & u \geq 0 \quad \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where  $B_1$  is the unit ball in  $\mathbb{R}^N$  has been studied by several authors [2, 5, 9, 15, 23, 24]. In [5, 24] it is proved that there exists  $\bar{\lambda} > 0$  such that (1.2) has a positive radial solution if and only if  $0 < \lambda \leq \bar{\lambda}$ , and this solution belongs to  $C^2(B_1) \cap C^1(\bar{B}_1)$ . Moreover the radial solution  $u_\lambda$  is unique,  $u'_\lambda(1) < 0$  if  $0 < \lambda < \bar{\lambda}$  and  $u'_\lambda(1) = 0$  if  $\lambda = \bar{\lambda}$ . For  $\lambda > \bar{\lambda}$  (1.2) still possesses a radial solution, but it has compact support.

In this work we address the existence question for (1.1) in general smooth bounded domains. We prove

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**Theorem 1.1.** *For all  $\lambda > 0$  problem (1.1) has a nontrivial solution  $u_\lambda \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$ .*

We prove this result constructing solutions  $u_{\lambda,\varepsilon}$  to

$$(1.3) \quad \begin{cases} -\Delta u + \frac{u}{(u+\varepsilon)^{1+\beta}} = \lambda u^p, & u > 0 \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\varepsilon > 0$ , through the mountain pass theorem. Then we prove that  $u_{\lambda,\varepsilon}$  is bounded in  $L^\infty(\Omega)$  and then in  $C^{1,\mu}(\overline{\Omega})$  where  $\mu = \frac{1-\beta}{1+\beta}$ , uniformly as  $\varepsilon \rightarrow 0$ . Finally we show that for a fixed  $\lambda$ ,  $\lim_{\varepsilon \rightarrow 0} u_{\lambda,\varepsilon}$  is a nontrivial solution of (1.1).

We then study the asymptotic behavior of the solutions  $u_\lambda$  as  $\lambda \rightarrow +\infty$ . For this it convenient to remark the following. Let  $\bar{u}$  be the radial solution of (1.2) corresponding to  $\lambda = \bar{\lambda}$  and extended by zero outside  $B_1$ . Set

$$w(x) = \bar{\lambda}^{\frac{1}{p+\beta}} \bar{u}(\bar{\lambda}^{-\frac{1+\beta}{2(p+\beta)}} x), \quad x \in \mathbb{R}^N.$$

Then, since  $\bar{u}$  has vanishing gradient on  $\partial B_1$  we see that  $w$  satisfies

$$(1.4) \quad -\Delta w + w^{-\beta} \chi_{[w>0]} = w^p \quad \text{in } \mathbb{R}^N.$$

We will call  $w$  the radial ground state of (1.4).

**Theorem 1.2.** *For  $\lambda > 0$  sufficiently large, the solution  $u_\lambda$  of (1.1) obtained as the limit  $u_\lambda = \lim_{\varepsilon \rightarrow 0} u_{\lambda,\varepsilon}$  in Theorem 1.1, has the form*

$$u_\lambda(x) = \lambda^{-\frac{1}{p+\beta}} w(\lambda^{\frac{1+\beta}{2(p+\beta)}}(x - x_\lambda))$$

for some point  $x_\lambda \in \Omega$ . Moreover

$$\text{dist}(x_\lambda, \partial\Omega) \rightarrow \max_{x \in \Omega} \text{dist}(x, \partial\Omega)$$

as  $\lambda \rightarrow \infty$ .

We note that given a point  $x_0 \in \Omega$ , when  $\lambda > 0$  is sufficiently large the function

$$u(x) = \lambda^{-\frac{1}{p+\beta}} w(\lambda^{\frac{1+\beta}{2(p+\beta)}}(x - x_0))$$

is a solution of (1.1). However, Theorem 1.2 asserts that the solution constructed as the limit of least energy solutions of the approximation (1.3) selects the position of its maximum as the one that maximizes the distance to the boundary. This is because the energy functional associated to (1.3) has an expansion in terms of  $\varepsilon$  and  $\lambda$  which in the leading order has a term that penalizes the distance to the boundary.

The above phenomenon is similar to what happens in other equations. For instance Flucher and Wei [8] studied

$$(1.5) \quad \begin{cases} -\varepsilon^2 \Delta u = g(u), & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$(1.6) \quad g(u) = \begin{cases} (u-1)^p & \text{for } u \geq 1 \\ 0 & \text{for } u < 1, \end{cases}$$

and  $1 < p < \frac{N+2}{N-2}$  and  $N > 2$ . For every  $\varepsilon > 0$  small, they proved existence of a positive solution  $u_\varepsilon$ , which is of mountain pass type. The boundary of the core

$A_\varepsilon = [u_\varepsilon > 1]$  is a free boundary. They proved also that as  $\varepsilon \rightarrow 0$  the mountain pass solution  $u_\varepsilon$  has a unique maximum point  $x_\varepsilon$  which converges to a harmonic center of  $\Omega$  and  $A_\varepsilon$  is asymptotically spherical. To achieve these results they established energy estimates, for which they needed to characterize the kernel of the linearized operator  $Lv := \Delta v + g'(w)v$  where  $w$  is the ground state solving (1.5) in  $\mathbb{R}^N$ .

Concentration at a point that maximizes the distance to the boundary was proved by Ni and Wei [19] for the least energy solution  $u_\varepsilon$  of

$$(1.7) \quad \begin{cases} -\varepsilon^2 \Delta u + u = f(u), & u > 0 \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for a nonlinearity  $f$  that includes  $f(u) = u^p$  with  $1 < p < \frac{N+2}{N-2}$ . They have used similar techniques used by Ni and Takagi [17, 18] for the same equation with boundary condition  $\partial u / \partial \nu = 0$  on  $\Omega$ . The argument in [19] is based on precise energy estimates of  $u_\varepsilon$  and requires to know uniqueness and non-degeneracy of the ground state

$$(1.8) \quad \begin{cases} -\Delta w + w = f(w), & w > 0 \quad \text{in } \mathbb{R}^N \\ w(x) \rightarrow 0 & \text{as } |x| \rightarrow 0. \end{cases}$$

Here we adopt the strategy presented in the work of del Pino and Felmer [6], which does not require uniqueness nor nondegeneracy of  $w$ . The estimates of the least energy of the associated functional are obtained by comparison to the least energy of solutions in balls.

Finally, we prove

**Theorem 1.3.** *Let  $u_\lambda$  denote the solution obtained as the limit  $u_\lambda = \lim_{\varepsilon \rightarrow 0} u_{\lambda, \varepsilon}$  in Theorem 1.1. Then for  $\lambda > 0$  small enough the solution  $u_\lambda$  is positive in  $\Omega$ .*

**1.2. The case  $p = 1$ .** We consider now

$$(1.9) \quad \begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u, & u \geq 0 \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is readily seen that if  $\lambda \leq \lambda_1$ , where  $\lambda_1$  denotes the first eigenvalue of  $-\Delta$  with Dirichlet boundary condition, then a solution is identically 0. Indeed, let  $\varphi_1 > 0$  be an eigenfunction of  $-\Delta$  associated to  $\lambda_1$ . Then multiplying the equation by  $\varphi_1$  yields

$$\lambda_1 \int_{\Omega} u \varphi_1 + \int_{[u>0]} u^{-\beta} \varphi_1 = \lambda \int_{\Omega} u \varphi_1$$

If  $u \not\equiv 0$  then  $\lambda > \lambda_1$ .

The radial case in a ball is studied in [2], where it is shown that there exist  $\lambda^*$  and  $\bar{\lambda}$  with  $\lambda_1 \leq \lambda^* < \bar{\lambda}$  such that there exists a positive radial solution if and only if  $\lambda \in (\lambda^*, \bar{\lambda}]$ . In the case of dimension  $N = 1$  and  $\Omega = (-1, 1)$  the results of [2] imply that  $\bar{\lambda} = (\frac{\pi}{1+\beta})^2$  and  $\lambda^* \geq (\frac{\pi}{2})^2$ . He also shows that the solution corresponding to  $\bar{\lambda}$  has vanishing gradient on the boundary of the ball. Similar results for radial solutions in an ball are obtained in [15]. They prove that there is a positive radial solution, provided  $\lambda_1 < \lambda < (1 + \frac{2(1-\beta)}{N(1+\beta)})\lambda_1(B)$ , where  $\lambda_1(B)$  is the first eigenvalue of  $-\Delta$  with Dirichlet boundary condition on the unit ball  $B$ .

We have

**Theorem 1.4.** *If  $\lambda > \lambda_1$ , then (1.9) has a nontrivial solution.*

We prove this theorem by letting  $p \rightarrow 1$  in the solution  $u_{\varepsilon, \lambda, p}$  of (1.3) and then  $\varepsilon \rightarrow 0$ .

Positive solutions as in [2, 15] can be obtained in a general domain for  $\lambda > \lambda_1$  and  $\lambda$  close to  $\lambda_1$  through the implicit function theorem.

**Theorem 1.5.** *There exists  $\bar{\lambda} > \lambda_1$  such that for  $\lambda_1 < \lambda < \bar{\lambda}$  there exists a unique nontrivial solution to (1.9), which is positive.*

In the above theorem the solution has the behavior  $u_\lambda(x) = c(\lambda - \lambda_1)^{-\frac{1}{1+\beta}} \varphi_1(x)(1 + o(1))$  as  $\lambda \rightarrow \lambda_1$  where  $\varphi_1$  is the positive eigenfunction of  $-\Delta$  with Dirichlet boundary condition normalized such that  $\int_\Omega \varphi_1^2 = 1$ , and  $c$  is given by

$$c^{1+\beta} = \int_\Omega \varphi_1^{1-\beta}.$$

We conjecture that when  $\lambda$  is sufficiently large a concentration phenomenon similar to the one for the case  $p > 1$  described before takes place. Namely we believe that the solution  $u_\lambda$  obtained through the approximation (1.3) takes the form of a rescaled version of a radial solution, centered at a point, which asymptotically maximizes the distance to the boundary.

The organization of the paper is the following. In Section 2 we present local and up to the boundary estimates in  $C^{1,\mu}$  for solutions of (1.3), which are uniform in  $\varepsilon$ . Using these estimates we prove Theorems 1.1 and 1.4 in Section 3. We need to recall a few properties of the ground state  $w$  of (1.4) which we do in Section 4. In Section 5 we obtain some preliminary results on the asymptotic behavior as  $\lambda \rightarrow \infty$  of the solution  $u_\lambda$  of (1.1), constructed in Theorem 1.1. Then Section 6 contains estimates of the energy of mountain pass solutions of (1.3) when the domain is a ball. Then in Section 7 we perform the proof of Theorem 1.2. In Section 8 we prove Theorem 1.3. Finally we prove Theorem 1.5 in Section 9.

## 2. ESTIMATES IN $C^{1,\mu}$

In this section we shall obtain a local estimate for solutions  $u$  to the perturbed equation (1.3). Actually we will obtain estimates for a slightly more general situation, useful in later sections.

Let  $D \subseteq \mathbb{R}^N$  be an open set with smooth boundary and consider the equation

$$(2.1) \quad \begin{cases} -\Delta u + a(x) \frac{u}{(u + \varepsilon)^{1+\beta}} = f(x) & \text{in } D \cap B_2(0) \\ u = 0 & \text{on } \partial D \cap B_2(0) \end{cases}$$

where  $0 < \beta < 1$ ,  $a, f \in L^\infty(D)$ ,  $a \geq 0$ ,  $\varepsilon > 0$ .

**Proposition 2.1.** *Let  $M > 0$ ,  $a \geq 0$  and  $a, f \in L^\infty(D)$ . Then there exists a constant  $C > 0$  such that for every  $u \in H^1(D \cap B_2(0)) \cap C(\overline{D} \cap B_2(0))$ ,  $u \geq 0$  satisfying (2.1) and*

$$(2.2) \quad \|u\|_{L^\infty(D \cap B_2(0))} \leq M,$$

we have

$$\|\nabla u(x)\|_{C^\mu(\overline{D \cap B_1(0)})} \leq C$$

where  $\mu = \frac{1-\beta}{1+\beta}$ .

The constant  $C$  depends on  $M, \|f\|_{L^\infty}, \|a\|_{L^\infty}, N, \beta$  and the smoothness of  $D$ , but is independent of the solution and  $\varepsilon$ .

Another related estimate that we will need is the following

**Proposition 2.2.** *Suppose that  $u \in H^1(B_{2R}(0))$  satisfies*

$$(2.3) \quad -\Delta u + a(x) \frac{u}{(u + \varepsilon)^{1+\beta}} = f(x) \quad \text{in } B_{2R}(0)$$

where  $0 < \beta < 1$ ,  $a, f \in L^\infty(B_{2R}(0))$ ,  $a \geq 0$ ,  $\varepsilon > 0$ . Then

$$|\nabla u|^2 \leq C u^{1-\beta} \quad \text{in } B_R(0)$$

where  $C$  depends only on  $R, \|u\|_{L^\infty(B_{2R}(0))}, \|f\|_{L^\infty(B_{2R}(0))}, \|a\|_{L^\infty(B_{2R}(0))}$ .

We emphasize here that the estimates above do not depend on  $\varepsilon > 0$ , which will be kept fixed in all this section.

The arguments are based on the work of Phillips [21], where optimal interior regularity is obtained for minimizers of a certain functional. The difference with that work is that here we do not assume that solutions are minimizers, and we deal also with regularity up to the boundary, following techniques introduced in [3, 4].

First, we introduce some notation. For  $\varepsilon > 0$  we define

$$(2.4) \quad g_\varepsilon(u) = \begin{cases} \frac{u}{(u+\varepsilon)^{1+\beta}} & \text{if } u \geq 0 \\ 0 & \text{if } u \leq 0. \end{cases}$$

and

$$(2.5) \quad G_\varepsilon(u) = \int_0^u g_\varepsilon(t) dt.$$

Let

$$x_0 \in D \cap B_1(0) \quad \text{and} \quad \alpha = \frac{2}{\beta + 1}.$$

Without loss of generality we may assume in what follows that  $x_0 = 0$ . Given  $r > 0$  let

$$u_r(x) = r^{-\alpha} u(rx)$$

which is defined for

$$x \in D_r = \frac{1}{r} D.$$

By the smoothness of  $D$  for a small  $r > 0$  one can construct a smooth domain  $V_r$  such that :

$$D_r \cap B_{3/4}(0) \subseteq V_r \subseteq D_r \cap B_1(0).$$

The smoothness of  $V_r$  can be made independent of  $r$  if we restrict  $0 < r < R$  for some  $R > 0$ .

The proof of Proposition 2.1 involves the construction of a suitable lower barrier.

**Lemma 2.3.** *There exists  $r_1 > 0, m_0 > 0$  such that if  $0 < r < r_1$  and  $h \in C(\partial V_r)$ ,  $h \geq 0$  satisfies*

$$\int_{\partial V_r} h \geq m_0,$$

then there exists  $w > 0$  satisfying

$$-\Delta w + w^{-\beta} \leq 0 \quad \text{in } V_r, \quad w = h \quad \text{on } \partial V_r$$

and

$$(2.6) \quad w(x) \geq c_1 \left( \int_{\partial V_r} h \right) \text{dist}(x, \partial V_r) \quad \text{for all } x \in V_r.$$

Here  $r_1, m_0, c_1$  are fixed positive constants depending only on  $\beta, N$  and the smoothness of  $D$ .

The function  $w$  depends on  $x_0$  and  $r$ , but for simplicity we will write just  $w$ . For the construction of  $w$  see [3]. The scaling  $r$  is going to be in the range  $0 < r \leq r_0$  where  $r_0$  is given by:

$$(2.7) \quad r_0 = \max \left\{ \left( \frac{u(x_0)}{C_1} \right)^{1/\alpha}, \left( \frac{u(x_0)}{C_1 \text{dist}(x_0, \partial D)} \right)^{1/(\alpha-1)} \right\},$$

where  $C_1 > 0$  is a universal constant to be specified later. It will be convenient to note that

$$\left( \frac{u(x_0)}{C_1} \right)^{1/\alpha} \geq \left( \frac{u(x_0)}{C_1 \text{dist}(x_0, \partial D)} \right)^{1/(\alpha-1)}$$

that is,  $r_0 = \left( \frac{u(x_0)}{C_1} \right)^{1/\alpha}$ , if and only if

$$C_1 \text{dist}(x_0, \partial D)^\alpha \geq u(x_0).$$

To proceed further we need the following consequence of standard elliptic estimates.

**Lemma 2.4.** *Suppose  $v \in H^1(V_r)$  satisfies*

$$-\Delta v \leq h \quad \text{in } V_r, \quad v = 0 \quad \text{on } \partial V_r \cap B_{3/4}(0),$$

where  $h \in L^\infty(V_r)$ . Then

$$v(y) \leq \bar{C} \text{dist}(y, \partial V_r) \left( \|h\|_{L^\infty(V_r)} + \int_{\partial V_r} |v| \right) \quad \text{for all } y \in B_{1/2}(0),$$

the constant  $\bar{C}$  depends only on  $r$ .

Note that  $u_r$  satisfies

$$-\Delta u_r + a_r(x) g_{\varepsilon, r}(u_r) = f_r(x) \quad \text{in } D_r$$

where

$$\begin{aligned} a_r(x) &= a(r(x + x_0)), & f_r(x) &= r^{2-\alpha} f(r(x + x_0)) \\ g_{\varepsilon, r}(v) &= r^{2-\alpha} g_\varepsilon(r^\alpha v), \end{aligned}$$

**Lemma 2.5.** *Assume  $0 < r \leq r_0$ . Then*

$$\int_{\partial V_r} u_r \geq m_0.$$

**Proof.** Using Lemma 2.4 we have

$$\int_{V_r} u_r \geq \frac{u_r(0)}{\bar{C} \text{dist}(0, \partial V_r)} - r^{2-\alpha} \|f\|_{L^\infty}.$$

On the other hand from the definitions

$$\frac{1}{C_0} \min(\text{dist}(x_0, \partial D), r) \leq r \text{dist}(0, \partial V_r) \leq C_0 \min(\text{dist}(x_0, \partial D), r)$$

for some constant  $C_0$  depending only on the geometry of  $D$ . This and  $u_r(0) = r^{-\alpha}u(x_0)$  yields

$$\begin{aligned} \int_{\partial V_r} u_r &\geq \frac{u(x_0)}{C_0 \bar{C} r^{\alpha-1} \min(\text{dist}(x_0, \partial D), r)} - r^{2-\alpha} \|f\|_{L^\infty} \\ &\geq \frac{u(x_0)}{C_0 \bar{C} r_0^{\alpha-1} \min(\text{dist}(x_0, \partial D), r_0)} - r_0^{2-\alpha} \|f\|_{L^\infty} \end{aligned}$$

Suppose that  $C_1 \text{dist}(x_0, \partial D)^\alpha \leq u(x_0)$ . Then  $r_0 = \left( \frac{u(x_0)}{C_1 \text{dist}(x_0, \partial D)} \right)^{1/(\alpha-1)}$  and we deduce

$$r_0^{\alpha-1} = \frac{u(x_0)}{C_1 \text{dist}(x_0, \partial D)} \geq \text{dist}(x_0, \partial D)^{\alpha-1}$$

so that  $r_0 \geq \text{dist}(x_0, \partial D)$ . It follows that

$$(2.8) \quad \int_{\partial V_r} u_r \geq \frac{C_1}{C_0 \bar{C}} - r_0^{2-\alpha} \|f\|_{L^\infty} \geq \frac{C_1}{C_0 \bar{C}} - r_0^{2-\alpha} \|f\|_{L^\infty}.$$

Suppose now that  $C_1 \text{dist}(x_0, \partial D)^\alpha \geq u(x_0)$ . Then  $r_0 = \left( \frac{u(x_0)}{C_1} \right)^{1/\alpha}$  and we deduce  $r_0 \leq \text{dist}(x_0, \partial D)$ . Hence (2.8) is still valid in this case.

To conclude we need to verify that  $r_0$  has an upper bound. If  $r_0 = \left( \frac{u(x_0)}{C_1} \right)^{1/\alpha}$  from (2.2) it follows that  $r_0 \leq (M/C_1)^{1/\alpha}$ . If  $r_0 = \left( \frac{u(x_0)}{C_1 \text{dist}(x_0, \partial D)} \right)^{1/(\alpha-1)}$  an upper bound for  $u(x_0)/\text{dist}(x_0, \partial D)$  follows from Lemma 2.4. By choosing  $C_1$  large we see from (2.8) that

$$\int_{\partial V_r} u_r \geq m_0, \quad \text{for all } 0 < r \leq r_0.$$

□

The main point in the proof of Proposition 2.1 is the following:

**Proposition 2.6.** *Assume  $0 < r \leq r_0$  and let  $w$  be the function constructed in Lemma 2.3 with  $w = u_r$  on  $\partial V_r$ . Then*

$$u_r \geq w \quad \text{in } V_r.$$

To prove the above result consider the problem

$$(2.9) \quad \begin{cases} -\Delta v + a_r(x)g_{\varepsilon,r}(v) = f_r(x) & \text{in } V_r \\ v = u_r & \text{on } \partial V_r \end{cases}$$

where we regard  $u_r$  as a given boundary data. Notice that  $v = u_r$  is a solution to (2.9) and that the solutions of (2.9) are critical points of the functional

$$\begin{aligned} J_r(v) &= \int_{V_r} \left( \frac{1}{2} |\nabla v|^2 + a_r(x)G_{\varepsilon,r}(v) - f_r(x)v \right) dx \\ v &\in H_0^1(V_r) + u_r, \end{aligned}$$

where

$$G_{\varepsilon,r}(v) = \int_0^v g_{\varepsilon,r}(s) ds.$$

**Lemma 2.7.** *Suppose  $u_1, u_2$  are subsolutions of (2.9) such that*

$$u_1 \leq u_2 \quad \text{on } \partial V_r$$

and

$$\int_{\partial V_r} u_1 \geq m_0.$$

Let  $w$  be the function constructed in Lemma 2.3 with  $w = u_1$  on  $\partial V_r$  and assume that

$$u_1 \geq w \quad \text{in } V_r.$$

Then

$$J_r(\max(u_1, u_2)) \leq J_r(u_2) + \left( -\frac{1}{2} + \frac{C}{m_0^{1+\beta}} \right) \int_{V_r} |\nabla(\max(u_1, u_2) - u_2)|^2.$$

**Proof.** Let  $u = \max(u_1, u_2)$ , which satisfies

$$(2.10) \quad \begin{cases} -\Delta u + a_r(x)g_{\varepsilon,r}(u) \leq f_r(x) & \text{in } V_r \\ u \leq u_r & \text{on } \partial V_r. \end{cases}$$

Multiplying (2.10) by  $u - u_2 \geq 0$  and integrating by parts we obtain

$$(2.11) \quad \int_{V_r} \nabla u \cdot \nabla(u - u_2) dx + \int_{V_r} a_r(x)g_{\varepsilon,r}(u)(u - u_2) dx \leq \int_{V_r} f_r(x)(u - u_2) dx.$$

On the other hand

$$\begin{aligned} J_r(u) - J_r(u_2) &= -\frac{1}{2} \int_{V_r} |\nabla(u - u_2)|^2 dx + \int_{V_r} \nabla u \cdot \nabla(u - u_2) dx \\ &\quad + \int_{V_r} a_r(x) (G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2)) dx \\ &\quad - \int_{V_r} f_r(x)(u - u_2) dx. \end{aligned}$$

Combining with (2.11) we find

$$(2.12) \quad \begin{aligned} J_r(u) - J_r(u_2) &= -\frac{1}{2} \int_{V_r} |\nabla(u - u_2)|^2 dx \\ &\quad + \int_{V_r} a_r(x) [G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u - u_2)] dx. \end{aligned}$$

But we have

$$(2.13) \quad G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u - u_2) \leq C(r^2 + u^{-1-\beta})(u - u_2)^2.$$

Before proving this we note here that  $G_{\varepsilon,r}(v) = r^{2-2\alpha}G_\varepsilon(r^\alpha v)$ . Since  $0 \leq g_\varepsilon(u) \leq u^{-\beta}$  we see that

$$(2.14) \quad 0 \leq G_{\varepsilon,r}(v) \leq \frac{v^{1-\beta}}{1-\beta} \quad \text{for } v \geq 0.$$

Now,

$$G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u - u_2) = \frac{1}{2} \frac{\partial g_{\varepsilon,r}}{\partial v}(\xi)(u - u_2)^2$$

for some  $\xi \in [u_2, u]$ . But from  $\frac{\partial g_{\varepsilon,r}}{\partial v}(\xi) = r^2 \frac{\partial g_{\varepsilon}}{\partial u}(r^\alpha \xi)$  and the inequality

$$\left| \frac{\partial g_{\varepsilon}}{\partial u}(u) \right| = \left| \frac{\varepsilon - \beta u}{(u + \varepsilon)^{2+\beta}} \right| \leq C u^{-1-\beta} \quad \text{for all } u \geq 0, \varepsilon > 0,$$

we have

$$\left| \frac{\partial g_{\varepsilon,r}}{\partial v}(\xi) \right| \leq C \xi^{-1-\beta}.$$

If  $u \leq 2u_2$  then  $\xi \geq u_2 \geq u/2$  and

$$\left| \frac{\partial g_{\varepsilon,r}}{\partial v}(\xi) \right| \leq C u^{-1-\beta}.$$

If, on the other hand,  $u > 2u_2$  then  $u \leq 2(u - u_2)$  and hence using (2.14) and  $g_{\varepsilon,r} \geq 0$  we find

$$G_{\varepsilon,r}(u) - G_{\varepsilon,r}(u_2) - g_{\varepsilon,r}(u)(u - u_2) \leq \frac{u^{1-\beta}}{1-\beta} \leq C u^{-1-\beta} u^2 \leq C u^{-1-\beta} (u - u_2)^2.$$

This proves (2.13) and combining with (2.12) we obtain

$$J_r(u) - J_r(u_2) \leq -\frac{1}{2} \int_{V_r} |\nabla(u - u_2)|^2 dx + C \|a\|_{L^\infty} \int_{V_r} u^{-1-\beta} (u - u_2)^2 dx$$

But  $u \geq u_1 \geq w$  and  $w$  satisfies (2.6) we have

$$\int_{V_r} u^{-1-\beta} (u - u_2)^2 dx \leq C m_0^{-1-\beta} \int_{V_r} \text{dist}(x, \partial V_r)^{-1-\beta} (u - u_2)^2 dx.$$

By Hölder's and Hardy's inequality

$$\int_{V_r} u^{-1-\beta} (u - u_2)^2 dx \leq C m_0^{-1-\beta} \int_{V_r} |\nabla(u - u_2)|^2.$$

Hence

$$J_r(u) \leq J_r(u_2) + \left( -\frac{1}{2} + \frac{C}{m_0^{1+\beta}} \right) \int_{V_r} |\nabla(u - u_2)|^2.$$

□

### Proof of Proposition 2.6.

Throughout this proof we assume  $0 < r \leq r_0$ . By Lemma 2.5 we can apply Lemma 2.3. Henceforth we let  $w$  be the function constructed in Lemma 2.3 with  $w = u_r$  on  $\partial V_r$ .

**Step 1.** If  $r > 0$  is small enough then (2.9) has a unique solution.

Suppose that there exists a sequence  $r_j \rightarrow 0$  and different solutions  $v_j^1, v_j^2$  to the problem (2.9). Let  $w_j = v_j^1 - v_j^2$ . Then  $w_j \not\equiv 0$  and satisfies

$$(2.15) \quad \begin{cases} -\Delta w_j + b_j(x) w_j = 0 & \text{in } V_r \\ w_j = 0 & \text{on } \partial V_r \end{cases}$$

where  $b_j$  is given by

$$b_j(x) = a_{r_j}(x) \frac{\partial g_{\varepsilon,r_j}}{\partial v}(\xi_j(x))$$

and  $\xi_j(x)$  is in between  $v_j^1(x)$  and  $v_j^2(x)$ . But

$$\frac{\partial g_{\varepsilon,r_j}}{\partial v}(v) = r_j^2 \frac{\varepsilon - \beta r_j^\alpha v}{(r_j^\alpha v + \varepsilon)^{2+\beta}}.$$

Since  $\varepsilon > 0$  is fixed and  $a_j$  is uniformly bounded in  $L^\infty$  we have

$$b_j \rightarrow 0 \quad \text{uniformly in } V_r \text{ as } j \rightarrow \infty.$$

Hence the operator in (2.15) becomes coercive as  $j \rightarrow \infty$  and therefore  $w_j \equiv 0$  for large  $j$ .

**Step 2.** Suppose  $u \in H_0^1(V_r) + u_r$  is a minimizer of  $J_r$  on this set. Then

$$u \geq w \quad \text{in } V_r.$$

To prove this, we apply Lemma 2.7 with  $u_1 = w$  and  $u_2 = u$ . Taking  $m_0$  large we see that if  $u$  is a minimizer then  $\max(u, w) \equiv u$ , that is,  $u \geq w$  in  $V_r$ .

**Step 3.** The functional  $J_r$  has unique minimum in  $H_0^1(V_r) + u_r$ , which we will write as  $\tilde{u}_r$ . By the previous step

$$\tilde{u}_r \geq w \quad \text{in } V_r.$$

Indeed, the existence of at least one minimizer of  $J_r$  follows from the lower semi-continuity and coercivity of the functional in  $H_0^1(V_r) + u_r$ . If  $u_1, u_2$  are minimizers, then  $u_1 \geq w$  and  $u_2 \geq w$  by the previous step. Taking  $m_0$  large we deduce, applying Lemma 2.7, that  $u_1 \leq u_2$  and  $u_2 \leq u_1$ .

**Step 4.** The function  $u_r$  is the minimizer of  $J_r$  on the set  $H_0^1(V_r) + u_r$ . By the previous steps we conclude that

$$u_r \geq w \quad \text{in } V_r.$$

Indeed, problem (2.9) has a unique solution for small  $r > 0$ . Thus  $u_r$  is the minimizer of  $J_r$  for  $r > 0$  small. If  $0 < r \leq r_0$  the linear operator  $DJ_r(\tilde{u}_r)$  is coercive on  $H_0^1(V_r)$ . Indeed

$$(DJ_r(\tilde{u}_r)\varphi, \varphi) = \int_{V_r} (|\nabla\varphi|^2 + a_r(x)g'_{\varepsilon,r}(\tilde{u}_r)\varphi^2) dx, \quad \text{for all } \varphi \in H_0^1(V_r).$$

But  $g_{\varepsilon,r}(v) \geq -\beta v^{-1-\beta}$  and therefore

$$(DJ_r(\tilde{u}_r)\varphi, \varphi) \geq \int_{V_r} (|\nabla\varphi|^2 - \beta\|a_r\|_{L^\infty}\tilde{u}_r^{-1-\beta}\varphi^2) dx$$

From  $\tilde{u}_r \geq w$ , (2.6) and Lemma 2.5

$$\tilde{u}_r \geq c_1 m_0 \text{dist}(x, \partial V_r) \quad \text{for all } x \in V_r.$$

Thus

$$(DJ_r(\tilde{u}_r)\varphi, \varphi) \geq \int_{V_r} (|\nabla\varphi|^2 - \beta\|a_r\|_{L^\infty}(c_1 m_0)^{-1-\beta} \text{dist}(x, \partial V_r)^{-1-\beta} \varphi^2) dx$$

If we take  $m_0$  large enough we can apply Hardy's inequality and obtain the coercivity of  $DJ_r(\tilde{u}_r)$  on  $H_0^1(V_r)$ . Hence the branch of minimizers cannot bifurcate if  $0 < r \leq r_0$ , and since the branches  $u_r$  and  $\tilde{u}_r$  coincide for small  $r$  we must have  $u_r = \tilde{u}_r$  for  $0 < r \leq r_0$ .  $\square$

**Proof of Propositions 2.1 and 2.2.** By Proposition 2.6 and (2.6) we have established

$$u_r \geq c_1 \left( \int_{\partial V_r} u_r \right) \text{dist}(x, \partial V_r) \quad \text{for all } x \in V_r$$

and for all  $0 < r \leq r_0$ , where  $r_0$  is given by (2.7). The estimates for  $\nabla u$  can then be proved exactly in the same way as in Lemmas 10 and 11 and Theorem 3 of [3].  $\square$

## 3. EXISTENCE OF SOLUTIONS: PROOF OF THEOREMS 1.1 AND 1.4

Given  $1 < p < \frac{N+2}{N-2}$  we define the following functionals in  $H_0^1(\Omega)$

$$(3.1) \quad J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{1-\beta} \int_\Omega u_+^{1-\beta} - \frac{\lambda}{p+1} \int_\Omega |u|^{p+1}$$

and

$$(3.2) \quad J_{\lambda,\varepsilon}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_\Omega G_\varepsilon(u) - \frac{\lambda}{p+1} \int_\Omega |u|^{p+1}$$

where  $G_\varepsilon$  is defined in (2.5). Note that

$$G_\varepsilon(u) = \int_0^u g_\varepsilon(s) ds = \frac{\beta u + \varepsilon}{\beta(1-\beta)(u+\varepsilon)^\beta} - \frac{\varepsilon^{1-\beta}}{\beta(1-\beta)} \quad \text{for all } u \geq 0.$$

Let  $A > 0$  be sufficiently large and fixed to ensure

$$\frac{1}{2} \int_\Omega |\nabla(A\varphi_1)|^2 - \frac{\lambda}{p+1} \int_\Omega (A\varphi_1)^{p+1} < 0$$

for all  $\lambda > \lambda_1$ . Then

$$(3.3) \quad J_{\lambda,\varepsilon}(A\varphi_1) < 0$$

for all  $\varepsilon > 0$  and  $\lambda > \lambda_1$ .

**Lemma 3.1.** *Let  $1 < p < \frac{N+2}{N-2}$  and  $\lambda > 0$ . Then*

$$(3.4) \quad \begin{cases} -\Delta u + g_\varepsilon(u) = \lambda u^p & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

has a positive solution  $u_{\varepsilon,\lambda,p}$ .

**Proof.** We solve (3.4) using the mountain pass theorem for the functional  $J_{\lambda,\varepsilon}$  defined in (3.2). Note that since  $G_\varepsilon \geq 0$  this functional satisfies:

$$\text{there exist } \rho > 0, c > 0 \text{ such that } J_{\lambda,\varepsilon}(u) \geq c \quad \forall \|u\|_{H_0^1(\Omega)} = \rho.$$

This and (3.3) give the geometric condition for the mountain pass theorem, and the Ambrosetti-Rabinowitz condition, namely

$\exists \theta > 2$  such that

$$\theta \left( \frac{\lambda}{p+1} u^{p+1} - G_\varepsilon(u) \right) \leq \lambda u^p - g_\varepsilon(u) \text{ for sufficiently large } |u|$$

is satisfied since the term that dominates in the nonlinearity for large  $u$  is  $u^p$ . Therefore there exists a critical point  $u$  of  $J_{\lambda,\varepsilon}$  in  $H_0^1(\Omega)$ . By standard regularity theory  $u$  is  $C^2(\bar{\Omega})$  and satisfies

$$-\Delta u + g_\varepsilon(u) = \lambda |u|^p \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

We claim that  $u > 0$  in  $\Omega$ . To prove this it suffices to verify that  $u \geq 0$  in  $\Omega$ . Suppose on the contrary that  $\omega = \{x \in \Omega : u(x) < 0\}$  is non-empty. Then

$$-\Delta u = \lambda |u|^p > 0 \quad \text{in } \omega, \quad u = 0 \quad \text{on } \partial\omega,$$

and we deduce  $u > 0$  in  $\omega$ , a contradiction. Thus we have produced a positive solution  $u$  of (3.4) for every  $\lambda > 0$ .  $\square$

**Proof of Theorem 1.1.** By Lemma 3.1 we know that (3.4) has a solution  $u_{\lambda,\varepsilon} > 0$ , which we write for short  $u_\varepsilon$ . We claim that there exists a constant  $C$  independent of  $\varepsilon$  such that

$$(3.5) \quad \|u_\varepsilon\|_{L^\infty(\Omega)} \leq C.$$

To prove this, we use a blow-up argument. Assuming that  $m_\varepsilon \equiv \max_{\overline{\Omega}} u_\varepsilon \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ , define

$$v_\varepsilon(x) = \frac{u_\varepsilon(m_\varepsilon^{-\frac{p-1}{2}} x)}{m_\varepsilon}, \quad x \in \Omega_\varepsilon = m_\varepsilon^{\frac{p-1}{2}} \Omega.$$

Then  $0 \leq v_\varepsilon \leq 1$  and satisfies

$$(3.6) \quad \begin{cases} -\Delta v_\varepsilon + \frac{1}{m_\varepsilon^{1+\beta}} \frac{v_\varepsilon}{(v_\varepsilon + \varepsilon/m_\varepsilon)^{p+\beta}} = \lambda v_\varepsilon & \text{in } \Omega_\varepsilon \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

For a subsequence we may assume that  $\Omega_\varepsilon \rightarrow U$  where  $U = \mathbb{R}^N$  or  $U$  is a half space. By Proposition 2.1,  $v_\varepsilon$  is bounded in  $C^{1,\mu}(B_R(0) \cap \overline{\Omega}_\varepsilon)$  norm for every  $R > 0$ , where  $\mu = \frac{1-\beta}{1+\beta}$ . It follows that  $v_\varepsilon$  converges uniformly as  $\varepsilon \rightarrow 0$  on compact sets of  $\overline{U}$  to some function  $v$  which is  $C^1(\overline{U})$  and satisfies  $\|v\|_{L^\infty(U)} = 1$  and  $v(0) = 1$ . Using test functions with support in the open set  $[v > 0]$  we see that  $-\Delta v = v^p$  in the set  $[v > 0]$ . Using the strong maximum principle we deduce that actually  $v > 0$  in all  $U$  and hence

$$\begin{cases} -\Delta v = v^p & \text{in } U \\ v = 0 & \text{on } \partial U \text{ if } \partial U \neq \emptyset. \\ 0 \leq v \leq 1. \end{cases}$$

By the results due to Gidas and Spruck [11, 12] we conclude  $v \equiv 0$ , which contradicts  $v(0) = 1$ . This proves (3.5).

In addition we also have a fixed lower bound for  $\|u_\varepsilon\|_{L^\infty(\Omega)}$ . Indeed, if  $x_0$  is a maximum of  $u_\varepsilon$  then

$$\frac{u_\varepsilon(x_0)}{(u_\varepsilon(x_0) + \varepsilon)^{1+\beta}} \leq \lambda u_\varepsilon(x_0)^p.$$

This implies that

$$u_\varepsilon(x_0) \geq c > 0,$$

Since  $u_\varepsilon$  is uniformly bounded in  $L^\infty(\Omega)$ , applying Proposition 2.1 we have that  $u_\varepsilon$  is uniformly bounded in  $C^{1,\mu}(\overline{\Omega})$ . Therefore up to a subsequence  $u_\varepsilon$  converges in  $C^1(\overline{\Omega})$  to a non-zero function  $u \in C^{1,\mu}(\overline{\Omega})$ . We need to show that  $u$  is a solution to (1.1). Since  $\nabla u = 0$  on the set  $\Omega \setminus [u > 0]$  we then need to prove that

$$(3.7) \quad \int_{[u>0]} \nabla u \nabla \varphi = \int_{[u>0]} (-u^{-\beta} + \lambda u) \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

First we remark that using test functions supported in the open set  $[u > 0]$  we find that  $u$  satisfies

$$(3.8) \quad -\Delta u + u^{-\beta} = \lambda u^p \quad \text{in } [u > 0].$$

Then observe that integrating (3.4) one obtains

$$\int_{\Omega} \frac{u_\varepsilon}{(u_\varepsilon + \varepsilon)^{1+\beta}} \leq \lambda \int_{\Omega} u_\varepsilon^p \leq C$$

since  $u_\varepsilon$  is bounded in  $L^\infty(\Omega)$ . By Fatou's lemma we find

$$\chi_{[u>0]}u^{-\beta} \in L^1(\Omega).$$

Take  $\varphi \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty([u > 0])$ . Then, using (3.8) we have

$$\int_{\Omega} \eta \nabla u \nabla \varphi + \int_{\Omega} \varphi \nabla u \nabla \eta = \int_{\Omega} (-u^{-\beta} + \lambda u^p) \varphi \eta.$$

Let  $h \in C^\infty(\mathbb{R})$  be such that  $h(t) = 0$  for  $t \leq 1$  and  $h(t) = 1$  for  $t \geq 2$ . Given  $\varepsilon > 0$  we take  $\eta_\varepsilon = h(u/\varepsilon)$ . By dominated convergence

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \eta_\varepsilon \nabla u \nabla \varphi = \int_{[u>0]} \nabla u \nabla \varphi$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} (-u^{-\beta} + \lambda u) \varphi \eta_\varepsilon = \int_{[u>0]} (-u^{-\beta} + \lambda u^p) \varphi.$$

Let  $S$  be the support of  $\varphi$ . Then by Proposition 2.2 there exists some constant  $C$  such that

$$|\nabla u|^2 \leq C u^{1-\beta} \quad \text{in } S.$$

Hence in  $S$

$$|\nabla u \nabla \eta_\varepsilon| = \frac{1}{\varepsilon} |h'(u/\varepsilon)| |\nabla u|^2 \leq \frac{C}{\varepsilon} u^{1-\beta} \chi_{[\varepsilon < u < 2\varepsilon]} \leq C u^{-\beta} \chi_{[\varepsilon < u < 2\varepsilon]}.$$

and it follows that

$$\left| \int_{\Omega} \varphi \nabla u \nabla \eta_\varepsilon \right| \leq C \|\varphi\|_{L^\infty(\Omega)} \int_{[u>0]} u^{-\beta} \chi_{[\varepsilon < u < 2\varepsilon]} \rightarrow 0$$

since  $\chi_{[\varepsilon < u < 2\varepsilon]} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $\chi_{[u>0]}u^{-\beta} \in L^1(\Omega)$ . This establishes (3.7).  $\square$

**Proof of Theorem 1.4.** By Lemma 3.1 we know that (3.4) has a solution  $u_{\lambda, \varepsilon, p} > 0$ . We let first  $p \rightarrow 1$ . For this we claim that if  $\lambda > \lambda_1$  then there exists  $p_0 = p_0(\varepsilon, \lambda) > 1$  such that

$$(3.9) \quad \|u_{\lambda, \varepsilon, p}\|_{L^\infty(\Omega)} \leq C_\varepsilon \quad \text{for } 1 < p < p_0(\varepsilon, \lambda).$$

To prove (3.9), let us write  $u = u_{\lambda, \varepsilon, p}$ . Multiply the equation by  $\varphi_1$  and integrate

$$\lambda \int_{\Omega} u^p \varphi_1 = \int_{\Omega} u \varphi_1 + g_\varepsilon(u) \varphi_1 \leq \frac{\lambda_1}{p} \int_{\Omega} u^p \varphi_1 + \frac{\lambda_1}{p'} + \max(g_\varepsilon) \|\varphi_1\|_{L^1}.$$

which yields

$$\left( \lambda - \frac{\lambda_1}{p} \right) \int_{\Omega} u^p \varphi_1 \leq \frac{\lambda_1}{p'} + C_\varepsilon.$$

This implies

$$\int_{\Omega} u^p \varphi_1 \leq C_\varepsilon$$

if we take  $p > \lambda_1/\lambda$ . By a bootstrap argument, if  $p - 1$  is small, we can reach the estimate  $\|u\|_{L^\infty} \leq C_\varepsilon$ . The constant  $C_\varepsilon$  is independent of  $p$ .

As in the proof of Theorem 1.1 we also have

$$\|u_{\lambda, \varepsilon, p}\|_{L^\infty(\Omega)} \geq c,$$

with  $c > 0$  independent of  $p$  and  $\varepsilon$ .

By Proposition 2.1  $u_\varepsilon = \lim_{p \rightarrow 1} u_{\lambda, \varepsilon, p}$  exists and is a nontrivial solution of

$$(3.10) \quad \begin{cases} -\Delta u + g_\varepsilon(u) = \lambda u, & u \geq 0 \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

By the strong maximum principle  $u_\varepsilon > 0$  in  $\Omega$ .

We claim that there exists a constant  $C$  independent of  $\varepsilon$  such that

$$(3.11) \quad \|u_\varepsilon\|_{L^\infty} \leq C.$$

To prove this, we use a similar blow-up argument as in the proof of Theorem 1.1, except that now the domain stays fixed. Assuming that  $m_\varepsilon \equiv \max_{\bar{\Omega}} u_\varepsilon \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ . Define

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{m_\varepsilon} \quad x \in \Omega.$$

Then  $0 \leq v_\varepsilon \leq 1$  and satisfies

$$(3.12) \quad \begin{cases} -\Delta v_\varepsilon + \frac{1}{m_\varepsilon^{1+\beta}} \frac{v_\varepsilon}{(v_\varepsilon + \varepsilon/m_\varepsilon)^{1+\beta}} = \lambda v_\varepsilon & \text{in } \Omega \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

By Proposition 2.1,  $v_\varepsilon$  is bounded in  $C^{1,\mu}(\bar{\Omega})$ ,  $\mu = \frac{1-\beta}{1+\beta}$ , and therefore along some subsequence it converges in  $C^1(\bar{\Omega})$  to a function  $v \in C^{1,\mu}(\bar{\Omega})$ . Since  $\|v_\varepsilon\|_{L^\infty(\Omega)} = 1$  we also have  $\|v\|_{L^\infty(\Omega)} = 1$ . Taking test functions with support in  $[v > 0]$  it follows that  $v$  satisfies

$$-\Delta v = \lambda v \quad \text{in } [v > 0].$$

Since  $v \geq 0$  in  $\Omega$  and  $v \not\equiv 0$  by the strong maximum principle we deduce that  $v > 0$  in  $\Omega$ . This yields a contradiction with  $\lambda > \lambda_1$  and establishes (3.11).

Since  $u_\varepsilon$  is uniformly bounded in  $L^\infty(\Omega)$ , applying Proposition 2.1 we have that  $u_\varepsilon$  is uniformly bounded in  $C^{1,\mu}(\bar{\Omega})$ . Therefore up to a subsequence  $u_\varepsilon$  converges in  $C^1(\bar{\Omega})$  to a function  $u \in C^{1,\mu}(\bar{\Omega})$ . We may now prove that  $u$  is a solution to (1.9) following the same steps as in the proof of Theorem 1.1.  $\square$

#### 4. GROUND STATES

We recall that there exists a radial solution with compact support of the equation

$$(4.1) \quad -\Delta w + \chi_{[w>0]} w^{-\beta} = w^p \quad \text{in } \mathbb{R}^N.$$

More precisely, consider

$$(4.2) \quad -\Delta u + u^{-\beta} = u^p, \quad u > 0 \quad \text{in } B_R, \quad u = 0 \quad \text{on } \partial B_R$$

where  $1 < p < \frac{N+2}{N-2}$  ( $p > 1$  if  $N = 2$ ) and  $R > 0$ . In [5, Corollary 1.2] (see also [24]) it is proved that there exists  $\bar{R} > 0$  such that (4.2) has a radial solution  $u$  if and only if  $0 < R \leq \bar{R}$ , and it is unique in the class of radial solutions. Moreover the radial solution to (4.2) with  $R = \bar{R}$  has vanishing gradient on the boundary and hence satisfies (4.1). For  $0 < R < \bar{R}$  the radial solution  $u$  to (4.2) satisfies  $u'(R) < 0$ .

Let us write  $w$  the solution of (4.2) with  $R = \bar{R}$ . We call this function the radial ground state of (4.1).

For  $u \in H^1(\mathbb{R}^N) \cap L^{1-\beta}(\mathbb{R}^N)$  define

(4.3)

$$J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u|^2 + \frac{u^{1-\beta}}{1-\beta} - \frac{u^{p+1}}{p+1} \right), \quad M(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^{1-\beta} - u^{p+1}).$$

**Proposition 4.1.** *Let  $R_1 > \bar{R}$ . Suppose  $u \in H_0^1(B_{R_1}(0))$ ,  $u \neq 0$  solves (4.1) in the sense  $\chi_{[u>0]} u^{-\beta} \in L^1(\mathbb{R}^N)$  and*

$$(4.4) \quad \int_{\mathbb{R}^N} \nabla u \nabla \varphi + \chi_{[u>0]} u^{-\beta} \varphi = \int_{\mathbb{R}^N} u^p \varphi \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N),$$

and satisfies

$$J(u) \leq J(\varphi)$$

for all  $\varphi \in H_0^1(B_{R_1}(0))$  such that  $M(\varphi) = 0$ . Then up to translation  $u = w$ .

**Proof.** Let  $u^*$  denote the Schwarz symmetrization of  $u$ . Then  $u^*$  is radially symmetric, radially nonincreasing and  $u^* \in H_0^1(B_R(0))$ . It also satisfies  $\int_{\mathbb{R}^N} (u^*)^{p+1} = \int_{\mathbb{R}^N} u^{p+1}$ ,  $\int_{\mathbb{R}^N} (u^*)^{1-\beta} = \int_{\mathbb{R}^N} u^{1-\beta}$  and

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2$$

with equality if and only if  $u = u^*$  (after translating). For these properties see for example [16]. Choose  $t_0 > 0$  such that  $M(t_0 u^*) = 0$ . Since  $u$  solves (4.1) it satisfies  $M(u) = 0$ . Note that there is a unique  $t > 0$  such that  $M(tu) = 0$  and this number is the one that maximizes  $t \mapsto J(tu)$ . Therefore

$$J(u) = \sup_{t \geq 0} J(tu).$$

Thus

$$J(t_0 u^*) = \int_{B_{R_1}(0)} \left( \frac{t_0^2}{2} |\nabla u^*|^2 + \frac{t_0^{1-\beta}}{1-\beta} (u^*)^{1-\beta} - \frac{t_0^{p+1}}{p+1} (u^*)^{p+1} \right) \leq J(t_0 u) \leq J(u)$$

with strict inequality unless  $\int_{\mathbb{R}^N} |\nabla u^*|^2 = \int_{\mathbb{R}^N} |\nabla u|^2$ , that is,  $u = u^*$  after translation. Since  $u$  minimizes  $J$  with respect to functions  $\varphi \in H_0^1(B_{R_1}(0))$  with  $M(\varphi) = 0$  we deduce that  $J(t_0 u^*) = J(u)$ . Therefore  $u = u^*$  after translating, which means  $u$  is a radial solution of (4.1). Let  $0 < R_0 \leq R_1$  be such that  $u > 0$  in  $B_{R_0}$  and  $u(r) = 0$  for  $r \geq R_0$ . Then  $u'(R_0) = 0$ . By the uniqueness of  $\bar{R}$  and the solution we find  $R_0 = \bar{R}$  and  $u = w$ .  $\square$

## 5. ASYMPTOTIC BEHAVIOR AS $\lambda \rightarrow +\infty$ , PART 1

**Proposition 5.1.** *Let  $u_\lambda$  be the solution to*

$$\begin{cases} -\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

constructed in Theorem 1.1 or Theorem 1.4 where  $1 \leq p < \frac{N+2}{N-2}$ . Then for some  $C > 0$

$$(5.1) \quad \lambda^{-\frac{1}{p+\beta}} \leq \|u_\lambda\|_{L^\infty(\Omega)} \leq C \lambda^{-\frac{1}{p+\beta}} \quad \text{for all large } \lambda.$$

**Proof.** Let  $x_\lambda \in \Omega$  be a point where  $u_\lambda$  attains its maximum. The first inequality in the statement follows from

$$u(x_\lambda)^{-\beta} - \lambda u(x_\lambda)^p = \Delta u(x_\lambda) \leq 0.$$

Let

$$m_\lambda = \|u_\lambda\|_{L^\infty(\Omega)}$$

and assume by contradiction that for some sequence  $\lambda \rightarrow +\infty$  we have

$$(5.2) \quad m_\lambda \lambda^{\frac{1}{p+\beta}} \rightarrow +\infty.$$

Define

$$v_\lambda(x) = \frac{u_\lambda\left(x_\lambda + \lambda^{-1/2} m_\lambda^{\frac{1-p}{2}} x\right)}{m_\lambda}.$$

Then  $v_\lambda$  satisfies

$$-\Delta v_\lambda + \frac{1}{\lambda m_\lambda^{p+\beta}} \chi_{[v_\lambda > 0]} v_\lambda^{-\beta} = v_\lambda^p \quad \text{in } \Omega_\lambda$$

where  $\Omega_\lambda = \lambda^{1/2} m_\lambda^{\frac{p-1}{2}} (\Omega - x_\lambda)$ . If  $m_\lambda \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$  then  $\lambda^{1/2} m_\lambda^{\frac{p-1}{2}} \rightarrow +\infty$  as well. If  $m_\lambda$  is bounded, then  $\lambda m_\lambda^{p-1} = \lambda m_\lambda^{p+\beta} m_\lambda^{-1-\beta} \rightarrow +\infty$  by (5.2). Thus in all cases  $\lambda^{1/2} m_\lambda^{\frac{p-1}{2}} \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ . For a subsequence we may assume that  $\Omega_\lambda \rightarrow U$  where either  $U = \mathbb{R}^N$  or  $U$  is a half space.

Observe that  $0 \leq v_\lambda \leq 1$  and  $v_\lambda(0) = 1$ . For a fixed  $\lambda > 0$ ,  $v_\lambda$  is the limit of  $v_{\lambda,\varepsilon}$  as  $\varepsilon \rightarrow 0$  which are uniformly bounded solutions of

$$\begin{cases} -\Delta v + \frac{1}{\lambda m_\lambda^p} g_\varepsilon(m_\lambda v) = v^p & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

to which Proposition 2.1 can be applied. Therefore  $v_\lambda$  is bounded in the  $C^{1,\mu}(B_R(0) \cap \overline{\Omega}_\lambda)$  norm for every  $R > 0$ . It follows that  $v_\lambda$  converges uniformly as  $\lambda \rightarrow +\infty$  on a bounded set of  $U$  to some function  $v$  which is  $C^1(\overline{U})$  and satisfies  $\|v\|_{L^\infty(U)} = 1$  and  $v(0) = 1$ .

Using test functions with support in the open set  $[v > 0]$  we see that  $-\Delta v = v^p$  in the set  $[v > 0]$ . Using the strong maximum principle we deduce that actually  $v > 0$  in all  $U$  and hence

$$\begin{cases} -\Delta v = v^p & \text{in } U \\ v = 0 & \text{on } \partial U \text{ if } \partial U \neq \emptyset. \\ 0 \leq v \leq 1. \end{cases}$$

If  $1 < p < \frac{N+2}{N-2}$  by the results Gidas and Spruck [11, 12] we conclude  $v \equiv 0$ . If  $p = 1$  then also  $v \equiv 0$  because otherwise  $v$  is a positive supersolution for the operator  $-\Delta - \lambda_1(R)$  in every large ball  $B_R(x) \subseteq U$  where the first eigenvalue with Dirichlet boundary condition is  $O(R^{-2})$ . Thus  $v \equiv 0$  which contradicts  $v(0) = 1$ .  $\square$

Define

$$\mu_\lambda = \inf_{\gamma \in \Gamma_\lambda} \sup_{t \in [0,1]} J_\lambda(\gamma(t))$$

where

$$\Gamma_\lambda = \{ \gamma : [0, 1] \rightarrow H_0^1(\Omega) : \gamma \text{ is continuous, } \gamma(0) = 0, \gamma(1) = A\varphi_1 \}$$

and

$$\mathcal{N}_\lambda = \left\{ u \in H_0^1(\Omega) : u \neq 0 \text{ and } \int_\Omega |\nabla u|^2 = \int_\Omega \left( \lambda \frac{|u|^{p+1}}{p+1} - \frac{u_+^{1-\beta}}{1-\beta} \right) \right\}$$

In the above definition of  $\Gamma_\lambda$  the constant  $A$  is fixed such that  $J_{\lambda,\varepsilon}(A\varphi_1) < 0$  for all  $\varepsilon > 0$  and all  $\lambda > \lambda_1$ .

The following lemma is standard for continuous nonlinearities  $f(u)$  satisfying the classical hypotheses for the existence of mountain pass solutions, see [1], and the condition  $f(u)/u$  increasing, see [7, 26]. The nonlinearity  $f(u) = \lambda u^p - u^{-\beta}$  satisfies the last assumption, but since it is discontinuous we provide the proof.

**Lemma 5.2.** *Let  $J_\lambda$  be the functional defined in (3.1). We have*

$$J_\lambda(u_\lambda) = \mu_\lambda = \inf_{\mathcal{N}_\lambda} J_\lambda.$$

**Proof.** Let  $u_\varepsilon$  denote the solution of (3.4) constructed in Lemma 3.1 with the mountain pass theorem. Multiplying equation (3.4) by  $u_\varepsilon$  and integrating we have

$$\int_\Omega (|\nabla u_\varepsilon|^2 + g_\varepsilon(u_\varepsilon)u_\varepsilon - \lambda u_\varepsilon^{p+1}) = 0.$$

Since  $g_\varepsilon(u)u \rightarrow u_+^{1-\beta}$  uniformly for  $u$  on compact sets of  $\mathbb{R}$  we have

$$\int_\Omega (|\nabla u_\varepsilon|^2 + u_\varepsilon^{1+\beta} - \lambda u_\varepsilon^{p+1}) = o(\varepsilon)$$

where  $o(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Thus there exists  $t(\varepsilon) = 1 + o(\varepsilon)$  with the property  $t(\varepsilon) u_\varepsilon \in \mathcal{N}_\lambda$ . Hence

$$\inf_{\mathcal{N}_\lambda} J_\lambda \leq J_\lambda(t(\varepsilon) u_\varepsilon).$$

But

$$J_\lambda(t(\varepsilon) u_\varepsilon) = J_\lambda(u_\varepsilon) + o(\varepsilon) = J_{\lambda,\varepsilon}(u_\varepsilon) + o(\varepsilon)$$

where  $J_{\lambda,\varepsilon}$  is the functional defined in (3.2). By construction of  $u_\varepsilon$

$$J_{\lambda,\varepsilon}(u_\varepsilon) = \mu_{\lambda,\varepsilon}$$

where

$$\mu_{\lambda,\varepsilon} = \inf_{\gamma \in \Gamma_\lambda} \sup_{t \in [0,1]} J_{\lambda,\varepsilon}(\gamma(t)).$$

Thus

$$\inf_{\mathcal{N}_\lambda} J_\lambda \leq \mu_{\lambda,\varepsilon} + o(\varepsilon).$$

For a fixed  $\gamma \in \Gamma_\lambda$

$$\mu_{\lambda,\varepsilon} \leq \sup_{t \in [0,1]} J_{\lambda,\varepsilon}(\gamma(t))$$

and letting  $\varepsilon \rightarrow 0$

$$\limsup_{\varepsilon \rightarrow 0} \mu_{\lambda,\varepsilon} \leq \sup_{t \in [0,1]} J_\lambda(\gamma(t)).$$

Therefore

$$\limsup_{\varepsilon \rightarrow 0} \mu_{\lambda,\varepsilon} \leq \mu_\lambda$$

and we deduce

$$\inf_{\mathcal{N}_\lambda} J_\lambda \leq \mu_\lambda.$$

To prove the converse let  $u \in \mathcal{N}_\lambda$ . Given  $c_1 > 0$ ,  $c_2 \geq 0$ ,  $c_3 > 0$  we consider the function

$$f(t) = c_1 \frac{t^2}{2} + c_2 \frac{t^{1-\beta}}{1-\beta} - c_3 \frac{t^{p+1}}{p+1} \quad \text{for } t > 0.$$

Note that

$$\frac{f'(t)}{t} = c_1 + c_2 t^{-\beta-1} - c_3 t^{p-1}$$

is a decreasing function with limit  $+\infty$  as  $t \rightarrow 0$  and  $-\infty$  as  $t \rightarrow +\infty$ . Thus  $f$  has a unique critical point, which corresponds to a maximum and is non-degenerate. Thus there is a unique  $t^*(u) > 0$  which is critical point of

$$t \mapsto J_\lambda(tu)$$

and hence  $t^*(u) = 1$ . Thus  $J_\lambda(tu) \leq J_\lambda(t^*(u)u)$  for all  $t \geq 0$ . Let  $t_1 > t^*(u)$  be large such that  $J_\lambda(t_1u) < 0$ . We take as  $\gamma$  the path that connects 0 with  $t_1u$  with a straight line and then  $t_1u$  with  $A\varphi_1$  on the affine space  $\{s_1(t_1u) + s_2A\varphi_1 : s_1, s_2 \in \mathbb{R}\}$  along which  $J_\lambda$  is negative. Then  $\max_{t \in [0,1]} J_\lambda(\gamma(t)) = J_\lambda(u)$ .  $\square$

**Lemma 5.3.** *If  $1 < p < \frac{N+2}{N-2}$  we have*

$$(5.3) \quad \int_{\Omega} |\nabla u_\lambda|^2 \leq C\lambda^{-q}, \quad \lambda \int_{\Omega} u_\lambda^{p+1} \leq C\lambda^{-q}, \quad \int_{\Omega} u_\lambda^{1-\beta} \leq C\lambda^{-q}$$

where  $q = \frac{2+N+(N-2)\beta}{2(p+\beta)}$ .

**Proof.** Let  $\varphi = a\varphi_0$  where  $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$ ,  $\varphi_0 \geq 0$  and  $\varphi_0 \not\equiv 0$ . We choose the constant  $a > 0$  such that

$$\int_{\mathbb{R}^N} (|\nabla \varphi|^2 + \varphi^{1-\beta} - \varphi^{p+1}) = 0.$$

For the next calculation we assume that  $0 \in \Omega$ , so that the support of  $\varphi_\lambda(x) = \lambda^{-\frac{1}{p+\beta}} \varphi(\lambda^{\frac{1+\beta}{2(p+\beta)}} x)$  is contained in  $\Omega$  for  $\lambda$  sufficiently large. Then

$$\begin{aligned} \int_{\Omega} |\nabla \varphi_\lambda|^2 &= \lambda^{-\frac{2-N-(N-2)\beta}{2(p+\beta)}} \int_{\mathbb{R}^N} |\nabla \varphi|^2 \\ \int_{\Omega} \varphi_\lambda^{1-\beta} &= \lambda^{-\frac{2-N-(N-2)\beta}{2(p+\beta)}} \int_{\mathbb{R}^N} \varphi^{1-\beta} \\ \int_{\Omega} \varphi_\lambda^{p+1} &= \lambda^{-\frac{2-N-(N-2)\beta}{2(p+\beta)}} \int_{\mathbb{R}^N} \varphi^{p+1}. \end{aligned}$$

Hence  $\varphi_\lambda \in \mathcal{N}_\lambda$  and since  $J_\lambda(\varphi_\lambda) = C\lambda^{-\frac{2-N-(N-2)\beta}{2(p+\beta)}}$  we obtain

$$J_\lambda(u_\lambda) \leq C\lambda^{-\frac{2-N-(N-2)\beta}{2(p+\beta)}}.$$

Therefore

$$\begin{aligned} C\lambda^{-\frac{2-N-(N-2)\beta}{2(p+\beta)}} &\geq J_\lambda(u_\lambda) = \int_{\Omega} \frac{1}{2} |\nabla u_\lambda|^2 + \frac{u_\lambda^{1-\beta}}{1-\beta} - \lambda \frac{u_\lambda^{p+1}}{p+1} \\ &= \int_{\Omega} \lambda \left( \frac{1}{2} - \frac{1}{p+1} \right) u_\lambda^{p+1} + \left( \frac{1}{1-\beta} - \frac{1}{2} \right) u_\lambda^{1-\beta} \\ &\geq c\lambda \int_{\Omega} u_\lambda^{p+1} \geq c \int_{\Omega} |\nabla u_\lambda|^2. \end{aligned}$$

$\square$

Let  $x_\lambda \in \Omega$  be a maximum point of  $u_\lambda$ . Let us introduce the rescaled function

$$v_\lambda(x) = \lambda^{\frac{1}{p+\beta}} u_\lambda \left( x_\lambda + \lambda^{-\frac{1+\beta}{2(p+\beta)}} x \right)$$

so that  $v_\lambda$  solves

$$(5.4) \quad -\Delta v_\lambda + \chi_{[v_\lambda > 0]} v_\lambda^{-\beta} = v_\lambda^p \quad \text{in } \Omega_\lambda, \quad v_\lambda = 0 \quad \text{on } \partial\Omega_\lambda,$$

where  $\Omega_\lambda = \lambda^{\frac{1+\beta}{2(p+\beta)}} (\Omega - x_\lambda)$ .

**Lemma 5.4.** *For a large  $\lambda > 0$  the solution  $u_\lambda$  has support in a ball  $B_{R_\lambda}(x_\lambda)$  with  $R_\lambda = C\lambda^{-\frac{1+\beta}{2(p+\beta)}}$ .*

**Proof.** By (5.1)

$$1 \leq \|v_\lambda\|_{L^\infty(\Omega_\lambda)} \leq C$$

and in particular  $1 \leq v_\lambda(0) \leq C$ . By Proposition 2.1 for every  $R > 0$  there exists  $C_R > 0$  such that for every  $x \in \Omega_\lambda$

$$\|\nabla v_\lambda\|_{L^\infty(B_R(x) \cap \Omega_\lambda)} \leq C_R$$

and therefore

$$(5.5) \quad \|\nabla v_\lambda\|_{L^\infty(\Omega_\lambda)} < +\infty.$$

On the other hand by (5.3) there is some constant such that

$$(5.6) \quad \int_{\Omega_\lambda} v_\lambda^{p+1} \leq C.$$

Define  $FB_\lambda = \partial\{x \in \Omega_\lambda : v_\lambda(x) > 0\}$ . There exists  $c, c' > 0$  such that if  $x_0 \in FB_\lambda$  then for every  $0 < r \leq c$

$$(5.7) \quad \text{there exists } x_r \in \Omega \text{ with } |x_r - x_0| = r \text{ such that } v_\lambda(x_r) \geq c' r^\alpha.$$

This is standard, see e.g. [22], but to be self-contained we give the explanation. By (5.5) we can select  $c > 0$  such that

$$v_\lambda(y)^{p+\beta} \leq \frac{1}{2} \quad \text{for all } y \in \Omega_\lambda, |y - x_0| \leq 2c.$$

Then on the set  $\{v_\lambda > 0\} \cap B_{2c}(x)$

$$\Delta(v_\lambda^{1+\beta}) = (1+\beta)(1 - v_\lambda^{p+\beta}) + (1+\beta)\beta v_\lambda^{\beta-1} |\nabla v_\lambda|^2 \geq \frac{1+\beta}{2}.$$

Let  $z(y) = \frac{1+\beta}{4N} |x - \bar{x}|^2$  where  $\bar{x}$  is a point close to  $x_0$ , say  $|\bar{x} - x_0| \leq c/2$ , such that  $v_\lambda(\bar{x}) > 0$ . Let  $0 < r \leq c$ . If  $v_\lambda^{1+\beta} \leq z$  for  $y \in \partial B_r(\bar{x}) \cap \Omega_\lambda$  then by the maximum principle  $v_\lambda(\bar{x}) \leq z(\bar{x}) = 0$ , which is not possible. Hence there is a point  $y \in \partial B_r(\bar{x}) \cap \Omega_\lambda$  such that

$$v_\lambda(y)^{1+\beta} \geq z(y).$$

Letting  $\bar{x} \rightarrow x_0$  establishes the assertion.

We observe that  $\{x \in \Omega_\lambda : v_\lambda(x) > 0\}$  is connected. Otherwise let  $\omega$  denote the connected component containing the origin, and assume that  $v_\lambda > 0$  for some point outside  $\omega$ . Let  $v_1 = v\chi_\omega$  and  $v_2 = v\chi_{\Omega \setminus \omega}$ , which are nontrivial solution to (5.4). Let

$$J(v) = \int_{\Omega_\lambda} \frac{1}{2} |\nabla v|^2 + \frac{v^{1-\beta}}{1-\beta} - \frac{v^{p+1}}{p+1}.$$

By the characterization of Lemma 5.2

$$J(v_\lambda) \leq J(v_1) \quad \text{and} \quad J(v_\lambda) \leq J(v_2).$$

Since  $J(v_\lambda) = J(v_1) + J(v_2)$  we must have  $J(v_1) \leq 0$  and  $J(v_2) \leq 0$ , which implies  $J(v_\lambda) = 0$ . But this and the fact that  $v_\lambda$  is a solution gives

$$\int_{\Omega_\lambda} \left( \frac{1}{2} - \frac{1}{p+1} \right) v_\lambda^{p+1} + \left( \frac{1}{1-\beta} - \frac{1}{2} \right) v_\lambda^{1-\beta} = 0$$

which implies  $v_\lambda = 0$ , a contradiction.

Using the previous properties and (5.5) and (5.6) we can now show that the support of  $v_\lambda$  is contained in a ball of fixed radius. Suppose  $x_k \in FB_\lambda$ ,  $k = 1, \dots, m$  are at distance at least  $2c$  from each other. By (5.7) there exists  $y_k \in \Omega$  with  $|y_k - x_k| = c$  such that

$$v_\lambda(y_k) \geq \tilde{c} > 0.$$

By the uniform Lipschitz bound we deduce that

$$\int_{\Omega_\lambda} v_\lambda^{p+1} \geq mc''$$

for some  $c'' > 0$ . Thus from the upper bound (5.6) we deduce  $m \leq C$  for some constant  $C$  independent of  $\lambda$ . This shows that the support of  $v_\lambda$  is contained in ball  $B_M(0)$  with  $M$  bounded independently of  $\lambda$  and this establishes the lemma.  $\square$

## 6. ENERGY ESTIMATES IN A BALL

We work here with  $1 < p < \frac{N+2}{N-2}$  and  $0 < \beta < 1$ . Given  $\rho > 0$  and  $\varepsilon > 0$  we consider the equation

$$(6.1) \quad \begin{cases} -\Delta w + \frac{w}{(w+\varepsilon)^{1+\beta}} = w^p, & w > 0 \quad \text{in } B_\rho \\ w = 0 & \text{on } \partial B_\rho \end{cases}$$

and its associated functional  $J_{\rho,\varepsilon} : H_0^1(B_\rho) \rightarrow \mathbb{R}$  defined by

$$J_{\rho,\varepsilon}(u) = \int_{B_\rho} \frac{1}{2} |\nabla u|^2 + G_\varepsilon(u) - \frac{(u^+)^{p+1}}{p+1}.$$

This functional has a least positive critical value  $c_{\rho,\varepsilon}$ . Since the nonlinearity  $f(u) = u^p - \frac{u}{(u+\varepsilon)^{1+\beta}}$  satisfies that  $f(u)/u$  is strictly increasing for  $u > 0$ , then  $c_{\rho,\varepsilon}$  can be characterized by

$$(6.2) \quad c_{\rho,\varepsilon} = \inf_{u \in H_0^1(B_\rho), u \neq 0} \sup_{t \geq 0} J_{\rho,\varepsilon}(tu),$$

see [7, 26].

For  $\varepsilon > 0$  we also consider the equation

$$(6.3) \quad \begin{cases} -\Delta w + \frac{w}{(w+\varepsilon)^{1+\beta}} = w^p, & w > 0 \quad \text{in } \mathbb{R}^N \\ w(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases}$$

Critical points of the functional

$$J_{\mathbb{R}^N,\varepsilon}(u) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 + G_\varepsilon(u) - \frac{(u^+)^{p+1}}{p+1}, \quad u \in H^1(\mathbb{R}^N)$$

give rise to solutions of (6.3). This functional has a least positive critical value  $c_\varepsilon$  characterized by

$$(6.4) \quad c_\varepsilon = \inf_{u \in H^1(\mathbb{R}^N), u \neq 0} \sup_{t \geq 0} J_{\mathbb{R}^N, \varepsilon}(tu),$$

The main result in this section is

**Proposition 6.1.** *There are  $R_1 > 0$ ,  $C > 0$  such that for every  $\delta > 0$  there is  $\varepsilon_0 > 0$  such that for all  $\rho \geq R_1$*

$$\begin{aligned} c_\varepsilon + e^{-2(\varepsilon^{-1-\beta}-C)^{1/2}(\rho-R_1)-C \log \rho - C \log(1/\varepsilon)-C} &\leq c_{\rho, \varepsilon} \leq \\ &\leq c_\varepsilon + e^{-2(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1)+C \log \rho + C \log(1/\varepsilon)+C} \end{aligned}$$

Solutions of (6.1) with energy  $c_{\rho, \varepsilon}$  are called least energy solutions, and similarly for solutions of (6.3) with energy  $c_\varepsilon$ . Using the Schwarz symmetrization we can show that the existence of a radial, decreasing least energy solution of (6.1) that we call  $w_{\rho, \varepsilon}$  and a radial, decreasing least energy solution  $w_\varepsilon$  of (6.3).

**Lemma 6.2.** *Let  $R_0 > 0$ ,  $\sigma > 0$  and  $\bar{w}$  be the solution to*

$$(6.5) \quad \bar{w}'' = \frac{1}{2} \frac{\bar{w}}{(w + \varepsilon)^{1+\beta}} \quad r > R_0$$

$$(6.6) \quad \bar{w}(R_0) = \sigma, \quad \lim_{r \rightarrow +\infty} \bar{w}(r) = 0.$$

*If  $\sigma > 0$  is sufficiently small, only depending on  $\beta$ , there is  $c_0 > 0$  such that*

$$(6.7) \quad \bar{w}(R_0 + 1) \leq e^{-c_0 \varepsilon^{-(1+\beta)/2}}.$$

**Proof.** We write in this proof  $w$  instead of  $\bar{w}$ . The solution to (6.5), (6.6) is obtained by multiplying (6.5) by  $w'$  and integrating:

$$\frac{1}{2}(w')^2 = \frac{1}{2}G_\varepsilon(w)$$

which gives the relation

$$(6.8) \quad \int_{w(r)}^\sigma \frac{1}{\sqrt{G_\varepsilon(s)}} ds = r - R_0.$$

Using the definition of  $g_\varepsilon$  we see that

$$g_\varepsilon(u) = \frac{u}{(u + \varepsilon)^{1+\beta}} \geq \frac{u}{(2\varepsilon)^{1+\beta}} \quad \text{for } 0 \leq u \leq \varepsilon$$

and therefore

$$G_\varepsilon(u) \geq \frac{u^2}{2^{2+\beta} \varepsilon^{1+\beta}} \quad \text{for } 0 \leq u \leq \varepsilon.$$

Similarly

$$g_\varepsilon(u) \geq \frac{1}{2^{2+\beta} u^\beta} \quad \text{for } u \geq \varepsilon$$

and

$$G_\varepsilon(u) \geq \frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{u^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)} \quad \text{for } u \geq \varepsilon.$$

By choosing  $\sigma > 0$  small we get  $w(R_0 + 1) \leq \varepsilon$ . Indeed, assume  $w(R_0 + 1) \geq \varepsilon$ . Then the equation (6.8) and the estimates for  $G_\varepsilon$  imply that

$$\int_\varepsilon^\sigma \left( \frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{s^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)} \right)^{-1/2} ds \geq 1$$

But

$$(6.9) \quad \int_{\varepsilon}^{\sigma} \left( \frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{s^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)} \right)^{-1/2} ds \leq C \int_{\varepsilon}^{\sigma} \frac{1}{s^{(1-\beta)/2}} ds \leq C\sigma^{(1+\beta)/2} \leq \frac{1}{2},$$

if we choose  $\sigma > 0$  small. This gives a contradiction.

Since  $w(R_0 + 1) \leq \varepsilon$  we find from (6.8) and the estimates for  $G_{\varepsilon}$  that

$$(6.10) \quad C\varepsilon^{(1+\beta)/2} \int_{w(R_0+1)}^{\varepsilon} \frac{1}{s} ds + \int_{\varepsilon}^{\sigma} \left( \frac{\varepsilon^{1-\beta}}{2^{2+\beta}} + \frac{s^{1-\beta} - \varepsilon^{1-\beta}}{2^{1+\beta}(1-\beta)} \right)^{-1/2} ds \geq 1$$

where  $C > 0$ . Then (6.9) and (6.10) yield

$$C\varepsilon^{(1+\beta)/2} \log \left( \frac{\varepsilon}{w(R_0 + 1)} \right) \geq \frac{1}{2}.$$

This estimate implies the inequality (6.7).  $\square$

**Lemma 6.3.** *Let  $w$  be a radial, decreasing solution  $w$  of (6.1) or (6.3). There is a fixed  $R_1 > 0$  such that for every  $\delta > 0$  there is  $\varepsilon_0 > 0$  such that for*

$$w(r) \leq e^{-(\varepsilon^{-1-\beta} - \delta)^{1/2}(r-R_1)} \quad \text{for all } r \geq R_1 \text{ and all } 0 < \varepsilon \leq \varepsilon_0.$$

**Proof.** We first remark that there is an a priori bound  $M$  for every radial, decreasing solution  $w$  of (6.3), that is,  $w(r) \leq M$  for all  $r \geq 0$ . Moreover given a  $\sigma > 0$  there is some  $R_0 > 0$  such that for every radial solution  $w$  of (6.3)

$$(6.11) \quad w(r) \leq \sigma \quad \text{for all } r \geq R_0.$$

We apply this property with  $\sigma > 0$  given in Lemma 6.2 and find  $R_0$  such that (6.11) holds. Let  $\bar{w}$  be the function constructed in Lemma 6.2. Then, taking  $\sigma > 0$  smaller if necessary, we see that  $\bar{w}$  is a supersolution to (6.3) and we deduce  $w(r) \leq \bar{w}(r)$  for all  $r \geq R_0$ . In particular

$$w(R_0 + 1) \leq e^{-c_0\varepsilon^{-(1+\beta)/2}},$$

where  $c_0 > 0$ . Set  $R_1 = R_0 + 1$ . Let  $\delta > 0$  be given and define

$$\bar{w}_2(r) = e^{-c_0\varepsilon^{-(1+\beta)/2}} e^{-(\varepsilon^{-1-\beta} - \delta)^{1/2}(r-R_1)}.$$

Then  $\bar{w}_2$  satisfies

$$\bar{w}_2'' = (\varepsilon^{-1-\beta} - \delta)\bar{w}_2,$$

and we can arrange the constants so that  $\bar{w}_2$  is a supersolution. Indeed,

$$\Delta \bar{w}_2 + \bar{w}_2^p - \frac{\bar{w}_2}{(\bar{w}_2 + \varepsilon)^{1+\beta}} \leq -\frac{\delta}{2}\bar{w}_2 + \bar{w}_2^p + \bar{w}_2 \left( \varepsilon^{-1-\beta} - \frac{\delta}{2} - \frac{1}{(\bar{w}_2 + \varepsilon)^{1+\beta}} \right)$$

Provided  $\bar{w}_2^{p-1} \leq \delta/2$  and  $\bar{w}_2/\varepsilon \leq 1/2$  we have

$$\Delta \bar{w}_2 + \bar{w}_2^p - \frac{\bar{w}_2}{(\bar{w}_2 + \varepsilon)^{1+\beta}} \leq \bar{w}_2 \left( -\frac{\delta}{2} + O\left(\frac{\bar{w}_2}{\varepsilon^{2+\beta}}\right) \right) \leq 0$$

if also  $\bar{w}_2 \leq C\delta\varepsilon^{2+\beta}$ . But since  $\bar{w}_2(r) \leq e^{-c_0\varepsilon^{-(1+\beta)/2}}$  for  $r \geq R_1$  we can achieve the inequalities by taking  $\varepsilon > 0$  small, depending on  $\delta$ . By comparison

$$w(r) \leq \bar{w}_2(r) \quad \forall r \geq R_1.$$

$\square$

**Lemma 6.4.** *Let  $w$  be a radial, decreasing solution of (6.1). There are fixed numbers  $R_2 > 0$ ,  $C > 0$  and  $\varepsilon_0 > 0$  such that*

$$w(\rho - \varepsilon^{-(1+\beta)/2}) \geq e^{-(\varepsilon^{-(1+\beta)/2} + C)(\rho - R_2) - C} \quad \text{for all } r \geq R_2 \text{ and all } 0 < \varepsilon \leq \varepsilon_0.$$

**Proof.** There is a uniform lower bound for  $w_{\rho,\varepsilon}(0)$ . Indeed, at the origin  $\Delta w_{\rho,\varepsilon} \leq 0$  and the equation yields  $w_{\rho,\varepsilon}(0)^{p-1}(w_{\rho,\varepsilon} + \varepsilon)^{1+\beta} \geq 1$  and this implies  $w_{\rho,\varepsilon}(0) \geq c$  for some  $c > 0$  independent of  $\varepsilon \in (0, 1]$  and of  $\rho \geq 1$ . The uniform estimate for the gradient Proposition 2.2 imply that if we fix  $R_2 > 0$  small, then  $w_{\rho,\varepsilon}(R_2) \geq c/2$  for all  $0 < \varepsilon \leq 1$  and all  $\rho \geq 1$ .

Let  $\underline{w}(r)$  be the solution of

$$\underline{w}'' + \frac{N-1}{R_2} \underline{w}' = \frac{1}{\varepsilon^{1+\beta}} \underline{w}, \quad r \in (0, \rho)$$

$$\underline{w}(R_2) = c, \quad \underline{w}(\rho) = 0.$$

Then  $\underline{w}$  is explicitly given by

$$\underline{w}(r) = c \frac{e^{\lambda_+(\rho-r)} - e^{\lambda_-(\rho-r)}}{e^{\lambda_+(\rho-R_2)} - e^{\lambda_-(\rho-R_2)}}, \quad r \in [0, \rho]$$

where

$$\lambda_{\pm} = -\frac{N-1}{2R_2} \pm \sqrt{\left(\frac{N-1}{2R_2}\right)^2 + \varepsilon^{-1-\beta}}.$$

Moreover,  $\underline{w}$  is a subsolution of (6.1). Therefore  $w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2}) \geq \underline{w}(\rho - \varepsilon^{(1+\beta)/2})$ , so

$$w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2}) \geq c \frac{e^{\lambda_+ \varepsilon^{(1+\beta)/2}} - e^{\lambda_- \varepsilon^{(1+\beta)/2}}}{e^{\lambda_+(\rho-R_2)} - e^{\lambda_-(\rho-R_2)}}.$$

Since

$$\lambda_{\pm} = \pm \varepsilon^{-(1+\beta)/2} - \frac{N-1}{2R_2} + O\left(\frac{\varepsilon^{(1+\beta)/2}}{R_2^2}\right)$$

the conclusion follows.  $\square$

**Proof of Proposition 6.1.** We prove first the upper estimate. Let  $w_\varepsilon$  denote a radial, decreasing least energy solution of (6.3). Let  $\rho > 0$  and  $v_\rho$  be the solution to

$$(6.12) \quad \Delta v_\rho = \frac{1}{\varepsilon^{1+\beta}} v_\rho \quad \text{in } B_\rho \setminus \bar{B}_{\rho-1}$$

with  $v_\rho(\rho-1) = w_\varepsilon(\rho-1)$  and  $v_\rho(\rho) = 0$ . Define

$$\bar{w}_{\rho,\varepsilon}(r) = \begin{cases} w_\varepsilon(r) & \text{if } 0 \leq r \leq \rho-1 \\ v_\rho(r) & \text{if } \rho-1 \leq r \leq \rho. \end{cases}$$

Then

$$c_{\rho,\varepsilon} = J_{\rho,\varepsilon}(w_{\rho,\varepsilon}) = \max_{t \geq 0} J_{\rho,\varepsilon}(tw_{\rho,\varepsilon}) \leq \max_{t \geq 0} J_{\rho,\varepsilon}(t\bar{w}_{\rho,\varepsilon}) = J_{\rho,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon})$$

where  $t_{\rho,\varepsilon} > 0$  is the unique  $t$  where the last maximum is attained. Since  $w_{\rho,\varepsilon} \rightarrow w_\varepsilon$  as  $\rho \rightarrow +\infty$  uniformly for  $\varepsilon > 0$  small, we have  $t_{\rho,\varepsilon} \rightarrow 1$  as  $\rho \rightarrow +\infty$ , uniformly for

$\varepsilon > 0$  small. Then

$$\begin{aligned} J_{\rho,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon}) &\leq \int_{B_{\rho-1}} \frac{1}{2}t_{\rho,\varepsilon}^2 |\nabla w_\varepsilon|^2 + G_\varepsilon(t_{\rho,\varepsilon}w_\varepsilon) - \frac{(t_{\rho,\varepsilon}w_\varepsilon)^{p+1}}{p+1} \\ &\quad + \int_{B_\rho \setminus B_{\rho-1}} \frac{1}{2}t_{\rho,\varepsilon}^2 |\nabla v_\rho|^2 + G_\varepsilon(t_{\rho,\varepsilon}v_\rho) \end{aligned}$$

We have the following expansion for  $G_\varepsilon$

$$(6.13) \quad G_\varepsilon(u) = \frac{u^2}{2\varepsilon^{1+\beta}} + O\left(\frac{u^3}{\varepsilon^{2+\beta}}\right),$$

where  $O\left(\frac{u^3}{\varepsilon^{2+\beta}}\right)$  is uniform for  $u \leq \varepsilon/2$ . Let  $\delta > 0$  be given and  $\varepsilon_0 > 0$ ,  $R_1 > 0$  be as in Lemma 6.3. Let us work with  $\rho \geq R_1 + 2$ . Then we have  $v_\rho(r) \leq e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}}$  for  $r \in [\rho-1, \rho]$ . Thus by taking  $\varepsilon > 0$  small we have

$$\begin{aligned} J_{\rho,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon}) &\leq \int_{\mathbb{R}^N} \frac{1}{2}t_{\rho,\varepsilon}^2 |\nabla w_\varepsilon|^2 + G_\varepsilon(t_{\rho,\varepsilon}w_\varepsilon) - \frac{(t_{\rho,\varepsilon}w_\varepsilon)^{p+1}}{p+1} \\ &\quad + \int_{B_\rho \setminus B_{\rho-1}} \left( \frac{1}{2}t_{\rho,\varepsilon}^2 |\nabla v_\rho|^2 + t_{\rho,\varepsilon}^2 \frac{v_\rho^2}{2\varepsilon^{1+\beta}} + Ct_{\rho,\varepsilon}^3 \frac{v_\rho^3}{\varepsilon^{2+\beta}} \right) \end{aligned}$$

since also  $\frac{1}{2}t_{\rho,\varepsilon}^2 |\nabla w_\varepsilon|^2 + G_\varepsilon(t_{\rho,\varepsilon}w_\varepsilon) - \frac{(t_{\rho,\varepsilon}w_\varepsilon)^{p+1}}{p+1} \geq 0$  for  $r \geq \rho-1$ . Therefore

$$\begin{aligned} c_{\rho,\varepsilon} &\leq J_{\mathbb{R}^N,\varepsilon}(t_{\rho,\varepsilon}w_\varepsilon) - \frac{1}{2}t_{\rho,\varepsilon}^2 a_N \rho^{N-1} v'_\rho(\rho-1) v_\rho(\rho-1) + C \frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho^3 \\ &\leq c_\rho - \frac{1}{2}t_{\rho,\varepsilon}^2 \int_{\partial B_{\rho-1}} v'_\rho v_\rho + C \frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho^3. \end{aligned}$$

To estimate the last integral note that  $v_\rho \leq w_\varepsilon$  for  $\rho-1 \leq r \leq \rho$  because  $w_\varepsilon$  is a supersolution of (6.12). Hence

$$\int_{B_\rho \setminus B_{\rho-1}} v_\rho^3 \leq C \rho^{N-1} e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1)}$$

so

$$C \frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho^3 \leq e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1) + C_1 \log \rho + C_2 \log(1/\varepsilon) + C_3}.$$

To estimate the boundary integral let  $z$  be the solution to

$$\begin{cases} \Delta z = 0 & \text{in } B_\rho \setminus \bar{B}_{\rho-1} \\ z(\rho-1) = 1, \quad z(\rho) = 0. \end{cases}$$

Then

$$\int_{\partial B_{\rho-1}} v'_\rho = \int_{\partial B_{\rho-1}} v_\rho z' - \frac{1}{\varepsilon^{1+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho z$$

As before

$$\frac{1}{\varepsilon^{1+\beta}} \int_{B_\rho \setminus B_{\rho-1}} v_\rho z \leq e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1) + C_1 \log \rho + C_2 \log(1/\varepsilon) + C_3}.$$

so

$$c_{\rho,\varepsilon} \leq c_\rho + e^{-2(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1) + C_1 \log \rho + C_2 \log(1/\varepsilon) + C_3}.$$

To prove the lower bound, let  $w_{\rho,\varepsilon}$  be a radial, decreasing least energy solution of (6.1). We let also  $\tilde{v}_\rho$  be the solution of

$$\Delta \tilde{v}_\rho = \frac{1}{\varepsilon^{1+\beta}} \tilde{v}_\rho \quad \text{in } \mathbb{R}^N \setminus \bar{B}_{\rho-\varepsilon(1+\beta)/2}$$

with  $\tilde{v}_\rho(\rho - \varepsilon(1+\beta)/2) = w_{\rho,\varepsilon}(\rho - \varepsilon(1+\beta)/2)$  and  $\lim_{r \rightarrow +\infty} \tilde{v}_\rho(r) = 0$ . Define

$$\bar{w}_{\rho,\varepsilon}(r) = \begin{cases} w_{\rho,\varepsilon}(r) & \text{if } 0 \leq r \leq \rho - \varepsilon(1+\beta)/2 \\ \tilde{v}_\rho(r) & \text{if } r \geq \rho - \varepsilon(1+\beta)/2. \end{cases}$$

The for all  $t > 0$

$$\begin{aligned} c_{\rho,\varepsilon} &\geq J_{\rho,\varepsilon}(tw_{\rho,\varepsilon}) \geq J_{\mathbb{R}^N,\varepsilon}(t\bar{w}_{\rho,\varepsilon}) \\ &+ \int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} \left( \frac{1}{2} t^2 |\nabla w_{\rho,\varepsilon}|^2 + G_\varepsilon(tw_{\rho,\varepsilon}) - \frac{(tw_{\rho,\varepsilon})^{p+1}}{p+1} \right) \\ &- \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon(1+\beta)/2}} \left( \frac{1}{2} t^2 |\nabla \tilde{v}_\rho|^2 + G_\varepsilon(t\tilde{v}_\rho) \right). \end{aligned}$$

We take now  $t = t_{\rho,\varepsilon}$  such that  $J_{\mathbb{R}^N,\varepsilon}(t_{\rho,\varepsilon}\bar{w}_{\rho,\varepsilon}) \geq c_\varepsilon$ . Since  $w_{\rho,\varepsilon} \rightarrow w_\varepsilon$  as  $\rho \rightarrow +\infty$  uniformly for  $\varepsilon > 0$  small, we have  $t_{\rho,\varepsilon} \rightarrow 1$  as  $\rho \rightarrow +\infty$ , uniformly for  $\varepsilon > 0$  small.

We use the expansion (6.13). Let  $\delta > 0$  be given and  $\varepsilon_0 > 0$ ,  $R_1 > 0$  be as in Lemma 6.3. Let us work with  $\rho \geq R_1 + 2$ . Then we have  $w_{\rho,\varepsilon}(r) \leq e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}}$  for  $\rho - 1 \leq r \leq \rho$ . Thus by taking  $\varepsilon > 0$  small we can estimate

$$\begin{aligned} &\int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} G_\varepsilon(t_{\rho,\varepsilon}w_{\rho,\varepsilon}) - \frac{(t_{\rho,\varepsilon}w_{\rho,\varepsilon})^{p+1}}{p+1} \geq \\ &\geq \int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} t_{\rho,\varepsilon}^2 \frac{w_{\rho,\varepsilon}^2}{2\varepsilon^{1+\beta}} - Ct_{\rho,\varepsilon}^3 \frac{w_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} - \frac{(t_{\rho,\varepsilon}w_{\rho,\varepsilon})^{p+1}}{p+1} \\ &= \frac{t_{\rho,\varepsilon}^2}{2\varepsilon^{1+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} w_{\rho,\varepsilon}^2 \left( 1 - \frac{2\varepsilon^{1+\beta}}{p+1} t_{\rho,\varepsilon}^{p-1} w_{\rho,\varepsilon}^{p-1} \right) - \frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} w_{\rho,\varepsilon}^3 \\ &\geq \frac{t_{\rho,\varepsilon}^2}{2\varepsilon^{1+\beta}} m_{\rho,\varepsilon} \int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} w_{\rho,\varepsilon}^2 - \frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon(1+\beta)/2}} w_{\rho,\varepsilon}^3 \end{aligned}$$

where

$$m_{\rho,\varepsilon} = \max_{\rho-\varepsilon(1+\beta)/2 \leq r \leq \rho} \left( 1 - \frac{2\varepsilon^{1+\beta}}{p+1} t_{\rho,\varepsilon}^{p-1} w_{\rho,\varepsilon}^{p-1} \right).$$

We also have  $\tilde{v}_\rho(r) \leq e^{-(\varepsilon^{-1-\beta}-\delta)^{1/2}}$  for  $r \geq \rho - 1$ . Then for  $\varepsilon > 0$  small we can estimate

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon(1+\beta)/2}} \left( \frac{1}{2} t^2 |\nabla \tilde{v}_\rho|^2 + G_\varepsilon(t\tilde{v}_\rho) \right) \leq \\ &\leq \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon(1+\beta)/2}} \left( \frac{1}{2} t^2 |\nabla \tilde{v}_\rho|^2 + t_{\rho,\varepsilon}^2 \frac{\tilde{v}_\rho^2}{2\varepsilon^{1+\beta}} + Ct_{\rho,\varepsilon}^3 \frac{\tilde{v}_\rho^3}{\varepsilon^{2+\beta}} \right) \end{aligned}$$

Let  $z$  be the solution to

$$\Delta z = \frac{m_{\rho,\varepsilon}}{\varepsilon^{1+\beta}} z \quad \text{in } B_\rho \setminus \bar{B}_{\rho-\varepsilon(1+\beta)/2}$$

with  $z(\rho - \varepsilon^{(1+\beta)/2}) = w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2})$  and  $z(\rho) = 0$ . Then,

$$\begin{aligned} c_{\rho,\varepsilon} &\geq c_\varepsilon + \frac{t_{\rho,\varepsilon}^2}{2} \int_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \left( |\nabla z|^2 + \frac{m_{\rho,\varepsilon}}{\varepsilon^{1+\beta}} z^2 \right) \\ &\quad - \frac{t_{\rho,\varepsilon}^2}{2} \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \left( |\nabla \tilde{v}_\rho|^2 + \frac{1}{\varepsilon^{1+\beta}} \tilde{v}_\rho^2 \right) \\ &\quad - \frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 - \frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \tilde{v}_\rho^3 \\ &= c_\varepsilon - \frac{t_{\rho,\varepsilon}^2}{2} w_{\rho,\varepsilon}(\rho - \varepsilon^{(1+\beta)/2}) \int_{\partial B_{\rho-\varepsilon^{(1+\beta)/2}}} \left( \frac{\partial z}{\partial \nu} - \frac{\partial \tilde{v}_\rho}{\partial \nu} \right) \\ &\quad - \frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 - \frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \tilde{v}_\rho^3 \end{aligned}$$

By similar estimates as for the upper bound

$$\frac{t_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{B_\rho \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} w_{\rho,\varepsilon}^3 \leq e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1)+C_1 \log \rho + C_2 \log(1/\varepsilon) + C_3}$$

and

$$\frac{Ct_{\rho,\varepsilon}^3}{\varepsilon^{2+\beta}} \int_{\mathbb{R}^N \setminus B_{\rho-\varepsilon^{(1+\beta)/2}}} \tilde{v}_\rho^3 \leq e^{-3(\varepsilon^{-1-\beta}-\delta)^{1/2}(\rho-R_1-1)+C_1 \log \rho + C_2 \log(1/\varepsilon) + C_3}$$

Using suitable barriers one can prove

$$\tilde{v}'_\rho(\rho - \varepsilon^{(1+\beta)/2}) \geq -\varepsilon^{-(1+\beta)/2} - \frac{N-1}{2(\rho - \varepsilon^{(1+\beta)/2})} + O\left(\frac{\varepsilon^{(1+\beta)/2}}{\rho^2}\right)$$

and

$$z'(\rho - \varepsilon^{(1+\beta)/2}) \leq -\frac{m_{\rho,\varepsilon}^{1/2}}{\varepsilon^{(1+\beta)/2}} \frac{e^{m_{\rho,\varepsilon}^{1/2}} + e^{-m_{\rho,\varepsilon}^{1/2}}}{e^{m_{\rho,\varepsilon}^{1/2}} - e^{-m_{\rho,\varepsilon}^{1/2}}}.$$

This implies

$$\tilde{v}'_\rho(\rho - \varepsilon^{(1+\beta)/2}) - z'(\rho - \varepsilon^{(1+\beta)/2}) \geq \varepsilon^{-(1+\beta)/2} \left( m_{\rho,\varepsilon}^{1/2} \frac{e^{m_{\rho,\varepsilon}^{1/2}} + e^{-m_{\rho,\varepsilon}^{1/2}}}{e^{m_{\rho,\varepsilon}^{1/2}} - e^{-m_{\rho,\varepsilon}^{1/2}}} - 1 \right) - \frac{C}{\rho}.$$

Since  $m_{\rho,\varepsilon} \rightarrow 1$  as  $\rho \rightarrow +\infty$  uniformly for  $\varepsilon > 0$  we deduce

$$c_{\rho,\varepsilon} \geq c_\varepsilon + e^{-2(\varepsilon^{-1-\beta}-C)^{1/2}(\rho-R_1-1)-C \log \rho - C \log(1/\varepsilon) - C}.$$

□

## 7. ASYMPTOTIC BEHAVIOR AS $\lambda \rightarrow +\infty$ , PART 2

In this section we prove Theorem 1.2 following the argument of [6]. For  $\lambda > 0$  and  $\varepsilon > 0$ , let  $u_{\lambda,\varepsilon}$  denote the solution of (1.3) obtained through the mountain pass theorem, as in the proof of Theorem 1.1. Let  $x_{\lambda,\varepsilon} \in \Omega$  denote a point where  $u_{\lambda,\varepsilon}$  attains its maximum. It will be convenient to introduce the rescaled functions

$$v_{\lambda,\varepsilon}(x) = \lambda^{\frac{1}{p+\beta}} u_{\lambda,\varepsilon} \left( x_{\lambda,\varepsilon} + \lambda^{-\frac{1+\beta}{2(p+\beta)}} x \right) \quad x \in \Omega_{\lambda,\varepsilon},$$

where  $\Omega_{\lambda,\varepsilon} = \lambda^{\frac{1+\beta}{2(p+\beta)}} (\Omega - x_{\lambda,\varepsilon})$ . Then  $v_{\lambda,\varepsilon}$  solves

$$(7.1) \quad -\Delta v + \frac{v}{(v + \varepsilon \lambda^{\frac{1}{p+\beta}})^{1+\beta}} = v^p \quad \text{in } \Omega_{\lambda,\varepsilon}, \quad v = 0 \quad \text{on } \partial\Omega_{\lambda,\varepsilon}.$$

Associated to (7.1) we have the functional

$$J_{\lambda,\varepsilon}(v) = \int_{\Omega_{\lambda,\varepsilon}} \left( \frac{1}{2} |\nabla v|^2 + G_{\varepsilon \lambda^{\frac{1}{p+\beta}}}(v) - \frac{(v^+)^{p+1}}{p+1} \right), \quad v \in H_0^1(\Omega_{\lambda,\varepsilon})$$

with least energy

$$c_{\lambda,\varepsilon} = \inf_{v \in H_0^1(\Omega_{\lambda,\varepsilon}), v \neq 0} \sup_{t \geq 0} J_{\lambda,\varepsilon}(tv)$$

Let  $c_\varepsilon$  be the value defined in (6.4). Let

$$d_{\lambda,\varepsilon} = \text{dist}(x_{\lambda,\varepsilon}, \partial\Omega_{\lambda,\varepsilon})$$

and

$$d_{max} = \max_{x \in \Omega} \text{dist}(x, \partial\Omega)$$

so that

$$\max_{x \in \Omega_{\lambda,\varepsilon}} \text{dist}(x, \partial\Omega_{\lambda,\varepsilon}) = \lambda^{\frac{1+\beta}{2(p+\beta)}} d_{max}.$$

We will prove that

$$(7.2) \quad c_{\lambda,\varepsilon} \leq c_\varepsilon + \exp \left( -2(\varepsilon^{-1-\beta} \lambda^{-\frac{1+\beta}{p+\beta}} - C)^{1/2} (\lambda^{\frac{1+\beta}{2(p+\beta)}} d_{max} - R_1) + C \log(1/\varepsilon) + C \log(\lambda) + C \right)$$

and

$$(7.3) \quad c_\varepsilon + \exp \left( -2(\varepsilon^{-(1+\beta)/2} \lambda^{-\frac{1+\beta}{2(p+\beta)}} + C)(d_{\lambda,\varepsilon} + o(1) - R_2) - C \log(1/\varepsilon) - C \log(\lambda) - C \right) \leq c_{\lambda,\varepsilon}$$

where  $o(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , and  $C, R_1, R_2$  are constants. Inequalities (7.2) and (7.3) imply that

$$(7.4) \quad \liminf_{\varepsilon \rightarrow 0} \text{dist}(x_{\lambda,\varepsilon}, \partial\Omega) \geq d_{max} - C \lambda^{-\frac{1+\beta}{2(p+\beta)}}$$

for some constant  $C$ .

As in the proof of Theorem 1.1 up to a subsequence  $u_{\lambda,\varepsilon}$  converges in  $C^1(\bar{\Omega})$  as  $\varepsilon \rightarrow 0$  to a non-zero function  $u_\lambda \in C^{1,\mu}(\bar{\Omega})$  which solves (1.1). Let  $x_\lambda \in \Omega$  be a maximum point of  $u_\lambda$ . Then (7.4) implies that

$$\text{dist}(x_\lambda, \partial\Omega) \geq d_{max} - C \lambda^{-\frac{1+\beta}{2(p+\beta)}}.$$

By Lemma 5.4, for a large  $\lambda > 0$  the solution  $u_\lambda$  has support contained in a ball  $B_{R_\lambda}(x_\lambda)$  with  $R_\lambda = C \lambda^{-\frac{1+\beta}{2(p+\beta)}}$  for some constant  $C$ . Thus by taking  $\lambda > 0$  large,  $u_\lambda$  has compact support in  $\Omega$  and hence solves

$$-\Delta u + \chi_{[u>0]} u^{-\beta} = \lambda u^p \quad \text{in } \mathbb{R}^N.$$

Let

$$v_\lambda(x) = \lambda^{\frac{1}{p+\beta}} u_\lambda \left( x_\lambda + \lambda^{-\frac{1+\beta}{2(p+\beta)}} x \right)$$

so that  $v_\lambda$  solves

$$-\Delta v_\lambda + \chi_{[v_\lambda>0]} v_\lambda^{-\beta} = v_\lambda^p \quad \text{in } \mathbb{R}^N$$

Let  $J, M$  be defined by (4.3). By Lemma 5.2

$$J(v) \leq J(\varphi)$$

for all  $\varphi \in H_0^1(\Omega_\lambda)$  where  $\Omega_\lambda = \lambda^{\frac{1+\beta}{2(p+\beta)}}(\Omega - x_\lambda)$ , which satisfies  $M(\varphi) = 0$ . By Proposition 4.1,  $v = w$  because we have fixed the maximum of  $v$  at the origin. This concludes the proof of Theorem 1.2, up to (7.2) and (7.3).

The proof of (7.2) and (7.3) are similar to those in [6], except for one estimate of the energy. For completeness we give the details. To prove the upper bound (7.2) let  $\bar{x}_{\lambda,\varepsilon} \in \Omega_{\lambda,\varepsilon}$  be a point that realizes the maximum distance to  $\partial\Omega_{\lambda,\varepsilon}$ . Since the least energy values for the functional  $J_{\lambda,\varepsilon}$  in  $\Omega_{\lambda,\varepsilon}$  and the ball with center  $\bar{x}_{\lambda,\varepsilon}$  and radius  $\lambda^{-\frac{1+\beta}{2(p+\beta)}}d_{max}$  are ordered we have

$$c_{\lambda,\varepsilon} \leq c_{\rho,\varepsilon} \lambda^{\frac{1}{p+\beta}}$$

where  $c_{\rho,\varepsilon}$  is defined in (6.2) and  $\rho = \lambda^{-\frac{1+\beta}{2(p+\beta)}}d_{max}$ . By Proposition 6.1 we have deduce (7.2).

We now derive (7.3). For simplicity of the notation we write  $v = v_{\lambda,\varepsilon}$ . Up to subsequence we can assume that  $x_{\lambda,\varepsilon} \rightarrow x_\lambda \in \bar{\Omega}$  as  $\varepsilon \rightarrow 0$ . Then

$$dist(x_{\lambda,\varepsilon}, \partial\Omega) \rightarrow dist(x_\lambda, \partial\Omega) \quad \text{as } \varepsilon \rightarrow 0.$$

Let  $R_{\lambda,\varepsilon} = dist(x_{\lambda,\varepsilon}, \partial\Omega)$  and  $R_0 = dist(x_\lambda, \partial\Omega)$ . Let  $\delta > 0$  be given and take  $R'_0 > 0$  such that

$$vol(B_{R'_0}(x_\lambda)) = vol(\Omega \cap B_{R_0+\delta}(x_\lambda)).$$

Now let  $0 < \delta' < \delta$  be such that  $R'_0 < R_0 + \delta'$ . Let  $\eta$  be a  $C^\infty(\mathbb{R})$  cut-off function such that  $\eta(s) = 1$  for  $0 \leq s \leq R_{\lambda,\varepsilon} + \delta'$ ,  $\eta(s) = 0$  for  $s \geq R_{\lambda,\varepsilon} + \delta$ , and with uniformly bounded gradient. We set  $\eta_\lambda(s) = \eta(\lambda^{-\frac{1+\beta}{2(p+\beta)}}s)$  and

$$\tilde{v}(x) = v(x) \eta_\lambda(|x|), \quad x \in \Omega_{\lambda,\varepsilon}.$$

We claim that for every  $t \in [0, 2]$

(7.5)

$$\begin{aligned} J_{\lambda,\varepsilon}(t\tilde{v}) &\leq J_{\lambda,\varepsilon}(t\tilde{v}) + \exp\left(-2((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}((R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta)}} - R_1)\right) \\ &\quad + C \log(1/\varepsilon) + C \log \lambda \end{aligned}$$

Indeed,

$$\begin{aligned} J_{\lambda,\varepsilon}(t\tilde{v}) &= \int_{\Omega_{\lambda,\varepsilon}} \left( \frac{t^2}{2} |\nabla \tilde{v}|^2 + G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(t\tilde{v}) - \frac{t^{p+1}}{p+1} \tilde{v}^{p+1} \right) \\ &\leq J_{\lambda,\varepsilon}(tv) + t^2 \int_{\Omega_{\lambda,\varepsilon}} \eta_\lambda(|x|) |\eta'_\lambda(|x|) v |\nabla v| + \frac{t^2}{2} \int_{\Omega_{\lambda,\varepsilon}} v^2 |\eta'_\lambda(|x|)|^2 \\ &\quad + \int_{\Omega_{\lambda,\varepsilon}} \left( G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv) - G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv\eta_\lambda(|x|)) \right) \\ &\quad + \frac{t^{p+1}}{p+1} \int_{\Omega_{\lambda,\varepsilon}} v^{p+1} (1 - \eta_\lambda(|x|)^{p+1}) \end{aligned}$$

Using the same supersolutions as in Lemmas 6.2 and 6.3 we find  $R_1$  and  $\varepsilon_0 > 0$  such that

$$(7.6) \quad v(x) \leq \exp\left(-((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}(|x| - R_1)\right)$$

for all  $x \in \Omega_{\lambda,\varepsilon}$  with  $|x| \geq R_1$ , and all  $0 < \varepsilon \leq \varepsilon_0$ . This implies also a similar estimate for the gradient, namely

$$|\nabla v(x)| \leq \exp\left(-((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}(|x| - R_1) + C \log(1/\varepsilon) + C \log \lambda\right)$$

for all  $x \in \Omega_{\lambda,\varepsilon}$  with  $|x| \geq R_1$ , and all  $0 < \varepsilon \leq \varepsilon_0$ . We can write

$$G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv) = \frac{v^2}{2\varepsilon^{1+\beta}\lambda^{\frac{1+\beta}{p+\beta}}} + O\left(\frac{v^3}{\varepsilon^{2+\beta}\lambda^{\frac{2+\beta}{p+\beta}}}\right)$$

provided  $\frac{v}{\varepsilon\lambda^{\frac{1}{p+\beta}}} \leq 1/2$ . Using (7.6) we see that this holds in  $\Omega_{\lambda,\varepsilon} \setminus B_{r(\lambda,\varepsilon)}(0)$  where  $r(\lambda,\varepsilon) = (R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta)}}$ . So we may estimate

$$\begin{aligned} \int_{\Omega_{\lambda,\varepsilon}} \left( G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv) - G_{\varepsilon\lambda^{\frac{1}{p+\beta}}}(tv\eta_\lambda(x)) \right) &\leq \frac{Ct^3}{\varepsilon^{2+\beta}\lambda^{\frac{2+\beta}{p+\beta}}} \int_{\Omega_{\lambda,\varepsilon} \setminus B(0, R_{\lambda,\varepsilon}\lambda^{\frac{1+\beta}{2(p+\beta)}})} v^3 \\ &\leq \exp\left(-3((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}(r(\lambda,\varepsilon) - R_1) + C \log(1/\varepsilon) + C \log \lambda\right) \end{aligned}$$

Similarly we find the following estimates

$$\begin{aligned} \frac{t^{p+1}}{p+1} \int_{\Omega_{\lambda,\varepsilon}} v^{p+1}(1 - \eta_\lambda(|x|)^{p+1}) \\ \leq \exp\left(- (p+1)((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}(r(\lambda,\varepsilon) - R_1) + C \log(1/\varepsilon) + C \log \lambda\right), \\ t^2 \int_{\Omega_{\lambda,\varepsilon}} \eta_\lambda(|x|)|\eta'_\lambda(|x|)v|\nabla v| \\ \leq \exp\left(-2((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}(r(\lambda,\varepsilon) - R_1) + C \log(1/\varepsilon) + C \log \lambda\right) \end{aligned}$$

and

$$\begin{aligned} \frac{t^2}{2} \int_{\Omega_{\lambda,\varepsilon}} v^2|\eta'_\lambda(|x|)|^2 \\ \leq \exp\left(-2((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}(r(\lambda,\varepsilon) - R_1) + C \log(1/\varepsilon) + C \log \lambda\right) \end{aligned}$$

which combined imply (7.5).

Let  $R'_{\lambda,\varepsilon} > 0$  be such that

$$\text{vol}(B_{R'_{\lambda,\varepsilon}}) = \text{vol}(\Omega \cap B_{R_{\lambda,\varepsilon}+\delta}).$$

Using the Schwarz symmetric rearrangement we find

$$J_{\lambda,\varepsilon}(t\tilde{v}) \geq J_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}}(t\tilde{v}^*)$$

where  $c_{\rho,\varepsilon}$  is defined in (6.2),  $\rho = R'_{\lambda,\varepsilon}\lambda^{\frac{1+\beta}{2(p+\beta)}}$  and  $\tilde{v}^*$  is the radially decreasing rearrangement of  $\tilde{v}$ . We choose now  $t \in [0, 2]$  such that  $J_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}}(t\tilde{v}^*)$  is maximized,

and find, using (7.5) and Proposition 6.1,

$$\begin{aligned}
c_{\lambda,\varepsilon} &\geq c_{\rho,\varepsilon\lambda^{\frac{1}{p+\beta}}} - \exp\left(-2((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}((R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta)}} - R_1) \right. \\
&\quad \left. + C \log(1/\varepsilon) + C \log \lambda\right) \\
&\geq c_\varepsilon + \exp\left(-2(\varepsilon^{-1-\beta} - C)^{1/2}((R_{\lambda,\varepsilon} + \delta)\lambda^{\frac{1+\beta}{2(p+\beta)}} - R_2) - C \log \rho - C \log(1/\varepsilon) - C\right) \\
&\quad - \exp\left(-2((\varepsilon\lambda^{\frac{1}{p+\beta}})^{-1-\beta} - 1)^{1/2}((R_{\lambda,\varepsilon} + \delta')\lambda^{\frac{1+\beta}{2(p+\beta)}} - R_1) + C \log(1/\varepsilon) + C \log \lambda\right) \\
&\geq c_\varepsilon + \exp\left(-2(\varepsilon^{-1-\beta} - C)^{1/2}((R_{\lambda,\varepsilon} + \delta)\lambda^{\frac{1+\beta}{2(p+\beta)}} - R_2) - C \log \rho - C \log(1/\varepsilon) - C\right)
\end{aligned}$$

for  $\varepsilon > 0$  small. This establishes (7.3).  $\square$

### 8. PROOF OF THEOREM 1.3

Let  $u_\lambda$  be the solution constructed in Theorem 1.1 and let

$$v_\lambda = \lambda^{\frac{1}{p-1}} u_\lambda.$$

Then  $v_\lambda$  satisfies

$$(8.1) \quad \begin{cases} -\Delta v + \lambda^{\frac{1+\beta}{p-1}} v^{-\beta} \chi_{\{v>0\}} = v^p, & v \geq 0 \quad \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us recall the classical equation

$$(8.2) \quad \begin{cases} -\Delta v = v^p, & v \geq 0 \quad \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

which arises as the limit of (8.1) as  $\lambda \rightarrow 0$ . It is well known that (8.2) admits a nontrivial solution, for example by minimizing the  $H_0^1(\Omega)$  norm on the unit sphere of  $L^p(\Omega)$  or by using the mountain pass theorem [1]. If  $\Omega$  is a ball, it is furthermore known that positive solutions of that (8.1) are radial [10] and that there is a unique nontrivial solution, which is furthermore nondegenerate.

We will first prove

**Lemma 8.1.** *Suppose  $\Omega$  is the unit ball in  $\mathbb{R}^N$ . Then the solution  $v_\lambda$  to (8.1) converges in the  $C^1(\overline{B}_1)$  sense to the unique positive solution of (8.2)*

**Proof.** Let  $v_0$  denote the unique nontrivial solution of (8.2). We can show that for  $\lambda > 0$  sufficiently close to 0, (8.1) in the ball has a solution close to  $v_0$ . One can construct this solution of the form  $v = v_0 + w$ , where  $w$  has now to satisfy, assuming  $w$  is small in  $C^1(\overline{B}_1)$ ,

$$-\Delta w - pv_0^{p-1}w = (v_0 + w)^p - v_0^p - pv_0^{p-1}w - \lambda^{\frac{1+\beta}{p-1}}(v_0 + w)^{-\beta} \quad \text{in } \Omega$$

$$w = 0 \quad \text{on } \partial\Omega.$$

Using the nondegeneracy of  $v_0$  we may set up a fixed point argument, similar to the one in the proof of Theorem 1.5 and deduce that (8.1) has a solution  $v_\lambda$  for  $\lambda > 0$ . By construction  $v_\lambda \rightarrow v_0$  as  $\lambda \rightarrow 0$ . But radial solutions to (8.1) are unique, see [5], and this proves the result.  $\square$

**Proof of Theorem 1.3.** An argument similar to that of the proof of Theorem 1.1 shows that  $\|v_\lambda\|_{L^\infty(\Omega)}$  remains bounded as  $\lambda \rightarrow 0$ . Therefore, thanks to Proposition 2.1,  $v_\lambda$  is also bounded in  $C^{1,\mu}(\overline{\Omega})$  where  $\mu = \frac{1-\beta}{1+\beta}$ . Up to subsequence  $v_\lambda$  then converges to a function  $v \in C^1(\overline{\Omega})$ . We have to discard the possibility that  $v = 0$ .

Let

$$J_\lambda(v) = \int_\Omega \left( \frac{1}{2} |\nabla v|^2 + \frac{\lambda^{\frac{1+\beta}{p-1}}}{1-\beta} v_+^{1-\beta} - \frac{1}{p+1} |v|^{p+1} \right), \quad v \in H_0^1(\Omega)$$

so that its critical points give rise to solutions of (8.1). The solution  $u_\lambda$  constructed in Theorem 1.1 is a least energy solution and therefore

$$J_\lambda(v_\lambda) = \sup_{t \geq 0} J_\lambda(t\lambda v).$$

Let  $R > 0$  be such that  $\text{vol}(B_R(0)) = \text{vol}(\Omega)$  and let  $v_\lambda^*$  be the Schwarz symmetrization of  $v$ . Then for every (???)  $t \geq 0$

$$J_\lambda(v_\lambda) \geq J_\lambda(tv_\lambda) \geq J_\lambda(tv_\lambda^*)$$

and therefore

$$J_\lambda(v_\lambda) \geq \inf_{v \in H_0^1(B_R(0)), v \neq 0} \sup_{t \geq 0} J_\lambda(v, B_R(0)) = J_\lambda(v_{rad,\lambda}; B_R(0))$$

where  $v_{rad,\lambda}$  is the unique radial solution of (8.1) in the ball  $B_R(0)$ . By Lemma 8.1  $v_{rad,\lambda} \rightarrow v_{rad,0}$  as  $\lambda \rightarrow 0$  in  $C^1(\overline{B_R(0)})$ , where  $v_{rad,0}$  is the unique nontrivial solution of (8.2). Therefore there is  $c > 0$  such that

$$J_\lambda(v_\lambda) \geq c$$

for  $\lambda > 0$ . It follows that  $v = \lim_{\lambda \rightarrow 0} v_\lambda$  can not be identically zero. Testing the equation with functions supported on the set  $[v > 0]$  we see that

$$-\Delta v = v^p \quad \text{in } [v > 0],$$

But this and the Hopf lemma imply that  $[v > 0] = \Omega$ , that is,  $v > 0$ . Since  $v_\lambda \rightarrow v$  in  $C^1(\overline{\Omega})$ , we deduce that in fact  $v_\lambda > 0$  for  $\lambda > 0$  small.  $\square$

## 9. POSITIVE SOLUTIONS FOR $p = 1$ AND $\lambda$ CLOSE TO $\lambda_1$

We look for a solution of (1.9) of the form

$$(9.1) \quad u = \varepsilon^{-\frac{1}{1+\beta}} \varphi \quad \text{for } \varepsilon = \lambda - \lambda_1 > 0 \text{ small.}$$

A calculation shows that  $u$  is a solution of (1.9) if and only if  $\varphi$  solves

$$(9.2) \quad \begin{cases} -\Delta \varphi + \varepsilon \varphi^{-\beta} = \lambda \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial \Omega \end{cases}$$

which we write as

$$(9.3) \quad \begin{cases} -\Delta \varphi - \lambda_1 \varphi = -\varepsilon \varphi^{-\beta} + \varepsilon \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial \Omega \end{cases}$$

We are going to find a solution for (9.3) of the form  $\varphi = c\varphi_1 + z$  with  $z$  orthogonal to  $\varphi_1$  in  $L^2(\Omega)$ , and  $z$  small compared to  $\varphi_1$  so that  $(c\varphi_1 + z)^{-\beta} \leq c'\delta^{-\beta}$ , for some constant  $c' > 0$ . Inserting the expression of  $\varphi$  into (9.2), one finds

$$(9.4) \quad \begin{cases} -\Delta z - \lambda_1 z = \varepsilon[-(c\varphi_1 + z)^{-\beta} + c\varphi_1 + z] & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

For a function  $h : \Omega \rightarrow \mathbb{R}$  we introduce the norm

$$\|h\|_\beta = \sup_{x \in \Omega} |h(x)|\delta(x)^\beta$$

where

$$\delta(x) = \text{dist}(x, \partial\Omega).$$

We recall, see [13], that if  $\|h\|_\beta < \infty$  then the problem

$$\begin{cases} \Delta u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution  $u \in C(\overline{\Omega}) \cap C^{1,\nu}(\Omega)$  (every  $0 < \nu < 1$ ). Moreover by the results of Gui and Lin [14] one also has

$$\|u\|_{C^{1,1-\beta}(\overline{\Omega})} \leq C\|h\|_\beta.$$

We need the following:

**Lemma 9.1.** *If  $h : \Omega \rightarrow \mathbb{R}$  satisfies  $\|h\|_\beta < \infty$  and  $\int_\Omega h\varphi_1 = 0$  then there is a solution  $u$  to*

$$(9.5) \quad \begin{cases} -\Delta u - \lambda_1 u = h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with  $\int_\Omega u\varphi_1 = 0$  and

$$(9.6) \quad \|u\|_{C^{1,1-\beta}(\overline{\Omega})} \leq C\|h\|_\beta.$$

**Proof.** If  $h \in L^\infty(\Omega)$  and  $\int_\Omega h\varphi_1 = 0$  then there is a solution  $u \in C^{1,\nu}(\overline{\Omega})$  (every  $0 < \nu < 1$ ) to (9.5) with  $\int_\Omega u\varphi_1 = 0$ . Thus it is sufficient to prove (9.6) in this situation and then proceed by density.

First we prove that there is  $C > 0$  depending only on  $\Omega, \beta$  such that

$$(9.7) \quad |u(x)| \leq C\|h\|_\beta \delta(x)^{-N} \quad \text{for all } x \in \Omega.$$

By standard arguments, if  $u$  solves (9.5) and  $\int_\Omega u\varphi_1 = 0$ , then

$$\|u\|_{L^1(\Omega)} \leq C\|h\|_{L^1(\Omega)}$$

with  $C$  independent of  $u$  and  $h$ . Now let  $x_0 \in \Omega$  and  $r = \delta(x_0)/2$ . Solve

$$\begin{aligned} -\Delta v - \lambda_1 v &= h & \text{in } B_r(x_0) \\ v &= 0 & \text{on } \partial B_r(x_0). \end{aligned}$$

Then, by standard elliptic estimates

$$|v(x_0)| \leq Cr^2 \sup_{x \in B_r(x_0)} |h(x)| \leq r^{2-\beta} \|h\|_\beta$$

and

$$\|v\|_{L^1(B_r(x_0))} \leq Cr^2 \|h\|_{L^1(B_r(x_0))} \leq Cr^{N+2-\beta} \|h\|_\beta$$

where  $C$  does not depend on  $h$  or  $r$ . Define  $w = u - v$ . Then

$$\Delta w + \lambda_1 w = 0 \quad \text{in } B_r(x_0).$$

Again by elliptic estimates

$$\begin{aligned} |w(x_0)| &\leq \frac{C}{r^N} \int_{B_r(x_0)} |w| \leq \frac{C}{r^N} \int_{B_r(x_0)} |v| + \frac{C}{r^N} \int_{B_r(x_0)} |u| \\ &\leq Cr^{2-\beta} \|h\|_\beta + Cr^{-N} \|u\|_{L^1(\Omega)} \leq Cr^{-N} \|h\|_\beta. \end{aligned}$$

This proves (9.7).

Fix a  $\tau \in (0, 1)$  and let  $\eta > 0$  be small so that  $\delta(x)$  is smooth in the region

$$N_\eta = \{x \in \Omega : \delta(x) < \eta\}.$$

Let  $\bar{u} = k \|h\|_\beta \delta^\tau$ . Then

$$\begin{aligned} -\Delta \bar{u} - \lambda_1 \bar{u} &= k \|h\|_\beta (-\tau(\tau-1) |\nabla \delta|^2 \delta^{\tau-2} - \tau \delta^{\tau-1} \Delta \delta - \lambda_1 \delta^\tau) \\ &\geq |h| \quad \text{in } N_\eta \end{aligned}$$

if we take  $\eta > 0$  small and  $k$  sufficiently large. Now we fix  $\eta$ . By increasing  $k$  if necessary we have  $|u| \leq \bar{u}$  on  $\partial N_\eta$ . By the maximum principle, which is valid for the operator  $-\Delta - \lambda_1$  in  $N_\eta$ , we deduce that  $|u| \leq \bar{u}$  in  $N_\eta$ . Thus we have obtained

$$|\Delta u| \leq \lambda_1 |u| + |h| \leq C \|h\|_\beta \delta^{-\beta}.$$

By the estimates of Gui and Lin [14] we deduce (9.6).  $\square$

**Proof of Theorem 1.5.** Let  $0 < \underline{c} < \bar{c}$  be arbitrary constants, to be chosen later on. We are going to prove that for every  $c \in [\underline{c}, \bar{c}]$  there is a solution  $z$  in an appropriate space and  $\mu \in \mathbb{R}$  to the problem

$$(9.8) \quad \begin{cases} -\Delta z - \lambda_1 z = \varepsilon [-(c\varphi_1 + z)^{-\beta} + c\varphi_1 + z] + \mu\varphi_1 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

if  $\varepsilon > 0$  is sufficiently small.

To do this consider the spaces  $E = \{h : \Omega \rightarrow \mathbb{R} : \|h\|_\beta < \infty, h \perp \varphi_1\}$ ,  $F = \{u \in C^{0,1}(\bar{\Omega}) : u \perp \varphi_1\}$  where the orthogonality is with respect to the  $L^2$  inner product. Define  $S : E \rightarrow F$  by  $Sh = u$  with  $u$  the solution of (9.5) such that  $\int_\Omega u \varphi_1 = 0$ . By Lemma 9.1,  $S$  is a linear bounded operator.

For  $\rho > 0$  define  $B_\rho := \{z \in C^{0,1}(\bar{\Omega}) : \|z\|_{C^{0,1}} \leq \rho, z = 0 \text{ on } \partial\Omega\}$ , where  $\|z\|_{C^{0,1}(\bar{\Omega})}$  is the smallest Lipschitz constant of  $z$ .

Let  $A > 0, B > 0$  be such that  $A\delta \leq \varphi_1 \leq B\delta$  in  $\Omega$ . If  $\rho > 0$  is such that  $\rho < \underline{c}A$  and  $h \in B_\rho$  we have that  $\|(c\varphi_1 + h)^{-\beta}\|_\beta < +\infty$ . Thus, for such  $\rho$  we can define  $\Phi : B_\rho \rightarrow C^{0,1}(\bar{\Omega})$  by

$$\Phi(h) := S(\varepsilon [-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h] + \mu\varphi_1)$$

where  $\mu \in \mathbb{R}$  is such that

$$\varepsilon \int_\Omega [-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h] \varphi_1 + \mu \int_\Omega \varphi_1^2 = 0.$$

Thus  $z = \Phi(h)$  solves

$$\begin{cases} -\Delta z - \lambda_1 z = \varepsilon [-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h] + \mu\varphi_1 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

with  $\int_\Omega z \varphi_1 = 0$ .

For  $\varepsilon > 0$  small enough  $\Phi$  is a contraction in  $B_\rho$ . Indeed, if  $h \in B_\rho$ , then

$$\|\Phi(h)\|_{C^{0,1}} \leq \|S\| (\varepsilon \|-(c\varphi_1 + h)^{-\beta} + c\varphi_1 + h\|_\beta + \mu \|\varphi_1\|_\beta) \leq \rho$$

for  $\varepsilon$  small enough, and

$$\|\Phi(z_1) - \Phi(z_2)\| \leq \varepsilon \|S\| \|(c\varphi_1 + z_2)^{-\beta} - (c\varphi_1 + z_1)^{-\beta} + z_1 - z_2\| \leq \varepsilon C \|z_1 - z_2\|.$$

Applying the Banach fixed point theorem, for  $c \in [\underline{c}, \bar{c}]$  there is a solution  $z(c) \in B_\rho$  of (9.8) with a corresponding  $\mu(c) \in \mathbb{R}$ , provided  $\varepsilon > 0$  is small. Since  $\Phi$  is continuous in  $c$ , by the fixed point characterization of  $z(c)$  we deduce that it is continuous with respect to  $c$ , and hence  $c \mapsto \mu(c)$  is also continuous.

We proceed to show that if we take  $\underline{c} > 0$  small and  $\bar{c} > 0$  is large, then for some  $c \in [\underline{c}, \bar{c}]$  we have  $\mu(c) = 0$ . We have the expression

$$\mu(c) = \frac{\int_{\Omega} -(c\varphi_1 + z(c))^{-\beta} \varphi_1 + (c\varphi_1 + z(c)) \varphi_1}{\int_{\Omega} \varphi_1^2}.$$

Let us work with  $\underline{c} > 0$  small so that

$$-\underline{c}^{-\beta} (A/2 + B)^{-\beta} A \int_{\Omega} \delta^{1-\beta} + \underline{c} (A/2 + B) B \int_{\Omega} \delta^2 < 0.$$

and with  $\rho < A\underline{c}/2$ . Since  $z \in B_\rho$  we have  $|z| \leq \rho\delta$ , and then  $c\varphi_1 + z \leq (\frac{A}{2} + B)c\delta$ . Hence

$$[-(c\varphi_1 + z)^{-\beta} + c\varphi_1 + z] \varphi_1 \leq -(\frac{A}{2} + B)^{-\beta} c^{-\beta} A \delta^{1-\beta} + (\frac{A}{2} + B) c B \delta^2$$

This implies  $\mu(\underline{c}) < 0$ .

Similarly, using that  $c\varphi + z \geq (cA - \rho)\delta \geq cA\delta/2$ , we see that taking  $\bar{c} > 0$  large enough  $\mu(\bar{c}) > 0$ .  $\square$

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