Relative equilibria in continuous stellar dynamics

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(joint work with Juan Campos and Manuel del Pino)

Consider the gravitational Vlasov-Poisson system

$$\left\{\begin{array}{l}\partial_t f+v\cdot\nabla_x f-\nabla_x\phi\cdot\nabla_v f=0\;,\\ \phi=-\frac{1}{4\pi\,|\cdot|}*\rho\;,\quad \rho=\displaystyle\int_{\mathbb{R}^3} f\,dv\;.\end{array}\right.\tag{1}$$

We look for time-periodic solutions which are in rotation at constant angular velocity ω. Replacing $x=(x',x^3)$ and $v=(v',v^3)$ respectively by $(e^{i\,\omega\,t}\,x',x^3)$ and $(i\,\omega\,x'+e^{i\,\omega\,t}\,v',v^3)$ and using complex notations so that $x',\ v'\in\mathbb{R}^2\approx\mathbb{C}$, Problem (1) becomes

$$\left\{\begin{array}{l}\partial_t f+v\cdot\nabla_x f-\nabla_x\phi\cdot\nabla_v f-\omega^2\,x'\cdot\nabla_{v'}f+2\,\omega\,i\,v'\cdot\nabla_{v'}f=0\;,\\ \phi=-\frac{1}{4\pi\,|\cdot|}*\rho\;,\quad \rho=\displaystyle\int_{\mathbb{R}^3} f\,dv\;,\end{array}\right.\tag{2}$$

where we have abusively used the same notations for the potential ϕ and the distribution function f, for sake of simplicity. A *relative equilibrium* of (1) is a stationary solution of (2) and can be obtained by considering critical points of the *free energy* functional

$$\mathcal{F}[f]=\iint_{\mathbb{R}^3\times\mathbb{R}^3}\beta(f)\,dx\,dv+\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\big(|v|^2-\omega^2\,|x'|^2\big)f\,dx\,dv-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx$$

for some arbitrary convex function β, under a mass constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,dx\,dv=M$. Notice that as soon as $\omega\neq 0$, $\mathcal{F}$ is not bounded from below anymore. A typical example of a function β, corresponding to the *polytropic gas model*, is $\beta(f)=\frac{1}{q}\,\kappa_q^{q-1}\,f^q$ for some $q\in(9/7,\infty)$ and some positive constant κ_q. Any relative equilibrium takes the form $f(x,v)=\gamma\big(\lambda+\frac{1}{2}\,|v|^2+\phi(x)-\frac{1}{2}\,\omega^2\,|x'|^2\big)$ where $\gamma(s)=\kappa_q^{-1}\,(-s)_+^{1/(q-1)}$ and λ is constant on each component of the support of f. The problem is now reduced to solve a nonlinear Poisson equation, namely

$$\Delta\phi=g\big(\lambda+\phi(x)-\tfrac{1}{2}\,\omega^2\,|x'|^2\big)\quad\text{if } x\in\operatorname{supp}(\rho)$$

and $\Delta\phi=0$ otherwise, with $g(\mu)=(-\mu)_+^p$ and $p=\frac{1}{q-1}+\frac{3}{2}$, if κ_q is appropriately chosen. Assuming that the solution has N disjoint connected components K_i, denoting by λ_i the value of λ on K_i and by χ_i the characteristic function of K_i, we end up looking for a positive solution $u=-\phi$ of

$$-\Delta u=\sum_{i=1}^N\rho_i^\omega\quad\text{in } \mathbb{R}^3\;,\quad \rho_i^\omega=\big(u-\lambda_i+\tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^p\,\chi_i$$

under the asymptotic boundary condition $\lim_{|x|\to\infty}u(x)=0$. We define the mass and the center of mass associated to each component by $m_i=\int_{\mathbb{R}^3}\rho_i^\omega\,dx$ and $\xi_i^\omega=\frac{1}{m_i}\int_{\mathbb{R}^3}x\,\rho_i^\omega\,dx$ respectively. The main result in [1] goes as follows.

Theorem 1. *Let* $N \geq 2$ *and* $p \in (3/2, 3) \cup (3, 5)$. *For almost any masses* m_i, $i = 1, \ldots N$, *and for any sufficiently small* $\omega > 0$, *there exist at least* $[2^{N-1}(N-2)+1]\,(N-2)\,!$ distinct *stationary solutions* f_ω *of* (2) *which are such that*

$$\int_{\mathbb{R}^3} f_\omega \, dv = \sum_{i=1}^{N} \rho_i^\omega + o(1)$$

where $o(1)$ *means that the remainder term uniformly converges to* 0 *as* $\omega \to 0_+$ *and identically vanishes away from* $\cup_{i=1}^{N} B_R(\xi_i^\omega)$, *for some* $R > 0$, *independent of* ω.

With the above notations, for all $i = 1, \ldots N$, *we have that*

$$\rho_i^\omega(x - \xi_i^\omega) = \lambda_i^p \, \rho_*\big(\lambda_i^{(p-1)/2}\, x\big) + o(1)$$

where ρ_* *is non-negative, radially symmetric, non-increasing, compactly supported function, depending only on* p, *and* λ_i *is such that* $m_i = \lambda_i^{(3-p)/2} \int_{\mathbb{R}^3} \rho_* \, dx + o(1)$.

The points ξ_i^ω *are such that* $\xi_i^\omega = \omega^{-2/3}\,(\zeta_i^\omega, 0)$ *where, for any* $i = 1, \ldots N$, $\zeta_i^\omega \in \mathbb{R}^2$ *converges as* $\omega \to 0$ *to a critical point of*

$$\mathcal{V}(\zeta_1, \ldots \zeta_N) = \frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i \, m_j}{|\zeta_i - \zeta_j|} + \frac{1}{2} \sum_{i=1}^{N} m_i \, |\zeta_i|^2 \, .$$

This theorem relies on a classification of relative equilibria for the N-body problems which has been established mostly by J.I. Palmore. See [1, Theorem 4] for a summary of these results. Here *distinct* solutions means that one solution cannot be deduced from another one by a simple scaling or by a rotation. The strategy is to find critical points of

$$\mathsf{J}\,[u] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx - \frac{1}{p+1} \sum_{i=1}^{N} \int_{\mathbb{R}^3} \big(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p+1} \chi_i \, dx \, ,$$

by using the solution of

$$-\Delta w_* = (w_* - 1)_+^p =: \rho_* \quad \text{in } \mathbb{R}^3$$

as "building brick" on each of the connected components. With $W_\xi := \sum_{i=1}^{N} w_i$, $w_i(x) = \lambda_i \, w_*\big(\lambda_i^{(p-1)/2}\,(x - \xi_i)\big)$ and $\xi = (\xi_1, \ldots \xi_N)$, we want to solve the problem

$$\Delta\phi + \sum_{i=1}^{N} p\big(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p-1} \chi_i \, \phi = -\mathsf{E} - \mathsf{N}[\phi]$$

with $\lim_{|x|\to\infty} \phi(x) = 0$, where $\mathsf{E} = \Delta W_\xi + \sum_{i=1}^{N} \big(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^p \chi_i$ and $\mathsf{N}[\phi]$ is a nonlinear correction. A lengthy computation shows that

$$\mathsf{J}\,[W_\xi] = \sum_{i=1}^{N} \lambda_i^{(5-p)/2} \, \mathsf{e}_* - \omega^{2/3}\,\mathcal{V}(\zeta_1, \ldots \zeta_N) + O(\omega^{4/3})$$

where $\mathsf{e}_* = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} (w-1)_+^{p+1} \, dx$ and $\zeta_i = \omega^{2/3}\,\xi_i'$ if the points ξ_i are such that, for a large, fixed $\mu > 0$, and all small $\omega > 0$, we have $|\xi_i| < \mu\,\omega^{-2/3}$

and $|\xi_i - \xi_j| > \mu^{-1}\,\omega^{-2/3}$. To localize each K_i in a neighborhood of ξ_i, we impose the orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\,\partial_{x_j} w_i\,\chi_i\,dx = 0 \quad \forall\,i = 1,\,2\ldots N,\ j = 1,\,2,\,3\,, \tag{3}$$

to the price of Lagrange multipliers. Fixed point methods allow to find a constrained solution ϕ. Since $\xi \mapsto \mathsf{J}\,[W_\xi]$ is a finite dimensional function, if $\xi_i = (\zeta_i, 0)$ is such that $(\zeta_1, \ldots \zeta_N)$ is in a neighborhood of a non-degenerate critical point of $\mathcal{V}$, we can find a critical point ϕ for which the Lagrange multipliers associated to (3) are all equal to zero. This completes the scheme of the proof, up to a last technicality. All above computations have been done in terms of fixed Lagrange multipliers (corresponding to the mass constraints associated to each K_i). These constraints still need to be inverted (in order to fix the masses), thus introducing an additional restriction, namely $p \neq 3$.

In this approach, relative equilibria have been obtained in an asymptotic regime in which each component of the distribution function behaves like a minimizer of the free energy when $\omega = 0$, slightly perturbed by the other components, and can be seen at large scale like *pseudo-particles.* These pseudo-particles are located close to the relative equilibria of the N-body problem which are obtained when the centrifugal force in the rotating frame equilibrates the force of gravitation. In the rotating frame, the centrifugal force gives rise to an harmonic potential in the variable x', with negative sign, which competes with the nonlinearity. The nonlinearity indeed tends to aggregate the mass into spherically symmetric functions.

Such *symmetry breaking* phenomena due to rotation effects have been investigated in [2] in the so-called *flat* case, which is slightly simpler (no x_3 variable) to the price of a nonlocal interaction. In such a case, a different branch of solutions has been considered, which originates from the radial solution corresponding to $\omega = 0$ and gets deformed as $|\omega|$ increases. These solutions can be defined as minimizers, provided their support is restricted to a well chosen ball. It is probably not very difficult to find similar solutions in the full three-dimensional setting, although they will certainly be harder to compute numerically. It would then be of interest to understand if such solutions can co-exists with the ones found in Theorem 1 and to extend them as ω increases as a branch of solutions depending on ω. If solutions co-exist, and after restricting the support of the solutions to a large but finite ball, comparing their energy would definitely provide a new insight into the physics of gravitating systems. This is also a very nice problem of symmetry breaking, for which almost nothing is known in case of a nonlocal nonlinearity such as the one corresponding to the Newtonian potential found by solving the attractive Poisson equation.

References

[1] J. Campos, M. del Pino, and J. Dolbeault. Relative equilibria in continuous stellar dynamics. *Communications in Mathematical Physics*, 300:765–788, 2010. 10.1007/s00220-010-1128-2.

[2] J. Dolbeault and J. Fernández. Localized minimizers of flat rotating gravitational systems. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 25(6):1043–1071, 2008.