

# MODULI SPACE THEORY FOR THE ALLEN-CAHN EQUATION IN THE PLANE

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## 1. INTRODUCTION

In this paper, we are interested in the space of solutions which are defined in the plane (i.e. entire solutions) of some class of semilinear elliptic equations and whose prototype is the Allen-Cahn equation

$$(1.1) \quad \Delta u + u - u^3 = 0.$$

An entire solution of (1.1), which is monotone in one direction, is known to depend only on one variable and, after a suitable rigid motion, it has the form

$$u_0(x, y) := \tanh\left(\frac{x}{\sqrt{2}}\right).$$

This result, originally conjectured by de Giorgi, was proven in [9] by Ghoussoub and Gui. In particular, the nodal set of a solution which is monotone in one direction is simply given by a straight line.

In [8], the authors of the present paper, together with J. Wei, have constructed new examples of entire solutions of (1.1). Let us recall this result in more details since it serves as a motivation for the present paper. For any  $k \geq 2$ , it is proven in [8] that there exist solutions of (1.1) whose nodal sets are, away from a compact, asymptotic at infinity to  $2k$ -oriented affine half lines. Moreover, it follows from the construction in [8] that these solutions are not isolated and in fact that they depend on a finite number of continuous parameters. Beside these, other entire solutions of (1.1) are known to exist. For example, solutions with dihedral symmetry have been constructed in [7] (see also [2], [10] and [5]).

All the solutions which have been constructed so far share the same structure at infinity, namely, their nodal sets is, away from a compact, asymptotic to a finite number of oriented half affine lines. Moreover, they tend to either  $\pm 1$  at infinity, away from a neighborhood of the nodal set where they are asymptotic to suitable translated and rotated copies of  $\pm u_0$ . In the present paper, we analyze the structure of the space of all entire solutions of (1.1) which, at infinity, are asymptotic to  $2k$  copies of  $\pm u_0$  (modulo the action of some rigid motion). We show that, under some nondegeneracy assumption, this space is a smooth manifold whose dimension is equal to  $2k$ , in agreement with the number of degrees of freedom which are available in the construction proposed in [8]. We also prove that the solutions constructed in [8] are smooth points in this space.

**1.1. The heteroclinic solution.** We start by briefly reviewing the material needed for our analysis and for a precise statement of our results. We assume that we are given a *double well* potential  $F$ . Namely, a function  $F$  which is even, at least of

class  $\mathcal{C}^2$ , is positive and which has only two zeros at the points  $\pm 1$ . We further assume that the zeros of  $F$  are nondegenerate minima. Hence,

$$(1.2) \quad F(\pm 1) = 0, \quad F''(\pm 1) \neq 0 \quad \text{and} \quad F(t) > 0 \quad \text{for all} \quad t \neq \pm 1.$$

It will be convenient to define

$$\alpha := \sqrt{F''(\pm 1)}.$$

The unique solution of

$$(1.3) \quad \ddot{u}_0 - F'(u_0) = 0,$$

which tends to  $-1$  as  $x$  tends to  $-\infty$ , tends to  $+1$  as  $x$  tends to  $+\infty$ , and which is equal to 0 when  $x = 0$ , will be called the *heteroclinic solution* (here  $\cdot$  denotes differentiation with respect to the variable  $x$ ). It is given implicitly by the identity

$$x = \int_0^{u_0(x)} \frac{ds}{\sqrt{2F(s)}},$$

which follows from the fact that

$$(1.4) \quad \dot{u}_0^2 - 2F(u_0) = 0.$$

Observe that this last equality also implies that the function  $u_0$  is strictly increasing.

**Example 1.1.** *A typical (or classical) example is the one where the function  $F$  is given by*

$$(1.5) \quad F(t) = \frac{1}{4}(1 - t^2)^2,$$

in which case we have explicitly

$$u_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right).$$

**Remark 1.1.** *For the sake of simplicity, we have assumed that the function  $F$  is even. However, most of the results and technics of the present paper do not require this assumption but, since notations become quite involved in the general case, we have chosen to restrict our attention to the case where the potential  $F$  is an even function.*

We collect some basic information about the spectrum of the operator

$$L := -\partial_x^2 + F''(u_0),$$

which arises as the linearized operator of (1.3) about  $u_0$  and which is acting on functions defined on  $\mathbb{R}$ . All the informations we need are included in the :

**Lemma 1.1.** *The spectrum of the operator  $L$  is the union of a finite or possibly infinite number of eigenvalues  $(\mu_j)_{j \geq 0}$*

$$\mu_0 = 0 < \mu_1 < \dots < \mu_n < \dots < \alpha^2 := F''(\pm 1),$$

*and the continuous spectrum which is given by  $[\alpha^2, \infty)$ .*

*Proof.* The fact that the continuous spectrum is equal to  $[\alpha^2, \infty)$  is standard. The fact that the bottom eigenvalue is 0 follows directly from the fact that the equation for  $u_0$  is autonomous and hence the function  $\partial_x u_0$  is in the  $L^2$ -kernel of  $L$ . Since this function is positive, it has to be the eigenfunction associated to the lowest eigenvalue of  $L$ . Finally, the operator  $L$  is of limit-point type the sequence of eigenvalues is increasing (we refer to [6], [3] or [22] for more details).  $\square$

**Remark 1.2.** *The number of eigenvalues of  $L$  included in  $[0, \alpha^2)$  might be finite or infinite. For notational convenience, in the case where the sequence of the eigenvalues of the operator  $L$  included in  $(-\infty, \alpha^2)$  is finite equal to  $\mu_0 < \mu_1 \dots < \mu_n$ , we agree that  $\mu_j := \alpha^2$ , for all  $j \geq n + 1$ . As such, the sequence  $(\mu_j)_{j \geq 0}$  is defined for all  $j \geq 0$  and, if we define the Rayleigh quotient*

$$R(v) := \frac{\int_{\mathbb{R}} (|\partial_x v|^2 + F''(u_0) v^2) dx}{\int_{\mathbb{R}} v^2 dx},$$

*the values of the sequence  $(\mu_j)_{j \geq 0}$  coincide with the values of*

$$\sup_{\dim E=i} \inf_{v \in H^1(\mathbb{R})} \{R(v) : v \perp_{L^2} w, \quad \forall w \in E\}.$$

*as  $i \geq 0$  varies.*

**Example 1.2.** *In the case where the nonlinearity is given by (1.5), the spectrum of  $L$  is explicitly known. The eigenvalues are given by*

$$\mu_0 = 0, \quad \text{with associated eigenfunction} \quad w_0(x) = \frac{1}{\cosh^2(\frac{x}{\sqrt{2}})}$$

*and*

$$\mu_1 = \frac{3}{2}, \quad \text{with associated eigenfunction} \quad w_1(x) = \frac{\sinh(\frac{x}{\sqrt{2}})}{\cosh^2(\frac{x}{\sqrt{2}})},$$

*while the bottom of the continuous spectrum is  $\alpha^2 = 2$ . For a proof of this fact, we refer to [19].*

We will denote by  $\Pi_j$  the  $L^2(\mathbb{R})$ -orthogonal projection over the  $j$ -th eigenspace of  $L$ . Since the eigenspace associated to the eigenvalue 0 is spanned by the function  $\partial_x u_0$ , we have the explicit formula

$$\Pi_0(w) = \frac{1}{\|\partial_x u_0\|_{L^2}^2} \left( \int_{\mathbb{R}} \partial_x u_0 w dx \right) \partial_x u_0.$$

## 2. STATEMENT OF THE RESULT

In this paper, we are interested in the space of solutions of the equation

$$(2.1) \quad \Delta u - F'(u) = 0,$$

which are defined in  $\mathbb{R}^2$ . For example, given  $r \in \mathbb{R}$  and a unit vector  $\mathbf{e} \in S^1$ , we can define

$$u(\mathbf{x}) := u_0(\mathbf{x} \cdot \mathbf{e}^\perp - r),$$

where  $\perp$  denotes the rotation of angle  $\pi/2$  in  $\mathbb{R}^2$ . Clearly  $u$  is a solution of (2.1) and this reflects the invariance of (2.1) under the action of the group of rigid motions of  $\mathbb{R}^2$ . We will say that  $\pm u$  are the *model solutions* whose nodal set is the affine line  $s \mapsto r \mathbf{e}^\perp + s \mathbf{e}$ .

As already mentioned in the introduction, the class of solutions we are interested in are solutions of (2.1) which are defined in the entire space and whose nodal set is, away from some compact, the union of  $2k$  curves which are asymptotic to  $2k$ -oriented half affine lines. Moreover, as one goes to infinity along one of these half affine lines, the solution converges to one of the two model solutions with this affine

line as nodal set. We will make this definition quantitatively precise in the next paragraph. In any case, the set of all such solutions will be denoted by  $\mathcal{M}_{2k}$  and the main result of this paper asserts that, under some nondegeneracy assumption, the set  $\mathcal{M}_{2k}$  is a smooth manifold of dimension  $2k$ .

**2.1. Geometric description of the solutions.** As promised, we give a precise description of the solutions we are interested in. This requires some preliminary definitions. At the heart of the description of the nodal set of the solutions is the set  $\Lambda$  of *oriented affine lines* in  $\mathbb{R}^2$ . Any element  $\lambda \in \Lambda$  can be uniquely written as

$$\lambda := r \mathbf{e}^\perp + \mathbb{R} \mathbf{e},$$

for some  $r \in \mathbb{R}$  and some unit vector  $\mathbf{e} \in S^1$ , which defines the orientation of the line. Recall that we denote by  $\perp$  the rotation of angle  $\pi/2$  in  $\mathbb{R}^2$ . Clearly,  $\Lambda$  is diffeomorphic to  $\mathbb{R} \times S^1$  and writing  $\mathbf{e} = (\cos \theta, \sin \theta)$ , we get local coordinates  $(r, \theta)$  in  $\Lambda$ . Observe that the affine lines are oriented and hence we do not identify the line corresponding to  $(r, \theta)$  and the line corresponding to  $(-r, -\theta)$ . There is also a natural symplectic structure on  $\Lambda$  which, in these local coordinates, is given by

$$\omega := dr \wedge d\theta.$$

Note that the map  $J$  defined by

$$J \partial_\theta = -\partial_r \quad \text{and} \quad J \partial_r = \partial_\theta,$$

(which corresponds to the rotation by  $\pi/2$  in the tangent space) induces an almost complex structure on  $\Lambda$ . This map, together with the 2-form  $\omega$  induces the natural metric on  $\Lambda$

$$g = dr^2 + d\theta^2.$$

More generally, for all  $k' \geq 1$ , let us denote by  $\Lambda^{k'}$  the set of  $k'$ -tuples of oriented affine lines in  $\mathbb{R}^2$ . This set is clearly diffeomorphic to  $\mathbb{R}^{k'} \times (S^1)^{k'}$  and, again, there exists a natural symplectic structure on  $\Lambda^{k'}$  which, in local coordinates  $(r_1, \dots, r_{k'}, \theta_1, \dots, \theta_{k'})$ , can be written as

$$(2.2) \quad \omega_{k'} := dr_1 \wedge d\theta_1 + \dots + dr_{k'} \wedge d\theta_{k'}.$$

The almost complex structure and the metric on  $\Lambda^{k'}$  can be introduced in the same way this was done for  $\Lambda$ .

We have the definition :

**Definition 2.1.** A  $k'$ -tuple of oriented affine lines  $\lambda = (\lambda_1, \dots, \lambda_{k'}) \in \Lambda^{k'}$  is said to be ordered if each  $\lambda_j$  can be written as

$$(2.3) \quad \lambda_j := r_j \mathbf{e}_j^\perp + \mathbb{R} \mathbf{e}_j,$$

for some  $r_j \in \mathbb{R}$  and some unit vector  $\mathbf{e}_j \in S^1$  which can be written as  $\mathbf{e}_j = (\cos \theta_j, \sin \theta_j)$  with

$$\theta_1 < \theta_2 < \dots < \theta_{k'} < 2\pi + \theta_1.$$

We will denote by  $\Lambda_{ord}^{k'}$  the set of  $k'$ -tuples of ordered, oriented affine lines and we will denote by

$$\theta_\lambda := \frac{1}{2} \min\{\theta_2 - \theta_1, \dots, \theta_{k'} - \theta_{k'-1}, 2\pi + \theta_1 - \theta_{k'}\},$$

the half of the minimum of the angles between two consecutive oriented affine lines  $\lambda_1, \dots, \lambda_{k'}$ .

Assume that we are given a  $k'$ -tuple of oriented affine lines  $\lambda = (\lambda_1, \dots, \lambda_{k'})$  as in (2.3). It is easy to check that for all  $R > 0$  large enough and for all  $j = 1, \dots, k'$ , there exists  $s_j \in \mathbb{R}$  such that :

(i) The point  $\mathbf{x}_j := r_j \mathbf{e}_j^\perp + s_j \mathbf{e}_j$  belongs to the circle  $\partial B_R$ , with  $R > 0$ .

(ii) The half lines

$$(2.4) \quad \lambda_j^+ := \mathbf{x}_j + \mathbb{R}^+ \mathbf{e}_j,$$

are disjoint and included in  $\mathbb{R}^2 \setminus B_R$ .

(iii) The infimum of the distance between two distinct half lines  $\lambda_i^+$  and  $\lambda_j^+$  is larger than 4.

The set of half affine lines  $\lambda_1^+, \dots, \lambda_{k'}^+$  together with the circle  $\partial B_R$  induce a decomposition of  $\mathbb{R}^2$  into  $k' + 1$  slightly overlapping connected components

$$\mathbb{R}^2 = \Omega_0 \cup \Omega_1 \cup \dots \cup \Omega_{k'},$$

where

$$\Omega_0 := B_{R+1},$$

and, for  $j = 1, \dots, k'$ ,

$$(2.5) \quad \Omega_j := \{ \mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| > R - 1 \text{ and } \text{dist}(\mathbf{x}, \lambda_j^+) < \text{dist}(\mathbf{x}, \lambda_i^+) + 1, \forall i \neq j \},$$

where  $\text{dist}(\mathbf{x}, \lambda_j)$  denotes the distance to  $\lambda_j^+$ . Observe that  $\Omega_j$  contains  $\lambda_j^+$  when  $j > 0$ .

We define  $\mathbb{I}_0, \mathbb{I}_1, \dots, \mathbb{I}_{k'}$  to be a smooth partition of unity of  $\mathbb{R}^2$  which is subordinate to the above decomposition of  $\mathbb{R}^2$ . Hence

$$\sum_{j=0}^{k'} \mathbb{I}_j \equiv 1,$$

and the support of  $\mathbb{I}_j$  is included in  $\Omega_j$ , for  $j = 0, \dots, k'$ . Without loss of generality, we can also assume that  $\mathbb{I}_0 \equiv 1$  in

$$\Omega'_0 := B_{R-1},$$

and  $\mathbb{I}_j \equiv 1$  in

$$\Omega'_j := \{ \mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| > R + 1 \text{ and } \text{dist}(\mathbf{x}, \lambda_j^+) < \text{dist}(\mathbf{x}, \lambda_i^+) - 1, \forall i \neq j \},$$

for  $j \neq 0$ . Finally, we assume that

$$\|\mathbb{I}_j\|_{C^2(\mathbb{R}^2)} \leq C.$$

We now take  $k' = 2k$ , for some  $k \geq 1$  and

$$\lambda = (\lambda_1, \dots, \lambda_{2k}) \in \Lambda_{ord}^{2k},$$

we write  $\lambda_j^\pm = \mathbf{x}_j + \mathbb{R}^\pm \mathbf{e}_j$  and we define

$$(2.6) \quad u_\lambda := \sum_{j=1}^{2k} (-1)^j \mathbb{I}_j u_0(\text{dist}^s(\cdot, \lambda_j)),$$

where

$$\text{dist}^s(\mathbf{x}, \lambda_j) := (\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{e}_j^\perp,$$

denotes the signed distance of  $\mathbf{x} \in \mathbb{R}^2$  to  $\lambda_j$ .

Observe that, by construction, the function  $u_\lambda$  is, away from a compact, asymptotic to copies of the model solutions  $\pm u_0$ , whose nodal set are the half affine lines  $\lambda_1^+, \dots, \lambda_{2k}^+$ . A simple computation shows that  $u_\lambda$  is not far from being a solution of (2.1) in the sense that  $\Delta u_\lambda - F'(u_\lambda)$  is a function which decays exponentially to 0 at infinity.

We are interested in solutions of (2.1) which are asymptotic to  $u_\lambda$  for some choice of  $\lambda \in \Lambda_{ord}^{2k}$ . More precisely, we have the :

**Definition 2.2.** Let  $\mathcal{M}_{2k}$  denote the set of solutions  $u$  of (2.1) which satisfy

$$(2.7) \quad u - u_\lambda \in W^{2,2}(\mathbb{R}^2),$$

for some  $\lambda \in \Lambda_{ord}^{2k}$ . We also define the decomposition operator  $\mathcal{J}$  by

$$\mathcal{J}(u) := (u - u_\lambda, \lambda) \in W^{2,2}(\mathbb{R}^2) \times \Lambda.$$

The topology on  $\mathcal{M}_{2k}$  is the one for which the operator  $\mathcal{J}$  is continuous.

The existence of the heteroclinic solution  $u_0$  shows that  $\mathcal{M}_2$  is non empty. Indeed, it is enough to consider

$$u(\mathbf{x}) = u_0(\mathbf{x} \cdot \mathbf{e}^\perp - r),$$

for any unit vector field  $\mathbf{e}$  and any  $r \in \mathbb{R}$ . As we already discussed in the introduction, the result of [8] provides solutions of (2.1) whose nodal set decomposes into  $k$  nearly parallel lines, very far from each other, this result together with the results in [7] and [2], imply that  $\mathcal{M}_{2k} \neq \emptyset$  for any  $k \geq 1$ . It is then natural to investigate the structure of  $\mathcal{M}_{2k}$ . The questions which are of interest are the following :

- (1) Among the functions  $u$  which can be written as

$$u = u_\lambda + v,$$

for some  $\lambda \in \Lambda_{ord}^{2k}$  and some  $v \in W^{2,2}(\mathbb{R}^2)$ , what are the solutions of (2.1) ? In particular, what is the dimension of the space of such solutions ?

- (2) Obviously, there exists a natural map

$$\mathcal{P} : \mathcal{M}_{2k} \longrightarrow \Lambda^{2k},$$

defined by

$$(2.8) \quad \mathcal{P}(u) := \lambda,$$

if  $u - u_\lambda \in W^{2,2}(\mathbb{R}^2)$ . What can be said about this map ?

**2.2. The results.** We keep the notations introduced above. Given  $k \geq 1$  and

$$\lambda = (\lambda_1, \dots, \lambda_{2k}) \in \Lambda_{ord}^{2k},$$

we write  $\lambda_j^\pm = \mathbf{x}_j + \mathbb{R}^\pm \mathbf{e}_j$  as in (2.4). We denote by  $\Omega_0, \dots, \Omega_{2k}$  the decomposition of  $\mathbb{R}^2$  associated to this  $2k$ -tuple of half affine lines and  $\mathbb{I}_0, \dots, \mathbb{I}_{2k}$  the partition of unity subordinate to this partition. Given  $\gamma, \delta \in \mathbb{R}$ , we define a weight  $\Gamma_{\gamma, \delta}$  such that

$$\Gamma_{\gamma, \delta}(\mathbf{x}) \sim 1,$$

in  $\Omega_0$  and, for  $j = 1, \dots, 2k$ ,

$$\Gamma_{\gamma, \delta}(\mathbf{x}) \sim e^{\gamma s} (\cosh r)^\delta.$$

in  $\Omega_j$ , where we write

$$\mathbf{x} = \mathbf{x}_j + r \mathbf{e}_j^\perp + s \mathbf{e}_j,$$

for some  $r \in \mathbb{R}$  and  $s > 0$  (observe that  $(r, s)$  are local coordinates which are well defined in each  $\Omega_j$ ). As usual, the notation  $f \sim g$  means that there exists some constant  $C > 1$  such that  $\frac{1}{C} |g| \leq |f| \leq C |g|$ . The explicit definition of the weight function  $\Gamma_{\gamma, \delta}$  is given by

$$(2.9) \quad \Gamma_{\gamma, \delta}(\mathbf{x}) := \mathbb{I}_0(\mathbf{x}) + \sum_{j=1}^{2k} \mathbb{I}_j(\mathbf{x}) e^{\gamma(\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{e}_j} \left( \cosh((\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{e}_j^\perp) \right)^\delta,$$

so that, by construction,  $\gamma$  is the rate of decay or blow up along the half lines  $\lambda_j^+$  and  $\delta$  is the rate of decay or blow up in the direction orthogonal to  $\lambda_j^+$ .

With this definition in mind, we define the weighted Lebesgue space

$$(2.10) \quad L_{\gamma, \delta}^2(\mathbb{R}^2) := \Gamma_{\gamma, \delta} L^2(\mathbb{R}^2),$$

and the weighted Sobolev space

$$(2.11) \quad W_{\gamma, \delta}^{2,2}(\mathbb{R}^2) := \Gamma_{\gamma, \delta} W^{2,2}(\mathbb{R}^2).$$

Observe that, even though this does not appear in the notations, the partition of unity, the weight and the induced weighted spaces all depend on the choice of  $\lambda \in \Lambda_{ord}^{2k}$ .

Our first result shows that, if  $u$  is a solution of (2.1) which is close to some  $u_\lambda$  (in  $W^{2,2}$  topology) then  $u - u_\lambda$  is exponentially decaying at infinity.

**Theorem 2.1** (Refined Asymptotics). *Assume that  $u$  is a solution of (2.1) and that  $\lambda \in \Lambda_{ord}^{2k}$  is chosen so that*

$$u - u_\lambda \in W^{2,2}(\mathbb{R}^2).$$

*Then, there exists  $\delta \in (0, \alpha)$  and  $\gamma > 0$  such that*

$$(2.12) \quad u - u_\lambda \in W_{-\gamma, -\delta}^{2,2}(\mathbb{R}^2).$$

*More precisely,  $\delta > 0$  and  $\gamma > 0$  can be chosen so that*

$$(2.13) \quad \gamma \in (0, \sqrt{\mu_1}), \quad \gamma^2 + \delta^2 < \alpha^2 \quad \text{and} \quad \alpha > \delta + \gamma \cot \theta_\lambda,$$

*where  $2\theta_\lambda$  is equal to the infimum of the angles between two consecutive oriented affine lines  $\lambda_1, \dots, \lambda_{2k}$  (see Definition 2.1).*

*In addition, there exists a neighborhood  $\mathcal{V}$  of  $u$  in  $\mathcal{M}_{2k}$  such that*

$$\mathcal{V} \ni \bar{u} \mapsto \mathcal{J}\bar{u} \in W_{-\gamma, -\delta}^{2,2}(\mathbb{R}^2) \times \Lambda,$$

*is continuous.*

Before, we can state the next result, we have to introduce the notion of *nondegeneracy* in this context.

**Definition 2.3.** *A function  $u \in \mathcal{M}_{2k}$  is said to be nondegenerate if the linearized operator*

$$-\Delta + F''(u),$$

*is injective in the space  $L_{-\gamma, \delta}^2(\mathbb{R}^2)$ , for some  $\gamma \in (0, \sqrt{\mu_1})$  and some  $\delta \in \mathbb{R}$  satisfying*

$$\gamma^2 + \delta^2 < \alpha^2.$$

As already mentioned, the existence of a family of solutions of (2.1) which belongs to  $\mathcal{M}_{2k}$  is guaranteed by the result in [8], for the equation

$$(2.14) \quad \Delta u + u - u^3 = 0.$$

We will prove in §9 that the solutions obtained in [8] are nondegenerate, this will imply that :

**Proposition 2.1.** *In the case where the nonlinearity is given by  $F(u) = \frac{1}{4}(1-u^2)^2$  (and the equation is given by (2.14)), for each  $k \geq 1$ ,  $\mathcal{M}_{2k}$  contains nondegenerate elements.*

For the sake of simplicity, when proving Proposition 2.1, we will restrict our attention to the special nonlinearity  $F(u) = \frac{1}{4}(1-u^2)^2$ , since the existence of solutions is proven for this special nonlinearity in (2.14). The method used in [8] extends in a straightforward manner to nonlinear equations of the type (2.1) and hence, it can be shown that for general nonlinearities,  $\mathcal{M}_{2k}$  also contains nondegenerate elements.

Checking whether a given solution is nondegenerate or not is a hard problem. For example, in [7], is built for the nonlinearity (1.5) a solution in  $\mathcal{M}_4$ , whose nodal set is the union of two perpendicular lines. The proof of the fact that this solution is nondegenerate is proven in [13].

The second result of this paper is the following :

**Theorem 2.2** (Dimension of the moduli space). *Assume that  $u \in \mathcal{M}_{2k}$  is nondegenerate. Then, close to  $u$ ,  $\mathcal{M}_{2k}$  is a smooth manifold of dimension  $2k$ .*

Near any nondegenerate elements of  $\mathcal{M}_{2k}$ , we also have some information about the mapping  $\mathcal{P}$ .

**Theorem 2.3.** *Assume that  $u \in \mathcal{M}_{2k}$  is nondegenerate. Then, there exists an open neighborhood of  $u$  in  $\mathcal{M}_{2k}$  whose image by  $\mathcal{P}$  is a Lagrangian submanifold of  $\Lambda^{2k}$  for the symplectic structure defined in (2.2).*

Geometrically the meaning of the mapping  $\mathcal{P}$  should be clear : Given any solution  $u \in \mathcal{M}_{2k}$ ,  $\mathcal{P}(u) \in \Lambda^{2k}$  corresponds to the choice of  $2k$  oriented affine lines which determine the asymptotics of the nodal set of  $u$  at infinity. Theorem 2.3 shows that there is in reality less *freedom* than what might be initially expected in selecting the asymptotes for the nodal sets of the solutions of (2.1). Indeed, at regular points of  $\mathcal{M}_{2k}$ , the image of  $\mathcal{P}$  is a  $2k$ -dimensional submanifold of  $\Lambda$  which is  $4k$ -dimensional.

Observe that the image of  $\mathcal{P}$  is naturally constrained. In fact, if  $u \in \mathcal{M}_{2k}$  and

$$\mathcal{P}(u) = (\lambda_1, \dots, \lambda_{2k}),$$

with

$$\lambda_j = r_j \mathbf{e}_j^\perp + \mathbb{R} \mathbf{e}_j,$$

it follows from [10] that

$$(2.15) \quad \sum_{j=1}^{2k} \mathbf{e}_j = 0.$$

Moreover, pushing further the analysis in [10], we can also prove that

$$(2.16) \quad \sum_{j=1}^{2k} r_j = 0.$$

We will refer to these equalities as the *balancing formulæ* and, for the sake of completeness, we will give a simple proof of these equalities in the Appendix A. Observe that (2.15) implies that the angle between two consecutive half lines is always less than or equal to  $\pi$  and that it can only be equal to  $\pi$  when  $k = 1$ . Therefore, if  $u \in \mathcal{M}_{2k}$  and  $\lambda = \mathcal{P}(u)$ , we always have

$$0 \leq \theta_\lambda \leq \pi/2.$$

The proofs of our results follow from the implicit function theorem in a suitably designed weighted function space. The results and the arguments are very much in the spirit of what has already been done in the study of the moduli spaces of complete non compact constant mean curvatures surfaces in Euclidean space or complete non compact constant scalar curvature metrics [14], [17] and [16].

**2.3. Comments and open problems.** The previous results raise some interesting questions and comments. The barrier construction in [7] generalizes to yield, for each  $k \geq 2$ , a solution with dihedral symmetry whose nodal set is given by the union of the oriented half lines

$$\lambda_j^\pm = \mathbb{R}^+ (\cos(j\pi/k), \sin(j\pi/k)),$$

for  $j = 1, \dots, 2k$ . These solutions whose existence is proven in [2], [10] and [5], are referred to as *saddle solutions*. In [13], it is proven that the saddle solution is nondegenerate when  $k = 1$ . Is it true that, for each  $k \geq 2$ , the  $k$ -th saddle solutions are nondegenerate in the sense of Definition 2.3 ?

We have strong evidence that, given  $k \geq 2$ , the solutions with  $k$  nearly parallel nodal lines constructed in [8], and the  $k$ -th saddle solutions belong to the same connected component of  $\mathcal{M}_{2k}$ .

Obviously, (2.1) is invariant under the action of translations and rotations of the plane. When  $k = 2$ , Theorem 2.3 implies that, near any nondegenerate element,  $\mathcal{M}_{2k}$  is 4-dimensional, but rigid motions already yield 3 degrees of freedom, therefore, there is only one free parameter which does not come from the action of rigid motions. This is in agreement with the result in [8], where solutions of (2.1) which are even under both the symmetry with respect to both the  $x$ -axis and the  $y$ -axis are constructed. Is it true that all elements of  $\mathcal{M}_2$  are even with respect to two perpendicular lines ? Let us mention that, when  $k \geq 3$ , solutions without any symmetries have been constructed in [8].

There are no example of two distinct solutions of (2.1) whose nodal sets are asymptotic to the same set of half affine lines at infinity. Is it true that the mapping  $\mathcal{P}$  is injective ? Assuming that any element of  $\mathcal{M}_{2k}$  is nondegenerate, the image of  $\mathcal{M}_{2k}$  by the mapping  $\mathcal{P}$  would then be an embedded Lagrangian submanifold in  $\Lambda^{2k}$ .

Finally, to end this introduction, we briefly describe the plan of the paper. We will start in §3 with the analysis of the mapping properties of the linearized operator about the *heteroclinic solution*. This, together with Fourier analysis, will allow us to derive in §4 the mapping properties of the linearized operator about the *model solution* when defined between appropriate weighted space in both the  $x$  and  $y$  directions. This part builds up on corresponding analysis for some class of elliptic operators defined on manifolds with cylindrical ends as in [15] (see also [21] for a concise exposition of some of these ideas). We will then proceed in §5 and §6 with the analysis of the linearized operator about *any*  $u \in \mathcal{M}_{2k}$ . Next, in §7, we

will prove the Refined Asymptotics theorem. Finally, in §8, we will set up the correct framework in which the implicit function theorem can be applied to prove Theorem 2.3. This part borrows ideas which have been used in [14] and which are by now classical. Section §9, is devoted to the proof of Proposition 2.1 and, in the last section §10, we give a proof of Theorem 2.3.

### 3. THE LINEARIZED OPERATOR ABOUT THE HETEROCLINIC SOLUTION

**3.1. The linearized operator about  $u_0$ .** We consider the heteroclinic solution  $u_0$  which has been defined in §1 and we recall that we have defined the operator

$$L := -\partial_x^2 + F''(u_0).$$

To simplify notations, we set

$$S := \{-\alpha \leq \dots, -\sqrt{\mu_j}, \dots, -\sqrt{\mu_1}, \sqrt{\mu_0} = 0, \sqrt{\mu_1}, \dots, \sqrt{\mu_j}, \dots \leq \alpha\}$$

where the  $(\mu_j)_{j \geq 0}$  are the eigenvalues of  $L$  (which form a finite or infinite sequence) and where we recall that we have defined

$$\alpha := \sqrt{F''(\pm 1)}.$$

Given any complex number  $\zeta := \gamma + i\xi \in \mathbb{C}$ , we define

$$L_\zeta := L - \zeta^2,$$

and we define  $\eta := \delta + i\mu \in \mathbb{C}$ , with  $\delta \geq 0$  by the identity

$$\eta^2 = \alpha^2 - \zeta^2.$$

Observe that we have

$$(3.1) \quad \delta^2 = \alpha^2 - \gamma^2 + \xi^2 + \mu^2 \geq \alpha^2 - \gamma^2.$$

The following result classifies the behavior of the solutions of the homogeneous problem  $L_\zeta W = 0$  at  $\pm\infty$ .

**Lemma 3.1.** *There exist two linearly independent (complex valued) functions  $W_\zeta^\pm$ , which are solutions of  $L_\zeta W_\zeta^\pm = 0$  in  $\mathbb{R}$ , such that :*

(i) *If  $\zeta \neq \pm\alpha$ , then  $W_\zeta^\pm(x) \sim e^{\pm\eta x}$ , as  $x$  tends to  $+\infty$ .*

(ii) *If  $\zeta = \pm\alpha$ , then  $W_\zeta^+(x) \sim 1$  and  $W_\zeta^-(x) \sim x$ , as  $x$  tends to  $+\infty$ .*

*Proof.* We give the proof of the result when  $\zeta \neq \pm\alpha$  the other case can be treated using similar arguments. When  $\zeta \neq \pm\alpha$ , one simply uses the fact that

$$(\partial_x^2 - \alpha^2 + \zeta^2) e^{\pm\eta x} = 0,$$

together with the fact that  $u_0$  tends to 1 exponentially fast at  $+\infty$ . The existence of  $W_\zeta^\pm$  follows at once from a standard perturbation argument for ordinary differential equations.  $\square$

**Definition 3.1.** *The indicial roots of  $L_\zeta$  are defined to be*

$$\pm\delta_\zeta := \pm\Re\sqrt{\alpha^2 - \zeta^2}.$$

Observe that

$$\delta_{\bar{\zeta}} = \delta_\zeta.$$

The following simple observation will be crucial :

**Lemma 3.2.** *The following holds :*

(i) *Assume that  $\gamma \in [-\alpha, \alpha]$ , then*

$$\min_{\xi \in \mathbb{R}} \delta_{\gamma+i\xi} = \sqrt{\alpha^2 - \gamma^2}.$$

(ii) *Assume that  $|\gamma| \geq \alpha$ , then*

$$\min_{\xi \in \mathbb{R}} \delta_{\gamma+i\xi} = 0.$$

*Proof.* This easily follows from (3.1). □

Given  $\delta \in \mathbb{R}$ , we define the weighted space

$$\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C}) := (\cosh x)^\delta L^2(\mathbb{R}, \mathbb{C}).$$

This weighted space is equipped with the natural norm

$$\|W\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})} := \|(\cosh x)^{-\delta} W\|_{L^2(\mathbb{R}, \mathbb{C})}.$$

We consider the unbounded operator

$$\begin{aligned} A_{\zeta, \delta} : \tilde{L}_\delta^2(\mathbb{R}, \mathbb{C}) &\longmapsto \tilde{L}_\delta^2(\mathbb{R}, \mathbb{C}), \\ W &\longmapsto L_\zeta W. \end{aligned}$$

The subscripts  $\zeta, \delta$  keep track of the weights  $\delta$  and the complex number  $\zeta$  which appears in  $A_{\zeta, \delta}$ . It is easy to check that the operator  $A_{\zeta, \delta}$  is an unbounded operator with dense domain equal to

$$\tilde{W}_\delta^{2,2}(\mathbb{R}, \mathbb{C}) := (\cosh x)^\delta W^{2,2}(\mathbb{R}, \mathbb{C}),$$

and also that it has closed graph. The following result is by now well known and follows from example from [15] (see also [21]) :

**Proposition 3.1.** *Assume that the operator  $A_{\zeta, \delta}$  is injective for some  $\delta \in (-\delta_\zeta, \delta_\zeta)$ , then  $A_{\zeta, \delta}$  is an isomorphism for any  $\delta \in (-\delta_\zeta, \delta_\zeta)$ .*

*Proof.* As mentioned, the proof of this result follows from the general theory developed in [15] or also in [21]. Since we are dealing with a second order ordinary differential equation, a direct proof is also available. We give here the two proofs.

*Proof 1.* Using the standard Hermitian product on  $L^2(\mathbb{R}, \mathbb{C})$ , we can identify the dual of  $\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})$  with  $\tilde{L}_{-\delta}^2(\mathbb{R}, \mathbb{C})$ . With this identification, the adjoint of  $A_{\zeta, \delta}$  can be identified with  $A_{\bar{\zeta}, -\delta}$ , since

$$\int_{\mathbb{R}} L_\zeta W \bar{V} dx = \int_{\mathbb{R}} W \overline{L_{\bar{\zeta}} V} dx,$$

for all smooth, complex valued functions  $V$  and  $W$  having compact support. Let us assume the  $\delta$  is not an indicial root of  $L_\zeta$  (namely  $\delta \neq \pm\delta_\zeta$ ). In this case, it follows from [15] or [21] that the operator  $A_{\zeta, \delta}$  is Fredholm and has closed range. Moreover, this operator is injective if and only if  $A_{\bar{\zeta}, -\delta}$  is surjective.

Now, let us assume that  $\delta < \delta_\zeta$ . Using Lemma 3.1, we check that any solution of  $L_\zeta W = 0$  which is bounded by  $(\cosh x)^\delta$  is in fact bounded by  $(\cosh x)^{-\delta_\zeta}$ . Hence, if the operator  $A_{\zeta, \delta}$  is injective for *some*  $\delta \in (-\delta_\zeta, \delta_\zeta)$  then it is injective for *any*  $\delta \in (-\delta_\zeta, \delta_\zeta)$ . Finally, assume that  $\delta \in (-\delta_\zeta, \delta_\zeta)$  and that  $A_{\zeta, \delta}$  is injective. Then,  $A_{\bar{\zeta}, -\delta}$  is also injective and, by the above duality argument,  $A_{\zeta, \delta}$  is also surjective. This completes the proof of the result.

*Proof 2.* We assume that  $\delta_\zeta > 0$  (otherwise there is nothing to prove) and we choose  $\delta \in (-\delta_\zeta, \delta_\zeta)$ . The space of solutions of the homogeneous problem  $L_\zeta W = 0$  is spanned by the two functions  $W_\zeta^\pm$  which have been defined in Lemma 3.1. By construction  $W_\zeta^-(x) \sim e^{-\eta x}$  at  $+\infty$  and hence it is bounded by a constant times  $(\cosh x)^\delta$  at  $+\infty$ . Since we assume that  $A_{\zeta, \delta}$  is injective, then necessarily we also have  $W_\zeta^-(x) \sim e^{-\eta x}$  at  $-\infty$ . Without loss of generality, we can assume that  $W_\zeta^+$  is constructed in such a way that  $W_\zeta^+(x) \sim e^{\eta x}$  at  $\pm\infty$ .

Let us denote by  $\mathbb{W}_\zeta$  the Wronskian of  $W_\zeta^\pm$ . Namely

$$\mathbb{W}_\zeta := W_\zeta^+ \partial_x W_\zeta^- - W_\zeta^- \partial_x W_\zeta^+.$$

Given  $V \in \tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})$ , it is easy to check that

$$W(x) := \frac{1}{\mathbb{W}_\zeta} \left( W_\zeta^-(x) \int_{-\infty}^x W_\zeta^+ V dx + W_\zeta^+(x) \int_x^{+\infty} W_\zeta^- V dx \right),$$

is well defined (this uses the fact that  $\delta < \delta_\zeta$ ) and is a solution of  $L_\zeta W = V$ . Moreover, direct estimates imply that  $W \in \tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})$  and also that

$$\|W\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})} \leq C \|V\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})},$$

for some constant  $C > 0$  independent of  $V$ . Details of the derivation of this estimate are postponed to Appendix B. This completes the proof of the result.  $\square$

The main result of this section is the following :

**Proposition 3.2.** *Assume that  $\gamma \in \mathbb{R} \setminus S$  and  $\delta \in \mathbb{R}$  are chosen so that*

$$\gamma^2 + \delta^2 < \alpha^2.$$

*Then, for all  $\xi \in \mathbb{R}$ , the operator*

$$\begin{aligned} A_{\gamma+i\xi, \delta} : \tilde{L}_\delta^2(\mathbb{R}, \mathbb{C}) &\longrightarrow \tilde{L}_\delta^2(\mathbb{R}, \mathbb{C}), \\ W &\longmapsto L_{\gamma+i\xi} W, \end{aligned}$$

*is an isomorphism and the norm of its inverse is bounded by a constant (depending on  $\gamma$  and  $\delta$ ) which does not depend on  $\xi \in \mathbb{R}$ . Moreover, there exists a constant  $C > 0$  (depending on  $\gamma$  and  $\delta$ ) such that*

$$\|W\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})} \leq \frac{C}{1 + |\xi|^2} \|L_{\gamma+i\xi} W\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})}.$$

*Proof.* We assume that  $\gamma \in \mathbb{R} \setminus S$  and we choose  $\delta' \in (-\delta_\zeta, 0]$ . We argue by contradiction. If  $A_{\zeta, \delta'}$  is not injective, then there exists a nontrivial solution of  $L_\zeta W = 0$  which belongs to  $\tilde{L}_{\delta'}^2(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C})$  and hence  $W$  is an eigenfunction of  $L$ . Indeed

$$LW = \zeta^2 W.$$

But the operator  $L$  being real and self adjoint, we find that necessarily  $\zeta^2 \in \mathbb{R}$ . This implies that  $\gamma \in S$ , which is in contradiction with our hypothesis. Therefore,  $A_{\zeta, \delta'}$  is injective and by the previous Proposition, we conclude that  $A_{\zeta, \delta'}$  is an isomorphism for any  $\delta' \in (-\delta_\zeta, \delta_\zeta)$ .

We assume that  $\gamma \in \mathbb{R} \setminus S$  and  $\delta \in \mathbb{R}$  are chosen so that  $\gamma^2 + \delta^2 < \alpha^2$ . In particular  $\gamma \in (-\alpha, \alpha)$  and it follows from (3.1) and Lemma 3.2 that

$$\delta \in (-\delta_\zeta, \delta_\zeta),$$

for all  $\xi \in \mathbb{R}$ . Therefore, we conclude that  $A_{\zeta, \delta}$  is an isomorphism and this property holds independently of  $\xi \in \mathbb{R}$ . It remains to estimate the norm of the inverse of  $A_{\zeta, \delta}$ .

First observe that  $\gamma$  and  $\delta$  are fixed and hence  $\xi$  is the only parameter allowed to vary. Clearly, the inverse of  $A_{\zeta, \delta}$  is bounded uniformly in  $\xi$  provided  $|\xi|$  stays in some compact subset. Therefore, we just have to estimate the norm of the inverse of  $A_{\zeta, \delta}$  when  $|\xi|$  is large. To handle this, we consider the equation

$$L_{\zeta} W = V,$$

which we integrate against  $(\cosh x)^{-2\delta} \bar{U}$ . Since we are working with complex valued functions, we have to consider the scalar product

$$\langle a, b \rangle = \frac{1}{2} (a\bar{b} + \bar{a}b).$$

After an integration by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}} (|\partial_x W|^2 - \delta \partial_x |W|^2 \tanh x + (F''(u_0) - \gamma^2 + \xi^2) |W|^2) (\cosh x)^{-2\delta} dx \\ = \int_{\mathbb{R}} \langle W, V \rangle (\cosh x)^{-2\delta} dx. \end{aligned}$$

A second integration by parts yields

$$\int_{\mathbb{R}} (|\partial_x W|^2 + (\xi^2 + B) |W|^2) (\cosh x)^{-2\delta} dx = \int_{\mathbb{R}} \langle W, V \rangle (\cosh x)^{-2\delta} dx,$$

where we have defined for short

$$B(x) := F''(u_0(x)) - \gamma^2 + \delta - \delta(2\delta + 1) (\tanh x)^2.$$

Now assume that  $\xi^2 > \|B\|_{L^\infty}$  and use Cauchy-Schwarz inequality to conclude that

$$(\xi^2 - \|B\|_{L^\infty})^2 \int_{\mathbb{R}} |W|^2 (\cosh x)^{-2\delta} dx \leq \int_{\mathbb{R}} |V|^2 (\cosh x)^{-2\delta} dx,$$

which provides the relevant estimate for  $|\xi|$  large enough. This completes the proof of the Proposition.  $\square$

**Remark 3.1.** *Observe that, given  $j \geq 0$ , the assumption  $\gamma \in \mathbb{R} \setminus S$  can be replaced by the assumption  $\gamma \neq \pm\sqrt{\mu_i}$  for all  $i \geq j + 1$ , but then the operator  $A_{\gamma+i\xi, \delta}$  has to be restricted to the closed subspace of  $\tilde{W}_{\delta}^{2,2}(\mathbb{R}, \mathbb{C})$ , which consists of functions  $w$  that satisfy the orthogonality condition*

$$\Pi_i(w) = 0, \quad \text{for } i = 0, \dots, j,$$

where  $\Pi_i$  is the projection onto the  $i$ -th eigenspace of the operator  $L$ .

#### 4. THE LINEARIZED OPERATOR ABOUT THE MODEL SOLUTION

We define

$$\mathcal{L} := -\partial_y^2 + L = -\Delta + F''(u_0).$$

We denote by  $(x, y)$  the coordinates of  $\mathbf{x} \in \mathbb{R}^2$ . Given  $\gamma, \delta \in \mathbb{R}$ , we define the weighted space

$$\tilde{L}_{\gamma, \delta}^2(\mathbb{R}^2) := e^{\gamma y} (\cosh x)^{\delta} L^2(\mathbb{R}^2),$$

and similarly, we define

$$\tilde{W}_{\gamma, \delta}^{2,2}(\mathbb{R}^2) := e^{\gamma y} (\cosh x)^{\delta} W^{2,2}(\mathbb{R}^2).$$

**Remark 4.1.** *From now on all functions spaces considered are spaces of real valued functions.*

These spaces are equipped with the natural norm induced by the weight  $e^{\gamma y} (\cosh x)^\delta$ . For example

$$\|w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)} := \|e^{-\gamma y} (\cosh x)^{-\delta} w\|_{L^2(\mathbb{R}^2)},$$

and so on.

**4.1. Global invertibility.** Given  $\gamma, \delta \in \mathbb{R}$ , we define the operator

$$\begin{aligned} \mathcal{A}_{\gamma,\delta} : \tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2) &\longrightarrow \tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2), \\ w &\longmapsto \mathcal{L} w, \end{aligned}$$

which is an unbounded operator with closed graph and dense domain given by  $\tilde{W}_{\gamma,\delta}^{2,2}(\mathbb{R}^2)$  (this fact follows easily from standard elliptic estimates).

Building on the analysis of the previous section we can prove that  $\mathcal{A}_{\gamma,\delta}$  is an isomorphism (from its domain into its range) provided the weights  $\gamma$  and  $\delta$  are carefully chosen. This is the content of the following proposition which borrows the arguments from [15] :

**Proposition 4.1.** *Assume that  $\gamma \in \mathbb{R} \setminus S$  and  $\delta \in \mathbb{R}$  satisfy*

$$\gamma^2 + \delta^2 < \alpha^2.$$

*Then, the operator  $\mathcal{A}_{\gamma,\delta}$  is an isomorphism.*

*Proof.* The proof is based on Fourier transform in the  $y$  variable. We want to solve the equation  $\mathcal{L} w = v$  when  $v \in \tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)$ . We write

$$w = e^{\gamma y} W \quad \text{and} \quad v = e^{\gamma y} V,$$

so that the equation we need to solve transforms into

$$(L - \gamma^2 - 2\gamma \partial_y - \partial_y^2) W = V.$$

Now, we perform the Fourier decomposition of  $W$  and  $V$  in the  $y$  variable. Let us denote by  $\hat{W}(\cdot, x)$  and  $\hat{V}(\cdot, x)$  the Fourier transforms of  $W(\cdot, x)$  and  $V(\cdot, x)$ . Then, our problem reduces to solving

$$L_\zeta \hat{W}(\cdot, \xi) = \hat{V}(\cdot, \xi),$$

which is a family of equations depending on the parameter  $\zeta = \gamma + i\xi$ . To solve this equation, we use the result of Proposition 3.2 and then take the inverse Fourier transform. Using Plancherel's theorem, we can write

$$\|w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} (\cosh x)^{-2\delta} |W|^2 dx dy = \int_{\mathbb{R}^2} (\cosh x)^{-2\delta} |\hat{W}|^2 dx d\xi.$$

The result of Proposition 3.2 then implies that

$$\|w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)}^2 \leq C \int_{\mathbb{R}^2} (\cosh x)^{-2\delta} |\hat{V}|^2 dx d\xi.$$

Using once more Plancherel's theorem, we conclude that

$$\|w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)}^2 \leq C \int_{\mathbb{R}^2} (\cosh x)^{-2\delta} |V|^2 dx dy = C \|v\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)}^2.$$

This completes the proof of the result.  $\square$

Observe that, as a byproduct, we have the inequality

$$(4.1) \quad \|w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)} \leq C \|\mathcal{L}w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)},$$

for some constant  $C > 0$  only depending on  $\gamma$  and  $\delta$ .

**Remark 4.2.** *As in Remark 3.1, given  $j \geq 0$ , we can modify the hypotheses of the above Proposition assuming only that  $\gamma \neq \pm\sqrt{\mu_i}$ , for all  $i \geq j+1$ , but then we have to consider the operator  $\mathcal{A}_{\delta,\gamma}$  restricted to the closed subspace of  $\tilde{W}_{\gamma,\delta}(\mathbb{R}^2)$  which consists of functions  $w$  satisfying*

$$\Pi_i(w(\cdot, y)) = 0, \quad i = 0, \dots, j,$$

for almost all  $y \in \mathbb{R}$ , where  $\Pi_i$  is the projection onto the  $i$ -th eigenspace of  $L$ .

Let  $G_{\zeta,\delta}$  denote the inverse of  $A_{\zeta,\delta}$  which is provided by Proposition 3.2. It is interesting to observe that the previous proof yields a representation formula for the solution of  $\mathcal{L}w = v$  which can be written formally as the integral over some contour in  $\mathbb{C}$ , namely

$$(4.2) \quad w(x, y) = \frac{1}{2i\pi} \int_{\Re \zeta = \gamma} e^{\zeta y} G_{\zeta,\delta} \left( \int_{\mathbb{R}} e^{-\zeta y'} v(x, y') dy' \right) d\zeta.$$

The estimate in Proposition 3.2 shows that the integral over  $\Re \zeta = \gamma$  is well defined.

**4.2. A priori estimates.** In this section we explain how *a priori* estimates can be obtained for solutions of

$$(\Delta - \Phi)w = v.$$

These estimates will be used in the subsequent proofs. Their derivation, which relies on suitable integration by parts, is quite flexible so that it can be easily adapted in different context. We will assume that the potential  $\Phi$  is a smooth function such that

$$(4.3) \quad \Phi \geq \alpha^2 - \varepsilon^2 > 0,$$

in  $\mathbb{R}^2$ , for some  $\varepsilon > 0$ .

We also assume that we are given a function  $\Gamma$  which is smooth, positive and for which

$$(4.4) \quad |\nabla \Gamma|^2 \leq \beta^2 \Gamma^2,$$

with  $\beta^2 < \alpha^2 - \varepsilon^2$ .

Typical examples of weight functions we will consider in applications are of the form

$$\Gamma(x, y) := e^{\delta x} e^{\gamma y}, \quad \Gamma(x, y) := e^{\delta x} (\cosh y)^\gamma$$

or

$$\Gamma(x, y) := (\cosh x)^\delta (\cosh y)^\gamma,$$

where  $\gamma$  and  $\delta$  are chosen so that

$$\delta^2 + \gamma^2 < \alpha^2 - \varepsilon^2.$$

It is straightforward to check that these functions satisfy (4.4) with  $\beta^2 = \delta^2 + \gamma^2$  (the fact that the latter does satisfy this inequality follows from the observation that  $|\sinh y| \leq \cosh y$ ).

Let us now explain how *a priori* estimates can be obtained. We assume that we are given functions  $w$  and  $v$  satisfying

$$(\Delta - \Phi)w = v,$$

in  $\mathbb{R}^2$ . We further assume that  $\Gamma w, \Gamma \nabla w, \Gamma \nabla^2 w \in L^2(\mathbb{R}^2)$  and  $\Gamma v \in L^2(\mathbb{R}^2)$ . We compute

$$-\int \Gamma^2 w (\Delta - \Phi) w \, d\mathbf{x} = \int w v \Gamma^2 \, d\mathbf{x}.$$

All integrations are understood over  $\mathbb{R}^2$ . Integration by parts yields

$$\int w^2 \Phi \Gamma^2 \, d\mathbf{x} + \int |\nabla w|^2 \Gamma^2 \, d\mathbf{x} = \int w v \Gamma^2 \, d\mathbf{x} - 2 \int \Gamma u \nabla w \nabla \Gamma \, d\mathbf{x}.$$

Using (4.3), and the inequality  $2ab \leq a^2 + b^2$  to estimate the last term, we get

$$(\alpha^2 - \varepsilon^2) \int w^2 \Gamma^2 \, d\mathbf{x} + \int |\nabla w|^2 \Gamma^2 \, d\mathbf{x} \leq \int w v \Gamma^2 \, d\mathbf{x} + \int |\nabla \Gamma|^2 w^2 \, d\mathbf{x} + \int |\nabla w|^2 \Gamma^2 \, d\mathbf{x}.$$

Thanks to (4.4) we conclude that

$$(\alpha^2 - \varepsilon^2 - \beta^2) \int w^2 \Gamma^2 \, d\mathbf{x} \leq \int w v \Gamma^2 \, d\mathbf{x},$$

and, using Cauchy-Schwarz inequality, we conclude that we have the *a priori* estimate

$$(4.5) \quad (\alpha^2 - \varepsilon^2 - \beta^2)^2 \int w^2 \Gamma^2 \, d\mathbf{x} \leq \int v^2 \Gamma^2 \, d\mathbf{x}.$$

Naturally, this argument require some care since all integrations by parts have to be justified but, in the case under study, this is standard and left to the reader.

Observe that this argument is quite flexible and can also be used when  $\mathbb{R}^2$  is replaced by a half space or a wedge, provided the function  $w$  is controlled on the boundary of the domain, for example if  $w$  vanishes on this boundary.

**4.3. Linear Decomposition Lemma.** We obtain some information about the solution of the equation  $\mathcal{L} w = v$  which is given by the result of Proposition 4.1.

The first result we prove is concerned with the asymptotic behavior of the solution  $w$  in the  $x$  variable when the function  $v$  has a better behavior than expected with respect to the function space we use for the inversion of  $\mathcal{L}$ . Here is the precise statement :

**Lemma 4.1.** *Assume that  $w \in \tilde{L}_{\gamma, \delta}^2(\mathbb{R}^2)$  and  $v \in \tilde{L}_{\gamma, \bar{\delta}}^2(\mathbb{R}^2)$  satisfy  $\mathcal{L} w = v$  with  $\bar{\delta} < \delta$ . Further assume that*

$$\delta^2 + \gamma^2 < \alpha^2 \quad \text{and} \quad \bar{\delta}^2 + \gamma^2 < \alpha^2.$$

*Then,  $w \in \tilde{L}_{\gamma, \bar{\delta}}^2(\mathbb{R}^2)$ .*

*Proof.* The proof is as follows. First, we choose  $x_0 > 0$  such that

$$\inf_{|x| \geq x_0} F''(u_0) \geq \alpha^2 - \varepsilon^2 > \delta^2 + \gamma^2,$$

for some  $\varepsilon > 0$ . Next, we take some cutoff function  $\chi$  which is identically equal to 0 for  $x < x_0$  and identically equal to 1 for  $x > x_0 + 1$ . We have

$$\mathcal{L}(\chi w) = \chi v + [\partial_x^2, \chi] w,$$

where

$$[\partial_x^2, \chi] w := \partial_x^2(\chi w) - \chi \partial_x^2 w.$$

We set

$$\mathbb{R}_{x_0}^2 = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \quad : \quad x \geq x_0\}.$$

Elliptic estimates imply that

$$\|w\|_{e^{\gamma y} e^{\delta x} W^{2,2}(\mathbb{R}_{x_0}^2)} \leq c(\|w\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)} + \|w\|_{L_{\gamma,\delta}^2(\mathbb{R}^2)}),$$

and hence, we conclude that

$$\|\chi v + [\partial_x^2, \chi] w\|_{e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2)} \leq c(\|v\|_{\tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)} + \|w\|_{L_{\gamma,\delta}^2(\mathbb{R}^2)}).$$

for some constant  $c > 0$  depending of  $x_0$ .

To proceed, we claim that there exists a function  $\bar{w} \in e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2, \mathbb{R})$  such that

$$(4.6) \quad \mathcal{L} \bar{w} = \mathcal{L}(\chi w),$$

in  $\mathbb{R}_{x_0}^2$  with  $\bar{w} = 0$  on  $\partial \mathbb{R}_{x_0}^2$ . To prove the claim, we first solve the equation in  $[x_0, x_1] \times [-y_0, y_0]$  under 0 Dirichlet boundary conditions, denote this solution by  $\bar{w}_{x_1, y_0}$ , and then let  $x_1$  and  $y_0$  tend to infinity. To pass to the limit, we use the *a priori* estimate of the previous section to get

$$\|\bar{w}_{x_1, y_0}\|_{e^{\gamma y} e^{\delta x} L^2([x_0, x_1] \times [-y_0, y_0])} \leq c \|\mathcal{L}(\chi w)\|_{e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2)},$$

where the constant depends neither on  $x_1 > x_0 + 1$  nor on  $y_0 > 1$ . Elliptic estimate together with this *a priori* estimate allows (up to subsequences) to pass to the limit as  $x_1$  and  $z_0$  tend to infinity. This completes the proof of the claim.

Finally, we use once more the *a priori* estimate to show that  $\chi w = \bar{w}$  in  $\mathbb{R}_{x_0}^2$ . Indeed, by construction  $\mathcal{L}(\bar{w} - \chi w) = 0$  and  $\bar{w} - \chi w \in e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2)$ . The *a priori* estimate of the previous section implies that

$$\|\bar{w} - \chi w\|_{e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2)} \leq c \|\mathcal{L}(\bar{w} - \chi w)\|_{e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2)} = 0.$$

Therefore, we conclude that  $w \in e^{\gamma y} e^{\delta x} L^2(\mathbb{R}_{x_0}^2)$ . A similar argument applies in

$$\mathbb{R}_{-x_0}^2 = \{\mathbf{x} = (x, y) \in \mathbb{R}^2 \quad : \quad x \leq -x_0\},$$

thus ending the proof.  $\square$

We define the function space

$$\mathbb{L}_{\gamma,\delta}^2(\mathbb{R}^2) := (\cosh y)^\gamma (\cosh x)^\delta L^2(\mathbb{R}^2).$$

Observe that when  $\gamma \geq 0$

$$\mathbb{L}_{-\gamma,\delta}^2(\mathbb{R}^2) \subset \tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2).$$

We are now concerned with the behavior of the solution  $w$  in the  $y$  variable when the function  $v$  has a better behavior than what could be expected from the function space we use for the inversion of  $\mathcal{L}$ .

Let  $\chi$  be a cutoff function which is identically 0 in  $(-\infty, -1)$  and identically 1 in  $(1, +\infty)$ . Then the following result holds :

**Lemma 4.2** (Linear Decomposition Lemma). *Let  $\mu_j < \mu_{j+1}$  be two consecutive eigenvalues of the operator  $L := -\partial_x^2 + F''(u_0)$ . We assume that  $\sqrt{\mu_j} < \gamma < \sqrt{\mu_{j+1}}$  and also that  $\gamma^2 + \delta^2 < \alpha^2$ . Further assume that  $w \in \tilde{L}_{\gamma,\delta}^2(\mathbb{R}^2)$  and*

$$\mathcal{L} w = v \in \mathbb{L}_{-\gamma,\delta}^2(\mathbb{R}^2).$$

Then, for  $i = 0, \dots, j$ , there exists  $w_i, \bar{w}_i$  in the  $i$ -th eigenspace of  $L$  and  $\bar{w} \in \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2)$  such that

$$w(x, y) = \bar{w}(x, y) + \chi(y) (w_0(x) + y \bar{w}_0(x)) + \sum_{i=1}^j \chi(y) (w_i(x) e^{\sqrt{\mu_i} y} + \bar{w}_i(x) e^{-\sqrt{\mu_i} y})$$

(we agree that the last sum is not present when  $j = 0$ ). Moreover,

$$\sum_{i=0}^j (\|w_i\|_{L^2(\mathbb{R})} + \|\bar{w}_i\|_{L^2(\mathbb{R})}) + \|\bar{w}\|_{\mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2)} \leq c \|v\|_{\mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2)}.$$

*Proof.* Recall that  $\Pi_i$  is the  $L^2$ -orthogonal projection over the  $i$ -th eigenspace of  $L$ . We decompose

$$w = w^\perp + \sum_{i=0}^j w_i^\parallel \quad \text{and} \quad v = v^\perp + \sum_{i=0}^j v_i^\parallel$$

where, for all  $i = 0, \dots, j$  and for almost every  $y \in \mathbb{R}$

$$\Pi_i(w^\perp) = 0, \quad \Pi_i(v^\perp) = 0,$$

and

$$w_i^\parallel = \Pi_i(w) \quad \text{and} \quad v_i^\parallel = \Pi_i(v).$$

Recall that, *mutatis mutandis*, the result of Proposition 4.1 holds when we restrict our attention the space of functions  $u$  satisfying

$$(4.7) \quad \Pi_i(w) = 0 \quad \text{for almost every } y \in \mathbb{R} \quad \text{and} \quad i = 0, \dots, j,$$

see Remark 4.2. The only difference being that the restriction on  $\gamma$  is now given by  $\gamma \neq \pm \sqrt{\mu_{i+1}}$  for all  $i \geq j$ .

Thanks to the analysis of the previous section, we can write

$$w^\perp(x, y) = \frac{1}{2\pi} \int_{\Re \zeta = \gamma} e^{-\zeta y} G_{\zeta, \delta} \left( \int_{\mathbb{R}} e^{\zeta y'} v^\perp(x, y') dy' \right) d\zeta.$$

Let us assume that, in addition,  $v^\perp$  has compact support (the general result will follow by density). Observe that the integrand depends analytically on  $\zeta$  in  $\{\zeta \in \mathbb{C} : \Re \zeta \in (-\sqrt{\mu_{j+1}}, \sqrt{\mu_{j+1}})\}$  since we are working under the assumption that all functions do satisfy (4.7). Therefore, Cauchy formula implies that

$$\frac{1}{2\pi} \int_{\Re \zeta = \gamma} e^{-\zeta y} G_{\zeta, \delta} \left( \int_{\mathbb{R}} e^{\zeta y'} v^\perp(x, y') dy' \right) d\zeta,$$

does not depend on  $\gamma \in (-\sqrt{\mu_{j+1}}, \sqrt{\mu_{j+1}})$ . Consequently, we have

$$w^\perp(x, y) = \frac{1}{2\pi} \int_{\Re \zeta = -\gamma} e^{-\zeta y} G_{\zeta, \delta} \left( \int_{\mathbb{R}} e^{\zeta y'} v^\perp(x, y') dy' \right) d\zeta,$$

which implies that

$$w^\perp \in L_{-\gamma, \delta}^2(\mathbb{R}^2) \cap L_{\gamma, \delta}^2(\mathbb{R}^2) = \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2).$$

It remains to prove the result for  $w_i^\parallel$ . Observe that this time the problem reduces to solving some ordinary differential equation

$$(-\partial_y^2 + \mu_j) w_i^\parallel = v_i^\parallel.$$

It is easy to check that the solution is explicitly given by  
(4.8)

$$w_i^{\parallel}(x, y) = \begin{cases} - \int_{-\infty}^y \int_{-\infty}^{y'} v_i^{\parallel}(x, y'') dy'' dy', & \text{when } i = 0, \\ \frac{1}{2\sqrt{\mu_i}} \int_{-\infty}^y (e^{\sqrt{\mu_i}(y'-y)} - e^{\sqrt{\mu_i}(y-y')}) v_i^{\parallel}(x, y') dy', & \text{when } 0 < i \leq j. \end{cases}$$

Moreover, it is easy to check that when  $0 < i \leq j$ , then  $\bar{u}_i^{\parallel}$  can be decomposed as

$$w_i^{\parallel} = \bar{w}_i^{\parallel} + \chi(y) (w_i e^{\sqrt{\mu_i}y} + \bar{w}_i e^{-\sqrt{\mu_i}y}),$$

where  $\bar{w}_i^{\parallel} \in \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2, \mathbb{R})$  and  $w_i, \bar{w}_i$  belong to the  $i$ -th eigenspace of  $L$ . A similar decomposition can be done when  $i = 0$ . The estimate in the statement of the result follows at once from the proof and is left to the reader.  $\square$

We will only use this result in the case where  $\gamma \in (0, \sqrt{\mu_1})$ , namely when  $j = 0$ . Let us examine this case more closely, giving at the same time another consequence of the Linear Decomposition Lemma. Keeping in mind the applications we are interested in, we fix  $\gamma \in (0, \sqrt{\mu_1})$  and  $\delta \in \mathbb{R}$  such that  $\delta^2 + \gamma^2 < \alpha^2$ . It will be useful to introduce an operator  $\mathbb{A}_{-\gamma, \delta}$ , defined by

$$(4.9) \quad \begin{array}{ccc} \mathbb{A}_{-\gamma, \delta} : \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2) \oplus \mathbb{D} & \longrightarrow & \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2), \\ w & \longmapsto & \mathcal{L} w, \end{array}$$

where

$$\mathbb{D} := \text{Span} \left\{ \begin{array}{ll} (x, y) \longmapsto \chi(y) \partial_x u_0(x), & (x, y) \longmapsto (1 - \chi(y)) \partial_x u_0(x), \\ (x, y) \longmapsto \chi(y) y \partial_x u_0(x), & (x, y) \longmapsto (1 - \chi(y)) y \partial_x u_0(x) \end{array} \right\},$$

is called the *deficiency space*.

We now prove that the operator  $\mathbb{A}_{-\gamma, \delta}$  is surjective and that it has a two dimensional kernel. Note that the deficiency space is 4 dimensional and that it can be decomposed into

$$\mathbb{D} = \mathbb{K} \oplus \mathbb{E},$$

where

$$\mathbb{K} = \text{Span} \{(x, y) \longmapsto \partial_x u_0(x), (x, y) \longmapsto y \partial_x u_0(x)\}.$$

and

$$\mathbb{E} = \text{Span} \{(x, y) \longmapsto \chi(y) \partial_x u_0(x), (x, y) \longmapsto \chi(y) y \partial_x u_0(x)\}.$$

We claim that the result of the Linear Decomposition Lemma implies that the unbounded operator

$$\begin{array}{ccc} \bar{\mathbb{A}}_{-\gamma, \delta} : \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2) \oplus \mathbb{E} & \longrightarrow & \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2), \\ w & \longmapsto & \mathcal{L} w, \end{array}$$

is an isomorphism. Indeed, if  $v \in \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2)$ , then Proposition 4.1 yields the existence of a solution  $w$  of  $\mathcal{L} w = v$  with  $u \in \tilde{L}_{\gamma, \delta}^2(\mathbb{R}^2)$ . Then, the Linear Decomposition Lemma implies that  $w \in \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2) \oplus \mathbb{E}$  and this proves that the operator  $\bar{\mathbb{A}}_{-\gamma, \delta}$  is surjective. Now, clearly  $\mathbb{K}$ . To proceed, we assume that  $w \in \mathbb{L}_{-\gamma, \delta}^2(\mathbb{R}^2) \oplus \mathbb{E}$  satisfies  $\mathcal{L} w = 0$ . Then,  $w \in \tilde{L}_{\gamma, \delta}^2(\mathbb{R}^2)$  and Proposition 4.1 implies that  $u \equiv 0$ ,

which completes the proof of the claim. As a by product, we have shown that the operator  $\mathbb{A}_{-\gamma,\delta}$  is surjective and has a 2-dimensional kernel equal to  $\mathbb{K}$ .

Naturally, all these considerations can be extended to the case where  $\gamma$  is chosen so that  $\sqrt{\mu_j} < \gamma < \sqrt{\mu_{j+1}}$ .

### 5. THE LINEARIZED OPERATOR ABOUT AN ELEMENT OF $\mathcal{M}_{2k}$

In this section, we first explain how the previous results can be put together to obtain similar results for the linearized operator about the function  $u_\lambda$  defined in (2.6), where  $\lambda := (\lambda_1, \dots, \lambda_{2k})$ . Next, we will show that this result extends for the linearized operator about any element of  $\mathcal{M}_{2k}$ .

We denote by  $\mathfrak{L}_0$  the linearized operator about  $u_\lambda$ , namely

$$\mathfrak{L}_0 = -\Delta + F''(u_\lambda).$$

Now, we derive an *a priori* estimate for the operator  $\mathfrak{L}_0$  in the space  $L^2_{\gamma,\delta}(\mathbb{R}^2)$  (see (2.10) for its definition). In the sequel we will also need the weighted space  $W^{2,2}_{\gamma,\delta}(\mathbb{R}^2)$  defined in (2.11). The following result follows from the previous analysis together with a careful use of cutoff functions :

**Proposition 5.1.** *Assume that  $\gamma \in \mathbb{R} \setminus S$ , and*

$$\delta^2 + \gamma^2 < \alpha^2.$$

*Then, there exists  $\bar{R} \geq R$  and  $c > 0$  such that, for all  $w, v \in L^2_{\gamma,\delta}(\mathbb{R}^2)$ , satisfying*

$$\mathfrak{L}_0 w = v,$$

*we have*

$$\|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} \leq c \left( \|v\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + \|w\|_{L^2(B_{\bar{R}})} \right).$$

*Proof.* First of all, observe that elliptic estimates immediately imply that

$$(5.1) \quad \|w\|_{W^{2,2}_{\gamma,\delta}(\mathbb{R}^2)} \leq c \left( \|v\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + \|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} \right).$$

Next, we keep the notation of § 2.1. In particular, the oriented half lines  $\lambda_j^+$ , which correspond to the asymptotes of the nodal curves of  $u_\lambda$ , are parameterized by

$$(0, \infty) \ni s \mapsto \mathbf{x}_j + s \mathbf{e}_j \in \mathbb{R}^2.$$

For notational purposes, it will be convenient to extend the sequence of oriented half lines  $\lambda_1^+, \dots, \lambda_{2k}^+$  periodically by setting  $\lambda_{j+2k}^+ := \lambda_j^+$ ,  $\mathbf{x}_{j+2k} := \mathbf{x}_j$  and  $\mathbf{e}_{j+2k} = \mathbf{e}_j$  for any  $j \in \mathbb{Z}$ .

We denote by  $\lambda_{j+\frac{1}{2}}^+$  the oriented half line bisecting the lines containing  $\lambda_j^+$  and  $\lambda_{j+1}^+$  parameterized by

$$s \in (0, \infty) \mapsto \mathbf{x}_{j+\frac{1}{2}} + s \mathbf{e}_{j+\frac{1}{2}} \in \mathbb{R}^2,$$

where

$$\mathbf{e}_{j+\frac{1}{2}} := \frac{\mathbf{e}_j + \mathbf{e}_{j+1}}{2},$$

and  $\mathbf{x}_{j+\frac{1}{2}} \in \partial B_R$  (this point is contained in the angular arc (with positive orientation) of  $\partial B_R$  starting at  $\mathbf{x}_j$  and ending at  $\mathbf{x}_{j+1}$ ).

We define the angular sector  $S_j$  which is the connected component of  $\mathbb{R}^2 \setminus (\lambda_{j-\frac{1}{2}}^+ \cup \lambda_{j+\frac{1}{2}}^+ \cup \partial B_R)$  containing  $\lambda_j^+$ . In the notation of §2.2,  $S_j = \Omega_j$ . Similarly, we define

the angular sector  $S_{j+\frac{1}{2}}$  which is the connected component of  $\mathbb{R}^2 \setminus (\lambda_j^+ \cup \lambda_{j+1}^+ \cup \partial B_R)$  containing  $\lambda_{j+\frac{1}{2}}^+$ .

We need to introduce two more angular sectors

$$S''_{j+1/2} \subset S'_{j+1/2} \subset S_{j+1/2},$$

which are both included in the connected component of  $\mathbb{R}^2 \setminus (\lambda_j^+ \cup \lambda_{j+1}^+ \cup \partial B_R)$  which contains  $\lambda_{j+\frac{1}{2}}^+$ . The sector  $S'_{j+1/2}$  is limited by  $\partial B_R$  and the two half-lines  $\lambda_{j+1/4}^+$  and  $\lambda_{j+3/4}^+$  respectively bisecting  $\lambda_j^+$  and  $\lambda_{j+1/2}^+$  for the former and  $\lambda_{j+1/2}^+$  and  $\lambda_{j+1}^+$  for the latter. The sector  $S''_{j+1/2}$  is limited by  $\partial B_R$  and the two half-lines  $\lambda_{j+3/8}^+$  and  $\lambda_{j+5/8}^+$  respectively bisecting  $\lambda_{j+1/4}^+$  and  $\lambda_{j+1/2}^+$  for the former and  $\lambda_{j+1/2}^+$  and  $\lambda_{j+3/4}^+$  for the latter.

Finally, for all  $j = 1, \dots, 2k$ , we define the cutoff function  $\mathbb{I}_{j+\frac{1}{2}}$  such that :

- (i)  $\mathbb{I}_{j+\frac{1}{2}} \equiv 0$  in the set  $\mathbb{R}^2 \setminus S'_{j+1/2}$ .
- (ii)  $\mathbb{I}_{j+\frac{1}{2}} \equiv 1$  in the sector  $S''_{j+1/2}$ .
- (iii)  $|\nabla^\ell \mathbb{I}_{j+\frac{1}{2}}(\mathbf{x})| \leq c(1 + |\mathbf{x}|)^{-\ell}$ , for  $\ell = 0, 1, 2$ .

Observe that (i)–(iii) imply that the support of  $\mathbb{I}_{j+\frac{1}{2}}$  is an expanding, wedge-like set centered around  $\lambda_{j+\frac{1}{2}}^+$  and which is contained in  $S'_{j+1/2}$ .

We write

$$\gamma \mathbf{e}_j + \delta \mathbf{e}_j^\perp = \tilde{\gamma}_{j+\frac{1}{2}} \mathbf{e}_{j+\frac{1}{2}} + \tilde{\delta}_{j+\frac{1}{2}} \mathbf{e}_{j+\frac{1}{2}}^\perp,$$

where  $\tilde{\gamma}_{j+\frac{1}{2}}^2 + \tilde{\delta}_{j+\frac{1}{2}}^2 = \gamma^2 + \delta^2$ . Elementary geometry implies that

$$\gamma \mathbf{e}_{j+1} - \delta \mathbf{e}_{j+1}^\perp = \tilde{\gamma}_{j+\frac{1}{2}} \mathbf{e}_{j+\frac{1}{2}} - \tilde{\delta}_{j+\frac{1}{2}} \mathbf{e}_{j+\frac{1}{2}}^\perp,$$

since  $\mathbf{e}_{j+1}$  and  $\mathbf{e}_j$  are symmetric with respect to the reflection leaving  $\mathbf{e}_{j+\frac{1}{2}}$  fixed and

We set

$$(5.2) \quad \Gamma_{\tilde{\gamma}_{j+\frac{1}{2}}, \tilde{\delta}_{j+\frac{1}{2}}}(\mathbf{x}) := e^{\tilde{\gamma}_{j+\frac{1}{2}}(\mathbf{x} \cdot \mathbf{e}_{j+\frac{1}{2}})} \left( \cosh(\mathbf{x} \cdot \mathbf{e}_{j+\frac{1}{2}}^\perp) \right)^{\tilde{\delta}_{j+\frac{1}{2}}}.$$

Observe that the weights  $\Gamma_{\gamma, \delta}$  and  $\Gamma_{\tilde{\gamma}_{j+\frac{1}{2}}, \tilde{\delta}_{j+\frac{1}{2}}}$  are equivalent in the sector  $S_{j+\frac{1}{2}}$ .

Now, if  $\mathfrak{L}_0 w = v$  then

$$\mathfrak{L}_0(\mathbb{I}_{j+\frac{1}{2}} w) = \mathbb{I}_{j+\frac{1}{2}} v + [\mathfrak{L}_0, \mathbb{I}_{j+\frac{1}{2}}] w,$$

where, as usual,

$$[\mathfrak{L}_0, \mathbb{I}_{j+\frac{1}{2}}] w := \mathfrak{L}_0(\mathbb{I}_{j+\frac{1}{2}} w) - \mathbb{I}_{j+\frac{1}{2}}(\mathfrak{L}_0 w).$$

Given  $\varepsilon > 0$  small enough, there exists  $\bar{R} \geq R > 0$  such that

$$F''(u_\lambda) \geq \alpha^2 - \varepsilon^2,$$

in the support of  $\mathbb{I}_{j+\frac{1}{2}}$  and away from  $B_{\bar{R}}$ . Making use of an argument similar to the one we used to obtain (4.5), we get

$$(5.3) \quad (\alpha^2 - \varepsilon^2 - \tilde{\beta}_{j+1/2}^2)^2 \int_{\mathbb{R}^2 \setminus B_{\bar{R}}} \Gamma_{-\tilde{\gamma}_{j+\frac{1}{2}}, -\tilde{\delta}_{j+\frac{1}{2}}}^2 |\mathbb{I}_{j+\frac{1}{2}} w|^2 dx \\ \leq C \left( \int_{\mathbb{R}^2 \setminus B_{\bar{R}}} \Gamma_{-\tilde{\gamma}_{j+\frac{1}{2}}, -\tilde{\delta}_{j+\frac{1}{2}}}^2 |\mathbb{I}_{j+\frac{1}{2}} v|^2 dx \right. \\ \left. + \int_{\mathbb{R}^2 \setminus B_{\bar{R}}} \Gamma_{-\tilde{\gamma}_{j+\frac{1}{2}}, -\tilde{\delta}_{j+\frac{1}{2}}}^2 |[\mathfrak{L}_0, \mathbb{I}_{j+\frac{1}{2}}] w|^2 dx \right. \\ \left. + \int_{\partial B_{\bar{R}}} \Gamma_{-\tilde{\gamma}_{j+\frac{1}{2}}, -\tilde{\delta}_{j+\frac{1}{2}}}^2 \left| \partial_r (\mathbb{I}_{j+\frac{1}{2}} w) \right| \mathbb{I}_{j+\frac{1}{2}} w dx \right),$$

where  $\tilde{\beta}_{j+1/2}^2 := \tilde{\delta}_{j+\frac{1}{2}}^2 + \tilde{\gamma}_{j+\frac{1}{2}}^2$ . Since  $\Gamma_{\gamma, \delta}$  and  $\Gamma_{\tilde{\gamma}_{j+\frac{1}{2}}, \tilde{\delta}_{j+\frac{1}{2}}}$  are equivalent in the sector  $S_{j+\frac{1}{2}}$ , the first term on the right hand side can be estimated by a constant (independent of  $\bar{R}$ ) times  $\|v\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)}$ . Using the fact that  $\mathbb{I}_{j+\frac{1}{2}}$  satisfies property (iii), we conclude that the second term on the right hand side is bounded by a constant (independent of  $\bar{R}$ ) times  $\bar{R}^{-1} \|w\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)}$ . Finally, using standard elliptic estimates together with a trace embedding, one can check that the third term on the right hand side can be estimated by a constant (depending on  $\bar{R}$ ) times  $\|w\|_{L^2(B_{\bar{R}+1})}$ . Using once more the fact that the weights  $\Gamma_{\gamma, \delta}$  and  $\Gamma_{\tilde{\gamma}_{j+\frac{1}{2}}, \tilde{\delta}_{j+\frac{1}{2}}}$  are equivalent in the sector we are working in, we conclude that

$$(5.4) \quad \|\mathbb{I}_{j+\frac{1}{2}} w\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)} \leq c \|v\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)} + c_{\bar{R}} \|w\|_{L^2(B_{\bar{R}+1})} + c \bar{R}^{-1} \|w\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)},$$

where  $c_{\bar{R}}$  depends on  $\bar{R}$  but  $c$  does not.

Now, let us estimate the functions  $\mathbb{I}_j w$ , for  $j = 1, \dots, 2k$ . We set

$$\mathcal{L}_j := -\Delta + F''((-1)^j u_0(\text{dist}^s(\cdot, \lambda_j))).$$

Without loss of generality, we can assume that the operator  $\mathcal{L}_j$  coincides with the operator  $\mathcal{L}$  defined in §4 (if this is not the case, it is enough to apply some rigid motion to achieve this). The function  $\mathbb{I}_j w$  satisfies

$$(5.5) \quad \mathcal{L}_j(\mathbb{I}_j w) = \mathbb{I}_j v + [\mathfrak{L}_0, \mathbb{I}_j] w + (\mathcal{L}_j - \mathfrak{L}_0)(\mathbb{I}_j w).$$

We can then apply the result of Proposition 4.1 and in particular we can make use of (4.1). To estimate the second term in (5.5), we use the definition of the  $\mathbb{I}_j$  (see section 2.1), together with elliptic estimates to get

$$\|[\mathfrak{L}_0, \mathbb{I}_j] w\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)} \leq c \|\mathbb{I}_{j+\frac{1}{2}} w\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)} + c_{\bar{R}} \|w\|_{L^2(B_{\bar{R}+1})},$$

where, as usual,  $c_{\bar{R}}$  depends on  $\bar{R}$  while  $c$  does not.

Using the definition of  $u_\lambda$  given in (2.6), one can check that there exists  $\gamma_0 > 0$  such that

$$|F''(u_\lambda) - F''((-1)^j u_0(\text{dist}^s(\cdot, \lambda_j)))| \leq c |u_\lambda - (-1)^j u_0(\text{dist}^s(\cdot, \lambda_j))| \leq c e^{-\gamma_0 \bar{R}},$$

in the sector where  $\mathbb{I}_j > 0$  and where  $|x| > \bar{R}$ . Therefore, we get

$$\|(\mathcal{L}_j - \mathfrak{L}_0)(\mathbb{I}_j w)\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)} \leq c_{\bar{R}} \|w\|_{L^2(B_{\bar{R}+1})} + c e^{-\gamma_0 \bar{R}} \|w\|_{L_{\gamma, \delta}^2(\mathbb{R}^2)}.$$

Collecting these estimates, we conclude that

$$(5.6) \quad \begin{aligned} \|\mathbb{I}_j w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} &\leq c \|v\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + c_{\bar{R}} \|w\|_{L^2(B_{\bar{R}+1})} \\ &+ c \|\mathbb{I}_{j+\frac{1}{2}} w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + c e^{-\gamma_0 \bar{R}} \|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)}. \end{aligned}$$

When  $j = 0$ , the corresponding estimate for  $\mathbb{I}_0 w$  is straightforward and left to the reader. In any case, (5.4) and (5.6) imply that

$$\begin{aligned} \|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} &\leq c \|v\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + c_{\bar{R}} \|w\|_{L^2(B_{\bar{R}+1})} \\ &+ c (e^{-\gamma_0 \bar{R}} + \bar{R}^{-1}) \|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)}, \end{aligned}$$

where  $c_{\bar{R}}$  depends on  $\bar{R}$  while  $c$  does not. The proof of the main estimate then follows by taking  $\bar{R}$  large enough.  $\square$

As a consequence of the previous Proposition, we obtain the :

**Corollary 5.1.** *Assume that  $u \in \mathcal{M}_{2k}$  and let*

$$\mathfrak{L} := -\Delta + F''(u),$$

*be the linearized operator about  $u$ . Let  $\gamma, \delta$  be chosen so that they satisfy the hypothesis of Proposition 5.1. Then, there exists  $\bar{R} \geq R$  and  $c > 0$  such that, for all  $w, v \in L^2_{\gamma,\delta}(\mathbb{R}^2)$ , satisfying*

$$\mathfrak{L} w = v,$$

*the following inequality holds*

$$\|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} \leq c \left( \|v\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + \|w\|_{L^2(B_{\bar{R}})} \right).$$

*Proof.* The proof of the Corollary is a simple consequence of classical perturbation arguments. Indeed, by definition, if  $u \in \mathcal{M}_{2k}$ , there exists  $\lambda \in \Lambda$  such that  $u - u_\lambda \in W^{2,2}(\mathbb{R}^2)$ . Thanks to Sobolev embedding, for any  $\varepsilon$ , there exists  $\bar{R}_\varepsilon \geq \bar{R}$  such that

$$\|F''(u) - F''(u_\lambda)\|_{L^\infty(\mathbb{R}^2 \setminus B_{\bar{R}_\varepsilon})} \leq \varepsilon.$$

Hence

$$\|(F''(u) - F''(u_\lambda)) w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} \leq c_\varepsilon \|w\|_{L^2(B_{\bar{R}_\varepsilon})} + \varepsilon \|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)},$$

for some constant  $c_\varepsilon > 0$  (depending on  $\varepsilon$ ). Since  $\mathfrak{L}_0 w = v + (F''(u_\lambda) - F''(u)) w$ , it follows from Proposition 5.1, that

$$\|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} \leq c \|v\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + c\varepsilon \|w\|_{L^2_{\gamma,\delta}(\mathbb{R}^2)} + c_\varepsilon \|w\|_{L^2(B_{\bar{R}_\varepsilon})},$$

where  $c > 0$  does not depend on  $\varepsilon$ . The result then follows at once, choosing  $\varepsilon$  small enough so that  $c\varepsilon \leq 1/2$ .  $\square$

Thanks to the previous Corollary, we are now in a position to prove the following important :

**Proposition 5.2.** *Assume that  $\gamma \in \mathbb{R} \setminus S$  and*

$$\delta^2 + \gamma^2 < \alpha^2.$$

*Then, the operator*

$$\begin{aligned} \mathfrak{A}_{\gamma,\delta} : L^2_{\gamma,\delta}(\mathbb{R}^2) &\longrightarrow L^2_{\gamma,\delta}(\mathbb{R}^2), \\ w &\longmapsto \mathfrak{L} w, \end{aligned}$$

*is Fredholm. In addition  $\mathfrak{A}_{\gamma,\delta}$  is injective if and only if  $\mathfrak{A}_{-\gamma,-\delta}$  is surjective.*

*Proof.* These are classical results of Fredholm theory and their proofs follow at once the proofs in the compact setting together with intensive use of Corollary 5.1. For example, to prove that the kernel of  $\mathfrak{A}_{\gamma,\delta}$  is finite dimensional, we consider the unit ball of  $\text{Ker } \mathfrak{A}_{\gamma,\delta}$  using the norm of  $L^2(B_{\bar{R}})$ . We use the result of Corollary 5.1 to show that

$$\mathcal{B}_1 := \{w \in \text{Ker } \mathfrak{A}_{\gamma,\delta} : \|w\|_{L^2(B_{\bar{R}})} = 1\},$$

the closed unit ball of  $\text{Ker } \mathfrak{A}_{\gamma,\delta}$  for the topology induced by  $L^2_{\gamma,\delta}(\mathbb{R}^2)$  is compact. Elliptic estimates then imply that it is also bounded in  $W^{2,2}_{\gamma,\delta}(\mathbb{R}^2)$  and Sobolev embedding implies that it is compact in  $L^2(B_{\bar{R}})$ , hence  $\text{Ker } \mathfrak{A}_{\gamma,\delta}$  has to be finite dimensional.

The proof that  $\mathfrak{A}_{\gamma,\delta}$  is closed is again based on Corollary 5.1 using similar arguments. For details we refer the reader to [21] where similar results are established. Finally, the last statement follows from standard results on unbounded operators (see for example [4]) together with the natural identification of the dual of  $L^2_{\gamma,\delta}(\mathbb{R}^2)$  with  $L^2_{-\gamma,-\delta}(\mathbb{R}^2)$ .  $\square$

## 6. LINEAR DECOMPOSITION LEMMA

In this section, we pursue our investigations of the properties of the operator  $\mathfrak{A}_{\gamma,\delta}$ . We assume that  $\mathfrak{L}$  is the linearized operator about an element  $u \in \mathcal{M}_{2k}$  such that  $u - u_\lambda \in W^{2,2}_{-\bar{\gamma},-\bar{\delta}}(\mathbb{R}^2)$ , for some  $\bar{\gamma}, \bar{\delta} > 0$  satisfying (2.13).

First, we derive a Linear Decomposition Lemma for the operator  $\mathfrak{L}$  in the spirit of the one we have obtained in § 4.3 and we use this result to compute the dimension of the kernel of the operator  $\mathfrak{A}_{\gamma,\delta}$ .

For  $j = 1, \dots, 2k$ , it will be convenient to define the vector fields

$$(6.1) \quad X_j(\mathbf{x}) := \mathbb{I}_j(\mathbf{x}) \mathbf{e}_j^\perp,$$

and

$$(6.2) \quad Y_j(\mathbf{x}) := \mathbb{I}_j(\mathbf{x}) (\mathbf{e}_j \cdot (\mathbf{x} - \mathbf{x}_j) \mathbf{e}_j^\perp - \mathbf{e}_j^\perp \cdot (\mathbf{x} - \mathbf{x}_j) \mathbf{e}_j).$$

To have a better understanding of these two vector fields, observe that the vector fields

$$\hat{X}_j(\mathbf{x}) := \mathbf{e}_j^\perp,$$

and

$$\hat{Y}_j(\mathbf{x}) := \mathbf{e}_j \cdot (\mathbf{x} - \mathbf{x}_j) \mathbf{e}_j^\perp - \mathbf{e}_j^\perp \cdot (\mathbf{x} - \mathbf{x}_j) \mathbf{e}_j,$$

are the Killing vector fields associated, respectively, to translations in the direction of the vector  $\mathbf{e}_j^\perp$  and rotation about  $\mathbf{x}_j$ , expressed in terms of coordinate vectors  $\{\mathbf{e}_j^\perp, \mathbf{e}_j\}$ .

Finally, we define the *deficiency space*

$$\mathfrak{D} := \text{Span} \{du(X_j), du(Y_j) : j = 1, \dots, 2k\}.$$

Since we assume that  $u - u_\lambda \in W^{2,2}_{-\bar{\gamma},-\bar{\delta}}(\mathbb{R}^2)$ , we have

$$(6.3) \quad \begin{aligned} du(X_j)(\mathbf{x}) - \partial_x u_0((\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{e}_j^\perp) &\in W^{1,2}_{-\gamma,-\delta}(\mathbb{R}^2), \\ du(Y_j) - \partial_x u_0((\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{e}_j^\perp) \mathbf{e}_j \cdot (\mathbf{x} - \mathbf{x}_j) &\in W^{1,2}_{-\gamma,-\delta}(\mathbb{R}^2), \end{aligned}$$

for any  $\gamma \in (0, \bar{\gamma})$  and  $\delta \in (0, \bar{\delta})$ .

The following result is a consequence of the Lemma 4.2.

**Proposition 6.1.** *Assume that  $\gamma \in (0, \sqrt{\lambda_1})$  and  $\delta > 0$  are fixed so that*

$$\gamma^2 + \delta^2 < \alpha^2,$$

*and  $\delta \in (0, \bar{\delta})$ ,  $\gamma \in (0, \bar{\gamma})$ . We further assume that  $\mathfrak{A}_{-\gamma, -\delta}$  is injective. Then, for all  $v \in L^2_{-\gamma, -\delta}(\mathbb{R}^2)$  there exists a function  $w \in L^2_{\gamma, \delta}(\mathbb{R}^2)$  solution of*

$$(6.4) \quad \mathfrak{L}w = v.$$

*which satisfies*

$$(6.5) \quad w \in L^2_{-\gamma, -\delta}(\mathbb{R}^2) \oplus \mathfrak{D}.$$

*Proof.* We notice that, thanks to Proposition 5.2, the existence of a solution  $w \in L^2_{\gamma, \delta}(\mathbb{R}^2)$  of (6.4) follows easily since  $\mathfrak{A}_{\gamma, \delta}$  is surjective and  $L^2_{-\gamma, -\delta}(\mathbb{R}^2) \subset L^2_{\gamma, \delta}(\mathbb{R}^2)$ . Therefore, it remains to show that  $w \in L^2_{-\gamma, -\delta}(\mathbb{R}^2) \oplus \mathfrak{D}$ .

We begin by showing that in those angular sectors which do not contain the ends  $\lambda_j^+$ , the function  $u$  grows exponentially at slightly slower rate than suggested *a priori*. To state this precisely we will use the notations introduced in the proof of Proposition 5.1.

Since we work with one sector at a time, we may well assume that

$$\lambda_{j+\frac{1}{2}}^+ \cap (\mathbb{R}^2 \setminus B_R) = \{\mathbf{x} = (0, y) : y \geq R\},$$

(this can always be achieved after a suitable rigid motion) and that the image of the angular sector  $S_{j+\frac{1}{2}} \cap (\mathbb{R}^2 \setminus B_R)$  under this rigid motion is the angular sector

$$A_{R, \beta} := \{\mathbf{x} = (x, y) : \beta|x| \leq y\} \cap (\mathbb{R}^2 \setminus B_R),$$

for some  $\beta > 0$ . Likewise the image of  $S'_{j+\frac{1}{2}} \cap (\mathbb{R}^2 \setminus B_R)$  is the sector  $A_{R, \frac{\beta}{2}}$ . Note that if  $\theta_{j+1} - \theta_j$  is the oriented angle between  $\lambda_j^+$  and  $\lambda_{j+1}^+$  then

$$\cot\left(\frac{\theta_{j+1} - \theta_j}{2}\right) = \frac{1}{\beta}.$$

We define

$$\mathbf{e}_{\pm\beta} := \frac{(1, \pm\beta)}{\sqrt{1 + \beta^2}}, \quad \mathbf{e}_{\pm\beta}^\perp := \frac{(\mp\beta, 1)}{\sqrt{1 + \beta^2}}.$$

and we let  $\mathbf{x}_R^\pm$  be the points of intersection of the nodal lines  $y = \pm\beta x$  with  $B_R$ . Consider the functions  $G_\varepsilon^\pm$  defined by

$$G_\varepsilon^\pm(\mathbf{x}) := (\cosh(\mathbf{e}_{\pm\beta} \cdot (\mathbf{x} - \mathbf{x}_R^\pm)))^\gamma (\cosh(\mathbf{e}_{\pm\beta}^\perp \cdot (\mathbf{x} - \mathbf{x}_R^\pm)))^{\delta - \varepsilon}.$$

Choosing  $M > 0$  large enough and  $\varepsilon > 0$  sufficiently small (so that  $\alpha^2 - \gamma^2 - (\delta - \varepsilon)^2 > 0$ ), we find that the function

$$w_{M, \varepsilon} = M(G_\varepsilon^+ + G_\varepsilon^-),$$

is a positive supersolution for the equation  $\mathfrak{L}w = v$  in the sector  $A_{R, \beta}$ . Then, by comparison principle and elementary geometric manipulations, we see that there exists  $M > 0$  and  $\varepsilon' \in (0, \varepsilon)$  such that

$$(6.6) \quad \begin{aligned} |w(\mathbf{x})| &\leq M (\cosh(\mathbf{e}_{+\beta} \cdot (\mathbf{x} - \mathbf{x}_R^+)))^{\gamma - \varepsilon'} (\cosh(\mathbf{e}_{+\beta}^\perp \cdot (\mathbf{x} - \mathbf{x}_R^+)))^{\delta - \varepsilon'} \\ &\quad + M (\cosh(\mathbf{e}_{-\beta} \cdot (\mathbf{x} - \mathbf{x}_R^-)))^{\gamma - \varepsilon'} (\cosh(\mathbf{e}_{-\beta}^\perp \cdot (\mathbf{x} - \mathbf{x}_R^-)))^{\delta - \varepsilon'} \end{aligned}$$

in the sector  $A_{R, \beta/2}$ . Thus, at least as far as the sector  $A_{R, \beta/2}$  is concerned, the function  $w$  is in a better space than one predicted initially.

Again, applying a suitable rigid motion, we may assume that the nodal line  $\lambda_j^+$  is contained in the  $y$ -axis,  $y > 0$ . In this case, the lines bisecting the angles between  $\lambda_j^+$  and  $\lambda_{j+1}^+$  and  $\lambda_j^+$  and  $\lambda_{j-1}^+$  become respectively

$$L_1 := \{(x, y) : y = -\beta_1 x, x < 0\} \quad \text{and} \quad L_2 := \{(x, y) : y = \beta_2 x, x > 0\},$$

where  $\beta_1, \beta_2 > 0$ . We denote by  $\tilde{A}_R$  the sector which is outside  $B_R$  and bounded by these lines. We consider a smooth cutoff functions  $\tilde{\mathbb{I}}$  whose support is equal to  $\tilde{A}_R$  and such that :

- (i)  $\tilde{\mathbb{I}}(x, y) = 1$  whenever  $-2\beta_1 x < y < 2\beta_2 x$ .
- (ii)  $\tilde{\mathbb{I}}(x, y) = 0$ , whenever  $y \leq -\beta_1 x$  or  $\beta_2 x \leq y$ .

The function  $\tilde{w} := \tilde{\mathbb{I}} w$  solves the equation

$$(-\Delta + F''(u_0)) \tilde{w} = \tilde{\mathbb{I}} v - [\Delta, \tilde{\mathbb{I}}] w + (F''(u_0) - F''(u)) \tilde{\mathbb{I}} w.$$

This equation can now be considered as an equation in the whole  $\mathbb{R}^2$  with the right hand side in  $L^2_{\gamma-\varepsilon'', \delta-\varepsilon''}(\mathbb{R}^2)$ , for some  $\varepsilon'' > 0$ . This latter fact follows from (6.6) together with elliptic estimates. Therefore, Lemma 4.2 and the discussion at the end of §4 can be used to prove that

$$\tilde{w} \in L^2_{\gamma-\varepsilon'', \delta-\varepsilon''}(\mathbb{R}^2) \oplus \mathbb{D}.$$

From this, considering all other ends in a similar manner, we get the existence of  $\bar{\varepsilon} > 0$  such that

$$w \in L^2_{\gamma-\bar{\varepsilon}, \delta-\bar{\varepsilon}}(\mathbb{R}^2) \oplus \mathfrak{D}.$$

This argument can be now iterated to conclude, after finitely many steps, we obtain (6.5) and the proof is complete.  $\square$

We introduce the following :

**Definition 6.1.** *A solution  $u \in \mathcal{M}_{2k}$  is said to be non degenerate if the linear operator  $\mathfrak{A}_{-\gamma, -\delta}$ , associated to  $-\Delta + F''(u)$ , is injective for all  $\delta, \gamma > 0$ .*

Observe that in fact, in order to show that  $u$  is non degenerate, it is enough to prove that  $\mathfrak{A}_{\gamma, \delta}$  is injective for *some*  $\gamma \in (0, \sqrt{\lambda_1})$  and  $\delta \in \mathbb{R}$  satisfying the hypothesis of Proposition 6.1. Then according to the result of this Proposition, the operator  $\mathfrak{A}_{-\gamma, -\delta}$  will be injective for *any*  $\delta, \gamma > 0$  satisfying these hypothesis.

We define the operator

$$(6.7) \quad \begin{aligned} \tilde{\mathfrak{A}}_{-\gamma, -\delta} : L^2_{-\gamma, -\delta}(\mathbb{R}^2) \oplus \mathfrak{D} &\longrightarrow L^2_{-\gamma, -\delta}(\mathbb{R}^2), \\ u &\longmapsto \mathfrak{L} u. \end{aligned}$$

Now, as at the end of §4, we compute the dimension of the kernel of the operator  $\tilde{\mathfrak{A}}_{-\gamma, -\delta}$ , using Proposition 6.1 in the same way we have used Lemma 4.2 to compute the dimension of the kernel of the operator  $\tilde{\mathfrak{A}}_{-\gamma, \delta}$ .

**Proposition 6.2.** *Assume that  $\mathfrak{A}_{-\gamma, -\delta}$  is injective for some  $\gamma \in (0, \bar{\gamma})$  and  $\delta \in (0, \bar{\delta})$  such that the hypothesis of Proposition 6.1 are satisfied. Then the operator  $\tilde{\mathfrak{A}}_{-\gamma, -\delta}$  is surjective and has a  $2k$  dimensional kernel.*

*Proof.* As already mentioned the proof is very close to the one already outlined at the end of §4. By assumption the operator  $\mathfrak{A}_{-\gamma,-\delta}$  is injective. Hence its dual  $\mathfrak{A}_{\gamma,\delta}$  will be surjective. Moreover, Proposition 6.1 implies that

$$\text{Ker}(\mathfrak{A}_{\gamma,\delta}) = \text{Ker}(\tilde{\mathfrak{A}}_{-\gamma,-\delta}),$$

and, by duality, we get

$$\dim \text{Ker}(\mathfrak{A}_{\gamma,\delta}) = \text{codim} \text{Im}(\mathfrak{A}_{-\gamma,-\delta}).$$

Thanks to Proposition 6.1), we also have

$$\dim \mathfrak{D} = \dim \text{Ker}(\mathfrak{A}_{\gamma,\delta}) + \text{codim} \text{Im}(\mathfrak{A}_{-\gamma,-\delta}).$$

From this it follows that

$$(6.8) \quad \dim \text{Ker}(\tilde{\mathfrak{A}}_{-\gamma,-\delta}) = \text{codim} \text{Im}(\mathfrak{A}_{-\gamma,-\delta}) = \frac{1}{2} \dim \mathfrak{D},$$

as claimed, since  $\dim \mathfrak{D} = 4k$ . □

The formula (6.8) is usually referred to as the *relative index formula*.

## 7. REFINED ASYMPTOTICS, THE PROOF OF THEOREM 2.1

We will use the results of the previous sections, and in particular Proposition 5.1 to prove Theorem 2.1. Thus we assume that  $u = u_\lambda + v \in \mathcal{M}_{2k}$ , and  $v \in W^{2,2}(\mathbb{R}^2)$  where  $\lambda = (\lambda_1, \dots, \lambda_{2k})$ . Let us denote

$$(7.1) \quad N(u) := -\Delta u + F'(u).$$

Then, we can write

$$N(u) = N(u_\lambda) + \mathfrak{L}_0 v + Q(v),$$

where

$$\mathfrak{L}_0 := -\Delta + F''(u_\lambda),$$

has already been defined in §5 and where

$$Q(v) := F'(u_\lambda + v) - F'(u_\lambda) - F''(u_\lambda)v,$$

collects the nonlinear terms.

Now, if  $N(u) = 0$  then, the function  $v$  solves

$$(7.2) \quad \mathfrak{L}_0 v = N(u_\lambda) - Q(v).$$

As we will see later, the function  $N(u_\lambda)$  belongs to  $L^2_{-\gamma,-\delta}(\mathbb{R}^2)$  for some  $\gamma \in (0, \bar{\gamma})$  and  $\delta \in (0, \bar{\delta})$ . This, together with the Linear Decomposition Lemma and the quadratic nature of the term  $Q(v)$  strongly suggests that the function  $v$  should belong to  $(W^{2,2}_{-\gamma,-\delta}(\mathbb{R}^2) \oplus \mathfrak{D}) \cap W^{2,2}(\mathbb{R}^2)$ . Then, the fact that the elements of the deficiency space  $\mathfrak{D}$  do not belong to  $W^{2,2}(\mathbb{R}^2)$  implies that  $v \in W^{2,2}_{-\gamma,-\delta}(\mathbb{R}^2)$ . Unfortunately, the situation turns out to be slightly more complicated since, to begin with, the only information we have is that  $v \in W^{2,2}(\mathbb{R}^2)$  and hence we do not have any exponential decay for the function  $v$ . In particular, we do not have any decay property for the term  $Q(v)$ .

Observe that, if we knew that  $v$  had some exponential decay at infinity, then a simple bootstrap argument would give us the optimal exponential decay for the function  $v$ , comparable with the one of the term  $N(u_\lambda)$ . So the main issue is to be able to start the bootstrap process. To this end, we will use a scaling argument inspired by an idea of L. Simon. In [11] a similar argument was used to show the

refined asymptotics for constant scalar curvature metrics with isolated singularities. One of the crucial ingredients in the proof will be the balancing formula (2.15)–(2.16).

**Lemma 7.1.** *Assume that  $\bar{\gamma} > 0$  and  $\bar{\delta} > 0$  are fixed such that  $\bar{\gamma}^2 + \bar{\delta}^2 < \alpha^2$  and*

$$(7.3) \quad \bar{\gamma} \cot\left(\frac{\theta_{j+1} - \theta_j}{2}\right) + \bar{\delta} < \alpha,$$

for  $j = 1, \dots, 2k$  (recall that  $\theta_j$  is the angle which defines the oriented half line  $\lambda_j^+$ ). Then,

$$|N(u_\lambda)| \leq C \Gamma_{-\hat{\gamma}, -\hat{\delta}},$$

for any  $\hat{\gamma} \in (0, \bar{\gamma})$  and  $\hat{\delta} \in (0, \bar{\delta})$  satisfying  $\gamma^2 + \delta^2 < \alpha^2$ , where  $\Gamma_{\gamma, \delta}$  is the weight function defined in (2.9).

*Proof.* We denote for brevity  $u_{0,j} := u_0(\text{dist}^s(\cdot, \lambda_j))$ . To begin with, we write

$$(7.4) \quad N(u_\lambda) = \sum_{j=1}^{2k} [-\Delta, (-1)^j \mathbb{I}_j] u_{0,j} + F'(u_\lambda) - \sum_{j=1}^{2k} (-1)^j \mathbb{I}_j F'(u_{0,j}).$$

To estimate the first of the terms on the right hand side, we consider the set

$$O_\ell := \{\mathbf{x} \in \mathbb{R}^2 : (\mathbb{I}_\ell + \mathbb{I}_{\ell+1})(\mathbf{x}) = 1\},$$

for some  $1 \leq \ell \leq 2k$ . Then, using the fact that  $\mathbb{I}_\ell + \mathbb{I}_{\ell+1} \equiv 1$  in  $O_\ell$ , we can write

$$\sum_{j=1}^{2k} [-\Delta, (-1)^j \mathbb{I}_j] u_{0,j} = (-1)^{\ell+1} (2 \nabla(u_{0,\ell} + u_{0,\ell+1}) \cdot \nabla \mathbb{I}_\ell + (u_{0,\ell} + u_{0,\ell+1}) \Delta \mathbb{I}_\ell).$$

in  $O_\ell$ . We recall now that the heteroclinic solution  $u_0$ , which is odd, satisfies the following asymptotic formula

$$|u_0(x) - 1| \leq C e^{-\alpha x},$$

for  $x \geq 0$ , with similar estimates for its derivatives. Thus, we get

$$|(u_{0,j} + u_{0,j+1})(\mathbf{x})| \leq c \sum_{\ell=0,1} e^{-\bar{\gamma}(\mathbf{x} - \mathbf{x}_{j+\ell}) \cdot \mathbf{e}_{j+\ell}} (\cosh((\mathbf{x} - \mathbf{x}_{j+\ell}) \cdot \mathbf{e}_{j+\ell}^{\perp}))^{-\bar{\delta}},$$

for  $\mathbf{x} \in O_\ell$ , provided  $\bar{\gamma} > 0$  and  $\bar{\delta} > 0$  satisfy (7.3). Since analogous estimates hold for the term involving the gradient of  $u_{0,j} + u_{0,j+1}$ , we conclude that,

$$\left\| \sum_{j=1}^{2k} [-\Delta, (-1)^j \mathbb{I}_j] u_{0,j} \right\|_{L^2_{-\gamma, -\delta}(\mathbb{R}^2)} \leq c,$$

for  $\gamma \in (0, \bar{\gamma})$ ,  $\delta \in (0, \bar{\delta})$ . The estimates of the other term in (7.4) can be obtained using similar arguments and we conclude that

$$\|N(u_\lambda)\|_{L^2_{-\gamma, -\delta}(\mathbb{R}^2)} \leq c.$$

This completes the proof if the result.  $\square$

For further use, we assume that  $\hat{\gamma}$  is chosen so that  $\hat{\gamma} \in (0, \sqrt{\mu_1})$ . For all  $R > 0$ , we define

$$(7.5) \quad \mathcal{E}(R) := \max \left( e^{-aR}, \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \right),$$

where the constant  $a > 0$  will be chosen close enough to 0. We will now state the main result of this section.

**Proposition 7.1.** *There exists  $\bar{R} > 0$  and  $r_* > 0$  such that for all  $R > \bar{R}$ , we have*

$$(7.6) \quad \mathcal{E}(R + r_*) \leq \frac{1}{2} \mathcal{E}(R).$$

Before we proceed with the proof of this Proposition, let us explain why the proof of Theorem 2.1 follows easily from it. Indeed, assuming we have already proven this Proposition, we can define

$$\bar{a} := \frac{1}{2r_*} \log 2 > 0,$$

then, using (7.6), we get for any  $R > \bar{R}$ ,

$$(7.7) \quad \mathcal{E}(R) \leq \mathcal{E}(\bar{R}) e^{-\bar{a}(R-\bar{R})}.$$

But, this implies that

$$\|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \leq \mathcal{E}(\bar{R}) e^{-\bar{a}(R-\bar{R})},$$

from which we get that  $u - u_\lambda \in L^2_{-\gamma, -\delta}(\mathbb{R}^2)$  for some  $\delta, \gamma > 0$ .

The proof of Proposition 7.1 involves several steps. Using appropriate barriers, we begin by showing that in those angular sectors which do not contain the ends  $\lambda_1^+, \dots, \lambda_{2k}^+$ , the function  $v := u - u_\lambda$  decays exponentially. To state this precisely, we use once again the notations used in the proof of Proposition 5.1.

Using appropriate barrier functions, we prove that the function  $u - u_\lambda$  has some exponential decay, away from the half affine lines  $\lambda_j^+$ . This is the content of the:

**Lemma 7.2.** *There exist constants  $a_0 > 0$ ,  $\delta_0 > 0$ ,  $C > 0$  and  $\bar{R}_0 > 0$  such that, for all  $R > \bar{R}_0$ , we have*

$$(7.8) \quad |u - u_\lambda| \leq C \left( e^{-a_0|x|} + (e^{-a_0(|x|-R)} + \Gamma_{0, -\delta_0}) \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \right),$$

in  $\mathbb{R}^2 \setminus B_R$ , where  $\Gamma_{\gamma, \delta}$  is the weight function defined in (2.9).

*Proof.* It is enough to work in one sector at a time, say for example  $S_{j+\frac{1}{2}}$ . After a rigid motion (and possible change of the origin), we may assume that

$$\lambda_{j+\frac{1}{2}}^+ \cap (\mathbb{R}^2 \setminus B_R) = \{\mathbf{x} = (0, y) : y \geq R\},$$

and that the image of the angular sector  $S_{j+\frac{1}{2}} \cap (\mathbb{R}^2 \setminus B_R)$  under this rigid motion is the angular sector

$$A_R := \{\mathbf{x} = (x, y) : \beta|x| \leq y\} \cap (\mathbb{R}^2 \setminus B_R),$$

for some  $\beta > 0$ .

We define the function

$$\begin{aligned} G_{\sigma, \tau}(\mathbf{x}) &:= \left( \varepsilon e^{\sigma(|x|-R)} + e^{-\sigma(|x|-R)} + e^{-\tau(y-\beta x)} + e^{-\tau(y+\beta x)} \right) \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \\ &+ \left( \varepsilon e^{\sigma|x|} + e^{-\sigma|x|} \right), \end{aligned}$$

where  $\sigma, \tau > 0$  are chosen close enough to 0. Observe that  $G_{\sigma, \tau} > 0$  in  $A_R$  and

$$(7.9) \quad G_{\sigma, \tau}(\mathbf{x}) \geq \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)},$$

on  $\partial A_R$ . Moreover, we have

$$(7.10) \quad \left(-\Delta + \frac{\alpha^2}{2}\right) G_{\sigma, \tau} \geq \frac{\alpha^2}{4} e^{-\sigma|\mathbf{x}|},$$

in  $A_R$  provided  $R$  is chosen large enough and  $\sigma, \tau > 0$  are chosen close enough to 0. We set  $v := u - u_\lambda$ . Observe that  $v$  satisfies the equation

$$(7.11) \quad (-\Delta + \Phi)v + N(u_\lambda) = 0,$$

where by definition

$$\Phi := \frac{F'(u) - F'(u_\lambda)}{u - u_\lambda},$$

whenever  $u - u_\lambda \neq 0$  and

$$\Phi := F''(u_\lambda),$$

whenever  $u = u_\lambda$ . Now, choosing  $\bar{R}$  large enough, we can ensure that  $\Phi \geq \alpha^2/4$  in the subset of  $A_R$  defined by

$$\bar{A}_R := \{\mathbf{x} \in A_R : \text{dist}(\mathbf{x}, \partial A_R) \geq \bar{R}\},$$

for all  $R \geq \bar{R}$ . This follows at once from the fact that  $u_\lambda$  converges uniformly to  $\pm 1$  away from the  $\lambda_j^+$  and the fact that  $u - u_\lambda$  tends to 0 uniformly at infinity.

Using the result of Lemma 7.1, we find that, for each  $\varepsilon > 0$ , a constant times  $G_{\sigma, \tau}$  is a positive supersolution for problem (7.11) in the sector  $A_R$  provided  $R$  is chosen large enough and  $\sigma, \tau > 0$  close enough to 0. Hence, we have

$$(7.12) \quad |v| \leq C G_{\sigma, \tau},$$

in  $A_R$ . The assertion of the Lemma follows from letting  $\varepsilon$  tend to 0 in this pointwise estimate.  $\square$

Next, we estimate the function  $v$  near the half lines  $\lambda_j^+$ . As already mentioned, there is no loss of generality in assuming that the half line  $\lambda_j^+$  coincides with the  $y$  axis, if this is not the case, this can be achieved using some proper rigid motion.

So we fix  $j$  and  $R > 0$  large enough. We define  $S_{j, R}$  to be the connected component of  $\mathbb{R}^2$  which contains the half line  $\lambda_j^+ \cap (\mathbb{R}^2 \setminus B_R)$  and which is bounded by the half lines  $\lambda_{j-\frac{1}{2}}^+ \cap (\mathbb{R}^2 \setminus B_R)$ ,  $\lambda_{j+\frac{1}{2}}^+ \cap (\mathbb{R}^2 \setminus B_R)$  and  $\partial B_R$ . The angle between two consecutive ends is strictly less than  $\pi$  and hence  $S_{j, R} \subset \mathbb{R} \times (0, \infty)$ . We define  $\check{\mathbb{I}}_{j, R}$  to be a cutoff function such that

$$(i) \quad \check{\mathbb{I}}_{j, R} \equiv 1 \text{ in the subset of } S_{j, R} \text{ such that } \text{dist}(\mathbf{x}, \partial B_R) \geq 1.$$

$$(ii) \quad \check{\mathbb{I}}_{j, R} \equiv 0 \text{ in } \mathbb{R}^2 \setminus S_{j, R}.$$

$$(iii) \quad |\nabla^\ell \check{\mathbb{I}}_{j, R}| \leq c, \text{ for } \ell = 0, 1 \text{ and } 2.$$

Finally, we define the function

$$v_{j, R} := \check{\mathbb{I}}_{j, R} (u - u_\lambda).$$

Since  $u$  is a solution of (2.1), one can check that the function  $v_{j,R}$  is a solution of

$$(7.13) \quad \begin{aligned} (-\Delta + F''(u_\lambda)) v_{j,R} &= \check{\mathbb{I}}_{j,R} N(u_\lambda) - [\Delta, \check{\mathbb{I}}_{j,R}] (u - u_\lambda) \\ &- \check{\mathbb{I}}_{j,R} (F'(u) - F'(u_\lambda) - F''(u_\lambda)(u - u_\lambda)). \end{aligned}$$

Since  $F$  is even, so is  $F''$ , and hence, close to  $\lambda_j^+$ ,  $F''(u_0)$  is equal to  $F''(u_\lambda)$ . Therefore, we can rewrite (7.13) as

$$(7.14) \quad \begin{aligned} (-\Delta + F''(u_0)) v_{j,R} &= \check{\mathbb{I}}_{j,R} N(u_\lambda) - [\Delta, \check{\mathbb{I}}_{j,R}] (u - u_\lambda) + (F''(u_0) - F''(u_\lambda)) v_{j,R} \\ &- \check{\mathbb{I}}_{j,R} (F'(u) - F'(u_\lambda) - F''(u_\lambda)(u - u_\lambda)). \end{aligned}$$

We will denote for short

$$\begin{aligned} h_{j,R}^0 &:= \check{\mathbb{I}}_{j,R} N(u_\lambda), \\ h_{j,R}^1 &:= -[\Delta, \check{\mathbb{I}}_{j,R}] (u - u_\lambda) + (F''(u_0) - F''(u_\lambda)) v_{j,R}, \end{aligned}$$

and

$$h_{j,R}^2 := -\check{\mathbb{I}}_{j,R} (F'(u) - F'(u_\lambda) - F''(u_\lambda)(u - u_\lambda)).$$

So that (7.14) becomes

$$(7.15) \quad (-\Delta + F''(u_0)) v_{j,R} = h_{j,R}^1 + h_{j,R}^2 + h_{j,R}^3$$

We decompose any function  $f$  defined in  $\mathbb{R}^2$  into

$$f = f^\parallel + f^\perp$$

where  $f^\parallel := \Pi_0(f)$  and  $f^\perp$  is  $L^2(\mathbb{R})$ -orthogonal to  $\partial_x u_0(x)$  for all  $y \in \mathbb{R}$ . Starting from (7.15) we decompose  $v_{j,R} = v_{j,R}^\parallel + v_{j,R}^\perp$ , and using natural notations, we obtain the system of two equations

$$(-\Delta + F''(u_0)) v_{j,R}^\parallel = h_{j,R}^{1,\parallel} + h_{j,R}^{2,\parallel} + h_{j,R}^{3,\parallel},$$

and

$$(-\Delta + F''(u_0)) v_{j,R}^\perp = h_{j,R}^{1,\perp} + h_{j,R}^{2,\perp} + h_{j,R}^{3,\perp}.$$

We now estimate each term in the decomposition of  $v_{j,R}$ . First, we have the :

**Lemma 7.3.** *There exist constants  $a_1 > 0$ ,  $C > 0$  and  $\delta_1 > 0$  such that, the following pointwise estimate holds*

$$|v_{j,R}^\perp| \leq C (e^{-a_1|x|} + e^{-a_1(|x|-R)}) \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} + \Gamma_{0,-\delta_1} \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)}^2.$$

*Proof.* This estimate follows at once from Remark 4.2 which ensures that, since we are working in the orthogonal complement of functions of the form  $f(y) \partial_x u_0(x)$ , we can choose  $\gamma \in (-\sqrt{\mu_1}, \sqrt{\mu_1})$  in Proposition 4.1 to estimate  $v_{j,R}^\perp$  in terms of the norm of  $h_{j,R}^{1,\perp}$ ,  $h_{j,R}^{2,\perp}$  and  $h_{j,R}^{3,\perp}$ .

To estimate the first term  $h_{j,R}^{1,\perp}$ , we make use of Lemma 7.1 and, applying Proposition 4.1 with  $\delta = -\hat{\delta}$ ,  $\gamma = -\hat{\gamma}$  we get a contribution to the estimate of  $v_{j,R}^\perp$  which is bounded by a constant times  $e^{-a_1|x|}$  provided  $a_1 > 0$  is chosen close enough to 0 (the constants  $\hat{\delta}$  and  $\hat{\gamma}$  are the constants defined in Lemma 7.1 and recall that we have agreed that  $\hat{\gamma} \in (0, \sqrt{\mu_1})$ ). To estimate the second term  $h_{j,R}^{2,\perp}$ , we use the result of Lemma 7.2 which ensures that  $u - u_\lambda$  and  $u_\lambda - u_0$ , restricted to the support of  $h_{j,R}^{2,\perp}$ , decay exponentially at infinity, and apply Proposition 4.1 with some  $\gamma < 0$  close enough to 0 and  $\delta = 0$ , we obtain that this term contributes to the estimate of  $v_{j,R}^\perp$  by a constant times  $e^{-\gamma R} \Gamma_{-\gamma,0} \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)}^2$ . Observe that

the main error comes from the action of the cutoff function when  $\mathbf{x} \in S_{j,R}$  satisfies  $|\mathbf{x}| \in [R, R+1]$ . Finally, to estimate the third term  $h_{j,R}^{3,\perp}$ , we apply Proposition 4.1 with  $\gamma = 0$  and  $\delta < 0$  close enough to 0. Details are left to the reader.  $\square$

Thanks to the previous Lemma, we have already proven that

**Corollary 7.1.** *There exists  $r_* > 0$ ,  $\bar{R} > 0$  and  $\zeta > 0$  such that*

$$\sup_{|\mathbf{x}| > R+r_*} |v_{j,R}^\perp| \leq \frac{1}{4} \mathcal{E}(R),$$

provided  $\mathcal{E}(R) \leq \zeta$  and  $R \geq \bar{R}$ .

It remains to understand the solution of

$$(-\Delta + F''(u_0)) v_{j,R}^\parallel = h_{j,R}^{1,\parallel} + h_{j,R}^{2,\parallel} + h_{j,R}^{3,\parallel}.$$

If we write

$$v_{j,R}^\parallel(x, y) = \phi_{j,R}(y) \partial_x u_0(x), \quad \text{and} \quad h_{j,R}^\ell(x, y) = \psi_{j,R}^\ell(y) \partial_x u_0(x),$$

for  $\ell = 0, 1, 2$ , we find that  $\phi_{j,R}$  is a solution of

$$-\partial_y^2 \phi_{j,R} = \psi_{j,R}^0 + \psi_{j,R}^1 + \psi_{j,R}^2.$$

Since  $v_{j,R}$  has support in  $\mathbb{R} \times (R, \infty)$ , we conclude that  $\phi_{j,R}$  has support in  $(R, \infty)$ . In particular,

$$\phi_{j,R}(y) = A + B(y - R) - \int_y^{+\infty} \int_t^{+\infty} (\psi_{j,R}^0(s) + \psi_{j,R}^1(s)) ds dt - \int_0^y \int_0^t \psi_{j,R}^2(s) ds dt.$$

We have the following :

**Lemma 7.4.** *There exist constants  $a_2 > 0$  and  $C > 0$  such that*

$$\left| \int_y^{+\infty} \int_t^{+\infty} (\psi_{j,R}^0(s) + \psi_{j,R}^1(s)) ds dt \right| \leq C \left( e^{-a_2 y} + e^{-a_2(y-R)} \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \right)$$

and

$$\left| \int_0^y \int_0^t \psi_{j,R}^2(s) ds dt \right| \leq C \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)}^2 (1 + (\max(0, y - R))^2),$$

for all  $y \geq 0$ .

*Proof.* The proof is a simple consequence of Lemma 7.1 and Lemma 7.2.  $\square$

It remains to estimate the parameters  $A$  and  $B$ . We start with the :

**Lemma 7.5.** *There exist a constant  $C > 0$  such that the following estimates hold*

$$|B| \leq C \left( e^{-a_2 R} + \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \right)^{1/2} \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)},$$

provided  $\|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \leq 1$ .

*Proof.* Using the fact that  $v_{j,R}^\parallel$  is bounded by a constant times  $\|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)}$  together with the result of Lemma 7.4, we find

$$|A| \leq C \left( e^{-a_2 R} + \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \right).$$

Once this crude estimate is obtained, we can obtain some precise estimate for  $B$ . To this aim, we again use the fact that  $v_{j,R}^{\parallel}$  is bounded by a constant times  $\|u - u_{\lambda}\|_{L^{\infty}(\mathbb{R}^2 \setminus B_R)}$  and using the result of Lemma 7.4, we get

$$(y - R)|B| \leq C \left( e^{-a_2 R} + \|u - u_{\lambda}\|_{L^{\infty}(\mathbb{R}^2 \setminus B_R)} + (y - R)^2 \|u - u_{\lambda}\|_{L^{\infty}(\mathbb{R}^2 \setminus B_R)}^2 \right),$$

for all  $y \geq R+1$  (here we also use the fact that we assume that  $\|u - u_{\lambda}\|_{L^{\infty}(\mathbb{R}^2 \setminus B_R)} \leq 1$ ). This inequality together with little algebra implies that

$$|B| \leq C \left( e^{-a_2 R} + \|u - u_{\lambda}\|_{L^{\infty}(\mathbb{R}^2 \setminus B_R)} \right)^{1/2} \|u - u_{\lambda}\|_{L^{\infty}(\mathbb{R}^2 \setminus B_R)}.$$

This completes the proof of the estimate.  $\square$

In order to derive an estimate for  $A$ , we appeal to the *balancing formula* (11.11) which follows from Lemma 11.1.

$$(7.16) \quad \int_{\partial\Omega} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) \cdot \nu \, ds = 0$$

for any Killing vector field  $X$  and any compact  $\Omega \subset \mathbb{R}^2$ . Here  $\nu$  is a outward pointing unit normal vector field to  $\partial\Omega$ . Of interest will be the case where  $X$  is the constant vector field

$$X_{\tau} := x \partial_y - y \partial_x,$$

associated to rotations centered at the origin.

Let us briefly digress and consider the case where the function  $u$ , solution of (2.1), is defined on  $\mathbb{R} \times (0, \infty)$  and can be expanded as

$$u(x, y) = u_0(x) + A \partial_x u_0(y) + v(x, y)$$

when  $x$  is close to 0, where  $A$  is small and where the function  $v$  is smaller (in a sense to be made precise). Then, (7.16) implies that

$$\int_{y=y_0} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) \cdot \nu \, ds,$$

does not depend on  $y_0$ . On the one hand, when  $u$  is replaced by  $u_0$ , this quantity is equal to 0, hence, assuming that  $u$  converges fast enough to  $u_0$  and letting  $y_0$  tend to  $\infty$ , we find that

$$\int_{y=y_0} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) \cdot \nu \, ds = 0.$$

On the other hand, inserting  $u = u_0 + A \partial_x u_0 + v$  in this equality and assuming that  $A$  and  $v$  are small, we get

$$\int_{y=y_0} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) \cdot \nu \, ds = A \int_{\mathbb{R}} \left( \frac{1}{2} |\partial_x u_0|^2 + F(u_0) \right) dx + \mathcal{O}(A^2) + \mathcal{O}(\|v\|).$$

Therefore, we conclude that

$$A \int_{\mathbb{R}} \left( \frac{1}{2} |\partial_x u_0|^2 + F(u_0) \right) dx + \mathcal{O}(A^2) + \mathcal{O}(\|v\|) = 0,$$

and this implies that  $A = \mathcal{O}(\|v\|)$ . This is this idea we will implement to derive a precise estimate for the constant  $A$ .

**Lemma 7.6.** *For all  $\varepsilon > 0$ ,  $a_3 > 0$  and there exists  $\zeta_\varepsilon > 0$ ,  $\bar{R}_\varepsilon > 0$  such that*

$$|A| \leq \varepsilon \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \leq C e^{-a_3 R} + \varepsilon \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)},$$

*provided  $R \geq \bar{R}_\varepsilon$  and  $\|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \leq \zeta_\varepsilon$ .*

*Proof.* Since we have assumed that the half line  $\lambda_j^+$  is equal to the  $y$  axis, we have  $u_\lambda = \pm u_0$  close to this axis and to fixe the ideas, let us assume that  $u_\lambda = u_0$ . Now, we define

$$u_{j,R} := u_\lambda + v_{j,R} = u + (1 - \check{\mathbb{I}}_{j,R})v$$

We set

$$h_{j,R} := -\Delta u_{j,R} + F'(u_{j,R}) = -\Delta(u - u_{j,R}) + (F'(u_{j,R}) - F'(u)).$$

So (7.16) has to be replaced by

$$\begin{aligned} \int_{\partial\Omega} \left( \left( \frac{1}{2} |\nabla u_{j,R}|^2 + F(u_{j,R}) \right) X - X(u_{j,R}) \nabla u_{j,R} \right) \cdot \nu ds \\ = \int_{\Omega} h_{j,R} X(u_{j,R}) dx. \end{aligned}$$

Writing  $u_{j,R} = u_0 + w$ , using the fact that  $u_0$  satisfies (7.16) and substracting the two equations, we obtain

$$\begin{aligned} \int_{\partial\Omega} \left( \frac{1}{2} (|\nabla(u_0 + w)|^2 - |\nabla u_0|^2 + F(u_0 + w) - F(u_0)) X \right. \\ \left. - (X(u_0 + w) \nabla(u_0 + w) - X(u_0) \nabla u_0) \right) \cdot \nu ds = \int_{\Omega} h_{j,R} X(u_{j,R}) dx. \end{aligned}$$

We specialize this to the case where  $X = x \partial_y - y \partial_x$  and  $\Omega := \mathbb{R} \times [y_0, y_1]$  and, letting  $y_1$  tend to  $\infty$ , we get

$$\int_{y=y_0} (\partial_x u_0 (\partial_x w - \partial_y w) + (F(u_0 + w) - F(u_0))) dx = \int_{y \geq y_0} h_{j,R} X(u_{j,R}) dx.$$

We decompose

$$u_{j,R} = u_0 + A \partial_x u_0 + \tilde{w}$$

and chose  $y_0 \geq R + 1$ . Thanks to the estimates already derived, we conclude, with little work, that there exosts  $a_3 > 0$  such that

$$\begin{aligned} |A| \leq C \left( e^{-a_3 R} + e^{-a_3(y_0 - R)} \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} + (y_0 - R)^2 \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)}^2 \right. \\ \left. + (y_0 - R) \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} (e^{-a_2 R} + \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)})^{1/2} \right), \end{aligned}$$

provided  $(y_0 - R) \|u - u_\lambda\|_{L^\infty(\mathbb{R}^2 \setminus B_R)} \leq 1$ . Then it is enough to choose  $r_0 := y_0 - R$  large enough so that  $C e^{-a_3(y_0 - R)} \leq \varepsilon$  and then we choose  $\bar{R}_\varepsilon > 0$  large enough and  $\zeta_\varepsilon > 0$  small enough so that  $r_0 \left( e^{-a_2 \bar{R}_\varepsilon} + \zeta_\varepsilon \right)^{1/2} \leq \varepsilon/2$ . This completes the proof of the result.  $\square$

Thanks to the last three Lemma, we get :

**Corollary 7.2.** *There exist  $r_* > 0$ ,  $\bar{R} > 0$  and  $\zeta > 0$  such that*

$$\sup_{|x| > R + r_*} |v_{j,R}^\parallel| \leq \frac{1}{4} \mathcal{E}(R),$$

*provided  $\mathcal{E}(R) \leq \zeta$  and  $R \geq \bar{R}$ .*

The result of Proposition 7.1 follows from Corollary 7.1 and Corollary 7.2. Observe that one can always choose  $\bar{R} > 0$  large enough so that  $\mathcal{E}(\bar{R}) \leq \zeta$  where  $\zeta$  is the least of the two corresponding constants appearing in the statements of Corollary 7.1 and Corollary 7.2.

## 8. THE IMPLICIT FUNCTION THEOREM

We now are in a position to prove Theorem 2.3. To this end, we will apply the implicit function theorem using the linear analysis derived in the previous sections. By now, this argument is rather standard and hence, we will only outline the main points of the proof, leaving the details to the reader.

We start by defining a smooth family of diffeomorphisms of  $\mathbb{R}^2$  as follows : Given  $\mathbf{a} = (a_1, \dots, a_{2k}) \in \mathbb{R}^{2k}$  and  $\mathbf{b} = (b_1, \dots, b_{2k}) \in \mathbb{R}^{2k}$ , we define  $\Phi_{\mathbf{a}, \mathbf{b}}$  to be the value at time 1 of the flow associated to the vector field

$$\Xi_{\mathbf{a}, \mathbf{b}} := \sum_{j=1}^{2k} (a_j X_j + b_j Y_j).$$

Here  $X_j, Y_j$  are the vector fields defined in (6.1) and (6.2). Recall that,  $X_j$  corresponds to a translation the direction transversal to the  $j$ -th end while  $Y_j$  corresponds to a rotation of the  $j$ -th end. The map  $\Phi_{\mathbf{a}, \mathbf{b}}$  is clearly a diffeomorphism provided all coefficients  $a_j$  and  $b_j$  are small enough. Also, observe that, in most of the sector where  $\mathbb{I}_j \equiv 1$ , the map  $\Phi_{\mathbf{a}, \mathbf{b}}$  is equal to a rigid motion and hence it commutes with  $\Delta$  in the sense that

$$\Delta(w \circ \Phi_{\mathbf{a}, \mathbf{b}}) = (\Delta w) \circ \Phi_{\mathbf{a}, \mathbf{b}}.$$

We choose  $\gamma, \delta > 0$  close enough to 0. Given a function  $v \in W_{-\gamma, -\delta}^{2,2}(\mathbb{R}^2)$  and  $\mathbf{a} \in \mathbb{R}^{2k}$  and  $\mathbf{b} \in \mathbb{R}^{2k}$ , we define

$$N(v, \mathbf{a}, \mathbf{b}) := \left( -\Delta((u+v) \circ \Phi_{\mathbf{a}, \mathbf{b}}) + F'((u+v) \circ \Phi_{\mathbf{a}, \mathbf{b}}) \right) \circ \Phi_{-\mathbf{a}, -\mathbf{b}},$$

where  $u \in \mathcal{M}_{2k}$  is fixed. It is easy to check that the nonlinear map  $N$  is well defined and smooth from a neighborhood of 0 in  $W_{-\gamma, -\delta}^{2,2}(\mathbb{R}^2) \times \mathbb{R}^{2k} \times \mathbb{R}^{2k}$  into  $L_{\gamma, \delta}^2(\mathbb{R}^2)$ . This essentially follows from the fact that  $\Phi_{\mathbf{a}, \mathbf{b}}$  is a rigid motion and in the regions where it is not, the functions involved are in  $L_{-\gamma, -\delta}^2(\mathbb{R}^2)$ .

Since  $\Phi_{0,0} = \text{Id}$  therefore we check directly that

$$(8.1) \quad D_v N_{(0,0,0)} = \mathfrak{L}.$$

Moreover, we have

$$\partial_{a_j} N_{(0,0,0)} = \mathfrak{L}(du(X_j)), \quad \text{and} \quad \partial_{b_j} N_{(0,0,0)} = \mathfrak{L}(du(Y_j)).$$

If we assume that  $u \in \mathcal{M}_{2k}$  is nondegenerate, the result of Proposition 6.2 implies that the full differential  $DN_{(0,0,0)}$  (which can be naturally identified with the operator  $\tilde{\mathfrak{A}}_{-\gamma, -\delta}$ ) is surjective and has a kernel whose dimension is  $2k$ . The implicit function theorem then implies that, close to  $u = 0$ ,  $\mathbf{a} = \mathbf{b} = 0$  the set of zeros of  $N$  is a smooth  $2k$  dimensional manifold. This completes the result of the Theorem.

9. NONDEGENERACY OF SOLUTIONS WITH NODAL SET WITH NEARLY PARALLEL COMPONENTS.

In this section, we will check that for each  $k \geq 1$ , the set  $\mathcal{M}_{2k}$  contains at least one nondegenerate element, thus proving Proposition 2.1. We recall that for simplicity we assume here that  $F(u) = \frac{1}{4}(1 - u^2)^2$  however all the results needed to prove the proposition generalize directly to the case of a general, smooth even nonlinearity as described in (1.2). Consequently we expect that the assertion of Proposition 2.1 holds in this more general framework as well.

The result in [8] implies the existence of a family of solutions  $\{u_\varepsilon\}_{0 < \varepsilon < \varepsilon_0} \subset \mathcal{M}_{2k}$  whose ends are nearly parallel, and are graphs over the  $y$ -axis for some function whose derivative is bounded by a constant times  $\varepsilon$ . What is more, the construction shows that  $\{\mathbf{x} \in \mathbb{R}^2 : u_\varepsilon(\mathbf{x}) = 0\}$ , the nodal set of  $u_\varepsilon$ , decomposes into exactly  $k$  disjoint curves whose mutual distances tends to  $\infty$  as  $\varepsilon$  tends to 0.

To state our result in precise way, we assume that we are given a solution  $\mathbf{q} := (q_1, \dots, q_k)$  of the *Toda system*

$$(9.1) \quad c_0 q_j'' = e^{\sqrt{2}(q_{j-1} - q_j)} - e^{\sqrt{2}(q_j - q_{j+1})},$$

for  $j = 1, \dots, k$ , where  $c_0 = \frac{\sqrt{2}}{24}$  and we agree that

$$q_0 \equiv -\infty \quad \text{and} \quad q_{k+1} \equiv +\infty.$$

The Toda system (9.1) is a classical example of integrable system which has been extensively studied. It models the dynamics of finitely many mass points on the line under the influence of an exponential potential. We recall some of the results which are concerned with the solvability of (9.1) and which will be needed for our purposes. We refer to [12] and [18] for the complete description of the theory (see also [20], [23], [1]). Of importance for us is the fact that solutions of (9.1) can be described (almost explicitly) in terms of  $2k$  parameters. Moreover, if  $\mathbf{q}$  is a solution of (9.1), then the long term behavior (i.e. long term scattering) of the  $q_j$  at  $\pm\infty$  is well understood and it is known that, for all  $j = 1, \dots, k$ , there exist  $a_j^+, b_j^+ \in \mathbb{R}$ ,  $a_j^-, b_j^- \in \mathbb{R}$  and  $\tau_0 > 0$ , all depending on the solution  $\mathbf{q}$ , such that

$$(9.2) \quad q_j(t) = a_j^\pm |t| + b_j^\pm + \mathcal{O}_{\mathcal{C}^\infty(\mathbb{R})}(e^{-\tau_0 |t|}),$$

as  $t$  tends to  $\pm\infty$ . Moreover,  $a_{j+1}^\pm > a_j^\pm$  for all  $j = 1, \dots, k-1$ .

Given  $\varepsilon > 0$ , we define the vector valued function  $\mathbf{q}_\varepsilon$ , whose components are given by

$$(9.3) \quad q_{j,\varepsilon}(x) := q_j(\varepsilon x) - \sqrt{2} \left( j - \frac{k+1}{2} \right) \log \varepsilon.$$

It is easy to check that the  $q_{j,\varepsilon}$  are again solutions of (9.1).

Observe that, according to the description of the asymptotics of the functions  $q_j$ , the graphs of the functions  $q_{j,\varepsilon}$  are asymptotic to a set of  $2k$  oriented half lines. In addition, for  $\varepsilon > 0$  small enough, these graphs are disjoint and in fact their mutual distance is given by  $-\sqrt{2} \log \varepsilon + \mathcal{O}(1)$  as  $\varepsilon$  tends to 0.

It will be convenient to agree that  $\chi^+$  (resp.  $\chi^-$ ) is a smooth cutoff function defined on  $\mathbb{R}$  which is identically equal to 1 for  $x > 1$  (resp. for  $x < -1$ ) and identically equal to 0 for  $x < -1$  (resp. for  $x > 1$ ) and additionally  $\chi^- + \chi^+ \equiv 1$ . With these cutoff functions at hand, we define the 4-dimensional space

$$(9.4) \quad D := \text{Span} \{x \mapsto \chi^\pm(x), x \mapsto x \chi^\pm(x)\},$$

and, for all  $\mu \in (0, 1)$  and all  $\tau \in \mathbb{R}$ , we define the space  $\mathcal{C}_\tau^{2,\mu}(\mathbb{R})$  of  $\mathcal{C}^{2,\mu}$  functions  $h$  which satisfy

$$\|h\|_{\mathcal{C}_\tau^{2,\mu}(\mathbb{R})} := \|(\cosh x)^\tau h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty.$$

Keeping in mind the above notations, the following is proven in [8]:

**Theorem 9.1.** *For all  $\varepsilon > 0$  sufficiently small, there exists an entire solution  $u_\varepsilon \in \mathcal{M}_{2k}$  of the Allen-Cahn equation*

$$(9.5) \quad \Delta u + u - u^3 = 0,$$

in  $\mathbb{R}^2$ , whose nodal set is the union of  $k$  disjoint curves  $\mathcal{C}_{1,\varepsilon}, \dots, \mathcal{C}_{k,\varepsilon}$ . These curves are the graphs of the functions

$$x \mapsto q_{j,\varepsilon}(x) + h_{j,\varepsilon}(\varepsilon x),$$

for some functions  $h_{j,\varepsilon} \in \mathcal{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D$  satisfying

$$\|h_{j,\varepsilon}\|_{\mathcal{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D} \leq C \varepsilon^a.$$

for some constants  $C, a, \tau, \mu > 0$  independent of  $\varepsilon > 0$ .

In other words, given a solution of the Toda system, we can find a one parameter family of  $2k$ -ended solutions of (9.5) which depend on a small parameter  $\varepsilon > 0$ . As  $\varepsilon$  tends to 0, the nodal sets of the solutions we construct become close to the graphs of the functions  $q_{j,\varepsilon}$ .

Using also Proposition 1.1 and Proposition 5.4 in [8] one can be more precise about the description of the solution  $u_\varepsilon$ . If  $\lambda_\varepsilon := (\lambda_{1,\varepsilon}^+, \dots, \lambda_{2k,\varepsilon}^+)$  denotes the half affine lines associated to the ends of  $u_\varepsilon$  and if  $u_{\lambda_\varepsilon}$  denotes the associated function defined in (2.6), we know that  $u_\varepsilon = u_{\lambda_\varepsilon} + v_\varepsilon$ , where  $v_\varepsilon \in W_{-\gamma\varepsilon, -\delta}^{2,2}(\mathbb{R}^2)$  with some  $\delta, \gamma > 0$  such that  $\delta^2 + \gamma^2\varepsilon^2 < \alpha^2$ . In addition we have

$$(9.6) \quad \|v_\varepsilon\|_{W_{-\gamma\varepsilon, -\delta}^{2,2}(\mathbb{R}^2)} \leq C \varepsilon^{\bar{a}},$$

with some  $\bar{a} > 0$ .

*Proof of Proposition 2.1.* Our goal is to show that there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  the solution  $u_\varepsilon$  of (9.5) which is given by Theorem 9.1 is nondegenerate. We fixe  $\bar{\gamma}, \delta > 0$  such that  $\delta^2 < \alpha^2$ . We claim that, for  $\varepsilon > 0$  sufficiently small, the unique solution of the equation

$$(-\Delta + F''(u_\varepsilon)) w = 0,$$

which belongs to  $L_{-\bar{\gamma}\varepsilon, -\delta}^2(\mathbb{R}^2)$ , is  $w \equiv 0$  (observe that  $\varepsilon^2 \bar{\gamma}^2 + \delta^2 < \alpha^2$  for  $\varepsilon > 0$  close enough to 0). To prove the claim, we will use an argument similar to the one used in the proof of the Linear Decomposition Lemma (i.e. Lemma 4.2).

To begin with, since the problem is linear we may well assume that

$$(9.7) \quad \|w\|_{L_{-\bar{\gamma}\varepsilon, -\delta}^2(\mathbb{R}^2)} = 1.$$

Then, elliptic estimates imply that  $u \in W_{-\bar{\gamma}\varepsilon, -\delta}^{2,2}(\mathbb{R}^2)$  and

$$\|w\|_{W_{-\bar{\gamma}\varepsilon, -\delta}^{2,2}(\mathbb{R}^2)} \leq C \|w\|_{L_{-\bar{\gamma}\varepsilon, -\delta}^2(\mathbb{R}^2)}.$$

For  $j = 1, \dots, k$ , we define  $\hat{\mathbb{I}}_j$  to be a smooth cutoff functions such that

$$\text{Support } \hat{\mathbb{I}}_j \subset \left\{ \mathbf{x} \in \mathbb{R}^2 : \text{dist}(\mathbf{x}, \mathcal{C}_{j,\varepsilon}) \leq \text{dist}(\mathbf{x}, \mathcal{C}_{i,\varepsilon}) - \frac{\sqrt{2}}{4} \log \varepsilon, \quad \forall i \neq j \right\},$$

and  $\hat{\mathbb{I}}_j(\mathbf{x}) \equiv 1$  when  $\text{dist}(\mathbf{x}, \mathcal{C}_{j,\varepsilon}) \leq \text{dist}(\mathbf{x}, \mathcal{C}_{i,\varepsilon})$  for  $i \neq j$ . Using (9.7), we can assume that for some  $j_0 \in \{1, \dots, k\}$ , we have

$$(9.8) \quad \|\hat{\mathbb{I}}_{j_0} w\|_{L^2_{-\bar{\gamma}\varepsilon, -\delta}(\mathbb{R}^2)} \geq \frac{1}{2k}.$$

We define  $w_0 := \hat{\mathbb{I}}_{j_0} w$ . Observe that

$$\mathfrak{L}_\varepsilon w_0 = [\Delta, \hat{\mathbb{I}}_{j_0}] w.$$

where  $\mathfrak{L}_\varepsilon := -\Delta + F''(u_\varepsilon)$ .

Let  $(t, s)$  be Fermi coordinates associated to  $\mathcal{C}_{j_0,\varepsilon}$ , so that  $s$  is the arc length on  $\mathcal{C}_{j_0,\varepsilon}$  and  $t(\mathbf{x}) := \text{dist}^s(\mathbf{x}, \mathcal{C}_{j_0,\varepsilon})$  denotes the signed distance to  $\mathcal{C}_{j_0,\varepsilon}$ . We change the coordinates from  $(x, y)$  to Fermi coordinates  $(t, s)$  (this change of variables is quite standard and we refer the reader to calculations in Section 5 of [8] for details).

Using (9.3), (9.6) together with the assumption on the function  $w$ , we check that

$$h_\varepsilon := (-\partial_s^2 - \partial_t^2 + F''(u_0(t))) w_0,$$

satisfies

$$(9.9) \quad \|h_\varepsilon\|_{L^2_{-\bar{\gamma}\varepsilon, -\delta}(\mathbb{R}^2)} \leq C \left( \varepsilon^\tau + \frac{1}{\log \frac{1}{\varepsilon}} \right) \|w\|_{L^2_{-\bar{\gamma}\varepsilon, -\delta}(\mathbb{R}^2)},$$

for some  $\tau > 0$ . Thanks to the Linear Decomposition Lemma (Lemma 4.2), we know that  $w_0 \in L^2_{-\bar{\gamma}\varepsilon, -\delta}(\mathbb{R}^2)$  and also that

$$\|w_0\|_{L^2_{-\bar{\gamma}\varepsilon, -\delta}(\mathbb{R}^2)} \leq C \left( \varepsilon^\tau + \frac{1}{\log \frac{1}{\varepsilon}} \right) \|w\|_{L^2_{-\bar{\gamma}\varepsilon, -\delta}(\mathbb{R}^2)}.$$

For  $\varepsilon$  small enough this inequality is not compatible with (9.8) and (9.7). Having reached a contradiction, we conclude that  $w_0 \equiv 0$  and hence  $w \equiv 0$ . This completes the proof of the result.  $\square$

## 10. THE LAGRANGIAN STRUCTURE AND THE PROOF OF THEOREM 2.3

We assume that we are given a function  $u$  which is a nondegenerate element of  $\mathcal{M}_{2k}$ . The tangent space  $T_u \mathcal{M}_{2k}$  is spanned by the functions  $w$  which satisfy

$$\mathfrak{L} w = 0.$$

According to the result of Proposition 6.1 and Proposition 6.2, we know that such a function  $w$  can be decomposed into

$$(10.10) \quad w = \sum_{j=1}^{2k} a_j du(X_j) + b_j du(Y_j) + \bar{w},$$

where  $a_j, b_j \in \mathbb{R}$  and  $\bar{w} \in W_{-\gamma, -\delta}^{2,2}(\mathbb{R}^2)$  for some  $\gamma, \delta > 0$ . Let us consider two such functions  $w^{(1)}, w^{(2)} \in T_u \mathcal{M}_{2k}$  (the coefficients and functions in the corresponding decompositions will be adorned with superscripts (1) and (2)). For all  $R > 0$ , we integrate  $(w^{(1)} \mathfrak{L} w^{(2)} - w^{(2)} \mathfrak{L} w^{(1)})$  over the ball of radius  $R$  to get

$$0 = \int_{B_R} (w^{(1)} \mathfrak{L} w^{(2)} - w^{(2)} \mathfrak{L} w^{(1)}) dx = \int_{\partial B_R} (w^{(1)} \partial_r w^{(2)} - w^{(2)} \partial_r w^{(1)}) ds$$

But, using (10.10) and letting  $R \rightarrow \infty$  we conclude that

$$\sum_{j=1}^{2k} (a_j^{(1)} b_j^{(2)} - a_j^{(2)} b_j^{(1)}) = 0.$$

This completes the proof of Theorem 2.3.

## 11. APPENDIX A

Let  $u$  be a solution of (2.1) in  $\mathbb{R}^2$ . Assume that  $X$  is a vector field in  $\mathbb{R}^2$ . If  $X = \sum_i X^i \partial_{x_i}$ ,  $Y = \sum_i Y^i \partial_{x_i}$  and  $f$  is a smooth function, we will use the following notations

$$\begin{aligned} X(f) &:= \sum_j X^j \partial_{x_j} f, & \nabla f &:= \sum_j \partial_{x_j} f \partial_{x_j}, \\ \operatorname{div} X &:= \sum_i \partial_{x_i} X^i, \end{aligned}$$

and

$$d^* X := \frac{1}{2} \sum_{i,j} (\partial_{x_i} X^j + \partial_{x_j} X^i) dx_i \otimes dx_j,$$

so that

$$d^* X(Y, Y) = \sum_{i,j} \partial_{x_i} X^j Y^i Y^j.$$

We claim that :

**Lemma 11.1.** *(Balancing formula) The following identity holds*

$$\operatorname{div} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) = \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) \operatorname{div} X - d^* X(\nabla u, \nabla u).$$

*Proof.* This follows from direct computation.  $\square$

Observe that (2.1) is invariant under the action of isometries of  $\mathbb{R}^2$ . Translations of  $\mathbb{R}^2$  correspond to the constant vector field

$$X := X_0$$

where  $X_0$  is a fixed vector, while rotations correspond to the vector field

$$X := x \partial_y - y \partial_x.$$

In either case, we have  $\operatorname{div} X = 0$  and  $d^* X = 0$ . Therefore, we conclude that

$$\operatorname{div} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) = 0,$$

for these two vector fields. The divergence theorem then implies that

$$(11.11) \quad \int_{\partial\Omega} \left( \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) \cdot \nu ds = 0,$$

where  $\nu$  is the (outward pointing) unit normal vector field to  $\partial\Omega$ .

We set

$$c_0 := \int_{-\infty}^{+\infty} \left( \frac{1}{2} (\partial_x u_0)^2 + F(u_0) \right) dx,$$

and

$$c_1 := \int_{-\infty}^{+\infty} \left( \frac{1}{2} (\partial_x u_0)^2 - F(u_0) \right) x dx.$$

We now assume that  $u \in \mathcal{M}_{2k}$  and we keep the notations introduced in §2. We use (11.11) with  $\Omega$  equal to the ball of radius  $R$  centered at the origin and we then let  $R$  tend to  $\infty$ . Taking  $X = X_0$  and using (1.4), we find with little work

$$c_0 \sum_{j=1}^{2k} \mathbf{e}_j \cdot X_0 = 0.$$

Taking  $X = x \partial_y - y \partial_x$ , we find

$$\sum_{j=1}^{2k} ((-1)^j c_1 - c_0 r_j) = 0.$$

Hence

$$\sum_{j=1}^{2k} r_j = 0.$$

This completes the proof of (2.15) and (2.16).

**Remark 11.1.** *Observe that these conservation laws also hold when the nonlinearity  $F$  is not even.*

## 12. APPENDIX B

We keep the notations of the proof of Proposition 3.1 and we define

$$\hat{W}(x) := \frac{1}{\mathbb{W}_\zeta} W_\zeta^-(x) \int_{-\infty}^x W_\zeta^+(y) V(y) dy.$$

Since  $|W_\zeta^\pm| \leq C e^{\pm \delta_\zeta x}$ , we can write

$$|\hat{W}(x)| \leq C \tilde{W}(x),$$

where

$$\tilde{W}(x) := e^{-\delta_\zeta x} \int_{-\infty}^x e^{\delta_\zeta y} |V(y)| dy.$$

Obviously, it is enough to prove that

$$(12.12) \quad \|\tilde{W}\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})} \leq C \|V\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})},$$

since this will imply that

$$\|\hat{W}\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})} \leq C \|V\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})}.$$

The result will then follow from this estimate, together with a similar estimate for the function

$$\check{W}(x) := \frac{1}{W_\zeta} W_\zeta^+(x) \int_x^{+\infty} W_\zeta^-(y) V(y) dy.$$

To prove (12.12), we argue as follows. First, the following pointwise estimate follows from Cauchy-Schwarz inequality

$$(12.13) \quad |(\cosh y)^{-\delta} \tilde{W}(y)| \leq c \|V\|_{\tilde{L}_\delta^2(\mathbb{R}, \mathbb{C})}$$

for all  $y \in \mathbb{R}$ .

Next, we estimate

$$\begin{aligned}
\int_{-\infty}^0 e^{2\delta x} \tilde{W}^2(x) dx &= \int_{-\infty}^0 e^{2(\delta-\delta_\zeta)x} \left( \int_{-\infty}^x e^{\delta_\zeta y} |V(y)| dy \right)^2 dx \\
&= \lim_{s \rightarrow -\infty} \left[ \frac{1}{2(\delta-\delta_\zeta)} e^{2\delta x} \tilde{W}^2(y) \right]_s^0 \\
&\quad - \frac{1}{\delta-\delta_\zeta} \int_{-\infty}^0 e^{2\delta x} V(x) \tilde{W}(x) dx \\
&\leq C (\|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}^2 + \|\tilde{W}\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}) \|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})},
\end{aligned}$$

where we have used (12.13) together with Cauchy-Schwarz inequality to obtain the last estimate. Using similar arguments, we also get

$$\int_0^{+\infty} e^{-2\delta x} \tilde{W}^2(x) dx \leq C (\|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}^2 + \|\tilde{W}\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}) \|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}.$$

Hence, we conclude that

$$\|\tilde{W}\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}^2 \leq C \left( \|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}^2 + \|\tilde{W}\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})} \|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})} \right),$$

which implies that

$$\|\tilde{W}\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})} \leq C \|V\|_{\tilde{L}_\delta^2(\mathbb{R},\mathbb{C})}.$$

This completes the proof of the estimate.

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