LARGE TIME BEHAVIOR FOR A NONLOCAL DIFFUSION EQUATION WITH ABSORPTION AND BOUNDED INITIAL DATA

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ABSTRACT. We study the large time behavior of nonnegative solutions of the Cauchy problem $u_t = \int J(x-y)(u(y,t)-u(x,t))\,dy - u^p$, $u(x,0)=u_0(x)\in L^\infty$, where $|x|^\alpha u_0(x)\to A>0$ as $|x|\to\infty$. One of our main goals is the study of the critical case $p=1+2/\alpha$ for $0<\alpha<N$, left open in previous articles, for which we prove that $t^{\alpha/2}|u(x,t)-U(x,t)|\to 0$ where U is the solution of the heat equation with absorption with initial datum $U(x,0)=C_{A,N}|x|^{-\alpha}$. Our proof, involving sequences of rescalings of the solution, allows us to establish also the large time behavior of solutions having more general nonintegrable initial data u_0 in the supercritical case and also in the critical case ($p=1+2/N$) for bounded and integrable u_0.

1. Introduction

Consider the following nonlocal evolution problem with absorption

$$\begin{cases} u_t = \int J(x-y)\,(u(y,t)-u(x,t))\;dy - u^p(x,t) & \text{in } \mathbb{R}^N\times(0,\infty),\\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $J\in C_0^\infty(\mathbb{R}^N)$ is radially symmetric, $J\geq 0$ with $\int J=1$ and $u_0\geq 0$ and bounded.

Equation (1.1) can be seen as a model for the density of a population at a certain point and given time. In fact, let u represent such density and the kernel $J(x-y)$ represent the probability distribution density of jumping from a point x to a point y. Then, using the symmetry of the kernel J, the diffusion term $\int J(x-y)(u(y,t)-u(x,t))\,dy$ represents the difference between the rate at which the population is arriving at the point x and the rate at which it is leaving x. The absorption term $-u^p$ represents a rate of consumption due to an internal reaction.

This diffusion operator or similar ones have been used to model several nonlocal diffusion processes in the last few years. See for instance [1, 2, 3, 4, 5, 11, 22]. In particular, nonlocal diffusions are of interest in biological and biomedical problems. Recently, these kind of nonlocal operators have also been used for image enhancement [12].

We are interested in the large time behavior of the solutions to (1.1) and how the space dimension N, the absorption exponent p and the assumptions on the initial data u_0 influence the result.

These kind of problems have been widely studied for the heat equation with absorption or, more generally, the porous medium equation or other diffusion equations. (See, for instance, [7, 14, 16, 17, 18, 23]).

In the case of semilinear problems, one possible approach is the direct use of the variations of constants formula associated to the knowledge of a fundamental solution of the purely diffusive linear part. This approach has been considered for problem (1.1) in [20, 21] and it allowed to

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study the supercritical cases, in which the presence of the absorption term does not influence the asymptotic behavior.

Another possible approach is to consider rescalings that leave the purely diffusive equation unchanged. This is done, for instance, for the porous medium or the evolutionary p-laplace equations.

At first sight, this last approach is not possible for (1.1) since the nonlocal diffusion equation is not invariant under any rescaling. Anyhow, it is easy to see that if u is a solution to

$$u_t = \int J(x-y)(u(y,t)-u(x,t))\,dy = Lu, \tag{1.2}$$

and, for $k>0$ and $f(k)$ any function of the parameter k,

$$u^k(x,t) = f(k)u(kx, k^2 t),$$

then u^k is a solution to the following equation

$$v_t = k^2 \int J_k(x-y)\big(v(y,t)-v(x,t)\big)\,dy, \tag{1.3}$$

where $J_k(x) = k^N J(kx)$. It is not difficult to prove that for a fixed smooth function v, the right hand side in (1.3) converges, as k goes to infinity, to $\mathfrak{a}\,\Delta v$ where $\mathfrak{a}$ is a constant that depends only on the kernel J of the nonlocal operator.

This fact has already been used –with $\varepsilon = k^{-1} \to 0$– in order to prove that the solutions of the rescaled problems set in a fixed bounded domain converge to the solution of the heat equation with diffusivity $\mathfrak{a}$. (See, for instance, [9, 10]).

Therefore, since k going to infinity for $u^k(x,1)$ amounts to t going to infinity for $u(x,t)$, it is not at all striking that the asymptotic behavior as t goes to infinity of the solution of (1.1) is the same as that of the solution of the equation obtained by replacing the nonlocal operator by $\mathfrak{a}\,\Delta$, as was proved in [20] when $u_0 \in L^\infty \cap L^1$ and $p > 1+2/N$ and in [21] when u_0 is bounded and $|x|^\alpha u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha \le N$ and $p > 1 + 2/\alpha$.

Both in [20] and [21] no rescaling was proposed and instead, the variations of constants formula was used. Nevertheless, such method only led to the study of the supercritical cases.

As stated above, the idea of the rescaling method is that the behavior of $u(x,t)$ as $t \to \infty$ is that of $u^k(x,1)$ as $k \to \infty$ (for a suitable choice of $f(k)$).

In order to prove the convergence of the functions $u^k(x,1)$ on compact sets of $\mathbb{R}^N$, it is necessary to establish compactness of a family of uniformly bounded solutions to the equation satisfied by u^k. In the case of the heat or the porous medium equations, this compactness follows from their regularizing effect.

The purpose of this paper is twofold. On one hand, to establish the large time behavior in the critical case $p = 1 + 2/\alpha$, $0 < \alpha < N$ that was left open in [21]. On the other hand, to show how to use the rescaling method in the present situation in which each u^k is a solution of a different equation. And moreover, how to obtain, in the present situation, the compactness of the family $\{u^k\}$ from uniform L^∞ bounds.

One very important issue that we had to overcome is the lack of a regularizing effect of equation (1.2) and its rescalings. In fact, $u(x,t)$ is exactly as smooth as $u_0(x)$. This problem is overcome by the observation that the fundamental solution of equation (1.2) can be decomposed as $e^{-t}\delta + W(x,t)$ with $W(x,t)$ smooth, as proved in [6]. Then, the variations of constants formula for the rescaled equations allowed us to decompose $u^k(x,t) = v^k(x,t) + h^k(x,t)$ with $v^k \to 0$

and h^k smooth. So, the limit as $k \to \infty$ of $u^k(x,t)$ for $t > 0$ is that of the smooth functions h^k, for which we can prove compactness by establishing uniform Holder estimates.

One of the main contributions of the present paper is a very sharp estimate on the space-time behavior of $W(x,t)$ (as well as its space and time derivatives). This estimate is invariant under the rescaling $W_k(x,t) = k^N W(kx, k^2 t)$, thus leading to estimates for the rescalings u^k of the solution u.

This study of the good part of the fundamental solution and the idea of how to use the rescaling method in order to study asymptotics related to the nonlocal diffusion equation (1.2) give insight into how to attack other semilinear problems related to this equation like blow up or quenching problems (determine blow up and quenching profiles, for instance) and thus they are of an independent interest.

In the present paper we apply this method in order to study the asymptotic behavior as time goes to infinity of the solution to (1.1) for general bounded initial data u_0. We find that this behavior is determined by the way u_0 decays at infinity, retrieving for (1.1) results that were known for the heat equation (see [14, 16, 18]). In particular, this method allows to treat the critical case $p = 1 + 2/\alpha$, $0 < \alpha < N$ left open in our previous paper [21]. Moreover, we also complete the results of [20] for integrable initial data by proving that –in the critical case $p = 1 + 2/N$– there holds that $t^{N/2} u(x,t) \to 0$ as $t \to \infty$ (as compared to the supercritical case $p > 1 + 2/N$ where a nontrivial limit is achieved).

Moreover, since our approach allows for very general initial data, we find results for both integrable and nonintegrable initial data that do not behave as a negative power at infinity, and we give some examples of application of our results to such cases at the end of this article.

Before presenting some examples of initial data to which our results apply let us introduce some notation.

Notation. We consider rescalings that depend on the behavior of u_0 at infinity. For $k > 0$ we denote by $v_k(x,t) = k^N v(kx, k^2 t)$ and, following the notation in [18], we denote by $v^k(x,t) = f(k) v(kx, k^2 t)$ where

$$f(k) := \frac{k^N}{\int_{B_k} u_0}.$$

Remark 1.1. *Since $u_0 \geq 0$, $u_0 \neq 0$, there exist $\kappa > 0$ and $x_0 \in \mathbb{R}^N$ such that $\int_{B_k(x_0)} u_0(x)\,dx \geq \kappa k^N$ for $k > 0$ small. Without loss of generality we will assume that $x_0 = 0$.*

We assume further.

Conditions on f

(F1) $u_0 \in L^\infty(\mathbb{R}^N)$ and there exists $B > 0$ such that $f(|x|) u_0(x) \leq B$.

(F2) For every $\delta > 0$, there exists $C_\delta > 0$ such that $f(k) \leq C_\delta f(l)$ if $k_0 \leq k \leq l\delta^{-1}$.

(F3) There exists $c_0 \geq 0$ such that $F(k) := f(k)^{1-p} k^2 \to c_0$ as $k \to \infty$.

Remark 1.2. *Assumption* **(F3)** *implies that $f(k) \geq c_1 k^{2/(p-1)}$ if $k \geq k_0$ and k_0 is large. Therefore, $f(k) \to \infty$ as $k \to \infty$.*

Examples By including the function f in our rescaled sequence we are able to deal with rather general initial conditions. It is also interesting to note that, for $u_0 \in L^\infty$ satisfying

$$|x|^\alpha u_0(x) \to A > 0 \qquad \text{as } |x| \to \infty \ \text{ with } 0 < \alpha < N, \tag{1.4}$$

the function f behaves like k^α as k tends to infinity. This is then, a rather usual rescaling in this case.

The following is a list of examples of initial data satisfying our assumptions.

Ex. 1 Assume $u_0 \in L^\infty(\mathbb{R}^N)$ and $|x|^\alpha u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$. Then,

$$f(k) \sim k^\alpha \qquad \text{so we take it to be equal,}$$
$$F(k) = k^{-\alpha(p-1)+2} \to \begin{cases} 0 & \text{if } p > 1 + \frac{2}{\alpha}, \\ 1 & \text{if } p = 1 + \frac{2}{\alpha}. \end{cases}$$

Ex. 2 Assume $u_0 \in L^\infty(\mathbb{R}^N)$ and $|x|^N u_0(x) \to A > 0$ as $|x| \to \infty$. Then,

$$f(k) \sim \frac{k^N}{\log k} \qquad \text{so we take it to be equal,}$$
$$F(k) = k^{-N(p-1)+2}(\log k)^{p-1} \to 0 \in \mathbb{R} \qquad \text{if } p > 1 + \frac{2}{N}.$$

Ex. 3 Assume $u_0 \in L^\infty(\mathbb{R}^N)$ and $\dfrac{|x|^\alpha}{\log|x|} u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$. Then,

$$f(k) \sim \frac{k^\alpha}{\log k} \qquad \text{so we take it to be equal,}$$
$$F(k) = k^{-\alpha(p-1)+2}(\log k)^{p-1} \to 0 \in \mathbb{R} \qquad \text{if } p > 1 + \frac{2}{\alpha}.$$

Ex. 4 Assume $u_0 \in L^\infty(\mathbb{R}^N)$ and $|x|^\alpha(\log|x|)u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$. Then,

$$f(k) \sim k^\alpha \log k \qquad \text{so we take it to be equal,}$$
$$F(k) = \frac{k^{-\alpha(p-1)+2}}{(\log k)^{p-1}} \to 0 \quad \text{if } p \geq 1 + \frac{2}{\alpha}.$$

Ex. 5 Assume $u_0 \in L^\infty(\mathbb{R}^N)$ and $\dfrac{|x|^N}{\log|x|} u_0(x) \to A > 0$ as $|x| \to \infty$. Then,

$$f(k) \sim \frac{k^N}{\log^2 k} \qquad \text{so we take it to be equal,}$$
$$F(k) = k^{-N(p-1)+2}(\log k)^{2(p-1)} \to 0 \quad \text{if } p > 1 + \frac{2}{N}.$$

Ex. 6 Assume $u_0 \in L^\infty(\mathbb{R}^N)$ and $|x|^N(\log|x|)u_0(x) \to A > 0$ as $|x| \to \infty$. Then,

$$f(k) \sim \frac{k^N}{\log\log k} \qquad \text{so we take it to be equal,}$$
$$F(k) = k^{-N(p-1)+2}(\log\log k)^{p-1} \to 0 \quad \text{if } p > 1 + \frac{2}{N}.$$

Ex. 7 Assume $u_0 \in L^1(\mathbb{R}^N)$. Then,

$$f(k) \sim k^N \qquad \text{so we take it to be equal,}$$
$$F(k) = k^{-N(p-1)+2} \to \begin{cases} 0 & \text{if } p > 1 + \frac{2}{N}, \\ 1 & \text{if } p = 1 + \frac{2}{N}. \end{cases}$$

Let us just mention that the function $f(k)$ is related to the rate of decay in time of the solution whereas $c_0 = \lim_{k\to\infty} F(k)$ turns out to be the coefficient in front of the absorption term in the equation satisfied by the limiting profile.

For our main results we refer to Section 4.

The paper is organized as follows. In Section 2 we construct barriers for the good part W of the fundamental solution of (1.2), as well as for its space and time derivatives. By using these barriers we obtain an upper bound for the solution u_L to (1.2) for all times. This result improves the one we had established for finite time intervals in [21] in case u_0 satisfies (1.4). Also, this bound on u_L implies that the rescaled functions u^k are uniformly bounded in $\mathbb{R}^N \times [\tau, \infty)$ for every $\tau > 0$.

In Section 3 we analyze the rescaled problems satisfied by the u^k's and prove that, under sequences, $\{u^k\}$ converges uniformly on compact sets to a solution U of the equation $U_t - \mathfrak{a}\Delta U = -c_0 U^p$, where $c_0 = \lim_{k\to\infty} F(k)$.

Moreover, we obtain a general result stating that we can determine the limit function U. In fact, we prove that if $u_0^k(x) \to \phi(x)$ as k tends to infinity in the sense of distributions and, for every $R > 0$ there holds that

$$||u^k||_{L^1(B_R\times(0,\tau))} \le C(R)\tau,$$

and either there exists $\gamma > 0$ such that

$$||u^k||_{L^p(B_R\times(0,\tau))} \le C(R)\tau^\gamma,$$

or, for every $\tau \le 1$, $R > 0$,

$$\lim_{k\to\infty} ||u^k||_{L^p(B_R\times(0,\tau))} = 0,$$

then U satisfies moreover,

$$U(x, 0) = \phi(x).$$

Then we establish, for the different cases of nonintegrable initial data u_0 considered in the examples, the desired L^1 and L^p bounds.

At the end of the section we analyze the case where u_0 is integrable (in which case it does not satisfy an L^p estimate as above) and determine U, both for p supercritical and for p critical.

Finally, in Section 4 we prove our main results on the asymptotic behavior of the solution for initial data comprised in the examples above. In particular, we obtain the behavior in the critical cases $p = 1 + 2/\alpha$ when u_0 satisfies (1.4), and $p = 1 + 2/N$ when $u_0 \in L^1 \cap L^\infty$, that were left open in [21] and [20] respectively.

2. Barriers and first asymptotic estimates

In this section we analyze in detail the smooth part W of the fundamental solution to the linear equation

$$u_t = \int J(x-y)(u(y,t)-u(x,t))\,dy = Lu. \tag{2.5}$$

We establish upper bounds for W and also for its space and time derivatives. These will be essential in Section 3, in order to prove the convergence of the rescaled sequence.

In [6] the authors observed that the fundamental solution of (2.5) can be written as

$$U(x,t) = e^{-t}\delta + W(x,t),$$

where δ is the Dirac measure and $W(x,t)$ is a smooth function. Then, in [15] the authors showed that

$$|W(x,t)| \le C_0 t^{-N/2}.$$

In [21] we established that W is a solution to the problem

$$\begin{cases} W_t(x,t) = \displaystyle\int J(x-y)\big(W(y,t)-W(x,t)\big)\,dy + e^{-t}J(x), \\ W(x,0)=0, \end{cases} \tag{2.6}$$

and used this fact to prove that $W \ge 0$ and to obtain estimates in $L^q(\mathbb{R}^N)$, which in turn allowed us to establish the asymptotic behavior of solutions to the nonlocal problem with absorption in the supercritical case.

In order to deal with the critical case, we will use the method of rescaled sequences for which we need the knowledge of the behavior of $W(x,t)$ as $|x| \to \infty$. We obtain this behavior from sharp barriers. We have,

Theorem 2.1. *Let W as above. There exists a constant $C>0$ depending only on J and N such that*

$$W(x,t) \le C\frac{t}{|x|^{N+2}}. \tag{2.7}$$

Proof. First observe that $W \in C^\infty(\mathbb{R}^N \times [0,\infty))$. Moreover, $W(x,t) \le v_1(x,t) := \|J\|_\infty t$. In fact, in any finite time interval, the function v_1 is a bounded supersolution of the problem (2.6) satisfied by W. Thus, the inequality follows from the comparison principle. (See [19], Proposition 2.2).

Now, let $v_2(x,t) = C\frac{t}{|x|^{N+2}}$. We will show that there is a constant C depending only on J and N such that v_2 is a supersolution of the Dirichlet problem

$$\begin{cases} v_t - Lv = e^{-t}J(x) & \text{in } \mathcal{A} := \{|x| \ge K\sqrt{t}\} \cap \{|x| \ge 2R\}, \\ v = W & \text{in the complement of } \mathcal{A}, \\ v(x,0)=0. \end{cases}$$

satisfied by W. Here R is such that B_R contains the support of J, and K is a large enough constant to be determined.

In fact, $v_2 \ge W$ in $\mathcal{A}^c$ if C is large since $W \le \|J\|_\infty t$ and $W \le C_0 t^{-N/2}$.

On the other hand, $v_{2t} = \frac{C}{|x|^{N+2}}$. In order to estimate Lv_2 we use Taylor's expansion to get,

$$\frac{1}{|y|^{N+2}} - \frac{1}{|x|^{N+2}} = -\frac{N+2}{|x|^{N+4}}\, x\cdot(y-x) - \frac{1}{2}\frac{N+2}{|x|^{N+4}}\,|y-x|^2 + \frac{1}{2}\frac{(N+2)(N+4)}{|x|^{N+6}}\,|x\cdot(y-x)|^2 + \int_0^1 O\Big(\frac{|y-x|^3}{|x+s(y-x)|^{N+5}}\Big)\,ds.$$

Now, since J is radially symmetric,

$$-\frac{N+2}{|x|^{N+4}}\sum_{i=1}^N x_i \int J(x-y)(y_i-x_i)\,dy = 0, \quad \frac{1}{2}\frac{N+2}{|x|^{N+4}}\int J(x-y)|y-x|^2\,dy = \frac{(N+2)N}{|x|^{N+4}}\,\mathfrak{a},$$

and

$$\frac{1}{2}\sum_{i,j=1}^N \frac{x_i x_j}{|x|^{N+6}}\big(-(N+2)(N+4)\big)\int J(x-y)(y_i-x_i)(y_j-x_j)\,dy = -\frac{(N+2)(N+4)}{|x|^{N+4}}\,\mathfrak{a},$$

where $\mathfrak{a} = \frac{1}{2N}\int J(x)|x|^2\,dx$.

On the other hand, $x+s(y-x) \geq |x| - |y-x| \geq \frac{1}{2}|x|$ if $|x| \geq 2R$ and $|y-x| \leq R$. Thus, if $|x| \geq 2R$,

$$\int\int_0^1 J(x-y) O\Big(\frac{|y-x|^3}{|x+s(y-x)|^{N+5}}\Big)\,ds\,dy \leq \frac{C_1}{|x|^{N+5}}.$$

Putting everything together we get, if $|x| \geq 2R$,

$$v_{2t} - Lv_2 \geq \frac{C}{|x|^{N+2}}\Big(1 - C_2\frac{t}{|x|^2}\Big).$$

So that, if $|x| \geq K\sqrt{t}$ with K large enough,

$$v_{2y} - Lv_2 \geq \frac{C/2}{|x|^{N+2}} \geq e^{-t}J(x),$$

if C is large enough since J is bounded and has compact support.

And again, the result follows from the comparison principle for super and subsolutions (See [19], Proposition 2.2 where the comparison principle of bounded, continuous super and subsolutions is proved in the case of $\mathbb{R}^N$. The same proof gives the result in an exterior domain). □

Now we find a barrier for the space derivatives of W.

Theorem 2.2. *Let W be as above. There exists a constant $C > 0$ depending only on J and N such that*

$$|\nabla W(x,t)| \leq C\frac{t}{|x|^{N+3}}. \tag{2.8}$$

Proof. We proceed as above. First, by differentiating the equation satisfied by W we find that $V_i = W_{x_i}$ is the solution to

$$\begin{cases} V_{it} - LV_i = e^{-t}J_{x_i}, \\ V_i(x,0) = 0. \end{cases} \tag{2.9}$$

As with W we find immediately that $|V_i| \le \|\nabla J\|_\infty t$. On the other hand, from the Fourier characterization of W it can be seen that $\|\nabla W(\cdot,t)\|_\infty \le C_0 t^{-\frac{N+1}{2}}$. (See, for instance, [15]). Therefore, for every $K>0$ there exists a constant C such that

$$\bar v(x,t) := C\frac{t}{|x|^{N+3}} \ge V_i \qquad \text{in } \mathcal{A}^c.$$

On the other hand, the same type of computation as the one in Theorem 2.1 yields that, if K is large enough, there exists C such that $\bar v$ is a supersolution to the Dirichlet problem

$$\begin{cases} v_t - Lv = e^{-t}J_{x_i}(x) & \text{in } \mathcal{A} := \{|x| \ge K\sqrt{t}\} \cap \{|x| \ge 2R\}, \\ v = W & \text{in the complement of } \mathcal{A}, \\ v(x,0) = 0. \end{cases}$$

satisfied by V_i.

We conclude that $V_i \le \bar v$.

Analogously, $-\bar v$ is a subsolution to this problem. Thus, $|V_i| \le \bar v$ and the theorem is proved. □

The estimate in (2.8) allows to derive an estimate of the L^1 norm of ∇W. In fact,

$$\begin{aligned} \int |\nabla W(x,t)|\,dx &= \int_{|x|\le\sqrt t} |\nabla W(x,t)|\,dx + \int_{|x|\ge\sqrt t} |\nabla W(x,t)|\,dx \\ &\le C_0 \int_{|x|\le\sqrt t} t^{-\frac{N+1}{2}}\,dx + C\int_{|x|\ge\sqrt t} \frac{t}{|x|^{N+3}}\,dx = C_{N,J}\Big[t^{-\frac{N+1}{2}}t^{\frac N2} + tt^{-\frac32}\Big] = C_{N,J}t^{-\frac12}. \end{aligned}$$

So, we obtain the estimate

$$\int |\nabla W(x,t)|\,dx \le C_1\, t^{-\frac12}, \tag{2.10}$$

with C_1 depending only on J and N.

We obtain similar results for W_t. We have,

Theorem 2.3. *Let W be as above. There exists a constant $C>0$ depending only on J and N such that*

$$|W_t(x,t)| \le e^{-t}J(x) + C\frac{t}{(1+|x|)^{N+4}}. \tag{2.11}$$

Proof. By differentiating the equation satisfied by W we obtain

$$(W_t)_t(x,t) = LW_t(x,t) - e^{-t}J(x).$$

On the other hand, from the equation for W we get

$$W_t(x,0) = J(x).$$

Let $V(x,t) = W_t(x,t) - e^{-t}J(x)$. Then,

$$\begin{cases} V_t - LV = e^{-t}(J*J - J), \\ V(x,0) = 0. \end{cases} \tag{2.12}$$

Since $|J*J-J| \le 2\|J\|_\infty$ we get a first estimate for V: $|V(x,t)| \le 2\|J\|_\infty t$.

We can derive the estimate $|V(x,t)| \le Ct^{-\frac{N+2}{2}}$ by differentiating W with respect to time in its Fourier representation.

Now, proceeding as above we see that there exist C and $\mathcal{K}$ large so that the function $C\frac{t}{|x|^{N+4}}$ is a supersolution of the following problem satisfied by V.

$$\begin{cases} v_t - Lv = e^{-t}(J*J-J) & \text{in } \mathcal{A} := \{|x| \geq K\sqrt{t}\} \cap \{|x| \geq 2R\}, \\ v = V & \text{in the complement of } \mathcal{A}, \\ v(x,0) = 0. \end{cases}$$

Analogously, $-C\frac{t}{|x|^{N+4}}$ is a subsolution to this problem. Therefore,

$$|V(x,t)| \leq C\frac{t}{|x|^{N+4}}.$$

So that,

$$|W_t(x,t)| \leq e^{-t}J(x) + |V(x,t)| \leq e^{-t}J(x) + C\frac{t}{|x|^{N+4}}.$$

Since, $|V(x,t)| \leq Ct$ we prove (2.11). □

From estimate (2.11) we obtain the following estimates

$$\|W_t(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq Ct^{-1}, \tag{2.13}$$

$$\|W_t(\cdot,t)\|_{L^1(\mathbb{R}^N)} \leq e^{-t} + Ct. \tag{2.14}$$

Finally, we construct a barrier for the solution u_L. We have,

Proposition 2.1. *Let u_L be the solution to* (2.5) *with initial datum u_0 satisfying our assumptions. There exists a constant C depending only on $\|u_0\|_\infty$, B, N, J such that*

$$f(t^{1/2})u_L(x,t) \leq C, \qquad f(|x|)u_L(x,t) \leq C. \tag{2.15}$$

In particular, if $\mu > 0$ is such that $k^\mu \leq Cf(k)$ if $k \geq 1$ there holds that

$$u_L(x,t) \leq \frac{C}{(1+t^{1/2}+|x|)^\mu}. \tag{2.16}$$

Let us recall that this is always the case when $\mu = \frac{2}{p-1}$.

Remark 2.1. (2.16) *improves the estimate found in* [21], *Proposition 2.1 where the estimate was proved in finite time intervals when u_0 satisfies* (1.4) *with $\alpha > 0$.*

Proof. We already know that $u_L(x,t) \leq \|u_0\|_\infty$. Let us begin with the time estimate. We have –by the estimates on W and our assumptions on u_0– that for t large,

$$\begin{aligned} f(t^{1/2})u_L(x,t) &= f(t^{1/2})e^{-t}u_0(x) + f(t^{1/2})\int W(x-y,t)u_0(y)\,dy \\ &\leq C + \frac{1}{\int_{B_{t^{1/2}}} u_0}\int_{|y|<t^{1/2}} t^{N/2}W(x-y,t)u_0(y)\,dy + \int_{|y|>t^{1/2}} W(x-y,t)f(t^{1/2})u_0(y)\,dy \\ &\leq C\Big(1+\int W(x-y,t)\,dy\Big) \leq C. \end{aligned}$$

On the other hand, since u_L is bounded and f is locally bounded, $f(t^{1/2})u_L(x,t) \leq C$ if t is bounded.

Now, we estimate,

$$\begin{aligned} f(|x|)u_L(x,t) &= e^{-t} f(|x|)u_0(x) + f(|x|)\int W(x-y,t)u_0(y)\,dy \\ &\le B + f(|x|)\int_{|y|<\frac12|x|} W(x-y,t)u_0(y)\,dy + f(|x|)\int_{|y|\ge\frac12|x|} W(x-y,t)u_0(y)\,dy \\ &= B + I + II. \end{aligned}$$

Observe that $f(2k) = \frac{2^N k^N}{\int_{B_{2k}} u_0} \le 2^N f(k)$. Thus,

$$II \le \int W(x-y,t) f(2|y|)u_0(y)\,dy \le 2^N B.$$

In order to estimate I we use the barrier of W. We have, since $|y| < \frac12|x|$ implies that $|x-y| > \frac12|x|$,

$$\begin{aligned} I = f(|x|)\int_{|y|<\frac12|x|} W(x-y,t)u_0(y)\,dy &\le C|x|^{-2}t\frac{1}{\int_{B_{|x|}} u_0}\int_{|y|<\frac12|x|} u_0(y)\,dy \\ &\le C\frac{t}{|x|^2} \le C \quad \text{if } |x|^2 > t. \end{aligned}$$

On the other hand, in the region $k_0^2 \le |x|^2 \le t$ there holds that

$$f(|x|)u_L(x,t) \le C_1 f(t^{1/2})u_L(x,t) \le C.$$

Finally, if $|x| \le k_0$, there holds that $f(|x|)u_L(x,t) \le C$. So, the proposition is proved. □

Remark 2.2. *When $u_0 \ge 0$ the solution u_L of the homogeneous equation* (2.5) *with initial data u_0 is non-negative. Thus, u_L is a supersolution to* (1.1) *and* 0 *is a subsolution to* (1.1)*. By the comparison principle we deduce that*

$$0 \le u(x,t) \le u_L(x,t),$$

for every solution u of (1.1)*. Hence, the estimates of the previous proposition hold with u_L replaced by u, that is,*

$$f(|x|)u(x,t) \le C \quad \text{and} \quad f(t^{1/2})u(x,t) \le C.$$

3. The rescaled problem

In this section we analyze the rescaled problem. This is, the one satisfied by the rescaled functions u^k. Using the bounds obtained in the previous section we are able to prove that the rescaled sequence $\{u^k\}$ has a convergent subsequence to a function U. We establish the equation satisfied by the limit function U, as well as the initial datum $U(x,0)$, depending on the conditions assumed on u_0. In the case of nonintegrable initial data u_0, in order to completely determine U, it is necessary to establish certain bounds on the L^1 and L^p norms of u^k. On the other hand, if u_0 is integrable, we proceed in a different way, as can be seen at the end of this section.

Let u be a solution of

$$\begin{cases} u_t = Lu - u^p & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

As defined in the introduction we denote by

$$u^k(x,t) = f(k)u(kx,k^2t), \text{ where } f(k) = \frac{k^N}{\int_{B_R} u_0}.$$

By Remark 2.2 we have that $f(t^{1/2})u(x,t) \le C$. So that, by our assumption **(F2)** on f, if $t \ge t_0$ there holds that

$$u^k(x,t) \le C_{t_0} f(k\sqrt{t})u(kx,k^2t) \le C_{t_0}.$$

The function u^k satisfies the following equation,

$$u^k_t = k^2 L_k u^k - F(k)(u^k)^p, \tag{3.2}$$

where $F(k) = f(k)^{1-p}k^2 \to c_0 \ge 0$ as $k \to \infty$ by our assumptions, and the operator L_k is defined by

$$L_k v(x) = \big(J_k * v\big)(x) - v(x) = k^N \int J\big(k(x-y)\big)\big(v(y)-v(x)\big)\,dy. \tag{3.3}$$

Our goal is to study the behavior of the sequence $\{u^k\}$. To do so, we will decompose u^k into an exponentially small part and another one, h^k, depending on W, the smooth part of the fundamental solution of the homogeneous linear problem.

Take $t_0 > 0$ and write $u(x,t) = e^{-(t-k^2t_0)}u(x,k^2t_0) + z(x,t)$. Then,

$$z_t - Lz = e^{-(t-k^2t_0)}\big(J * u(\cdot,k^2t_0)\big) - u^p,$$

and $z(x,k^2t_0) = 0$. After rescaling we have that $u^k(x,t) = e^{-k^2(t-t_0)}u^k(x,t_0) + z^k(x,t)$. Observe that

$$e^{-k^2(t-t_0)}u^k(x,t_0) \le C_{t_0}e^{-k^2(t-t_0)} \to 0,$$

as $k \to \infty$ uniformly in $t - t_0 \ge c > 0$, so that the asymptotic behavior of u^k for $k \to \infty$ is that of z^k.

By the variations of constants formula we get

$$z(x,t) = \int_{k^2t_0}^t S(t-s)\Big[e^{-(s-k^2t_0)}\big(J * u(\cdot,k^2t_0)\big)(x) - u^p(x,s)\Big]\,ds,$$

where $S(t)$ is the semigroup associated to the homogeneous equation $u_t - Lu = 0$.

Thus,

$$\begin{aligned} z(x,t) &= \int_{k^2t_0}^t e^{-(t-s)}\Big[e^{-(s-k^2t_0)}\big(J * u(\cdot,k^2t_0)\big)(x) - u^p(x,s)\Big]\,ds \\ &\quad + \int_{k^2t_0}^t \int W(x-y,t-s)\Big[e^{-(s-k^2t_0)}\big(J * u(\cdot,k^2t_0)\big)(y) - u^p(y,s)\Big]\,dy\,ds \\ &= (t-k^2t_0)e^{-(t-k^2t_0)}\big(J * u(\cdot,k^2t_0)\big)(x) - \int_{k^2t_0}^t e^{-(t-s)}u^p(x,s)\,ds + h(x,t), \end{aligned}$$

with

$$h(x,t) = \int_{k^2t_0}^t \int W(x-y,t-s)\Big[e^{-(s-k^2t_0)}\big(J * u(\cdot,k^2t_0)\big)(y) - u^p(y,s)\Big]\,dy\,ds.$$

Therefore,

$$z^k(x,t) = k^2(t-t_0)e^{-k^2(t-t_0)}f(k)\big(J*u(\cdot,k^2t_0)\big)(kx) - F(k)\int_{t_0}^t e^{-k^2(t-s)}(u^k)^p(x,s)\,ds + h^k(x,t).$$

There holds that,

$$f(k)\big(J*u(\cdot,k^2t_0)\big)(kx) = \int J(kx-y)f(k)u(y,k^2t_0)\,dy = \big(J_k*u^k(\cdot,t_0)\big)(x).$$

So that,

$$z^k(x,t) = k^2(t-t_0)e^{-k^2(t-t_0)}\big(J_k*u^k(\cdot,t_0)\big)(x) - F(k)\int_{t_0}^t e^{-k^2(t-s)}(u^k)^p(x,s)\,ds + h^k(x,t).$$

With similar computations we find that

$$h^k(x,t) = k^2\int_{t_0}^t\int W_k(x-y,t-s)e^{-k^2(s-t_0)}\big(J_k*u^k(\cdot,t_0)\big)(y)\,dy\,ds - F(k)\int_{t_0}^t\int W_k(x-y,t-s)(u^k)^p(y,s)\,dy\,ds,$$

where $W_k(x,t) = k^N W(kx,k^2t)$.

Let us now estimate the first term in the expansion of z^k. We have

$$k^2(t-t_0)e^{-k^2(t-t_0)}\big(J_k*u^k(\cdot,t_0)\big)(x) \le k^2(t-t_0)e^{-k^2(t-t_0)}C_{t_0} \to 0 \quad \text{as } k\to\infty,$$

uniformly in $t-t_0 \ge c > 0$.

For the second term, since $F(k)$ is bounded, we have the estimate

$$F(k)\int_{t_0}^t e^{-k^2(t-s)}(u^k)^p(x,s)\,ds \le C_{t_0}\int_{t_0}^t e^{-k^2(t-s)}\,ds \le C_{t_0}k^{-2}.$$

Therefore, the asymptotic behavior as $k\to\infty$ of u^k is that of h^k.

Since the functions h^k are smooth, we can show that the family $\{h^k\}$ is precompact in $C(\mathcal{K})$ with $\mathcal{K} \subset\subset \mathbb{R}^N\times(t_0,\infty)$ by finding uniform Holder estimates.

We begin with estimates in space.

Proposition 3.1. *Let $t_0>0$ and $T>2t_0$. There exists a constant $L>0$ such $|\nabla h^k(x,t)|\le L$ for $x\in\mathbb{R}^N$ if $t\in[2t_0,T]$.*

Proof. Recall that we have obtained an estimate for the L^1 norm of $\nabla W(\cdot,t)$ (cf. (2.10)). Now we derive an estimate for the L^1 norm of $\nabla W_k(x,t) = kk^N\nabla W(kx,k^2t)$. We have,

$$\|\nabla W_k(\cdot,t)\|_{L^1(\mathbb{R}^N)} \le Ck(k^2t)^{-1/2} = Ct^{-1/2}.$$

Differentiating the function h^k we obtain,

$$|\nabla h^k(x,t)| \le k^2\int_{t_0}^t\int |\nabla W_k(x-y,t-s)|\,e^{-k^2(s-t_0)}\big(J_k*u^k(\cdot,t_0)\big)(y)\,dy\,ds + F(k)\int_{t_0}^t\int |\nabla W_k(x-y,t-s)|(u^k)^p(y,s)\,dy\,ds = I + II.$$

There holds,

$$I \le k^2 C_{t_0} \int_{t_0}^{3t/4} e^{-k^2(s-t_0)}(t-s)^{-1/2}\,ds + k^2 C_{t_0} \int_{3t/4}^{t} e^{-k^2(s-t_0)}(t-s)^{-1/2}\,ds$$
$$\le C_{t_0}(t/4)^{-1/2} + C_{t_0}(t/4)^{1/2}k^2 e^{-k^2(\frac{3t}{4}-t_0)} \le C_{t_0,T}.$$

On the other hand,

$$II \le C_{t_0} \int_{t_0}^{t} (t-s)^{-1/2}\,ds \le C_{t_0,T}.$$

□

Finally we prove one of our main results, namely, that the sequence $\{h^k\}$ and therefore, the sequence $\{u^k\}$ is uniformly convergent on compact sets.

Theorem 3.1. *There exists a subsequence that we still call h^k which is uniformly convergent on every compact subset of $\mathbb{R}^N \times [2t_0, \infty)$.*

Proof. In order to prove the result, let us split h^k into two terms.

$$h^k(x,t) = k^2 \int_{t_0}^{t} \int W_k(x-y,t-s) e^{-k^2(s-t_0)} \big(J_k * u^k(\cdot,t_0)\big)(y)\,dy\,ds$$
$$- F(k) \int_{t_0}^{t} \int W_k(x-y,t-s)(u^k)^p(y,s)\,dy\,ds = H_0^k(x,t) + H^k(x,t).$$

with

$$H_0^k(x,t) = k^2 \int_{t_0}^{t} \int W_k(x-y,t-s) e^{-k^2(s-t_0)} \big(J_k * u^k(\cdot,t_0)\big)(y)\,dy\,ds,$$

and

$$H^k(x,t) = F(k) \int_{t_0}^{t} \int W_k(x-y,t-s)(u^k)^p(y,s)\,dy\,ds.$$

By the estimates of $\|W_t(\cdot,t)\|_{L^1(\mathbb{R}^N)}$ (cf. (2.13)) we get

$$\|W_{kt}(\cdot,t)\|_{L^1} \le Ct^{-1} \qquad \text{and} \qquad \|W_{kt}(\cdot,t)\|_{L^1} \le k^2 e^{-k^2 t} + Ck^4 t.$$

Therefore,

$$|H_{0\,t}^k(x,t)| \le k^2 \int_{t_0}^{t} \int |W_{kt}(x-y,t-s)|\, e^{-k^2(s-t_0)} \big(J_k * u^k(\cdot,t_0)\big)(y)\,dy\,ds$$
$$\le Ck^2 \int_{t_0}^{3t/4} (t-s)^{-1} C_{t_0} e^{-k^2(s-t_0)}\,ds + Ck^2 \int_{3t/4}^{t} k^2 e^{-k^2(t-s)} C_{t_0} e^{-k^2(s-t_0)}\,ds$$
$$+ Ck^2 \int_{3t/4}^{t} k^4 (t-s) C_{t_0} e^{-k^2(s-t_0)}\,ds$$
$$\le C_{t_0}(t/4)^{-1} + C_{t_0}(t/4)k^4 e^{-k^2(t-t_0)} + C_{t_0}(t/4)k^4 e^{-k^2(\frac{3t}{4}-t_0)} \le C_{t_0,T}.$$

Therefore, the sequence H_0^k has a subsequence that converges uniformly on compact subsets of $\mathbb{R}^N \times [2t_0, \infty)$.

In order to see that the same conclusion holds for the sequence H^k, let us define

$$R^k(x,t) := F(k) \int_{t_0}^{t} \int U_{\mathfrak{a}}(x-y,t-s)(u^k)^p(y,s)\,dy\,ds,$$

with $U_{\mathfrak{a}}$ the fundamental solution of the heat equation with diffusivity $\mathfrak{a}$. Then, for every $T > 0$,

$$\|R^k\|_{L^\infty(\mathbb{R}^N\times[t_0,T])} \le C_{t_0,T},$$

and

$$\|H^k - R^k\|_{L^\infty(\mathbb{R}^N\times[t_0,T])} \to 0 \quad \text{as} \quad k\to\infty.$$

In fact,

$$\|H^k(\cdot,t) - R^k(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le F(k)\int_{t_0}^t \|W_k(\cdot,t-s) - U(\cdot,t-s)\|_{q'}\|(u^k)^p(\cdot,s)\|_q\,ds.$$

Recall that $f(k) \ge ck^\beta$ with $\beta = 2/(p-1)$. Let us take $q > N/\beta p$. So that there holds,

$$\|(u^k)^p(\cdot,s)\|_q \le \Big(\int \frac{dx}{\big(s^{1/2}+|x|\big)^{\beta pq}}\Big)^{1/q} \le C_{t_0,q} \quad \text{for} \quad s\ge t_0.$$

On the other hand, since $U_{\mathfrak{a}}(x,t) = U_{\mathfrak{a}k}(x,t)$,

$$\begin{aligned}\|W_k(\cdot,t) - U_{\mathfrak{a}}(\cdot,t)\|_{q'} &= \|W_k(\cdot,t) - U_{\mathfrak{a}k}(\cdot,t)\|_{q'} = k^{N/q}\|W(\cdot,k^2t) - U_{\mathfrak{a}}(\cdot,k^2t)\|_{q'}\\ &\le k^{N/q}\,(k^2t)^{-(N+1)/2q} = k^{-1/q}t^{-(N+1)/2q} \qquad\qquad \text{(cf. [21])}.\end{aligned}$$

Let us choose q big enough so that we also have $q > (N+1)/2$. Then,

$$\|H^k(\cdot,t) - R^k(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le F(k)C_{q,t_0}k^{-1/q}\int_{t_0}^t (t-s)^{-(N+1)/2q}\,ds \le C_{q,t_0,T}\,k^{-\frac{1}{q}} \text{ if } t_0\le t\le T.$$

Hence, $\|H^k - R^k\|_{L^\infty(\mathbb{R}^N\times[t_0,T])} \le C_{q,t_0,T}\,k^{-\frac{1}{q}} \to 0 \quad \text{as} \quad k\to\infty$.

Finally, observe that R^k is a solution of the heat equation with diffusivity $\mathfrak{a}$ and uniformly bounded right hand side. In fact,

$$R^k_t(x,t) - \mathfrak{a}\Delta R^k(x,t) = F(k)(u^k)^p(x,t),$$

and

$$0 \le F(k)(u^k)^p(x,t) \le C_{t_0} \quad \text{if} \quad t\ge t_0.$$

Therefore, the family R^k is uniformly Holder continuous in $\mathbb{R}^N\times[2t_0,T]$ for every $T > 2t_0$. We conclude that there exists a subsequence that is uniformly convergent on every compact subset of $\mathbb{R}^N\times[2t_0,\infty)$. And the same conclusion then holds for the family H^k. □

Now that we know that $\{u^k\}$ has a convergent subsequence, we proceed in identifying the limit function U. As a first step, and using the assumptions on $F(k)$ in the introducion we establish the equation satisfied by U.

Proposition 3.2. *Let $k_n\to\infty$ be such that $u^{k_n}\to U$ uniformly on compact sets of $\mathbb{R}^N\times(0,\infty)$. Then, U is a solution to*

$$U_t - \mathfrak{a}\Delta U = -c_0U^p, \tag{3.4}$$

where $c_0 = \lim_{k\to\infty} F(k)$.

Proof. For simplicity we drop the subscript n. Let $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,\infty))$ and $\mathcal{K}$ a compact set containing its support. Then,

$$\int_0^\infty \int_{\mathbb{R}^N} U(x,t)\big(\varphi_t + \mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt = \int_0^\infty \int_{\mathbb{R}^N} u^k(x,t)\big(\varphi_t + k^2 L_k\varphi\big)(x,t)\,dx\,dt$$
$$+ \int_0^\infty \int_{\mathbb{R}^N} (U-u^k)(x,t)\big(\varphi_t + \mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt - \int_0^\infty \int_{\mathbb{R}^N} u^k(x,t)\big(k^2 L_k\varphi - \mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt$$
$$= -\int_0^\infty \int_{\mathbb{R}^N} \big(u_t^k - k^2 L_k u^k\big)(x,t)\varphi(x,t)\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^N} (U-u^k)(x,t)\big(\varphi_t + \mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt$$
$$-\int_0^\infty \int_{\mathbb{R}^N} u^k(x,t)\,O\big(k^{-3}\big)\chi_{\mathcal{K}}\,dx\,dt = F(k)\int_0^\infty \int_{\mathbb{R}^N} (u^k)^p(x,t)\varphi(x,t)\,dx\,dt$$
$$+ \int_0^\infty \int_{\mathbb{R}^N} (U-u^k)(x,t)\big(\varphi_t + \mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt - \int_0^\infty \int_{\mathbb{R}^N} u^k(x,t)\,O\big(k^{-3}\big)\chi_{\mathcal{K}}\,dx\,dt$$
$$\to\ c_0 \int_0^\infty \int_{\mathbb{R}^N} U^p(x,t)\varphi(x,t)\,dx\,dt \quad \text{as } k\to\infty.$$

So that,

$$\int_0^\infty \int_{\mathbb{R}^N} U(x,t)\big(\varphi_t + \mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt =\ c_0 \int_0^\infty \int_{\mathbb{R}^N} U^p(x,t)\varphi(x,t)\,dx\,dt$$

for every$\varphi \in C_0^\infty(\mathbb{R}^N \times (0,\infty))$. Thus,

$$U_t - \mathfrak{a}\Delta U = -c_0 U^p.$$

□

Finally, in order to identify the initial datum $U(x,0)$, we need to take into account the behavior of u_0 at infinity. Moreover, it is necessary to have some control on the L^1 and L^p norms of u^k on sets of the form $B_R \times (0,\tau)$ for $R,\tau>0$.

Proposition 3.3. *Let u be a solution to* (3.1) *and u^k its rescaling. Let $k_n \to \infty$ and assume $u^{k_n} \to U$ as $n\to\infty$. Assume that for every $R>0$ there exists C_R such that*

$$\int_0^\tau \int_{B_R} u^k(x,t)\,dx\,dt \le C_R\tau,$$

and either there exists $\gamma>0$ such that

$$F(k)\int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt \le C_R\tau^\gamma,$$

or else, for every $R,\tau>0$,

$$\lim_{k\to\infty} F(k)\int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt = 0.$$

Assume further that

$$u_0^k(x) \to \phi(x) \quad (k\to\infty) \quad \textit{in the sense of distributions.}$$

Then, there holds that U is the solution to

$$\begin{cases} U_t - \mathfrak{a}\Delta U = -c_0 U^p, \\ U(x,0) = \phi(x). \end{cases}$$

Proof. We proceed as in the proof of Proposition 3.2. We drop the subscript n. Let $\varphi \in C_0^\infty(\mathbb{R}^N \times [0,\infty))$, $R>0$ such that $\varphi(x,t)=0$ if $|x|>R$ and let $\varepsilon>0$. Let $\tau>0$, $k_0>0$ be such that for $k \ge k_0$,

$$\begin{aligned}
&\int_0^\tau \int_{B_R} u^k(x,t)\,dx\,dt + F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt<\varepsilon,\\
&\int_0^\tau \int_{B_R} U(x,t)\,dx\,dt + c_0\int_0^\tau\int_{B_R}U^p(x,t)\,dx\,dt<\varepsilon.
\end{aligned}\tag{3.5}$$

Then, estimating as in the previous proposition the integrals between τ and ∞ and using the estimates in (3.5) we get,

$$\begin{aligned}
&\Big|\int_0^\infty\int_{\mathbb{R}^N} U(x,t)\big(\varphi_t+\mathfrak{a}\Delta\varphi\big)(x,t)\,dx\,dt - \int_0^\infty\int_{\mathbb{R}^N} c_0U^p(x,t)\varphi(x,t)\,dx\,dt\\
&+\int_{\mathbb{R}^N}\phi(x)\varphi(x,0)\,dx\Big| \le C\varepsilon + \Big|\int_{\mathbb{R}^N}u_0^k(x)\varphi(x,0)\,dx-\int_{\mathbb{R}^N}\phi(x)\varphi(x,0)\,dx\Big|\\
&\le \bar C\varepsilon \quad \text{if } k \text{ is large.}
\end{aligned}$$

And the proposition is proved. □

In order to prove the bounds assumed in the previous proposition, and hence be able to completely determine U, we need to consider separate cases, according to the behavior of u_0 at infinity. We begin with the case where u_0 behaves as a power $-\alpha > -N$.

Lemma 3.1. *Let u be the solution to* (3.1)*. Assume $(1+|x|)^\alpha u(x,t)\le B$ and $(1+t)^{\alpha/2}u(x,t)\le B$ with $0<\alpha<N$. Then, for every $R>0$ there exists C_R such that, for $k\ge\tau^{-1/2}$,*

$$\begin{aligned}
&\int_0^\tau\int_{B_R} u^k(x,t)\,dx\,dt \le C_R\tau,\\
&F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le C_Rk^{-\alpha(p-1)+2}\tau\le C_R\tau && \textit{if } N-\alpha p>0,\\
&F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le C_Rk^{-\alpha(p-1)+2}\tau|\log\tau|\le C_R\tau|\log\tau| && \textit{if } N-\alpha p=0,\\
&F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le Ck^{-\alpha(p-1)+2}\tau^{\frac{N-\alpha p+2}{2}}\le C\tau^{\frac{N-\alpha p+2}{2}} && \textit{if } 0>N-\alpha p>-2,\\
&F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le Ck^{\alpha-N}\le C_R\tau^{\frac{N-\alpha}{2}} && \textit{if } N-\alpha p<-2,\\
&F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le Ck^{\alpha-N}\log(1+k^2\tau)\le C\tau^{(N-\alpha)/2} && \textit{if } N-\alpha p=-2.
\end{aligned}$$

Proof. We begin with the estimate of the integral of u^k.

$$\begin{aligned}
&\int_0^\tau\int_{B_R}u^k(x,t)\,dx\,dt = f(k)k^{-N-2}\int_0^{k^2\tau}\int_{B_{Rk}}u(x,t)\,dx\,dt\\
&\quad\le Ck^{\alpha-N-2}\int_0^{k^2\tau}\int_{B_{Rk}}\frac{1}{(1+|x|)^\alpha}\,dx\,dt\le Ck^{\alpha-N-2}k^2\tau(Rk)^{N-\alpha}\le C_R\tau.
\end{aligned}\tag{3.6}$$

Let us now estimate the integral of $(u^k)^p$. There holds,

$$
\begin{aligned}
F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt &= F(k)f(k)^p k^{-N-2}\int_0^{k^2\tau}\int_{B_{Rk}} u^p(x,t)\,dx\,dt \\
&= f(k)k^{-N}\int_0^{k^2\tau}\int_{B_{Rk}} u^p(x,t)\,dx\,dt.
\end{aligned}
\tag{3.7}
$$

We consider several cases.

`Case 1`: $N-\alpha p>0$
We have,

$$
\begin{aligned}
F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt &\le CF(k)k^{\alpha p-N-2}\int_0^{k^2\tau}\int_{B_{Rk}}\frac{1}{(1+|x|)^{\alpha p}}\,dx\,dt \\
&\le CF(k)k^{\alpha p-N-2}k^2\tau(Rk)^{N-\alpha p} = C_R k^{-\alpha(p-1)+2}\tau \le C_R\tau.
\end{aligned}
$$

`Case 2`: $N-\alpha p<0$
First,

$$
\int_{B_{Rk}} u^p(x,t)\,dx \le C\int_{|x|\le\sqrt{t}}(1+t)^{-\alpha p/2}\,dx + C\int_{\sqrt{t}<|x|<Rk}\frac{1}{(1+|x|)^{\alpha p}}\,dx \le C(1+t)^{\frac{N-\alpha p}{2}}. \tag{3.8}
$$

Assume $k\ge\tau^{-1/2}$. We consider 3 subcases.

(i) $-2<N-\alpha p<0$.

$$
\begin{aligned}
F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt &\le Ck^{\alpha-N}\int_0^{k^2\tau}(1+t)^{\frac{N-\alpha p}{2}}\,dt \le Ck^{\alpha-N}(k^2\tau)^{\frac{N-\alpha p+2}{2}} \\
&= Ck^{-\alpha(p-1)+2}\tau^{\frac{N-\alpha p+2}{2}} \le C\tau^{\frac{N-\alpha p+2}{2}}.
\end{aligned}
$$

(ii) $N-\alpha p=-2$.

$$
\begin{aligned}
F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt &\le Ck^{\alpha-N}\int_0^{k^2\tau}(1+t)^{-1}\,dt = Ck^{\alpha-N}\log(1+k^2\tau) \\
&= C(k^2\tau)^{(\alpha-N)/2}\log(1+k^2\tau)\tau^{(N-\alpha)/2} \le C\tau^{(N-\alpha)/2}.
\end{aligned}
$$

(iii) $N-\alpha p<-2$.

$$
F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le Ck^{\alpha-N}\int_0^{k^2\tau}(1+t)^{(N-\alpha p)/2}\,dt \le Ck^{\alpha-N} \le C\tau^{(N-\alpha)/2}.
$$

`Case 3`: $N-\alpha p=0$
Instead of (3.8) we have,

$$
\int_{B_{Rk}} u^p(x,t)\,dx \le C\Big(1+\log\frac{Rk}{\sqrt{t}}\Big).
$$

Thus,

$$F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt \le Ck^{\alpha-N}\int_0^{k^2\tau}\Big(1+\log\frac{Rk}{\sqrt{t}}\Big)\,dt$$
$$= Ck^{\alpha-N+2}\int_0^\tau\Big(1+\log\frac{R}{\sqrt{t}}\Big)\,dt \le C_R k^{-\alpha(p-1)+2}\tau|\log\tau| \le C_R\tau|\log\tau|.$$

□

Corollary 3.1. *Assume $|x|^\alpha u_0(x)\to A>0$ as $|x|\to\infty$ with $0<\alpha<N$. Let u be the solution to (3.1). Then, u satisfies the conclusions of Lemma 3.1*

Proof. By Remark 2.2, the estimates of Proposition 2.1 hold for u. In this case, since $f(k)\sim k^\alpha$ we take it to be equal and hence,

$$(1+|x|)^\alpha u(x,t)\le C \quad\text{and}\quad (1+t)^{\alpha/2}u(x,t)\le C.$$

Thus, u satisfies the assumptions of Lemma 3.1. □

Using Lemma 3.1, we are able to prove with almost no computations, the desired estimates in the other examples considered in the introduction. We use the ideas of Kamin-Ughi in [18].

Let us introduce some notation. For $\mu>0$, let

$$u^{k,\mu}(x,t)=k^\mu u(kx,k^2t). \tag{3.9}$$

Then,

$$F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt = f(k)k^{-\mu}k^{-\mu(p-1)+2}\int_0^\tau\int_{B_R}(u^{k,\mu})^p(x,t)\,dx\,dt. \tag{3.10}$$

We have,

Lemma 3.2. *Assume $|x|^\alpha(\log|x|)^\beta u_0(x)\to A>0$ as $|x|\to\infty$ with $0<\alpha\le N$, $\beta=0,1,-1$. Let $p>1+2/\alpha$. Then, for every $R>0$ there exists C_R such that,*

$$\int_0^\tau\int_{B_R}u^k(x,t)\,dx\,dt\le C_R\tau, \tag{3.11}$$

and either

$$\lim_{k\to\infty}F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt=0,$$

or else, there exists $\gamma>0$ such that

$$F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt\le C_R\tau^\gamma.$$

The same result holds when $0<\alpha<N$, $\beta=1$ and $p=1+2/\alpha$.

Proof. We begin with the case $\alpha=N$ and $\beta=0$. It is easy to see that, when $\tau<1$,

$$\int_0^\tau\int_{B_R}u^k(x,t)\,dx\,dt\le C\frac{\log(Rk)}{\log k}k^{-2}\int_0^{k^2\tau}e^{-t}\,dt + C\frac{\log(3Rk)}{\log k}k^{-2}k^2\tau + C_K k^{-4}\int_0^{k^2\tau}t\,dt\le C_R\tau.$$

In a similar way we can prove the L^1 estimates for the other choices of α and β.

We now focus on the second estimate. To avoid technical difficulties, we sketch the proof in the case $\alpha=N$, $\beta=0$. The other cases are done in a similar way.

Recall that in this case $f(k) \sim \frac{k^N}{\log k}$ for $k \geq 2$. Let $0 < \mu < N$ to be chosen later. Then, for $|x| \geq 1$, $t \geq 1$,

$$|x|^\mu u(x,t) \leq C \frac{|x|^N}{\log(1+|x|)} u(x,t) \leq B, \qquad t^{\mu/2} u(x,t) \leq C \frac{t^{N/2}}{\log(1+t)} u(x,t) \leq B.$$

So that, we can apply the results of Lemma 3.1 to $u^{k,\mu}$. Let us choose μ so close to N so that $\mu p > N+2 (> \mu + 2)$. By (3.10) and Lemma 3.1, Case 2, (iii), there holds,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt \leq C f(k) k^{-\mu} k^{\mu - N} = \frac{C}{\log k}.$$

With similar computations we can prove that, when $|x|^N (\log|x|) u_0(x) \to A > 0$ as $|x| \to \infty$,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt \leq \frac{C}{\log\log k}.$$

On the other hand, when $\frac{|x|^N}{\log|x|} u_0(x) \to A > 0$ as $|x| \to \infty$,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt \leq \frac{C}{\log^2 k}.$$

In the case where $0 < \alpha < N$ and $\beta = 1$ or $\beta = -1$, we can show by using similar arguments and choosing $\mu \in (0,\alpha)$ depending on N, p and α that there exists γ depending on all these parameters such that

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt \leq C_R \tau^\gamma.$$

Finally, let us consider the critical case $p = 1 + 2/\alpha$ when $0 < \alpha < N$ and $\beta = 1$. The L^1 estimate follows as in the previous cases. On the contrary, for the L^p estimate we have to consider several cases and prove that

$$\lim_{k\to\infty} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt = 0$$

without an explicit estimate as before. We will write down the proof in case $N - \alpha p < 0$ and leave the other cases to the reader. There holds,

$$\begin{aligned}
\int_{B_{Rk}} u^p(x,t)\,dx &\leq C \int_{|x|\leq\sqrt{t}} \frac{dx}{(2+t^{1/2})^{\alpha p} \log^p(2+t^{1/2})} + C \int_{\sqrt{t}\leq|x|\leq Rk} \frac{dx}{(2+|x|)^{\alpha p}\log^p(2+|x|)} \\
&\leq C \frac{(2+t^{1/2})^{N-\alpha p}}{\log^p(2+t^{1/2})} + C \int_{\sqrt{t}}^{Rk} \frac{(2+r)^{N-\alpha p-1}}{\log^p(2+r)}\,dr \leq C \frac{(2+t^{1/2})^{N-\alpha p}}{\log^p(2+t^{1/2})} \\
&+ C \frac{1}{\log^p(2+t^{1/2})} \int_{\sqrt{t}}^{Rk} (2+r)^{N-\alpha p-1}\,dr \leq C \frac{(2+t^{1/2})^{N-\alpha p}}{\log^p(2+t^{1/2})}.
\end{aligned}$$

Since $\alpha p = \alpha + 2$, there holds that $N - \alpha p = N - \alpha - 2 > -2$. We have,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x,t)\,dx\,dt \leq C k^{\alpha - N} \log k \int_0^{k^2\tau} \frac{(2+t^{1/2})^{N-\alpha p}}{\log^p(2+t^{1/2})}\,dt,$$

and this last integral goes to infinity as k goes to infinity. Thus,

$$\begin{aligned}
&\lim_{k\to\infty} F(k)\int_0^\tau\int_{B_R}(u^k)^p(x,t)\,dx\,dt\\
&\le \lim_{k\to\infty}\frac{C\int_0^{k^2\tau}\frac{(2+t^{1/2})^{N-\alpha p}}{\log^p(2+t^{1/2})}\,dt}{\frac{k^{N-\alpha}}{\log k}} = \lim_{k\to\infty}\frac{\frac{C2k\tau(2+k\tau^{1/2})^{N-\alpha p}}{\log^p(2+k\tau^{1/2})}}{\frac{(N-\alpha)k^{N-\alpha-1}}{\log k}-\frac{k^{N-\alpha-1}}{\log^2 k}}\\
&= \lim_{k\to\infty}\frac{C2k\tau(2+k\tau^{1/2})^{N-\alpha-2}}{k^{N-\alpha-1}}\frac{\log k}{\log^p(2+k\tau^{1/2})}\frac{1}{N-\alpha-\frac{1}{\log k}} = 0.
\end{aligned}$$

□

The case where $u_0 \in L^1(\mathbb{R}^N)$ has to be studied separately. In fact, if $u_0 \in L^1(\mathbb{R}^N)$, the limit U of u^k does not satisfy that $U(x,0) = \lim u_0^k = M_0\delta$ with $M_0 = \int u_0$. Therefore, estimates as the ones we have just stated do not hold. In order to establish our result we need the following lemma.

Lemma 3.3. *Let $u_0 \in L^\infty(\mathbb{R}^N)$. Assume further that $|x|^{N+2}u_0(x) \le B$. Let u be the solution to (3.1) and u^k its rescaling. Assume $u^{k_n} \to U$ with $k_n \to \infty$ as $n \to \infty$. Then, there exists a constant $C > 0$ such that*

$$U(x,t) \le C\frac{t}{|x|^{N+2}}.$$

In particular, for every $\mu > 0$,

$$\lim_{t\to 0} U(x,t) = 0 \qquad \textit{uniformly in } \ |x| \ge \mu.$$

Proof. We proceed as in the proof of Theorem 2.1. There holds that $u(x,t) \le e^{-t}u_0(x) + z(x,t)$ with z the solution to

$$\begin{cases} z_t - Lz = e^{-t}(J * u_0)(x), \\ z(x,0) = 0. \end{cases} \tag{3.12}$$

By the assumption on the growth of u_0 at infinity there holds that $v(x,t) = C\frac{t}{|x|^{N+2}}$ is a supersolution to (3.12) in $\frac{t}{|x|^2} \le K$, $|x| \ge 2$. On the other hand, $z(x,t) \le v(x,t)$ in the complement of this set since $z(x,t) \le Ct$ and $t^{N/2}z(x,t) \le C$ in this case ($u_0 \in L^1(\mathbb{R}^N)$).

Therefore, $u(x,t) \le e^{-t}u_0(x) + C\frac{t}{|x|^{N+2}}$. Rescaling, and recalling that in the present case $f(k) = k^N$ we have that $u^k(x,t) \le e^{-k^2t}u_0^k(x) + C\frac{t}{|x|^{N+2}}$. Passing to the limit as $k \to \infty$ with $x \ne 0$ we get $U(x,t) \le C\frac{t}{|x|^{N+2}}$. And the result follows. □

Proposition 3.4. *Let $u_0 \in L^1 \cap L^\infty$. Let u be the solution to (3.1) and u^k its rescaling. Assume for some sequence $k_n \to \infty$ there holds that $u^{k_n} \to U$ uniformly on compact sets of $\mathbb{R}^N \times (0,\infty)$.*

Assume $p > 1 + 2/N$. Then, $U(x,t) = MU_{\mathfrak{a}}(x,t)$ where $U_{\mathfrak{a}}$ is the fundamental solution of the heat equation with diffusivity $\mathfrak{a}$ and $M = \int u_0(x)\,dx - \int_0^\infty \int u^p(x,t)\,dx\,dt$.

When $p = 1 + 2/N$ there holds that $U \equiv 0$.

Proof. As before we drop the subscript n. We already know that U is a solution to

$$U_t - \mathfrak{a}\Delta U = -c_0 U^p.$$

Assume for simplicity that u_0 has compact support. Then, by Lemma 3.3 there exists $C > 0$ such that $U(x,t) \le C\frac{t}{|x|^{N+2}}$. Let $M(t) = \int U(x,t)\,dx$. We will see that $M(t) \equiv M$ is constant, and $U(x,t) \to M\delta$ as $t \to 0$ in the sense of distributions where δ is the Dirac delta. In fact, assume we already proved that $M(t)$ is constant. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Then, given $\varepsilon > 0$, if μ is small we get,

$$\begin{aligned}
\Big|\int U(x,t)\varphi(x)\,dx - M\varphi(0)\Big| &= \Big|\int U(x,t)\big[\varphi(x)-\varphi(0)\big]\,dx\Big| \\
&\le \int_{|x|<\mu} U(x,t)|\varphi(x)-\varphi(0)|\,dx + C\int_{|x|>\mu} \frac{t}{|x|^{N+2}}|\varphi(x)-\varphi(0)|\,dx \\
&\le \varepsilon \int U(x,t)\,dx + 2C\|\varphi\|_\infty \int_{|x|>\mu}\frac{t}{|x|^{N+2}}\,dx \\
&\le M\varepsilon + \bar{C}_\mu\, t < (M+1)\varepsilon \qquad \text{if } t \text{ is small.}
\end{aligned}$$

Now, integrating the equation satisfied by u^k and using the symmetry of J we obtain

(3.13)
$$\int u^k(x,t)\,dx = \int u_0^k(x)\,dx - F(k)\int_0^t\int (u^k)^p(x,s)\,dx\,ds = \int u_0(x) - \int_0^{k^2 t}\int u^p(x,s)\,dx\,ds.$$

Moreover,

$$\int_{B_K} u^k(x,t)\,dx \to \int_{B_K} U(x,t)\,dx \quad \text{for every } K>0, \tag{3.14}$$

and

$$\begin{aligned}
&\int_{|x|>K} u^k(x,t)\,dx \le e^{-k^2 t}\int_{|x|>Kk} u_0(x)\,dx + Ct\int_{|x|>K}\frac{dx}{|x|^{N+2}} < \varepsilon, \\
&\int_{|x|>K} U(x,t)\,dx \le Ct\int_{|x|>K}\frac{dx}{|x|^{N+2}} < \varepsilon,
\end{aligned} \tag{3.15}$$

if K is large.

Hence, we have that

$$M(t) = \int U(x,t)\,dx = \int u_0(x)\,dx - \int_0^\infty\int u^p(x,t)\,dx\,dt = M.$$

When $p > 1 + 2/N$ there holds that $c_0 = 0$, so that U is M times the fundamental solution of the heat equation with diffusivity $\mathfrak{a}$. On the other hand, if $p = 1 + 2/N$ there holds that $c_0 = 1$, so that

$$\int U(x,t)\,dx = \int U(x,\tau)\,dx - \int_\tau^t\int U^p(x,s)\,dx\,ds.$$

Therefore, $M(t)$ cannot be constant unless $U \equiv 0$. And the proposition is proved when u_0 has compact support.

For $u_0 \in L^\infty \cap L^1$ arbitrary the proof follows by an approximation argument. We leave the details to the reader. ☐

4. Main results

In this section we prove our main results. Namely, we establish the asymptotic behavior of solutions to

$$\begin{cases} u_t(x,t) = \int J(x-y)(u(y,t)-u(x,t))\,dy - u^p(x,t) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

depending on the values of p, N and the coefficient α of non-integrability of the initial data u_0. At the end of this section we also address the case where u_0 is integrable and bounded. As mentioned before, we have to distinguish between several cases.

We begin with the case $0 < \alpha < N$. We further divide the analysis between the supercritical case $p > 1 + 2/\alpha$ and the critical case $p = 1 + 2/\alpha$. In the previous paper [21], we were able to establish the asymptotic behavior only for the supercritical case. Now we present a different proof that actually allows us to obtain the result both in the critical and supercritical cases. Moreover, in the supercritical case, we prove the same type of result for more general non-integrable initial data.

Theorem 4.1. *Let $u_0 \in L^\infty$ be such that $|x|^\alpha u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$. Let $p \geq 1 + 2/\alpha$ and u the solution to (4.1). Then, for every $R > 0$,*

$$t^{\alpha/2}|u(x,t) - U(x,t)| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \leq R\sqrt{t},$$

where U is the solution to

$$\begin{cases} U_t - \mathfrak{a}\Delta U = -c_0 U^p, \\ U(x,0) = \dfrac{C_{A,N}}{|x|^\alpha}. \end{cases} \tag{4.2}$$

with $c_0 = 0$ if $p > 1 + 2/\alpha$ and $c_0 = 1$ if $p = 1 + 2/\alpha$.

In a similar way we get the following results: When $\frac{|x|^\alpha}{\log|x|} u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$ and $p > 1 + 2/\alpha$, for every $R > 0$ there holds that

$$t^{\alpha/2}\Big|\frac{u(x,t)}{\log t^{1/2}} - U(x,t)\Big| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \leq R\sqrt{t},$$

with U as above.

When $|x|^\alpha(\log|x|)u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$ and $p \geq 1 + 2/\alpha$, for every $R > 0$ there holds that

$$t^{\alpha/2}\Big|u(x,t)\log t^{1/2} - U(x,t)\Big| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \leq R\sqrt{t},$$

with U the solution to (4.2) with $c_0 = 0$ even in the critical case $p = 1 + 2/\alpha$.

Proof. We know that the family u^k is precompact in $C(\mathcal{K})$ for every compact set $\mathcal{K} \subset \mathbb{R}^N \times (0,\infty)$. Therefore, for every sequence $k_n \to \infty$ there exists a subsequence that we still call k_n such that u^{k_n} converges uniformly on every compact subset of $\mathbb{R}^N \times (0,\infty)$ to a function U.

On the other hand, it is easy to see that $u_0^k \to \frac{C_{A,N}}{|x|^\alpha}$ in the sense of distributions. Moreover, we are in the situation of Proposition 3.3. So that, U is the solution to (4.2). Thus, the whole family u^k converges to U as $k \to \infty$. In particular, $u^k(x,1) \to U(x,1)$ uniformly on compact sets of $\mathbb{R}^N$.

As the solution of (4.2) is invariant under the present rescaling. There holds that $U(y,1) = U^k(y,1)$. Thus, for every $R > 0$,

$$k^\alpha |u(ky,k^2) - U(ky,k^2)| \to 0 \quad \text{uniformly for } |y| \le R.$$

By calling $x = y\sqrt{t}$, $t = k^2$ we get the result.

When $\frac{|x|^\alpha}{\log|x|} u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$, it is easy to see that we still have that $u_0^k \to \frac{C_{A,N}}{|x|^\alpha}$ in the sense of distributions. So the result follows also in this case.

Analogously, when $|x|^\alpha (\log|x|) u_0(x) \to A > 0$ as $|x| \to \infty$ with $0 < \alpha < N$, we also have that $u_0^k \to \frac{C_{A,N}}{|x|^\alpha}$ in the sense of distributions, and the result follows. □

We now analyze the case $\alpha = N$. Once again, in the supercritical case $p > 1 + 2/N$, we prove the result for more general non-integrable initial data than the one considered in [21].

Theorem 4.2. *Let $u_0 \in L^\infty$ be such that $|x|^N u_0(x) \to A > 0$ as $|x| \to \infty$. Let $p > 1 + 2/N$. Then, for every $R > 0$,*

$$t^{N/2} \Big| \frac{u(x,t)}{\log t^{1/2}} - U(x,t) \Big| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \le R\sqrt{t},$$

where u is the solution to (4.1) and U is the solution to

$$\begin{cases} U_t - \mathfrak{a}\Delta U = 0, \\ U(x,0) = C_{A,N}\delta, \end{cases}$$

with δ Dirac's delta.

In a similar way we get the following result when $\frac{|x|^N}{\log|x|} u_0(x) \to A > 0$ as $|x| \to \infty$ and $p > 1 + 2/N$: for every $R > 0$,

$$t^{N/2} \Big| \frac{u(x,t)}{\log^2 t^{1/2}} - U(x,t) \Big| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \le R\sqrt{t},$$

with U as above.

When $|x|^N (\log|x|) u_0(x) \to A > 0$ as $|x| \to \infty$ and $p > 1 + 2/N$ we get: for every $R > 0$,

$$t^{N/2} \Big| \frac{u(x,t)}{\log\log t^{1/2}} - U(x,t) \Big| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \le R\sqrt{t},$$

with U as above.

Proof. We proceed as in the previous theorem. This time, $u_0^k \to C_{A,N}\delta$ in the sense of distributions. In fact,

$$\int u_0^k(x)\varphi(x)\,dx = \frac{1}{\log k} \int_{|x|<Kk} u_0(x)\varphi(x/k)\,dx$$

where K is such that $\varphi(x) = 0$ if $|x| > K$. On the other hand,

$$\frac{1}{\log k} \int_{B_{Kk}} u_0(x)\big(\varphi(x/k) - \varphi(0)\big)\,dx \le \frac{C}{k\log k} \int_{B_{Kk}} \frac{|x|}{(1+|x|)^N}\,dx = \frac{C}{\log k}.$$

And the result easily follows from this formula and the fact that

$$\frac{1}{\log Kk} \int_{|x|<Kk} u_0(x)\,dx \to C_{A,N}, \qquad \frac{\log Kk}{\log k} \to 1 \qquad \text{as } k \to \infty.$$

The other cases follow similarly. □

Finally we consider the case where u_0 is integrable and bounded. Such case has been studied by Pazoto and Rossi in [20] for non-critical values of p . Here we present a new proof the includes the critical case and therefore settles the question as far as integrable data is concerned.

Theorem 4.3. *Let $u_0 \in L^1 \cap L^\infty$ and $p \geq 1 + 2/N$. Let u be the solution to (4.1).*

First, assume $p > 1 + 2/N$. Then, for every $R > 0$,

$$t^{N/2}|u(x,t) - U(x,t)| \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \leq R\sqrt{t},$$

where U is the solution to

$$\begin{cases} U_t - \mathfrak{a}\Delta U = 0, \\ U(x,0) = M\delta, \end{cases} \tag{4.3}$$

with δ Dirac's delta and $M = \int u_0(x)\,dx - \int_0^\infty \int u^p(x,t)\,dx\,dt$.

Then, let $p = 1 + 2/N$. For every $R > 0$ there holds that

$$t^{N/2}u(x,t) \to 0 \quad \text{as } t \to \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}.$$

Proof. As in the previous theorems we know that $u^k(y,1) \to U(y,1)$ uniformly on compact sets of $\mathbb{R}^N \times (0,\infty)$ as $k \to \infty$. In the present situation, if $p > 1 + 2/N$, we know that U is the solution to (4.3). In particular, U is invariant under the present rescaling. When $p = 1 + 2/N$ we know that $U \equiv 0$. So, we get the result as in the previous theorems. □

Remark 4.1. *The same method allows to study the asymptotic behavior of the solution u_L of the equation without absorption.*

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