Inverse function theorems

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Ekeland's variational principle (1972)

Theorem

Let (X, d) be a complete metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous map, bounded from below:

$$\begin{array}{l} \{(x, a) \mid a \geq f(x)\} \text{ is closed in } X \times \mathbb{R} \\ f(x) \geq 0, \quad \forall x \end{array}$$

Suppose $f(0) < \infty$. Then for every $A > 0$, there exists some $\bar{x}$ such that:

$$\begin{array}{rcl} f(\bar{x}) & \leq & f(0) \\ d(\bar{x}.0) & \leq & A \\ f(x) & \geq & f(\bar{x}) - \dfrac{f(0)}{A} d(x, \bar{x}) \quad \forall x \end{array}$$

This is a Baire-type result: relies on completeness, no compactnes needed

First-order version

Suppose X is a Banach space, and $d(x_1, x_2) = \|x_1 - x_2\|$. Apply EVP to $x = \bar{x} + tu$ and let $u \to 0$. We get:

$$
\begin{aligned}
f(\bar{x} + tu) &\geq f(\bar{x}) - \frac{f(0)}{A} t \|u\| \quad \forall (t, u) \\
\lim_{t \to +0} \frac{1}{t}\left(f(\bar{x} + tu) - f(\bar{x})\right) &\geq -\frac{f(0)}{A} \|u\| \quad \forall u \\
\langle Df(x), u \rangle &\geq -\frac{f(0)}{A} \|u\| \quad \forall u, \text{ or } \|Df(x)\|^* \leq \frac{f(0)}{A}
\end{aligned}
$$

Corollary

Suppose F is everywhere finite and Gâteaux-differentiable. Then there is a sequence x_n such that:

$$
\begin{aligned}
f(x_n) &\to \inf f \\
\|Df(x_n)\|^* &\to 0
\end{aligned}
$$

An inverse function theorem in Banach spaces

Theorem

Let X and Y be Banach spaces. Let $F : X \to Y$ be continuous and Gâteaux-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded in a neighbourhood of 0:

$$\forall v \in Y, \quad DF(x) L(x) v = u$$
$$\sup \{ \|L(x)\| \mid \|x\| \leq R \} < m$$

Then, for every $\bar{y}$ such that

$$\|\bar{y}\| \leq \frac{R}{m}$$

there is some $\bar{x}$ such that:

$$\begin{aligned} \|\bar{x}\| &\leq m \|\bar{y}\| \\ F(\bar{x}) &= \bar{y} \end{aligned}$$

Proof

Consider the function $f : X \rightarrow R$ defined by:

$$f(x) = \|F(x) - \bar{y}\|$$

It is continuous and bounded from below, so that we can apply EVP with $A = m\|\bar{y}\|$. We can find $\bar{x}$ with:

$$\begin{aligned} f(\bar{x}) &\leq f(0) = \|\bar{y}\| \\ \|\bar{x}\| &\leq m\|\bar{y}\| \leq R \\ \forall x, \quad f(x) &\geq f(\bar{x}) - \frac{f(0)}{m\|\bar{y}\|}\|x - \bar{x}\| = f(\bar{x}) - \frac{1}{m}\|x - \bar{x}\| \end{aligned}$$

I claim $F(\bar{x}) = \bar{y}$.

Proof (ct'd)

Assume $F(\bar{x}) \neq \bar{y}$. The last equation can be rewritten:

$$\forall t \geq 0,\ \forall u \in X, \qquad \frac{f(\bar{x}+tu) - f(\bar{x})}{t} \geq -\frac{1}{m}\|u\|$$

Simplify matters by assuming X is Hilbert. Then:

$$\left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}\|}, DF(\bar{x})\, u\right) = \langle Df(\bar{x}), u\rangle \geq -\frac{1}{m}\|u\|$$

We now take $u = -L(\bar{x})(F(\bar{x}) - \bar{y})$, so that $DF(\bar{x})\, u = -(F(\bar{x}) - \bar{y})$. We get a contradiction:

$$\|F(\bar{x}) - \bar{y}\| \leq \frac{\|L(\bar{x})\|}{m}\|F(\bar{x}) - \bar{y}\| < \|F(\bar{x}) - \bar{y}\|$$

Fréchet spaces.

A Fréchet space X is *graded* if its topology is defined by an increasing sequence of norms:

$$\forall x \in X, \quad \|x\|_k \leq \|x\|_{k+1}, \quad k \geq 0$$

A point $x \in X$ is *controlled* if there is a constant $c_0(x)$ such that:

$$\|x\|_k \leq c_0(x)^k$$

Definitions

A graded Fréchet space is *standard* if, for every $x \in X$, there is a constant $c_3(x)$ and a sequence x_n of controlled vectors such that:

$$\forall k \quad \lim_{n\to\infty} \|x_n - x\|_k = 0$$

$$\forall n, \quad \|x_n\|_k \leq c_3(x) \|x\|_k$$

The graded Fréchet spaces $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \cap C^k(\bar{\Omega}, \mathbb{R}^d)$ and $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \cap H^k(\Omega, \mathbb{R}^d)$ are both standard.

Normal maps

We are given two Fréchet spaces X and Y, and a neighbourhood of zero $B = \{x \mid \|x\|_{k_0} \leq R\}$ in X

Definition

A map $F : X \to Y$ is *normal* over B if there are two integers $d_1, \ d_2$ and two non-decreasing sequences $m_k > 0, \ m'_k > 0$ such that:

1. $F(0) = 0$ and F is continuous on B
2. F is Gâteaux-differentiable on B and for all $x \in B$

$$\forall k \in \mathbb{N}, \ \|DF(x)\,u\|_k \leq m_k \left(\|u\|_{k+d_1} + \|F(x)\|_k \|u\|_{k_0}\right)$$

3. There exists a linear map $L(x) : Y \longrightarrow X$ such that:

$$\begin{aligned} \forall v &\in Y, \ DF(x)\,L(x)\,v = v \\ \forall k &\in \mathbb{N}, \ \sup_{x \in B} \|L(x)\,v\|_k < m'_k \|v\|_{k+d_2} \end{aligned}$$

An inverse function theorem

Theorem

Suppose Y is standard, and $F : X \to Y$ is normal over $B = \left\{x \mid \|x\|_{k_0} \leq R\right\}$. Then, for every y with

$$\|y\|_{k_0+d_2} \leq \frac{R}{m'_{k_0}}$$

there is some $x \in B$ such that:

$$\|x\|_{k_0} \leq m'_{k_0} \|y\|_{k_0+d_2} \text{ and } F(x) = y$$

Corollary (Lipschitz inverse)

For every y_1, y_2 *with* $\|y_i\|_{k_0+d_2} \leq m_{k_0}'^{-1} R$ *and every* $x_1 \in B$ *with* $F(x_1) = y_1$, *there is some* x_2 *with:*

$$\|x_2 - x_1\|_{k_0} \leq m_{k_0}' \|y_2 - y_1\|_{k_0+d_2} \text{ and } F(x_2) = y_2$$

Corollary (Finite regularity)

Suppose F *extends to a continuous map* $\bar{F} : X_{k_0} \to Y_{k_0-d_1}$. *Take some* $y \in Y_{k_0+d_2}$ *with* $\|y\|_{k_0+d_2} < R m_{k_0}'^{-1}$. *Then there is some* $x \in X_{k_0}$ *such that* $\|x\|_{k_0} < R$ *and* $\bar{F}(x) = y$.

Proof: step 1

Let $\bar{y}$ be given, with $\|y\|_{k_0+d_2} \leq \frac{R}{m'_{k_0}}$. Let $\beta_k \geq 0$ be a sequence with unbounded support satisfying:

$$\begin{aligned}\sum_{k=0}^{\infty} \beta_k m_k m'_{k+d_1} n^k &< \infty, \quad \forall n \in \mathbb{N}, \\ \frac{1}{\beta_{k_0+d_2}} \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k &\leq \frac{R}{m'_{k_0}}\end{aligned}$$

Set $\alpha_k := m'^{-1}_{k_0} \beta_{k+d_2}$ and define:

$$\|x\|_\alpha := \sum_{k=0}^{\infty} \alpha_k \|x\|_k \,, \quad X_\alpha = \{x \in X \mid \|x\|_\alpha < \infty\}$$

Then $X_\alpha \subsetneq X$ is a linear subspace, X_α is a Banach space and the identity map $X_\alpha \to X$ is continuous: So the restriction $F : X_\alpha \to Y$ is continuous.

Step 1 (ct'd)

Now consider the function $f : X_\alpha \longrightarrow \mathbb{R} \cup \{+\infty\}$ (the value $+\infty$ is allowed) defined by:

$$f(x) = \sum_{k=0}^{\infty} \beta_k \|F(x) - \bar{y}\|_k$$

f is lower semi-continuous, and $0 \le \inf f \le f(0) = \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \infty$. By the EVP there is a point $\bar{x} \in X_\alpha$ such that:

$$\begin{aligned} f(\bar{x}) &\le f(0) = \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k, \quad \|\bar{x}\|_\alpha \le \alpha_{k_0} R \\ f(x) &\ge f(\bar{x}) - \frac{f(0)}{\alpha_{k_0} R} \|x - \bar{x}\|_\alpha, \quad \forall x \in X_\alpha \end{aligned}$$

It follows that:

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha_k \|\bar{x}\|_k &\le \alpha_{k_0} R, \text{so } \|\bar{x}\|_{k_0} \le R \\ \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k &< \infty, \ \sum_{k=0}^{\infty} \beta_k \|F(\bar{x}) - \bar{y}\|_k < \infty, \ \sum_{k=0}^{\infty} \beta_k \|F(\bar{x})\|_k < \infty \end{aligned}$$

Proof: step 2

Assume then $F(\bar{x}) \neq \bar{y}$. If $u \in X_\alpha$, we can set $x = \bar{x} + tu$, replace f by its value and divide by t.

$$-\lim_{\substack{t \longrightarrow 0 \\ t>0}} \frac{1}{t} \left[\sum_{k=0}^{\infty} \beta_k \left\| \bar{y} - F(\bar{x} + tu) \right\|_k - \sum_{k=0}^{\infty} \beta_k \left\| \bar{y} - F(\bar{x}) \right\|_k \right] \leq A \sum_{k \geq 0} \alpha_k \left\| u \right\|$$

with $A = \sum \beta_k \left\| \bar{y} \right\|_k (\alpha_{k_0} R)^{-1} < 1$. We would like to go one step further:

$$-\sum_{k=0}^{\infty} \beta_k \left(\frac{F(\bar{x}) - \bar{y}}{\left\| F(\bar{x}) - \bar{y}_k \right\|_k}, DF(\bar{x})\, u \right)_k \leq A \sum_{k \geq 0} \alpha_k \left\| u \right\|_k$$

This program can be carried through (by repeated use of Lebesgue's dominated convergence theorem) if we take $u = u_n$, where $u_n = L(\bar{x})\, v_n$ and:

$$\begin{aligned} v_n &\rightarrow F(\bar{x}) - \bar{y}, \quad v_n \text{ controlled,} \\ \left\| v_n \right\|_k &\leq c_3 \left(F(\bar{x}) - \bar{y} \right) \left\| F(\bar{x}) - \bar{y} \right\|_k \end{aligned}$$

Proof: step 3

Plugging in $u_n = L(\bar{x})\, v_n$, we get:

$$
\begin{aligned}
-\sum_{k=0}^{\infty} \beta_k \left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}_k\|_k}, DF(\bar{x})\, L(\bar{x})\, v_n \right)_k &\leq A \sum_{k \geq 0} \alpha_k \|L(\bar{x})\, v_n\|_k \\
-\sum_{k=0}^{\infty} \beta_k \left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}_k\|_k}, v_n \right)_k &\leq A \sum_{k \geq 0} \alpha_k \|L(\bar{x})\, v_n\|_k
\end{aligned}
$$

Letting $n \longrightarrow \infty$, so $v_n \longrightarrow F(\bar{x}) - \bar{y}$, this becomes:

$$
\begin{aligned}
-\sum_{k=0}^{\infty} \beta_k \|F(\bar{x}) - \bar{y}_k\|_k &\leq A \sum_{k \geq 0} \alpha_k m_k \|F(\bar{x}) - \bar{y}_k\|_{k+d_2} \\
&= A \sum_{k \geq d_2} \beta_k \|F(\bar{x}) - \bar{y}_k\|_k
\end{aligned}
$$

which is a contradiction since $A < 1$

A setting for Nash-Moser

We work with a continuous family of norms and finite regularity:

$$\begin{aligned} X_{s_0} &\subset X_s \subset X_{s_1} \text{ for } s_0 < s < s_1 \\ s_0 &< s < s' < s_1 \Longrightarrow \|x\|_s \leq \|x\|_{s'} \end{aligned}$$

We assume that X_{s_0} admits a family of smoothing operators, i.e. for every $N \in \mathbb{N}$ a projection $\Pi_N : X_{s_0} \to X_{s_0}$ with

$$\begin{aligned} \|\Pi_N u\|_{s_0+d} &\leq C(d)\, N^d \|u\|_s, \quad \forall d \geq 0 \\ \|(I - \Pi_N)\, u\|_{s_0} &\leq C(d)\, N^{-d} \|u\|_{s_0+d}, \quad \forall d \geq 0 \end{aligned}$$

Typically $E_N = \Pi_N X_{s_0}$ is finite-dimensional

Roughly tame maps

We are given a neighbourhood of zero $B = \left\{x \mid \ \|x\|_{s_0} \leq R\right\}$ in X, and a number $d_1 \geq 0$

Definition

A map $F : X_{s_1} \rightarrow X_{s_0}$ is *roughly tame* over B if there exists non-negative numbers d_1, d_2, and c such that

1. $F(0) = 0$ and F is continuous and Gâteaux-differentiable from $B \cap X_{s+d_1}$ into X_s for every $s_0 \leq s \leq s_1$
2. For all $x \in B$, we have:
$$\|DF(x)\, u\|_s \leq m\left(\|u\|_{s+d_1} + \|x\|_{s+d_1}\|u\|_{s_0}\right), \quad s_0 \leq s \leq s_1$$
3. For each $N \in \mathbb{N}$, there exists a linear map $L_N(x)$ from E_N into itself such that:
$$\begin{aligned} \forall v \in \quad & E_N, \quad \Pi_N DF(x)\, L_N(x)\, v = v \\ \|L_N(x)\, v\|_s \quad \leq \quad & cN^{\,d_2}\left(\|v\|_s + \|x\|_s \|v\|_{s_0}\right), \quad s_0 \leq s \leq s_1 \end{aligned}$$

Nash-Moser theorem without smoothness

With Eric Séré we have proved the following (unpublished)

Theorem

Suppose F is roughly tame and:

$$s_1 - s_0 > 4\left(d_1 + d_2\right)$$

Then there is some $\varepsilon > 0$ and $\bar{s}$ with $s_0 < \bar{s} < s_1$ such that, if $\left\|y\right\|_{\bar{s}} \leq \varepsilon$, the equation $F\left(x\right) = y$ has a solution in $B \cap X_{s_0 + d_1}$

Note that typically $\bar{s} > s_0 + d_2$: there is an *additional loss of derivatives*, due to the presence of x in the estimates wrt to u. The proof goes by following the procedure of Berti, Bolle and Procesi, namely a Galerkin iteration where one solves *exactly* at each step the approximate equation on $\Pi_N X_{s_0}$. They do so by applying a "smooth" inverse function theorem (in finite dimension) at each step, and we simply substitute the non-smooth theorem I proved earlier.

With Jacques Fejoz, we are now trying to apply this result to KAM

Bibliography

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