Economic growth and sustainable development: how should we discount the future?

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Abstract

In 1987, the Brundtland Commission famously defined sustainable development as "development that meets the needs of the present without compromising the needs of the future". This paper is concerned with translating this definition in the framework of the neoclassical one-sector model of economic growth. We investigate and compare three possible criteria for sustainable development. The first one, which was introduced by Chichilnisky, the second one, which was introduced by Ekeland and Lazrak, and the third one, which goes back to Ramsey himself. We define and investigate equilibrium strategies. For the Chichilnisky criterion, there is a unique equilibrium strategy, which is just the optimal strategy for the neoclassical model. In the other two cases, there is a continuum of equilibrium strategies. We conclude that the most satisfying candidates for sustainable development are the equilibrium strategies for the third criterion (H-criterion).

1 Introduction

In the neoclassical model for economic growth, under the dynamics of capital accumulation:

$$\frac{dk}{dt} = f(k(t)) - c(t), \quad k(0) = k_0, \tag{1.1}$$

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where $k(t)$ is the capital level and $c(t)$ is the consumption at time t, one wants to maximize the total discounted utility from time 0 to ∞:

$$\int_0^\infty u(c(t), k(t))e^{-\delta t}dt, \tag{1.2}$$

where u is strictly concave with respect to its first variable and concave with respect to its second variable. We shall refer to $e^{-\delta t}$ as the discount factor, to δ as the interest rate, and to the whole integral as the Samuelson criterion, or shortly, S-criterion. Ramsey, in his seminal paper [Ram1], took $\delta = 0$, that is, he did not discount the future. Taking $\delta > 0$ is his own words, "a practice which is ethically indefensible, and arises merely from the weakness of imagination" [Ram1].

It seems that the practice of taking $\delta > 0$ arises from an influential paper of Samuelson [Sam1]. It is now the standard approach to economic growth: see for instance the textbooks of Blanchard and Fisher [BlF1], or Aghion and Bolton [AgB1]. However, this practice has been challenged from two sides. On the one hand, psychological evidence shows that individuals do not discount the future at a constant rate: individual behavior is better represented by the hyperbolic discount factor $(1 + kt)^{-1}$ than by the exponential $e^{-\delta t}$. Concern for intergenerational equity has led Ekeland and Lazrak [EkL2], following earlier work by Sumaila and Walters [SuW1], to propose discount factors which are linear combinations of exponentials, namely $\lambda e^{-\delta t} + \mu e^{-\rho t}$, which also lead to non-constant interest rates. It is by now well known that non-constant interest rates imply time inconsistency, so that optimal solutions are no longer implementable, and one has to look instead for equilibrium strategies: see the work of Karp-Lee [KaL1], [Kar1], [Kar2], and Ekeland-Lazrak [Eke2], [EkL1], [EkL2] to that effect.

A challenge to the classical Ramsey model has come from another direction as well. In two influential papers [Chi1], [Chi2], Chichilnisky has proposed an axiomatic approach to sustainable development, based on the twin ideas that there should be no dictatorship of the present and no dictatorship of the future. In the case of discrete time, $t = 0, 1, 2...$, she shows in [Chi2] that all welfare function $W : l^\infty \to \mathbb{R}$ which have neither dictatorship of the present nor dictatorship of the future must have the linear form:

$$W(u_0, u_1, ...) = \sum_{t=0}^{\infty} \lambda_t u_t + \varphi(u_0, u_1, ...), \tag{1.3}$$

where $\lambda_t > 0, \sum_{t=0}^{\infty} \lambda_t < +\infty$, and φ is a purely finitely additive measure. For instance, one may take $\varphi(u_0, u_1, ...) = \lim_{t\to\infty} u_t$ when this limit exists. In [Chi1] and [Chi2], she then suggests to use the following criterion

$$\int_0^\infty u(c(t), k(t))e^{-\delta t}dt + \alpha \lim_{t\to\infty} \tilde{u}(c(t), k(t)) \tag{1.4}$$

in the continuous time case - Henceforth we shall refer to (1.4) as the C-criterion.

It is well-known that, under broad conditions on f, the limit $\lim_{t\to\infty} f(t)$ exists if and only if the limit $\lim_{r\to 0} r\int_0^\infty e^{-rt} f(t)dt$ exists, and we then have

$$\lim_{t\to\infty} f(t) = \lim_{r\to 0} r \int_0^\infty e^{-rt} f(t)dt. \tag{1.5}$$

It is therefore tempting to consider the following criterion:

$$\int_0^\infty u(c(t), k(t))e^{-\delta t}dt + \alpha r \int_0^\infty \tilde{u}(c(t), k(t))e^{-rt}dt, \tag{1.6}$$

which we will henceforth call the E(r)-criterion. Note that, it is a linear combination of exponentials, a case already studied by Karp and Ekeland-Lazrak. One would think that equilibrium strategies for the E(r)-criterion converge to the equilibrium strategy for the C-criterion, but it turns out not to be the case.

In his doctoral thesis [Huy1], Thai Ha-Huy introduced the criterion

$$W(u_0, u_1 ...) = \sum_{t=0}^{\infty} a_t(u_t - \overline{u}), \tag{1.7}$$

where $\forall t, 0 < \underline{a} \le a \le \overline{a}$, $\sum_{t=0}^{\infty} a_t = +\infty$, $\bar{u} = \lim_{t\to\infty} u_t$ provided that $\bar{u}$ and the sum of the right hand side in (1.7) exist and are finite. The criterion (1.7) satisfies the two No-dictatorship Axioms. In the continuous time case, following (1.7), we also consider the following criterion function

$$\int_0^\infty u(c(t), k(t))e^{-\delta t}dt + \alpha \int_0^\infty [\tilde{u}(c(t), k(t)) - \tilde{u}(c_\infty, k_\infty)]dt, \tag{1.8}$$

where $c_\infty = \lim_{t\to\infty} c(t)$ and $k_\infty = \lim_{t\to\infty} k(t)$, and we will henceforth call it the H-criterion.

Our main results are as follows. There is no optimal solution for the growth model if the C-criterion is chosen: in other words, the supremum is not attained. There is a single equilibrium strategy, but it is quite disappointing, since it does not depend on α: it is just the optimal strategy in the classical Ramsey model. In other words, if one uses the C-criterion, one can just take $\alpha = 0$ and forget about the future.

For each fixed $r > 0$, we show that there is a continuum of equilibrium strategies for the E(r)-criterion: there is an non-empty open set I_r such that, for every $k_\infty \in I_r$, there is an equilibrium strategy converging to k_∞. Note that $I_0 := \lim_{r\to 0} I_r$ is also a non-empty open set. For each $k_\infty \in I_0$ and every $r > 0$ small enough, there is an equilibrium strategy for E(r)-criterion which converges to k_∞ (although this strategy may be defined only in a small interval around k_∞).

Finally, we consider the H-criterion. We show that there is some k_H such that, for every $k_\infty > k_H$, there is an H-equilibrium strategy converging to k_∞. So the H-criterion does not suffer from the drawbacks of the C-criterion (which leads down to the neoclassical optimal growth model) or the E(r)-criterion (where E(r)-equilibrium strategies do not converge when $r \to 0$). We think that the H-criterion is the best adapted to sustainable development, and we note that it is close to the original criterion chosen by Ramsey.

The structure of the paper is as follows. In Section 2, we study the optimal solutions when S-criterion or C-criterion is used. In Section 3, we describe the Ekeland-Lazrak and the Karp approach to equilibrium strategies, and we show they are equivalent. We characterize the equilibrium strategies by a system of two ODE in implicit form. The analysis in this section is new, because we deal with two utility functions $u(c,k)$ and $\tilde{u}(c,k)$ whereas the Ekeland-Lazrak and Karp papers have $u = \tilde{u}$, depending only on c. In Section 4, we solve these equations for the E(r)-criterion. In Section 5, we prove that there is a continuum of E(r)-equilibrium strategies and steady states for every $r > 0$, but when r goes to zero, a series of such E(r)-equilibrium strategies with the same steady state and different r do not converge. In Section 6, we directly treat the C-criterion. We will find that the only equilibrium strategy is the optimal strategy under the S-criterion because the decision maker cannot influence the limit term of the C-criterion. In Section 7, we treat the H-criterion, and we find a continuum of equilibrium strategies and steady states of the economy. The conclusions are summarized in Section 8. The paper ends with an Appendix containing the proofs.

2 The optimal solutions to the S-criterions and the C-criterion

2.1 The S-criterion

We want to find the HJB equation of the three optimal control under the three criterion (1.2), (1.4) and (1.6) respectively.

Firstly, we consider

$$\begin{cases} \max\limits_{c(\cdot)} \displaystyle\int_0^\infty u(c(t),k(t))e^{-\delta t}dt, \\ \dfrac{dk}{dt} = f(k(t)) - c(t), \quad k(0) = k_0. \end{cases} \tag{2.1}$$

Suppose the value function is

$$V(k_0) = \max_{c(t)} \left\{ \int_0^\infty u(c(t),k(t))e^{-\delta t}dt \middle| \frac{dk}{dt} = f(k(t)) - c(t), \quad k(0) = k_0 \right\}, \tag{2.2}$$

then it satisfies the HJB equation

$$\delta V(k) = \max_c \{u(c,k) + (f(k) - c)V'(k)\}. \tag{2.3}$$

Suppose $z = I(y, k)$ is the inverse function of $u_1'(z, k) = y$, that is to say, we have

$$u_1'(I(y, k), k) = y, \quad I(u_1'(z, k), k) = z, \tag{2.4}$$

where we denote $u_1'(c, k)$ and $u_2'(c, k)$ the derivatives of $u(c, k)$ with respect to c and k respectively. The existence and uniqueness of I are guaranteed by the strictly concavity of u with respect to its first argument. Then in (2.3), by the strictly concavity of u on its first argument, we have that the maximized argument c satisfies:

$$u_1'(c, k) - V'(k) = 0, \tag{2.5}$$

thus $c = I(V'(k), k)$ by (2.4), and plugging it into (2.3), we obtain

$$(f(k) - I(V'(k), k))V'(k) + u(I(V'(k), k), k) = \delta V(k). \tag{2.6}$$

Assume that, under the optimal feedback control $c(k) = I(V'(k), k)$, the capital level $k(t)$ converges to a limit k_∞. Then

$$f(k_\infty) = c(k_\infty) = I(V'(k_\infty), k_\infty), \tag{2.7}$$

and by (2.4), we have

$$V'(k_\infty) = u_1'(I(V'(k_\infty), k_\infty), k_\infty) = u_1'(f(k_\infty), k_\infty). \tag{2.8}$$

If V is C^2 near k_∞, we may differentiate (2.6) and let $k = k_\infty$, getting

$$\begin{aligned}
\delta V'(k_\infty) &= (f(k_\infty) - I(V'(k_\infty), k_\infty))V''(k_\infty) \\
&\quad + [f'(k_\infty) - I_1'(V'(k_\infty), k_\infty)V''(k_\infty) - I_2'(V'(k_\infty), k_\infty)]V'(k_\infty) \\
&\quad + u_1'(I(V'(k_\infty), k_\infty), k_\infty)[I_1'(V'(k_\infty), k_\infty)V''(k_\infty) + I_2'(V'(k_\infty), k_\infty)] \\
&\quad + u_2'(I(V'(k_\infty), k_\infty), k_\infty) \\
&= f'(k_\infty)V'(k_\infty) + u_2'(I(V'(k_\infty), k_\infty), k_\infty) \\
&= f'(k_\infty)V'(k_\infty) + u_2'(f(k_\infty), k_\infty),
\end{aligned} \tag{2.9}$$

where $I_1'(y, k)$ and $I_2'(y, k)$ are the derivatives of $I(y, k)$ with respect to y and k respectively, and where we used (2.7) and (2.8). With (2.8) and (2.9), we also have

$$f'(k_\infty) + \frac{u_2'(f(k_\infty), k_\infty)}{u_1'(f(k_\infty), k_\infty)} = \delta. \tag{2.10}$$

If the utility function u does not depend on its second argument k, i.e., if the capital k does not bring the direct utilities, then the condition (2.10) becomes the ordinary condition $f'(k_\infty) = \delta$ in [BlF1]. We have

Theorem 2.1. *Let*

$$K = \left\{ k > 0 \middle| f'(k) + \frac{u_2'(f(k), k)}{u_1'(f(k), k)} = \delta \right\}. \tag{2.11}$$

If there is an differentiable optimal solution $\sigma(k)$ of problem (2.1) such that $k(t)$ converges to k_∞, then

$$k_\infty \in K \tag{2.12}$$

and the value function V of (2.2) is C^2 and satisfies

$$\begin{cases} (f(k) - I(V'(k), k))V'(k) + u(I(V'(k), k), k) = \delta V(k), \\ V(k_\infty) = \dfrac{1}{\delta} u(f(k_\infty), k_\infty), \end{cases} \tag{2.13}$$

and the optimal strategy $\sigma(k)$ is given by:

$$\sigma(k) = I(V'(k), k), \tag{2.14}$$

where I is defined by (2.4).

Conversely, if there is a $k_\infty \in K$ of satisfying

$$\frac{u_2'(f(k_\infty), k_\infty)}{u_1'(f(k_\infty), k_\infty)}, \frac{u_{12}''(f(k_\infty), k_\infty)}{u_{11}''(f(k_\infty), k_\infty)} < \min\left\{1, \frac{u_1'(f(k_\infty), k_\infty) f''(k_\infty)}{(1+\delta) u_{11}''(f(k_\infty), k_\infty)}\right\}, \tag{2.15}$$

then system (2.13) has a c^2 solution in a neighborhood of k_∞. Moreover, $\sigma(k) = I(V'(k), k)$ is an optimal strategy and the solution $k(t)$ of (3.2) converges to k_∞.

This theorem is proved by Appendix A.

Remark. Condition (2.15) means the capital level has a small effect to the utility than the direct consumption. As we can see in Appendix A, if (2.15) does not hold, then the optimal control solution may not exist.

2.2 The C-criterion

Then, we consider

$$\begin{cases} \max\limits_{c(\cdot)} \displaystyle\int_0^\infty u(c(t), k(t)) e^{-\delta t} dt + \alpha \lim_{t \to +\infty} \tilde{u}(c(t), k(t)), \\ \dfrac{dk}{dt} = f(k(t)) - c(t), \quad k(0) = k_0. \end{cases} \tag{2.16}$$

Since f is strictly increasing with respect to k, the optimal solution does not exist.

Theorem 2.2. *There does not exist an optimal control of (2.16) such that the capital $k(t)$ converges to some finite level.*

Proof. Assume the optimal control is $c(t)$, and under this control, $k(t)$ converges to some $k_\infty < +\infty$. Then for t^* large enough, we define a new control $c^*(t)$: $c^*(t) = c(t)$, $\forall t \in [0, t^*]$ and choose proper values after time $t > t^*$ such that the corresponding capital $k^*(t)$ converges to some $k_\infty + 1$. Then we have

$$\begin{aligned}
W(c^*(\cdot), k^*(\cdot)) &= \int_0^\infty u(c^*(t), k^*(t))e^{-\delta t}dt + \alpha \lim_{t\to+\infty} \tilde{u}(c^*(t), k^*(t)) \\
&> \int_0^{t^*} u(c^*(t), k^*(t))e^{-\delta t}dt + \alpha\tilde{u}(f(k_\infty + 1), k_\infty + 1) \\
&= \int_0^{t^*} u(c(t), k(t))e^{-\delta t}dt + \alpha\tilde{u}(f(k_\infty + 1), k_\infty + 1) \\
&> \int_0^{\infty} u(c(t), k(t))e^{-\delta t}dt - \epsilon + \alpha\tilde{u}(f(k_\infty + 1), k_\infty + 1) \\
&> \int_0^{\infty} u(c(t), k(t))e^{-\delta t}dt + \alpha\tilde{u}(f(k_\infty), k_\infty) \\
&= W(c(\cdot), k(\cdot)), \qquad (2.17)
\end{aligned}$$

where the last equality holds because of the maximal value of $\tilde{u}(f(k), k)$ is strictly increasing and ϵ small enough. This is a contradiction. Thus, the result holds. ∎

The C-criterion exhibit time inconsistency, as we shall see later on, but we want to point out a special feature: if we consider it as an optimization problem from the point of view of time $t = 0$, that is, if we assume that the decision-maker at $t = 0$ can commit his successors, then the maximum is not attained.

3 The problem of time inconsistency

3.1 Preliminaries

We now use the E(r)-criterion, namely (1.6). More generally, we use $h(t)$ and $\tilde{h}(t)$ to represent the first discounted factor $e^{-\delta t}$ and the second discounted factor e^{-rt} respectively.

We consider the intertemporal decision problem(as it seen at time $t = 0$)

$$\begin{cases} \textit{object functional } J(c(\cdot)) = \displaystyle\int_0^\infty h(t)u(c(t), k(t))dt + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}(c(t), k(t))dt, \\ \dfrac{dk}{dt} = f(k(t)) - c(t), \quad k(0) = k_0. \end{cases} \qquad (3.1)$$

Because of time-inconsistency, problem (3.1) can no longer be seen as an optimization problem. There is no way for the decision-maker at time 0 to achieve what is, from her point of view, the first-best solution of the problem, and she must turn to a second-best policy: the best she can do is to guess what her successor are planning to do, and then to plan her own consumption $c(0)$ accordingly. In other word, we will be looking for a subgame-perfect equilibrium(cf. [EkL2] for more details) of a certain game.

To characterize the equilibrium policy, we first use the approach by I.Ekeland and A.Lazrak of [Eke2], and then we use the approach by L.Karp of [Kar2].

As in [Eke2], we restrict our analysis to convergent Markov strategies. We introduce the definition of Morkov strategy and convergent strategy.

Definition 3.1. *If the consumption policy depends only on the current capital stock and not on past history, current time or some extraneous factors, then the strategy is called a Markov strategy.*

A Markov strategy is given by $c = \sigma(k)$, where we often suppose $\sigma : R \to R$ is a continuously differentiable function. If we apply the strategy σ, the dynamics of capital accumulation from $t = 0$ are given by:

$$\frac{dk}{ds} = f(k(s)) - \sigma(k(s)), \quad k(0) = k_0 \tag{3.2}$$

Definition 3.2. *A Markov strategy σ is called a convergent Markov strategy or convergent strategy, if there is some steady state $\bar{k} > 0$ of σ, such that the solution $k = k(s)$ of (3.2) satisfies $\lim_{s\to\infty} k(s) = \bar{k}$ when the initial value k_0 is sufficiently close to $\bar{k}$.*

3.2 Definition and characterization of equilibrium strategies

3.2.1 The approach by Ekeland-Lazrak

Note that $u \neq \tilde{u}$ and u depends on the capital stock k as well as the current consumption c. Then the definition and characterization of equilibrium strategies become more general than in [EkL2].

Now we suppose σ is a convergent Markov strategy, has been announced and is public knowledge. The decision-maker at 0 has capital stock k_0. If all future decision-maker apply the strategy σ, the resulting future capital stock flow $k_b(t)$ obeys

$$\begin{cases} \dfrac{dk_b}{dt} = f(k_b(t)) - \sigma(k_b(t)), \quad t \geq 0, \\ k_b(0) = k_0. \end{cases} \tag{3.3}$$

Since all the decision maker face the similar problem only with different initial stocks, we can just consider the decision-maker at time 0. We suppose the decision-maker at time 0 can commit all the decision-makers in interval $[0, \epsilon]$ for $\epsilon > 0$ small enough. She expects all later ones to apply the same consumption policy, that is, to consume $\sigma(k)$ for decision-maker t when at that time the capital level is k. If she commits to another bundle c, as the continuity of u and $\tilde{u}$, the immediate utility flow during $[0, \epsilon]$ is $[u(c, k_0) + \alpha r\tilde{u}(c, k_0)]\epsilon + o(\epsilon)$ where $o(\epsilon)$ is a higher order term of ϵ and we omit it thereafter. At time ϵ, the resulting capital will be $k_0 + (f(k_0) - c)$, where we omit the higher order terms. From then on, the strategy σ will be applied which results in a capital stock k_c satisfying

$$\begin{cases} \dfrac{dk_c}{dt} = f(k_c(t)) - \sigma(k_c(t)), \ t \geq \epsilon, \\ k_c(\epsilon) = k_0 + (f(k_0) - c)\epsilon. \end{cases} \tag{3.4}$$

The capital stock k_c can be written as

$$k_c(t) = k_b(t) + k_i(t)\epsilon, \tag{3.5}$$

where

$$\begin{cases} \dfrac{dk_i}{dt} = (f'(k_b(t)) - \sigma'(k_b(t)))k_i(t), \ t \geq \epsilon, \\ k_i(\epsilon) = \sigma(k_0) - c. \end{cases} \tag{3.6}$$

Summing up the utility of decision-makers at $[0, \epsilon]$ and of the later decision-makers, we find that the total gain for the decision-maker at time 0 from consuming bundle c during the interval of length ϵ when she can commit, is

$$\begin{aligned} & u(c, k_0)\epsilon + \int_\epsilon^\infty h(s)u(\sigma(k_b(t) + \epsilon k_i(t)), k_b(t) + \epsilon k_i(t))dt \\ & + \alpha r \left[\tilde{u}(c, k_0)\epsilon + \int_\epsilon^\infty \tilde{h}(s)\tilde{u}(\sigma(k_b(t) + \epsilon k_i(t)), k_b(t) + \epsilon k_i(t))dt \right], \end{aligned} \tag{3.7}$$

and in the limit, when $\epsilon \to 0$, and the commitment span of the decision-maker vanishes, expanding this expression to the first order of ϵ leaves us with

$$\begin{aligned} & \int_0^\infty h(t)u(\sigma(k_b(t)), k_b(t))dt + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}(\sigma(k_b(t)), k_b(t))dt \\ & + \epsilon P(k_0, \sigma, c), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} P(k_0, \sigma, c) &= u(c, k_0) - u(\sigma(k_0), k_0) + \alpha r(\tilde{u}(c, k_0) - \tilde{u}(\sigma(k_0), k_0)) \\ & \quad + \int_0^\infty h(t)u_1'(\sigma(k_b(t)), k_b(t))\sigma'(k_b(t))k_i(t)dt \\ & \quad + \int_0^\infty h(t)u_2'(\sigma(k_b(t)), k_b(t))k_i(t)dt \\ & \quad + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_1'(\sigma(k_b(t)), k_b(t))\sigma'(k_b(t))k_i(t)dt \\ & \quad + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_2'(\sigma(k_b(t)), k_b(t))k_i(t)dt, \end{aligned} \tag{3.9}$$

where $u_1', u_2', \tilde{u}_1'$ and $\tilde{u}_2'$ are the partial derivatives of u with respect to the first and second variable, and $\tilde{u}$ with respect to the first and second variable respectively, and k_i solves the linear equation

$$\begin{cases} \dfrac{dk_i}{dt} = (f'(k_b(t)) - \sigma'(k_b(t)))k_i(t), \ t \geq 0, \\ k_i(0) = \sigma(k_0) - c. \end{cases} \tag{3.10}$$

The first term of (3.8) does not depend on the decision-maker at time 0, and the second term is the one that the decision-maker at time 0 will try to maximize. Thus, given that a strategy σ has been announced and that the current state is k_0, the decision-maker at time 0 faces the optimization problem

$$\max_c P(k_0, \sigma, c), \tag{3.11}$$

where $P(k_0, \sigma, c)$ is defined by (3.9).

Definition 3.3. *A convergent Markov strategy $\sigma : R \to R$ is an equilibrium strategy for the intertemporal decision problem (3.1), if for every $k_0 \in R$, the maximum in problem (3.11) is attained for $c = \sigma(k_0)$:*

$$\sigma(k_0) = arg\max_c P(k_0, \sigma, c). \tag{3.12}$$

Given a convergent Markov strategy σ, we shall deal with the Cauchy problem (3.3). The value $k_b(t)$ depends on current time t, initial data k_0 and the strategy σ. To stress this dependence, we write $k_b(t) = \mathcal{K}(t; k_0, \sigma)$ where $\mathcal{K}$ is the flow associated with the differential equation (3.3) defined by

$$\begin{cases} \dfrac{\partial \mathcal{K}(t; k_0, \sigma)}{\partial t} = f(\mathcal{K}(t; k_0, \sigma)) - \sigma(\mathcal{K}(t; k_0, \sigma)), \ t \geq 0, \\ \mathcal{K}(0; k_0, \sigma) = k_0. \end{cases} \tag{3.13}$$

We now characterizes the equilibrium strategy. The following theorem use two parts to characterizes it: a functional equation on the value function and an instantaneous optimality condition which determines the current consumption.

Theorem 3.4. *Let σ be an equilibrium continuously differentiable convergent Markov strategy, or shortly, equilibrium convergent Markov strategy, then the value function*

$$v(k_0) = \int_0^\infty h(t) u(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) dt + \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) dt \tag{3.14}$$

satisfies, for all k_0, the functional equation

$$\begin{aligned} v(k_0) = \quad & \int_0^\infty h(t) u(i \circ v'(\mathcal{K}(t; k_0, i \circ v')), \mathcal{K}(t; k_0, i \circ v')) dt \\ & + \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}(i \circ v'(\mathcal{K}(t; k_0, i \circ v')), \mathcal{K}(t; k_0, i \circ v')) dt \end{aligned} \tag{3.15}$$

and the instantaneous optimality condition

$$\bar{u}_1'(\sigma(k_0), k_0) = v'(k_0), \quad \sigma(k_0) = i(v'(k_0), k_0), \tag{3.16}$$

where

$$\bar{u}(c,k) = u(c,k) + \alpha r \tilde{u}(c,k), \tag{3.17}$$

and $\bar{u}_1'$ *is the partial derivative of* $\bar{u}$ *with respect to the first variable* c*, and* $z = i(y,k)$ *is the implicit function of* $\bar{u}_1'(z,k) = y$*, that is to say, we have*

$$\bar{u}_1'(i(y,k),k) = y, \quad i(\bar{u}_1'(z,k),k) = z, \tag{3.18}$$

and the existence and uniqueness of i *can be guaranteed by the strictly concavity of* $\bar{u}$ *on its first argument.*

Conversely, if a function v *is twice continuously differentiable, satisfies (3.15) and the strategy* $\sigma(k_0) = i(v'(k_0), k_0)$*, for convenient, we denote it by* $\sigma = i \circ v'$*, is convergent, then* σ *is an equilibrium strategy.*

Remark. Since $\partial^2 \bar{u}(c,k)/\partial c^2 = \partial^2 u(c,k)/\partial c^2 + \alpha r \partial^2 \tilde{u}(c,k)/\partial c^2 > 0$, then $\bar{u}(c,k)$ is strictly concave with respect to c, and the equation $\bar{u}_1'(z,k) = y$ with respect to z has a unique solution $z = z(y,k)$.

This theorem is proved in Appendix B.

The instantaneous relation (3.16) expresses the usual tradeoff between the utility derived from current consumption and the utility value of saving. Equation (3.15) is a fundamental characterization of the equilibrium strategies and it takes the form of a functional equation on v. The following theorem gives an alternative characterization, the differential equation, which resembles the usual Euler equation from the calculus of variation.

Theorem 3.5. *Following notations of Theorem 3.4, let* v *be a* C^2 *function such that the strategy* $\sigma = i \circ v'$ *converges to* $\bar{k}$*. Then* v *satisfies the integrated equation (3.15) if and only if it satisfies the following functional equation*

$$\begin{aligned} & -\int_0^\infty h'(t) u(i \circ v'(\mathcal{K}(t;k_0, i \circ v')), \mathcal{K}(i \circ v'; t, k_0)) dt \\ & -\alpha r \int_0^\infty \tilde{h}'(t) \tilde{u}(i \circ v'(\mathcal{K}(t;k_0, i \circ v')), \mathcal{K}(t;k_0, i \circ v')) dt \\ = \; & \bar{u}(i \circ v'(k_0), k_0) + v'(k_0)(f(k_0) - i \circ v'(k_0))] \end{aligned} \tag{3.19}$$

together with the boundary condition

$$v(\bar{k}) = u(f(\bar{k}), \bar{k}) \int_0^\infty h(t) dt + \alpha r \tilde{u}(f(\bar{k}), \bar{k}) \int_0^\infty \tilde{h}(t) dt \tag{3.20}$$

This theorem is proved by Appendix C.

From now on, we use k_∞ to denote the steady state of a economy under a convergent Markov strategy. We also have the following Lemma.

Lemma 3.6. *Let σ be a convergent Markov strategy and its steady state is k_∞, $h, \tilde{h} : [0,\infty] \to R$ be two C^1 functions with exponential decay at infinity as above and λ be a constant. Then*

$$I(k_0) = \int_0^\infty h(t)u(\sigma(\mathcal{K}(t;k_0,\sigma)), \mathcal{K}(t;k_0,\sigma))dt + \lambda \int_0^\infty \tilde{h}(t)\tilde{u}(\sigma(\mathcal{K}(t;k_0,\sigma)), \mathcal{K}(t;k_0,\sigma))dt \quad (3.21)$$

is equivalent to

$$\begin{cases} I'(k_0)(f(k) - \sigma(k)) + u(\sigma(k_0), k_0) + \lambda\tilde{u}(\sigma(k_0), k_0) \\ = -\displaystyle\int_0^\infty h'(t)u(\sigma(\mathcal{K}(t;k_0,\sigma)), \mathcal{K}(t;k_0,\sigma))dt - \lambda \int_0^\infty \tilde{h}'(t)\tilde{u}(\sigma(\mathcal{K}(t;k_0,\sigma)), \mathcal{K}(t;k_0,\sigma))dt, \\ I(k_\infty) = u(f(k_\infty), k_\infty) \displaystyle\int_0^\infty h(t)dt + \lambda\tilde{u}(f(k_\infty), k_\infty) \int_0^\infty \tilde{h}(t)dt. \end{cases} \quad (3.22)$$

Remark. Replacing $v(\cdot)$ by $I(\cdot)$, αr by λ, then the proof in Theorem 3.5 yield the proof of Lemma 3.6. Thus we omit the details.

We will apply these results in the next section.

3.2.2 The approach by Karp

We suppose again that the decision-maker at time 0 can commit all the decision-makers in interval $[0, \epsilon]$ for $\epsilon > 0$ small enough. She expects all later ones to apply the same consumption policy σ.

We also use $v(k_0)$ of (3.14) as our value function, then the only one thing the decision-maker at time 0 can do is to choose a proper consumption c and commit all the decision-makers in interval $[0, \epsilon]$ so that she can maximize her total discounted utilities, and under such condition, the total discounted utilities are $v(k_0)$:

$$v(k_0) = \max_c \left\{ [u(c, k_0) + \alpha r\tilde{u}(c, k_0)]\epsilon + \int_\epsilon^\infty [h(t)u(\sigma(k_c(t)), k_c(t)) + \alpha r\tilde{h}(t)\tilde{u}(\sigma(k_c(t)), k_c(t))]dt \right\}, \quad (3.23)$$

where $k_c(t)$ is defined by (3.4).

On the other hand, by the definition of the value function of (3.14), we have

$$\begin{aligned} v(k_c(\epsilon)) &= \int_0^\infty h(t)u(\sigma(\mathcal{K}(t;k_c(\epsilon),\sigma)),\mathcal{K}(t;k_c(\epsilon),\sigma))dt \\ &\quad +\alpha r\int_0^\infty \tilde{h}(t)\tilde{u}(\sigma(\mathcal{K}(t;k_c(\epsilon),\sigma)),\mathcal{K}(t;k_c(\epsilon),\sigma))dt \\ &= \int_\epsilon^\infty [h(t)u(\sigma(k_c(t+\epsilon)),k_c(t+\epsilon))+\alpha r\tilde{h}(t)\tilde{u}(\sigma(k_c(t+\epsilon)),k_c(t+\epsilon))]dt \\ &= \int_\epsilon^\infty [h(s-\epsilon)u(\sigma(k_c(s)),k_c(s))+\alpha r\tilde{h}(s-\epsilon)\tilde{u}(\sigma(k_c(s)),k_c(s))]ds, \end{aligned} \tag{3.24}$$

where we used (3.4) and (3.13) in the second equality and $s=t-\epsilon$ in the last equality.

By (3.23) and (3.24), we have

$$\begin{aligned} v(k_0) &= \max_c \Big\{[u(c,k_0)+\alpha r\tilde{u}(c,k_0)]\epsilon + v(k_c(\epsilon)) + \int_\epsilon^\infty [h(t)-h(t-\epsilon)]u(\sigma(k_c(t)),k_c(t))dt \\ &\quad +\alpha r\int_\epsilon^\infty [\tilde{h}(t)-\tilde{h}(t-\epsilon)]\tilde{u}(\sigma(k_c(t)),k_c(t))dt\Big\}, \end{aligned} \tag{3.25}$$

and plugging into $k_c(t)=k_b(t)+\epsilon k_i(t), t\geq\epsilon$ and $k_c(\epsilon)=k_0+(f(k_0)-c)\epsilon$ by (3.4), with the exponentially decay of $h(t)$ and $\tilde{h}(t)$, we have

$$\begin{aligned} v(k_0) &= \max_c \Big\{[u(c,k_0)+\alpha r\tilde{u}(c,k_0)]\epsilon + v(k_0) + v'(k_0)(f(k_0)-c)\epsilon \\ &\quad +\epsilon\int_\epsilon^\infty h'(t)u(\sigma(k_b(t)),k_b(t))dt + \epsilon\alpha r\int_\epsilon^\infty \tilde{h}'(t)\tilde{u}(\sigma(k_b(t)),k_b(t))dt + o(\epsilon)\Big\} \\ &= \max_c \Big\{v(k_0)+\epsilon\Big[u(c,k_0)+\alpha r\tilde{u}(c,k_0)+v'(k_0)(f(k_0)-c) \\ &\quad +\int_0^\infty h'(t)u(\sigma(k_b(t)),k_b(t))dt + \alpha r\int_0^\infty \tilde{h}'(t)\tilde{u}(\sigma(k_b(t)),k_b(t))dt\Big]+o(\epsilon)\Big\}. \end{aligned} \tag{3.26}$$

Let $\epsilon\to 0$ and notice $\bar{u}=u+\alpha r\tilde{u}$ by (3.17), we have

$$\begin{aligned} &-\int_0^\infty h'(t)u(\sigma(k_b(t)),k_b(t))dt - \alpha r\int_0^\infty \tilde{h}'(t)\tilde{u}(\sigma(k_b(t)),k_b(t))dt \\ &= \max_c\{\bar{u}(c,k_0)+v'(k_0)(f(k_0)-c)\}. \end{aligned} \tag{3.27}$$

Equation (3.27) was first obtained by Karp [KaL1], [Kar1], [Kar2]. Note that this is the HJB equation for an optimal control problem, namely

$$\begin{cases} \displaystyle\max_{c(\cdot)} \int_0^\infty e^{-rt}[u(c(t),k(t))+\alpha r\tilde{u}(c(t),k(t))-K(c(t))]dt, \\ \displaystyle\frac{dk}{dt}=f(k(t))-c(t), \quad k(0)=k_0, \end{cases} \tag{3.28}$$

where $K(k_0)=(\delta-r)\int_0^\infty e^{-\delta t}u(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt$ and we used $h(t)=e^{-\delta t}, \tilde{h}(t)=e^{-rt}$.

We now show that this approach is equivalent to the approach by Ekeland-Lazrak. In the right hand side of (3.27), the maximum is attained at:

$$c = arg\ \max_c \{\bar{u}(c, k_0) + v'(k_0)(f(k_0) - c)\} = i(v'(k_0), k_0), \tag{3.29}$$

thus we have

$$\begin{aligned} & -\int_0^\infty h'(t)u(\sigma(k_b(t)), k_b(t))dt - \alpha r \int_0^\infty \tilde{h}'(t)\tilde{u}(\sigma(k_b(t)), k_b(t))dt \\ = \ & \bar{u}(i(v'(k_0), k_0), k_0) + v'(k_0)(f(k_0) - i(v'(k_0), k_0)). \end{aligned} \tag{3.30}$$

Notice that $k_b(t) = \mathcal{K}(t; k_0, \sigma)$ of (3.13), then equation (3.30) is exactly the same as (3.19) in Theorem 3.5. Moreover, by (3.29), the equilibrium strategy is given by $\sigma(k_0) = i(v'(k_0), k_0)$, and which is exactly the instantaneous condition (3.16) as we obtained by the approach by Ekeland-Lazrak.

4 Equilibrium strategies for the E(r)-criterion

First, we state a proposition.

Proposition 4.1. *Let $v(k_0)$ be a C^2 function such that the strategy $\sigma(k_0) = i \circ v'(k_0) = i(v'(k_0), k_0)$ converges to k_∞. Then v satisfies (3.19) and (3.20) if and only if there exist a C^1 function $w(k_0)$ such that (v, w) satisfies the system*

$$\begin{cases} v'(k_0)(f(k_0) - \sigma(k_0)) + \bar{u}(\sigma(k_0), k_0) = av(k_0) + bw(k_0), \\ w'(k_0)(f(k_0) - \sigma(k_0)) + u(\sigma(k_0), k_0) - \alpha r \tilde{u}(\sigma(k_0), k_0) = bv(k_0) + aw(k_0), \end{cases} \tag{4.1}$$

where $a = \frac{\delta + r}{2}, b = \frac{\delta - r}{2}$ and with the boundary conditions

$$\begin{cases} v(k_\infty) = \dfrac{1}{\delta} u(f(k_\infty), k_\infty) + \alpha \tilde{u}(f(k_\infty), k_\infty), \\ w(k_\infty) = \dfrac{1}{\delta} u(f(k_\infty), k_\infty) - \alpha \tilde{u}(f(k_\infty), k_\infty). \end{cases} \tag{4.2}$$

This theorem is proved by Appendix D.

We define two sequences of continuous functions:

$$\underline{g}_r(k) = \frac{\delta u_1'(f(k), k) + \alpha r \delta \tilde{u}_1'(f(k), k) - (u_2'(f(k), k) + \alpha \delta \tilde{u}_2'(f(k), k))}{u_1'(f(k), k) + \alpha \delta \tilde{u}_1'(f(k), k)}, \tag{4.3}$$

$$\overline{g}_r(k) = \frac{\delta u_1'(f(k), k) + \alpha r^2 \tilde{u}_1'(f(k), k) - (u_2'(f(k), k) + \alpha r \tilde{u}_2'(f(k), k))}{u_1'(f(k), k) + \alpha r \tilde{u}_1'(f(k), k)}, \tag{4.4}$$

and make the following assumptions

$$\sup_k \frac{u_2'(f(k),k)}{u_1'(f(k),k)} \quad < \quad \delta, \tag{4.5}$$

$$\sup_k \frac{\tilde{u}_1'(f(k),k)}{u_1'(f(k),k)} \quad \leq \quad C, \tag{4.6}$$

for some constant $C > 0$. Then we have

$$\begin{aligned}
\overline{g}_r(k) - \underline{g}_r(k) &= \frac{\delta u_1'(f(k),k) - u_2'(f(k),k)}{u_1'(f(k),k) + \alpha r \tilde{u}_1'(f(k),k)} + \alpha r \frac{r\tilde{u}_1'(f(k),k) - \tilde{u}_2'(f(k),k))}{u_1'(f(k),k) + \alpha r \tilde{u}_1'(f(k),k)} \\
&\quad - \Big[\frac{\delta u_1'(f(k),k) - u_2'(f(k),k)}{u_1'(f(k),k) + \alpha\delta \tilde{u}_1'(f(k),k)} + \alpha\delta \frac{r\tilde{u}_1'(f(k),k) - \tilde{u}_2'(f(k),k)}{u_1'(f(k),k) + \alpha\delta \tilde{u}_1'(f(k),k)}\Big] \\
&= \alpha(\delta - r)\frac{[\delta - r - \frac{u_2'(f(k),k)}{u_1'(f(k),k)} + \frac{\tilde{u}_2'(f(k),k)}{\tilde{u}_1'(f(k),k)}]u_1'(f(k),k)\tilde{u}_1'(f(k),k)}{(u_1'(f(k),k) + \alpha r \tilde{u}_1'(f(k),k))(u_1'(f(k),k) + \alpha\delta \tilde{u}_1'(f(k),k))}.
\end{aligned} \tag{4.7}$$

Since u and $\tilde{u}$ are utility functions, $u_1', u_2', \tilde{u}_1', \tilde{u}_2' > 0$, and by (4.5), for r small enough, we have $\overline{g}_r(k) - \underline{g}_r(k) > 0$. Let $r \to 0$, both $\underline{g}_r(k)$ and $\bar{g}_r(k)$ have limit functions, we denote them by $\underline{g}_0(k)$ and $\overline{g}_0(k)$ respectively:

$$\underline{g}_0(k) \quad = \quad \lim_{r\to 0} \underline{g}_r(k) = \delta - \frac{u_2'(f(k),k) + \alpha\delta\tilde{u}_2'(f(k),k) + \alpha\delta^2\tilde{u}_1'(f(k),k)}{u_1'(f(k),k) + \alpha\delta\tilde{u}_1'(f(k),k)}, \tag{4.8}$$

$$\overline{g}_0(k) \quad = \quad \lim_{r\to 0} \bar{g}_r(k) = \delta - \frac{u_2'(f(k),k)}{u_1'(f(k),k)}. \tag{4.9}$$

We then state a Lemma:

Lemma 4.2. *Suppose the production function f is twice continuously differentiable in $\mathbb{R}^+$ and satisfies the Inada conditions, then there exist infinitely many k_∞ satisfying*

$$\underline{g}_0(k_\infty) < f'(k_\infty) < \overline{g}_0(k_\infty). \tag{4.10}$$

Moreover, for $r > 0$ small enough, there exists infinite many k_∞ satisfying

$$\underline{g}_r(k_\infty) < f'(k_\infty) < \overline{g}_r(k_\infty). \tag{4.11}$$

Proof. Consider the function $f'(k) - \overline{g}_0(k)$. Since f satisfies the Inada conditions, then $\lim_{k\to 0}(f'(k) - \overline{g}_0(k)) = +\infty$. On the other hand, for k large enough, we have $f'(k) - \overline{g}_0(k) = f'(k) - (\delta - \frac{u_2'(f(k),k)}{u_1'(f(k),k)}) \leq f'(k) - (\delta - \sup_k \frac{u_2'(f(k),k)}{u_1'(f(k),k)}) < 0$ by using (4.5). By the continuity of $f'(k) - \overline{g}_0(k)$, there exist a $k^* > 0$ such that $f'(k^*) - \overline{g}_0(k^*) = 0$ and $f'(k) - \overline{g}_0(k) < 0$ for any $k > k^*$. Since $\underline{g}_0(k^*) < \overline{g}_0(k^*) = f'(k^*)$, there exist an $\epsilon_1 > 0$ such that $\underline{g}_0(k) < f'(k)$ for any $k \in (k^*, k^* + \epsilon_1)$. However, for any $k \in (k^*, k^* + \epsilon_1)$,we have $\underline{g}_0(k) < f'(k) < \overline{g}_0(k)$,thus we find infinitely many k_∞ satisfying condition (4.10).

(4.11) can be obtained by (4.8), (4.9) and (4.10). ∎

Then we can establish our main theorem:

Theorem 4.3. *Suppose the production function f is three times continuously differentiable and satisfies the Inada conditions, and suppose the utility function u is strictly concave and $\tilde{u}$ is concave with respect to the first parameter c, and both are four times continuously differentiable for all variables, satisfying (4.5) and (4.6). For any $k_\infty > 0$ satisfying (4.11), and any point k_0 close enough to k_∞, there exists an equilibrium Markov strategy which converges to k_∞ from the initial capital k_0.*

The proof goes by showing that the system (4.1) with the boundary conditions (4.2) has a C^2 solution. This will be our first step. We then prove that the strategy is convergent.

In Appendix E, we prove the existence of the equilibrium Markov strategy and some additional properties. We change variables and transform the equations to two similar 3-dimensional systems, then we linearize the system and use center manifold theorem (cf. [Car1] for more details) to find the solutions. The center manifold theorem method was used already by I. Ekeland and A. Lazrak (cf. Theorem 5 of [EkL1]).

In Appendix F, we prove the convergence.

Then applying the Proposition 4.1, Theorem 3.5 and Theorem 3.4, we found a Markov strategy σ for k_∞ with any initial capital in the neighborhood of k_∞.

5 Letting $r \to 0$

Then let $r \to 0$ in the system (4.1), and we use $(v_0(k), w_0(k))$ to denote the solution of the limiting system. We have

$$\begin{cases} v_0'(k)(f(k) - \sigma_0(k)) + u(\sigma_0(k), k) = \dfrac{\delta}{2}(v_0(k) + w_0(k)), \\ w_0'(k)(f(k) - \sigma_0(k)) + u(\sigma_0(k), k) = \dfrac{\delta}{2}(v_0(k) + w_0(k)), \end{cases} \tag{5.1}$$

For the solution of (5.1), we have

Lemma 5.1. *If $(v_0(k), w_0(k))$ is a solution of (5.1) with boundary condition (4.2), then $v_0(k)$ is the solution of the system:*

$$\begin{cases} v_0'(k)(f(k) - \sigma_0(k)) + u(\sigma_0(k), k) + \alpha\delta\tilde{u}(\sigma_0(k_\infty), k_\infty) = \delta v_0(k), \\ v_0(k_\infty) = \dfrac{1}{\delta}u(f(k_\infty), k_\infty) + \alpha\tilde{u}(f(k_\infty), k_\infty), \end{cases} \tag{5.2}$$

and $w_0(k)$ satisfies

$$w_0(k) = v_0(k) - 2\alpha\tilde{u}(f(k_\infty), k_\infty), \quad k \in (k_\infty - \epsilon, k_\infty + \epsilon). \tag{5.3}$$

.

Conversely, if system (5.2) has a solution (v_0, w_0) satisfies (5.3), then $(v_0(k), w_0(k))$ is a solution of (5.1) with boundary condition (4.2).

Moreover, if system (5.2) has an unique convergent solution, then (5.1) with boundary condition (4.2) has an unique couple of convergent solutions.

Proof. From (5.1), $v_0(k) - w_0(k)$ must be constant as the similar discussion of Lemma 3.8, and by the boundary conditions (4.2), we obtain (5.3). Substituting it in the first equation of (5.1), and combining with the first boundary condition of (4.2), we obtain the system (5.2).

The uniqueness proof is similar to the proof of Theorem 2.1 (Appendix A). ∎

Theorem 5.2. *Given k_∞ satisfying (4.10), we have*

(1) for k_∞ satisfying (4.10), namely $\underline{g}_0(k_\infty) < f'(k_\infty) < \overline{g}_0(k_\infty)$, there is a sequence $\sigma_r, r \to 0$ which converges to k_∞ as an equilibrium strategy;

(2) there is no $\epsilon > 0$ and no sequence $r_n \to 0$ such that the sequence σ_{r_n} converges in $C^2[k_\infty - \epsilon, k_\infty + \epsilon]$.

Proof. If k_∞ satisfy (4.10), by (4.8) and (4.9), for $r > 0$ small enough, k_∞ must satisfy (4.11). Then by Theorem 4.3, such sequence σ_r exists.

Recall $v_r'(k) = \frac{\partial \bar{u}_r}{\partial c}(\sigma_r(k), k)$ of (3.16). If there exists a convergent subsequence of σ_r, namely σ_{r_n}, then there exists a C^2 function v satisfying

$$v(k) = \lim_{n\to\infty} v_{r_n}(k), \quad v'(k) = \lim_{n\to\infty} v_{r_n}'(k), \tag{5.4}$$

where $\lim_{n\to\infty} r_n = 0$. Then $v(k)$ satisfies equations (5.1). Thus, $v(k)$ is just the same function as $v_0(k)$ in Theorem 5.1, and then $v(k)$ satisfies (5.2). By the similar calculation in the proof of Theorem 2.1, we must have $k_\infty \in K$ of (2.11), and this is a contradiction with (4.10). ∎

Remark. For each r, we have found a continuum of equilibrium strategies. But quite disappointing, the sequence of the equilibrium strategies with different r does not converge. There may be two possibilities. The one is, when r goes to zero, the maximal interval of existence for the equilibrium strategy σ_r tends to a single point. The other is, there exists an interval such that for $r > 0$ small enough, the equilibrium strategy exists in such interval, but the limit of the equilibrium strategies do not converge. So considering the C-criterion as the limit of the E(r)-criterion when $r \to 0$ is not a good way to do the time inconsistency problem.

6 The equilibrium strategies for C-criterion

We are now directly deal with the C-criterion. Similarly with Definition 3.1 and Definition 3.2, we can define the convergent Markov strategies for the criterion (1.6).

Now we suppose σ is a convergent strategy, has been announced and is public knowledge. The decision-maker at 0 has capital stock k_0.

If all future decision-maker apply the strategy σ, then the total welfare of all the decision-makers in the point of view of decision-maker at 0 is

$$W(\sigma) = \int_0^\infty u(\sigma(k_b(t)), k_b(t))e^{-\delta t}dt + \alpha \lim_{t\to\infty} \tilde{u}(\sigma(k_b(t)), k_b(t)), \tag{6.1}$$

where $k_b(t)$ is given by (3.3).

Similarly with Section 3, we suppose the decision-maker at time 0 can commit all the decision-makers in interval $[0, \epsilon]$ for $\epsilon > 0$ small enough. She expects all later ones to apply the same public consumption policy σ, that is, to consume $\sigma(k_t)$ for decision-maker t, where k_t is denoted by the capital level of time t. Assume she commits to another consumption level c, thus the decision-maker at 0 change the strategy to

$$\sigma_\epsilon(t,k) = \begin{cases} c, \quad 0 \le t \le \epsilon, k \in I, \\ \sigma(k), \quad t > \epsilon, k \in I, \end{cases} \tag{6.2}$$

and the total welfare in the point of view of decision-maker at 0 becomes

$$W(\sigma_\epsilon) = \int_0^\infty u(\sigma_\epsilon(t, k_c(t)), k_c(t))e^{-\delta t}dt + \alpha \lim_{t\to\infty} \tilde{u}(\sigma_\epsilon(t, k_c(t)), k_c(t)), \tag{6.3}$$

where $k_c(t)$ is given by (3.4).

With $k_c(t) = k_b(t) + k_i(t)\epsilon$ of (3.5), expanding (6.3), we have

$$\begin{aligned} W(\sigma_\epsilon) &= \int_0^\epsilon u(\sigma_\epsilon(t, k_c(t)), k_c(t))e^{-\delta t}dt + \int_\epsilon^\infty u(\sigma_\epsilon(t, k_c(t)), k_c(t))e^{-\delta t}dt \\ &\quad +\alpha \lim_{t>\epsilon, t\to\infty} \tilde{u}(\sigma_\epsilon(t, k_c(t)), k_c(t)) \\ &= u(c, k_0)\epsilon + \int_\epsilon^\infty u(\sigma(k_b(t) + \epsilon k_i(t)), k_b(t) + \epsilon k_i(t))e^{-\delta t}dt \\ &\quad +\alpha \lim_{t>\epsilon, t\to\infty} \tilde{u}(\sigma(k_c(t)), k_c(t)) + o(\epsilon) \\ &= u(c, k_0)\epsilon + \int_\epsilon^\infty u(\sigma(k_b(t)), k_b(t))e^{-\delta t}dt \\ &\quad +\epsilon \int_\epsilon^\infty [u_1'(\sigma(k_b(t)), k_b(t))\sigma'(k_b(t)) + u_2'(\sigma(k_b(t)), k_b(t))]k_i(t)e^{-\delta t}dt \\ &\quad +\alpha \lim_{t\to\infty} \tilde{u}(\sigma(k_c(t)), k_c(t)) + o(\epsilon), \end{aligned} \tag{6.4}$$

where in the second inequality, we used (6.2), the definition of σ_ϵ. Subtracting (6.1) from (6.4), we have

$$
\begin{aligned}
W(\sigma_\epsilon) - W(\sigma) &= \epsilon u(c, k_0) - \int_0^\epsilon u(\sigma(k_b(t)), k_b(t)) e^{-\delta t} dt \\
&\quad + \epsilon \int_0^\infty [u_1'(\sigma(k_b(t)), k_b(t))\sigma'(k_b(t)) + u_2'(\sigma(k_b(t)), k_b(t))] k_i(t) e^{-\delta t} dt \\
&\quad + \alpha \lim_{t\to\infty} \tilde{u}(\sigma(k_c(t)), k_c(t)) - \alpha \lim_{t\to\infty} \tilde{u}(\sigma(k_b(t)), k_b(t)) + o(\epsilon) \\
&= \epsilon \Big[u(c, k_0) - u(\sigma(k_0), k_0) + \int_0^\infty u_1'(\sigma(k_b(t)), k_b(t))\sigma'(k_b(t)) k_i(t) e^{-\delta t} dt \\
&\quad + \int_0^\infty u_2'(\sigma(k_b(t)), k_b(t)) k_i(t) e^{-\delta t} dt \\
&\quad + \alpha \frac{\lim_{t\to\infty} \tilde{u}(\sigma(k_c(t)), k_c(t)) - \lim_{t\to\infty} \tilde{u}(\sigma(k_b(t)), k_b(t))}{\epsilon} \Big] + o(\epsilon). \qquad (6.5)
\end{aligned}
$$

By the requirement of Definition 6.1, we consider the capital dynamics (3.3) and (3.4) when $t \geq \epsilon$ and with $k_b(\epsilon), k_v(\epsilon)$ as the initial capital, then the limit

$$
R_\infty := \lim_{\epsilon\downarrow 0} \frac{\lim_{t\to\infty} k_c(t) - \lim_{t\to\infty} k_b(t)}{k_c(\epsilon) - k_b(\epsilon)} \qquad (6.6)
$$

exists and also the limit

$$
\begin{aligned}
k_{d\infty} &:= \lim_{\epsilon\downarrow 0} \frac{\lim_{t\to\infty} k_c(t) - \lim_{t\to\infty} k_b(t)}{\epsilon} \\
&= \lim_{\epsilon\downarrow 0} \frac{k_c(\epsilon) - k_b(\epsilon)}{\epsilon} \lim_{\epsilon\downarrow 0} \frac{\lim_{t\to\infty} k_c(t) - \lim_{t\to\infty} k_b(t)}{k_c(\epsilon) - k_b(\epsilon)} \\
&= (\sigma(k_0) - c) R_\infty \qquad (6.7)
\end{aligned}
$$

exists, where in the last equality, we used $k_c(t) = k_b(t) + \epsilon k_i(t)$ and (3.10). Therefore, the limit

$$
\begin{aligned}
P_\infty &:= \lim_{\epsilon\downarrow 0} \frac{\lim_{t\to\infty} \tilde{u}(\sigma(k_c(t)), k_c(t)) - \lim_{t\to\infty} \tilde{u}(\sigma(k_b(t)), k_b(t))}{\epsilon} \\
&= \Big[\tilde{u}_1'(\sigma(k_b(+\infty)), k_b(+\infty))\sigma'(k_b(+\infty)) \\
&\quad + \tilde{u}_2'(\sigma(k_b(+\infty)), k_b(+\infty)) \Big] \lim_{\epsilon\downarrow 0} \frac{\lim_{t\to\infty} k_c(t) - \lim_{t\to\infty} k_b(t)}{\epsilon} \\
&= \Big[\tilde{u}_1'(\sigma(k_b(+\infty)), k_b(+\infty))\sigma'(k_b(+\infty)) \\
&\quad + \tilde{u}_2'(\sigma(k_b(+\infty)), k_b(+\infty)) \Big] (\sigma(k_0) - c) R_\infty \qquad (6.8)
\end{aligned}
$$

exists and we define

$$
\begin{aligned}
P_0(\sigma, k_0, c) &= u(c, k_0) - u(\sigma(k_0), k_0) \\
&\quad + \int_0^\infty u_1'(\sigma(k_b(t)), k_b(t))\sigma'(k_b(t)) k_d(t) e^{-\delta t} dt + \int_0^\infty u_2'(\sigma(k_b(t)), k_b(t)) k_d(t) e^{-\delta t} dt \\
&\quad + \alpha \lim_{\epsilon\downarrow 0} \frac{\lim_{t\to\infty} \tilde{u}(\sigma(k_v(t)), k_v(t)) - \lim_{t\to\infty} \tilde{u}(\sigma(k_b(t)), k_b(t))}{\epsilon}. \qquad (6.9)
\end{aligned}
$$

Then (6.5) becomes

$$W(\sigma_\epsilon) - W(\sigma) = \epsilon P_0(\sigma, k_0, c) + o(\epsilon). \tag{6.10}$$

Then the differences appear. In the general case, since the decision maker cannot influence the term $\lim_{t\to+\infty} \tilde{u}(c(t), k(t))$, that is $R_\infty = 0$, then the limit capital will keep a constant level when the initial capital k_0 varies in a small interval. Therefore we come back to the classical Ramsey problem, and exactly, we have

Theorem 6.2. *Let σ be an equilibrium strategy. Suppose $I_+ \cup I_- \neq \emptyset$, then for $k_0 \in I_+ \cup I_-$, the limit capital $\mathcal{K}(+\infty; k_0, \sigma)$ (denoted by k_∞) of the dynamics $\mathcal{K}(t; k_0, \sigma)$ belongs to K of (2.11), i.e.,*

$$k_\infty = \mathcal{K}(+\infty; k_0, \sigma) \in K, \tag{6.11}$$

and the value function

$$v(k_0) = \int_0^\infty u(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) e^{-\delta t} dt + \alpha \lim_{t\to\infty} \tilde{u}(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)). \tag{6.12}$$

satisfies:

$$\begin{cases} v'(k)(f(k) - \sigma(k)) + u(\sigma(k), k) = \delta(v(k) - \alpha \tilde{u}(f(k_\infty), k_\infty)), \\ v(k_\infty) = \dfrac{1}{\delta} u(f(k_\infty), k_\infty) + \alpha \tilde{u}(f(k_\infty), k_\infty), \end{cases} \tag{6.13}$$

where σ satisfies (2.14).

Conversely, if there is a $k_\infty > 0$ satisfies (6.11), a function v is twice continuously differentiable, satisfies (6.13) and under the strategy $\sigma(k) = i(v'(k), k)$, $k(t)$ converges to k_∞, then such σ is an equilibrium strategy.

Remark. Comparing with Theorem 2.1 and Theorem 6.2, we have $v(k) = V(k) + \alpha \tilde{u}(f(k_\infty), k_\infty)$. Then by Theorem 2.1, under some proper conditions, the solution of the equation (6.13) exsits and so does the equilibrium strategy.

7 The equilibrium strategies for H-criterion

In the preceding sections, we saw that the decision maker acting within a finite interval cannot influence the limit term $\lim_{t\to\infty} \tilde{u}(c(t), k(t))$, so that the C-criterion reduces to the S-criterion. We now turn to the H-criterion, which does not have this weakness.

We firstly define the admissible strategy under H-criterion.

Definition 7.1. *Given an initial capital k_0, a convergent Markov strategy σ will be admissible at k_0 if there is an open interval $I \subseteq \mathbb{R}^+$ containing k_0 and a steady state of the economy k_∞ such that*

$$\int_0^\infty |k(t) - k_\infty| dt < \infty. \tag{7.1}$$

Moreover, for an open interval I, if for any $k \in I$, σ is admissible at k, we say that σ is admissible in I or I is an admissible interval of strategy σ, and denote it by (I, σ).

Remark. (1) When $k(t) = k_\infty$ for some finite t, the capital level will be unchanged thereafter. So the $k(t)$ monotonously converges to k_∞ as t goes to ∞. Therefore (7.1) can also be written as

$$\left| \int_0^\infty [k(t) - k_\infty] dt \right| < \infty. \tag{7.2}$$

(2) If σ is an admissible strategy, when t is large enough, we have

$$\begin{aligned} & |\tilde{u}(\sigma(k(t)), k(t)) - \tilde{u}(\sigma(k_\infty), k_\infty)| \\ \leq \quad & |\tilde{u}_1'(\sigma(k_\infty), k_\infty)\sigma'(k_\infty) + \tilde{u}_2'(\sigma(k_\infty), k_\infty) + 1| \cdot |k(t) - k_\infty|, \end{aligned} \tag{7.3}$$

by (7.1), we have

$$\int_0^\infty |\tilde{u}(\sigma(k(t)), k(t)) - \tilde{u}(\sigma(k_\infty), k_\infty)| dt < +\infty, \tag{7.4}$$

thus the value of the H-criterion is finite under the strategy σ.

In this section, we will restrict our analysis to admissible strategies and omit the interval I if there is no confusions.

As in Section 3, we suppose σ is a admissible strategy, has been announced and is public knowledge. The decision-maker at 0 has capital stock k_0.

As we already noted, the decision maker cannot control the steady state of the economy. That is, k_∞ is fixed in the general case. Then with a calculation similar to one we did in Section 3 and Section 4, we can characterize the equilibrium strategy as

Theorem 7.2. *Let $v(k_0)$ be a C^2 function such that the strategy $\sigma(k_0) = i \circ v'(k_0) = i(v'(k_0), k_0)$ converges to k_∞. Then σ is an equilibrium strategy and v is the relative value function if and only if the following two conditions hold:*

(1) There exist a C^1 function $w(k_0)$ such that (v, w) satisfies the system

$$\begin{cases} v'(k_0)(f(k_0) - \sigma(k_0)) + \bar{u}(\sigma(k_0), k_0) - \alpha\tilde{u}(\sigma(k_\infty), k_\infty) = \dfrac{\delta}{2}(v(k_0) + w(k_0)), \\ w'(k_0)(f(k_0) - \sigma(k_0)) + u(\sigma(k_0), k_0) - \alpha(\tilde{u}(\sigma(k_0), k_0) - \tilde{u}(\sigma(k_\infty), k_\infty)) = \dfrac{\delta}{2}(v(k_0) + w(k_0)), \end{cases} \tag{7.5}$$

where $\bar{u} = u + \alpha\tilde{u}$, and $i(y, k)$ is determined by

$$\bar{u}_1'(i(y,k),k) = y \text{ or } i(\bar{u}_1'(z,k),k) = z, \tag{7.6}$$

with the boundary conditions

$$\begin{cases} v(k_\infty) = \dfrac{1}{\delta}u(f(k_\infty), k_\infty), \\ w(k_\infty) = \dfrac{1}{\delta}u(f(k_\infty), k_\infty). \end{cases} \tag{7.7}$$

(2) (7.1) holds for the corresponding strategy.

This theorem is proved by Appendix G.

Similarly to Theorem 4.3, we have

Theorem 7.3. *Suppose the production function f is three times continuously differentiable and satisfies the Inada conditions, and suppose the utility function u is strictly concave and $\tilde{u}$ is concave with respect to the first parameter c, and both are four times continuously differentiable for all variables, satisfying (4.5) and (4.6). For any $k_\infty > 0$ such that*

$$f'(k_\infty) < \frac{\delta u_1'(f(k_\infty),k_\infty) - (u_2'(f(k_\infty),k_\infty) + \alpha\tilde{u}_2'(f(k_\infty),k_\infty))}{u_1'(f(k_\infty),k_\infty) + \alpha\tilde{u}_1'(f(k_\infty),k_\infty)}, \tag{7.8}$$

then there exist an $\epsilon > 0$ such that the system (7.5) with the boundary conditions (7.7) has a C^2 solution in the interval $(k_\infty - \epsilon, k_\infty + \epsilon)$. Therefore, for any $k_0 \in (k_\infty - \epsilon, k_\infty + \epsilon)$, there exists an equilibrium Markov strategy which converges to k_∞ from the initial capital k_0.

This theorem is proved by Appendix H.

Remark. The right hand side of (7.8) is less than $\delta - \frac{u_2'(f(k_\infty),k_\infty)}{u_1'(f(k_\infty),k_\infty)}$, thus the limit capital level is strictly greater than the level in the optimal control problem with S-criterion.

8 Conclusion

In the optimal control problem of the S-criterion, the set of the steady states is given by (2.11). For the sake of simplicity, let us assume that it is a singleton, $K = \{k_S\}$.

Let us now turn to the equilibrium strategies of the time inconsistency problem. For the E(r)-criterion, the set of the steady states is given by (4.11). Assume again $f'(k) = \underline{g}_r(k)$ (and also $f'(k) = \overline{g}_r(k)$) has an unique solution with respect to k, then the set is a bounded open interval $(\underline{k}_{E(r)}, \overline{k}_{E(r)})$. The lower bound $\underline{k}_{E(r)}$ of this interval is near k_S and $\lim_{r\to 0} \underline{k}_{E(r)} = k_S$. For the C-criterion, the set of the steady states is given by (2.11), then it is the same singleton $K = \{k_S\}$. For the H-criterion, the set of the steady states is given by (7.8). Assume similarly as above, the set of the steady states is an unbounded interval $(k_H, +\infty)$. The lower bound k_H is strictly greater than k_S.

9 Appendix

A Proof of Theorem 2.1

The value function V of (2.2) is C^2 can be obtained by the differentiability of σ and (2.14).

Now we want to prove that there is an optimal solution. For this, we have to use the method of Caratheodary, which on the HJB equation (see [Eke1]).

Firstly, we fix an point k_∞ in K of (2.11). Recall $\sigma(k) = I(V'(k), k)$ of (2.14), we rewrite (2.13) as a Paff system:

$$\begin{aligned} dV &= pdk, &(9.1)\\ p(f(k) - I(p,k)) + u(I(p,k),k) &= \delta V, &(9.2) \end{aligned}$$

with respect to the (k, p), and consider the initial-value problem:

$$V(k_\infty) = \frac{1}{\delta} u(f(k_\infty), k_\infty), \tag{9.3}$$

where k_∞ is fixed and satisfies (2.10).

Differentiating (9.2) leads to:

$$\begin{aligned} \delta dV &= (f(k) - I(p,k))dp + p[f'(k)dk - I_1'(p,k)dp - I_2'(p,k)dk] \\ &\quad + u_1'(I(p,k),k)[I_1'(p,k)dp + I_2'(p,k)dk] + u_2'(I(p,k),k)dk \\ &= (f(k) - I(p,k))dp + pf'(k)dk + u_2'(I(p,k),k)dk, \end{aligned} \tag{9.4}$$

where we used (2.4) in the last equality. Plugging into (9.1), we have

$$(f(k) - I(p,k))dp = (\delta p - pf'(k) - u_2'(I(p,k),k))dk. \tag{9.5}$$

We change the time so that $k(0) = k_\infty$ and by the requirement of the convergence of the capital $k(t)$ to k_∞ needs

$$f(k_\infty) - \sigma(k_\infty) = 0. \tag{9.6}$$

If we denote $p(0)$ by p_∞, then by (2.8), we have

$$p_\infty = p(0) = V'(k(0)) = V'(k_\infty) = u_1'(\sigma(k_\infty), k_\infty) = u_1'(f(k_\infty), k_\infty). \tag{9.7}$$

Furthermore, we have

$$
\begin{aligned}
& f(k(0)) - I(p(0), k(0)) \\
= \;& f(k_\infty) - I(u_1'(f(k_\infty), k_\infty), k_\infty) \\
= \;& 0, \qquad (9.8) \\
& \delta p(0) - p(0) f'(k(0)) - u_2'(I(p(0), k(0)), k(0)) \\
= \;& u_1'(f(k_\infty), k_\infty)[\delta - f'(k_\infty) - \frac{u_2'(f(k_\infty), k_\infty)}{u_1'(f(k_\infty), k_\infty)}] \\
= \;& 0, \qquad (9.9)
\end{aligned}
$$

where we used (2.4) and (2.10) respectively. Thus from (9.5), we have a degenerate system with respect to two functions $k(t)$ and $p(t)$:

$$
\begin{cases} \dfrac{dk}{dt} = f(k) - I(p, k), \\ \dfrac{dp}{dt} = \delta p - p f'(k) - u_2'(I(p, k), k), \end{cases} \tag{9.10}
$$

with initial condition

$$
(k(0), p(0)) = (k_\infty, p_\infty). \tag{9.11}
$$

Recall $u_1'(I(y, k), k) = y$ from (2.4) and differentiate it with respect to y, k respectively:

$$
u_{11}''(I(y, k), k) I_1'(y, k) = 1, \quad u_{11}''(I(y, k), k) I_2'(y, k) + u_{12}''(I(y, k), k) = 0, \tag{9.12}
$$

and then we have

$$
\begin{aligned}
I_1'(p_\infty, k_\infty) &= \frac{1}{u_{11}''(I(p_\infty, k_\infty), k_\infty)} = \frac{1}{u_{11}''(f(k_\infty), k_\infty)}, && (9.13) \\
I_2'(p_\infty, k_\infty) &= -\frac{u_{11}''(I(p_\infty, k_\infty), k_\infty)}{u_{11}''(I(p_\infty, k_\infty), k_\infty)} = -\frac{u_{12}''(f(k_\infty), k_\infty)}{u_{11}''(f(k_\infty), k_\infty)}, && (9.14)
\end{aligned}
$$

where we used $I(p_\infty, k_\infty) = I(V'(k_\infty), k_\infty) = \sigma(k_\infty) = f(k_\infty)$.

Denote

$$
e = (k - k_\infty, p - p_\infty) \tag{9.15}
$$

and linearize the two functions in the right hand side of system (9.10) near point (k_∞, p_∞), we have

$$\begin{aligned}
& f(k) - I(p,k) \\
= \;& f'(k_\infty)(k-k_\infty) + I_1'(p_\infty, k_\infty)(p-p_\infty) + I_2'(p_\infty, k_\infty)(k-k_\infty) + O(|e|^2) \\
= \;& (f'(k_\infty) + \frac{u_{12}''(f(k_\infty), k_\infty)}{u_{11}''(f(k_\infty), k_\infty)})(k-k_\infty) - \frac{1}{u_{11}''(f(k_\infty), k_\infty)}(p-p_\infty) + O(|e|^2), \qquad (9.16) \\
& \delta p - pf'(k) - u_2'(I(p,k),k) \\
= \;& \delta(p-p_\infty) - f'(k_\infty)(p-p_\infty) - p_\infty f''(k_\infty)(k-k_\infty) \\
& -u_{12}''(I(p_\infty, k_\infty), k_\infty)[I_1'(p_\infty, k_\infty)(p-p_\infty) + I_2'(p_\infty, k_\infty)(k-k_\infty)] \\
& -u_{22}''(I(p_\infty, k_\infty), k_\infty)(k-k_\infty) + O(|e|^2) \\
= \;& [-u_1'(f(k_\infty), k_\infty)f''(k_\infty), k_\infty) - u_{22}''(f(k_\infty), k_\infty) + \frac{u_{12}''(f(k_\infty), k_\infty)}{u_{11}''(f(k_\infty), k_\infty)}](k-k_\infty) \\
& +[\delta - f'(k_\infty) - \frac{u_{12}''(f(k_\infty), k_\infty)}{u_{11}''(f(k_\infty), k_\infty)}](p-p_\infty) + O(|e|^2), \qquad (9.17)
\end{aligned}$$

where we used (9.7), (9.13) and (9.14). For later convenience, we make some simple notations for the constants with respect to k_∞:

$$f_\infty' := f'(k_\infty), \tag{9.18}$$

$$f_\infty'' := f''(k_\infty), \tag{9.19}$$

$$u_{1\infty}' := u_1'(f(k_\infty), k_\infty), \tag{9.20}$$

$$u_{2\infty}' := u_2'(f(k_\infty), k_\infty), \tag{9.21}$$

$$u_{11\infty}'' := u_{11}''(f(k_\infty), k_\infty), \tag{9.22}$$

$$u_{12\infty}'' := u_{12}''(f(k_\infty), k_\infty), \tag{9.23}$$

$$u_{22\infty}'' := u_{22}''(f(k_\infty), k_\infty). \tag{9.24}$$

Then by (9.16) and (9.17), system (9.10) can linearized as:

$$\frac{d}{dt}\begin{pmatrix} k-k_\infty \\ p-p_\infty \end{pmatrix} = \begin{pmatrix} f_\infty' + \frac{u_{12\infty}''}{u_{11\infty}''}, & -\frac{1}{u_{11\infty}''} \\ -u_{1\infty}' f_\infty'' - u_{22\infty}'' + \frac{(u_{12\infty}'')^2}{u_{11\infty}''}, & \delta - f_\infty' - \frac{u_{12\infty}''}{u_{11\infty}''} \end{pmatrix} \begin{pmatrix} k-k_\infty \\ p-p_\infty \end{pmatrix}. \tag{9.25}$$

and we denote the coefficient matrix by A_∞:

$$A_\infty := \begin{pmatrix} f_\infty' + \frac{u_{12\infty}''}{u_{11\infty}''}, & -\frac{1}{u_{11\infty}''} \\ -u_{1\infty}' f_\infty'' - u_{22\infty}'' + \frac{(u_{12\infty}'')^2}{u_{11\infty}''}, & \delta - f_\infty' - \frac{u_{12\infty}''}{u_{11\infty}''} \end{pmatrix}. \tag{9.26}$$

The characteristic polynomial of system (9.25) then is

$$\lambda^2 - \delta\lambda - [\frac{u_{1\infty}' f_\infty'' + u_{22\infty}''}{u_{11\infty}''} + f_\infty'(\frac{u_{12\infty}''}{u_{11\infty}''} - \frac{u_{2\infty}'}{u_{1\infty}'}) - \frac{u_{2\infty}'}{u_{1\infty}'}\frac{u_{12\infty}''}{u_{11\infty}''}] = 0, \tag{9.27}$$

where we used $f'_\infty + \frac{u'_{2\infty}}{u'_{1\infty}} = \delta$ by (2.14). If the constant item of the above polynomial greater than zero, then it has no negative roots, and in the later proof, we know that the convergence of the capital cannot be guaranteed even though the solution exists, thus the admissible equilibrium strategies does not exist.

Let $\epsilon = \min\{\frac{u'_{1\infty} f''_\infty}{(1+\delta)u''_{11\infty}}, 1\}$, since $\frac{u'_{2\infty}}{u'_{1\infty}}, \frac{u''_{12\infty}}{u''_{11\infty}} < \epsilon$ by (2.15), we have

$$\begin{aligned} & \frac{u'_{1\infty} f''_\infty + u''_{22\infty}}{u''_{11\infty}} + f'_\infty\left(\frac{u''_{12\infty}}{u''_{11\infty}} - \frac{u'_{2\infty}}{u'_{1\infty}}\right) - \frac{u'_{2\infty}}{u'_{1\infty}} \frac{u''_{12\infty}}{u''_{11\infty}} \\ > \; & \frac{u'_{1\infty} f''_\infty}{u''_{11\infty}} - f'_\infty \epsilon - \epsilon^2 \\ > \; & \frac{u'_{1\infty} f''_\infty}{u''_{11\infty}} - (f'_\infty + 1)\epsilon \\ \geq \; & \frac{u'_{1\infty} f''_\infty}{u''_{11\infty}} - (\delta + 1)\epsilon \\ > \; & 0, \end{aligned} \tag{9.28}$$

where in the third inequality, we used $f'_\infty = \delta - \frac{u'_{2\infty}}{u'_{1\infty}} \leq \delta$ by (2.14). Then the characteristic polynomial (9.27) of the linearized system (9.25) has two solutions: one positive solution λ_+ and one negative solution λ_-. And the corresponding eigenvalue are given by

$$(f'_\infty + \frac{u''_{12\infty}}{u''_{11\infty}} - \lambda_\pm)dk - \frac{1}{u''_{11\infty}} dp = 0. \tag{9.29}$$

Thus (k_∞, p_∞) is a hyperbolic fixed point of both system (9.10) and (9.25), with a stable manifold $\mathcal{S}$ which corresponds to λ_- and an unstable manifold $\mathcal{U}$ which corresponds to λ_+. Choosing a smooth parametrization $(k_s(x), p_s(x))$ for the stable manifold $\mathcal{S}$. Then the tangent of $\mathcal{S}$ (indeed, it is a curve) at the fixed point is

$$\frac{dp_s}{dk_s}(k_\infty) = u''_{11\infty}(f'_\infty - \lambda_-) + u''_{12\infty}, \tag{9.30}$$

and plugging $k = k_s(x), p = p_s(x)$ into equation (9.2), we get a curve γ:

$$k = k_s(x), \quad V_s = \frac{p_s(x)(f(k_s(x)) - I(p_s(x), k_s(x))) + u(I(p_s(x), k_s(x)), k_s(x))}{\delta}. \tag{9.31}$$

Moreover, we have

$$\begin{aligned}
\frac{dV_s}{dk_s}(k_\infty) &= \frac{1}{\delta}\Big\{\frac{dp_s}{dk_s}(k_\infty)(f(k_\infty)-I(p_\infty,k_\infty))+p_\infty[f'(k_\infty)-I_1'(p_\infty,k_\infty)\frac{dp_s}{dk_s}(k_\infty)-I_2'(p_\infty,k_\infty)] \\
&\quad +u_1'(I(p_\infty,k_\infty),k_\infty)[I_1'(p_\infty,k_\infty)\frac{dp_s}{dk_s}(k_\infty)+I_2'(p_\infty,k_\infty)] \\
&\quad +u_2'(I(p_\infty,k_\infty),k_\infty)\Big\} \\
&= \frac{1}{\delta}[u_1'(f(k_\infty),k_\infty)f'(k_\infty)+u_2'(f(k_\infty),k_\infty)] \\
&= \frac{u_1'(f(k_\infty),k_\infty)}{\delta}[f'(k_\infty)+\frac{u_2'(f(k_\infty),k_\infty)}{u_1'(f(k_\infty),k_\infty)}] \\
&= u_1'(f(k_\infty),k_\infty),
\end{aligned} \tag{9.32}$$

where we used (9.7) and (9.8) in the second equality and (2.14) in the last equality. Then the curve γ is a graph and then we can find its function $V_s(k)$. And $V_s(k)$ satisfies the system (2.13) in a neighborhood of k_∞. Moreover, we have

$$\frac{d^2V_s}{{dk_s}^2}(k_\infty)=\frac{dp_s}{dk_s}(k_\infty)=u''_{11\infty}(f'_\infty-\lambda_-)+u''_{12\infty}, \tag{9.33}$$

thus $V_s(k)$ is C^2 at k_∞. And the C^2 property at other points near k_∞ can be directly obtained by (9.10) when $f(k)-I(p,k)\neq 0$ at these points (see Case 1 and Case 2 in Section 2.4 of [Eke1]).

The last thing we have to do is to show the solution $k(t)$ of (1.1) converges to k_∞. Linearizing the equation

$$\frac{dk}{dt}=f(k)-\sigma(k)=f(k)-I(V'(k),k) \tag{9.34}$$

gives

$$\begin{aligned}
\frac{d(k-k_\infty)}{dt} &= [f'(k_\infty)-I_1'(V'(k_\infty),k_\infty)V''(k_\infty)-I_2'(V'(k_\infty),k_\infty)](k-k_\infty) \\
&= [f'(k_\infty)-\frac{1}{u''_{11}(f(k_\infty),k_\infty)}\frac{dp_s}{dk_s}(k_\infty)+\frac{u''_{12}(f(k_\infty),k_\infty)}{u''_{11}(f(k_\infty),k_\infty)}](k-k_\infty) \\
&= \lambda_-(k-k_\infty),
\end{aligned} \tag{9.35}$$

where we used (9.13),(9.14) in the second equality and used (9.30) in the last equality. Then $\lambda_-<0$ implies that $k(t)$ converges to k_∞. ∎

B Proof of Theorem 3.4

Let us first cite some useful property about $\mathcal{K}$, see [Eke2] or Chapter 2 of [Har1] to get more details. First, we have

$$\mathcal{K}(s;\mathcal{K}(t;k_0,\sigma),\sigma)=\mathcal{K}(s+t;k_0,\sigma). \tag{9.36}$$

Next, we denote the solution of the standard linearized equation of (3.13) by $\mathcal{R}(t)$, in the sense, we have

$$\frac{d\mathcal{R}}{dt} = (f'(\mathcal{K}(t;k_0,\sigma)) - \sigma'(\mathcal{K}(t;k_0,\sigma)))\mathcal{R}(t), \quad \mathcal{R}(0) = 1,$$

and then we have

$$\mathcal{R}(t) = e^{\int_0^t (f'(\mathcal{K}(\sigma;s,k_0)) - \sigma'(\mathcal{K}(\sigma;s,k_0)))ds}. \tag{9.37}$$

Then the value $k_i(t)$ can be write as

$$k_i(t) = \mathcal{R}(t)(\sigma(k_0) - c), \tag{9.38}$$

and $\mathcal{K}$ has following properties:

$$\frac{\partial \mathcal{K}(t;k_0,\sigma)}{\partial k_0} = \mathcal{R}(t), \tag{9.39}$$

$$\frac{\partial \mathcal{K}(t;k_0,\sigma)}{\partial t} = f(\mathcal{K}(t;k_0,\sigma)) - \sigma(\mathcal{K}(t;k_0,\sigma)). \tag{9.40}$$

Firstly, let's now turn to the first part of the theorem. Given a Markov convergent strategy σ, we define the associated value function $v(k_0)$ as in formula (3.14). Its derivative with respect to k_0 is given by

$$\begin{aligned} v'(k_0) = & \int_0^\infty h(t)u_1'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\sigma'(\mathcal{K}(t;k_0,\sigma))\frac{\partial \mathcal{K}(t;k_0,\sigma)}{\partial k_0}dt \\ & + \int_0^\infty h(t)u_2'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\frac{\partial \mathcal{K}(t;k_0,\sigma)}{\partial k_0}dt \\ & + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_1'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\sigma'(\mathcal{K}(t;k_0,\sigma))\frac{\partial \mathcal{K}(t;k_0,\sigma)}{\partial k_0}dt \\ & + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_2'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\frac{\partial \mathcal{K}(t;k_0,\sigma)}{\partial k_0}dt \\ = & \int_0^\infty h(t)u_1'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\sigma'(\mathcal{K}(t;k_0,\sigma))\mathcal{R}(t)dt \\ & + \int_0^\infty h(t)u_2'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\mathcal{R}(t)dt \\ & + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_1'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\sigma'(\mathcal{K}(t;k_0,\sigma))\mathcal{R}(t)dt \\ & + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_2'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\mathcal{R}(t)dt. \end{aligned} \tag{9.41}$$

Since σ is an equilibrium strategy, the maximum of $P(k_0,\sigma,c)$ with respect to c must be attained at $c = \sigma(k_0)$, see Definition 3.3. Substituting $k_b(t) = \mathcal{K}(t;k_0,\sigma)$ and $k_i(t)$ in (9.38) into (3.9) and recall $\bar{u}(c,k) = u(c,k) + \alpha r \tilde{u}(c,k)$, we get

$$\begin{aligned} P(k_0, \sigma, c) &= \bar{u}(c, k_0) - \bar{u}(\sigma(k_0), k_0) \\ &\quad + \int_0^\infty h(t) u_1'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \sigma'(\mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t)(\sigma(k_0) - c) dt \\ &\quad + \int_0^\infty h(t) u_2'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t)(\sigma(k_0) - c) dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}_1'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \sigma'(\mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t)(\sigma(k_0) - c) dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}_2'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t)(\sigma(k_0) - c) dt. \end{aligned} \tag{9.42}$$

Since $\bar{u}$ is strictly concave and differentiable with respect to c which follows from properties of u and $\tilde{u}$, the necessary and sufficient condition to maximize $P(k_0, \sigma, c)$ with respect to c is the derivative with respect to c of the left hand side of (9.42) vanishes at $c = \sigma(k_0)$, that is:

$$\begin{aligned} \bar{u}_1'(c, k_0)\Big|_{c=\sigma(k_0)} &= \int_0^\infty h(t) u_1'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \sigma'(\mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t) dt \\ &\quad + \int_0^\infty h(t) u_2'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t) dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}_1'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \sigma'(\mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t) dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}_2'(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) \mathcal{R}(t) dt, \end{aligned} \tag{9.43}$$

which is precisely $v'(k_0)$, as we just wanted. Therefore, the equilibrium strategy satisfy

$$\bar{u}_1'(\sigma(k_0), k_0) = v'(k_0) \tag{9.44}$$

and we have $\sigma(k_0) = z(v'(k_0), k_0) := z \circ v'(k_0)$. Substituting back into equation (3.14), we get the functional equation (3.15). This prove the direct part of the theorem.

For the second part, it has been found in [Eke2], thus we omit it here. ∎

C Proof of Theorem 3.5

Let a function $v : \mathbb{R} \to \mathbb{R}$ be given. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(k_0) = v(k_0) - \int_0^\infty h(t) u(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) dt - \alpha r \int_0^\infty \tilde{h}(t) \tilde{u}(\sigma(\mathcal{K}(t; k_0, \sigma)), \mathcal{K}(t; k_0, \sigma)) dt, \tag{9.45}$$

where $\sigma(k_0) = i(v'(k_0), k_0)$. Consider $\psi(t, k_0)$ as the value of φ along the trajectory $t \to \mathcal{K}(t; k_0, \sigma)$

originating from k at time 0, that is

$$\begin{aligned}
\psi(t,k_0) &= \varphi(\mathcal{K}(t;k_0,\sigma)) \\
&= v(\mathcal{K}(t;k_0,\sigma)) - \int_0^\infty h(s)u(\sigma(\mathcal{K}(s;\mathcal{K}(t;k_0,\sigma),\sigma)),\mathcal{K}(s;\mathcal{K}(t;k_0,\sigma),\sigma))ds \\
&\quad -\alpha r\int_0^\infty \tilde{h}(s)\tilde{u}(\sigma(\mathcal{K}(s;\mathcal{K}(t;k_0,\sigma),\sigma)),\mathcal{K}(s;\mathcal{K}(t;k_0,\sigma),\sigma))ds \\
&= v(\mathcal{K}(t;k_0,\sigma)) - \int_0^\infty h(s)u(\sigma(\mathcal{K}(s+t;k_0,\sigma)),\mathcal{K}(s+t;k_0,\sigma))ds \\
&\quad -\alpha r\int_0^\infty \tilde{h}(s)\tilde{u}(\sigma(\mathcal{K}(s+t;k_0,\sigma)),\mathcal{K}(s+t;k_0,\sigma))ds \\
&= v(\mathcal{K}(t;k_0,\sigma)) - \int_t^\infty h(s-t)u(\sigma(\mathcal{K}(\sigma;s,k_0)),\mathcal{K}(\sigma;s,k_0))ds \\
&\quad -\alpha r\int_t^\infty \tilde{h}(s-t)\tilde{u}(\sigma(\mathcal{K}(\sigma;s,k_0)),\mathcal{K}(\sigma;s,k_0))ds, \qquad (9.46)
\end{aligned}$$

where we have used (9.36).

We compute the derivative of ψ with respect to t:

$$\begin{aligned}
\frac{\partial\psi(t,k_0)}{\partial t} &= v'(\mathcal{K}(t;k_0,\sigma))[f(\mathcal{K}(t;k_0,\sigma)) - \sigma(\mathcal{K}(t;k_0,\sigma))] \\
&\quad +u(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma)) \\
&\quad +\int_t^\infty h'(s-t)u(\sigma(\mathcal{K}(\sigma;s,k_0)),\mathcal{K}(\sigma;s,k_0))ds \\
&\quad +\alpha r\tilde{u}(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma)) \\
&\quad +\alpha r\int_t^\infty \tilde{h}'(s-t)\tilde{u}(\sigma(\mathcal{K}(\sigma;s,k_0)),\mathcal{K}(\sigma;s,k_0))ds \\
&= v'(\mathcal{K}(t;k_0,\sigma))[f(\mathcal{K}(t;k_0,\sigma)) - \sigma(\mathcal{K}(t;k_0,\sigma))] \\
&\quad +\bar{u}(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma)) \\
&\quad +\int_t^\infty h'(s-t)u(\sigma(\mathcal{K}(\sigma;s,k_0)),\mathcal{K}(\sigma;s,k_0))ds \\
&\quad +\alpha r\int_t^\infty \tilde{h}'(s-t)\tilde{u}(\sigma(\mathcal{K}(\sigma;s,k_0)),\mathcal{K}(\sigma;s,k_0))ds \\
&= v'(\mathcal{K}(t;k_0,i\circ v'))[f(\mathcal{K}(t;k_0,i\circ v')) - i\circ v'(\mathcal{K}(t;k_0,i\circ v'))] \\
&\quad +\bar{u}(i\circ v'(\mathcal{K}(t;k_0,i\circ v')),\mathcal{K}(t;k_0,i\circ v')) \\
&\quad +\int_0^\infty h'(s)u(i\circ v'(\mathcal{K}(s;\mathcal{K}(t;k_0,i\circ v'),i\circ v')),\mathcal{K}(s;\mathcal{K}(t;k_0,i\circ v'),i\circ v'))ds \\
&\quad +\alpha r\int_0^\infty \tilde{h}'(s)\tilde{u}(i\circ v'(\mathcal{K}(s;\mathcal{K}(t;k_0,i\circ v'),i\circ v')),\mathcal{K}(s;\mathcal{K}(t;k_0,i\circ v'),i\circ v'))ds,
\end{aligned}$$
$$(9.47)$$

where we used (9.36), (9.40), $h(0)=\tilde{h}(0)=1$, $\sigma = i\circ v'$ and $\bar{u} = u + \alpha r\tilde{u}$. If (3.19) holds, then the right hand side of the last equation is identically zero along the trajectory, so that $\psi(t,k_0)$ does not depend on t, thus $\psi(s,k_0)=\psi(t,k_0)$ for all $s,t\geq 0$. Letting $t\to\infty$ in the definition of ψ, we get

$$\begin{aligned}
\psi(s,k_0) &= \lim_{t\to\infty}\psi(t,k_0)\\
&= \lim_{t\to\infty}\Big\{v(\mathcal{K}(t;k_0,\sigma))-\int_0^\infty h(s)u(\sigma(\mathcal{K}(s+t;k_0,\sigma)),\mathcal{K}(s+t;k_0,\sigma))ds\\
&\quad -\alpha r\int_0^\infty \tilde{h}(s)\tilde{u}(\sigma(\mathcal{K}(s+t;k_0,\sigma)),\mathcal{K}(s+t;k_0,\sigma))ds\Big\}\\
&= v(\bar{k})-\int_0^\infty h(s)u(\sigma(\bar{k}),\bar{k})ds-\alpha r\int_0^\infty \tilde{h}(s)\tilde{u}(\sigma(\bar{k}),\bar{k})ds \qquad (9.48)
\end{aligned}$$

and hence, if (3.20) holds, then $\psi=\varphi\equiv 0$ and so equation (3.15) holds. Conversely, if $v(k)$ satisfies equation (3.15), then the same lines of reasoning shows that equation (3.19) and the boundary condition are satisfied. ∎

D Proof of Proposition 4.1

(3.19) and (3.20) implies (3.15) by Theorem 3.5. Thus for $h(t)=e^{-\delta t}$ and $\tilde{h}(t)=e^{-rt}$, by (3.15) we obtain

$$v(k_0)=\int_0^\infty e^{-\delta t}u(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt+\alpha r\int_0^\infty e^{-rt}\tilde{u}(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt. \qquad (9.49)$$

Then we define a new function w as follows

$$w(k_0)=\int_0^\infty e^{-\delta t}u(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt-\alpha r\int_0^\infty e^{-rt}\tilde{u}(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt. \qquad (9.50)$$

Note that $\sigma=i\circ v'$ is an equilibrium strategy that converges to k_∞. Applying Lemma 3.6 to v and w, we obtain (4.1) and (4.2).

$$\begin{aligned}
& v'(k_0)(f(k_0)-\sigma(k_0))+u(\sigma(k_0),k_0)+\alpha r\tilde{u}(\sigma(k_0),k_0)\\
=\ & \delta\int_0^\infty e^{-\delta t}u(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt+\alpha r^2\int_0^\infty e^{-rt}\tilde{u}(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))dt\\
=\ & \delta\frac{v(k_0)+w(k_0)}{2}+\alpha r^2\frac{v(k_0)-w(k_0)}{2\alpha r}\\
=\ & av(k_0)+bw(k_0) \qquad (9.51)
\end{aligned}$$

and

$$\begin{aligned}
v(k_\infty) &= u(f(k_\infty),k_\infty)\int_0^\infty e^{-\delta t}dt+\alpha r\tilde{u}(f(k_\infty),k_\infty)\int_0^\infty e^{-rt}dt\\
&= \frac{1}{\delta}u(f(k_\infty),k_\infty)+\alpha\tilde{u}(f(k_\infty),k_\infty), \qquad (9.52)
\end{aligned}$$

where we used $a=\frac{\delta+r}{2}, b=\frac{\delta-r}{2}$. Thus we obtain the first equation of (4.1) and first condition of (4.2), and the rest are similar.

Conversely, suppose v_1 and w_1 satisfy the equations (4.1) and the boundary conditions (4.2), with the convergent Markov strategy $\sigma_1 = i \circ v_1'$ converges to k_∞, that is:

$$\begin{cases} v_1'(k_0)(f(k_0) - \sigma_1(k_0)) + \bar{u}(\sigma_1(k_0), k_0) = av_1(k_0) + bw_1(k_0), \\ w_1'(k_0)(f(k_0) - \sigma_1(k_0)) + u(\sigma_1(k_0), k_0) - \alpha r \tilde{u}(\sigma_1(k_0), k_0) = bv_1(k_0) + aw_1(k_0) \end{cases} \tag{9.53}$$

with the boundary conditions

$$\begin{cases} v_1(k_\infty) = \frac{1}{\delta} u(f(k_\infty), k_\infty) + \alpha \tilde{u}(f(k_\infty), k_\infty), \\ w_1(k_\infty) = \frac{1}{\delta} u(f(k_\infty), k_\infty) - \alpha \tilde{u}(f(k_\infty), k_\infty). \end{cases} \tag{9.54}$$

Consider the following functions

$$\begin{aligned} v_2(k_0) &= \int_0^\infty e^{-\delta t} u(\sigma_1(\mathcal{K}(t; k_0, \sigma_1)), \mathcal{K}(t; k_0, \sigma_1)) dt \\ &\quad + \alpha r \int_0^\infty e^{-rt} \tilde{u}(\sigma_1(\mathcal{K}(t; k_0, \sigma_1)), \mathcal{K}(t; k_0, \sigma_1)) dt, \end{aligned} \tag{9.55}$$

$$\begin{aligned} w_2(k_0) &= \int_0^\infty e^{-\delta t} u(\sigma_1(\mathcal{K}(t; k_0, \sigma_1)), \mathcal{K}(t; k_0, \sigma_1)) dt \\ &\quad - \alpha r \int_0^\infty e^{-rt} \tilde{u}(\sigma_1(\mathcal{K}(t; k_0, \sigma_1)), \mathcal{K}(t; k_0, \sigma_1)) dt. \end{aligned} \tag{9.56}$$

Applying Lemma 3.5 with $I = v_2$ and $I = w_2$ respectively, we get

$$\begin{cases} v_2'(k_0)(f(k_0) - \sigma_1(k_0)) + \bar{u}(\sigma_1(k_0), k_0) = av_2(k_0) + bw_2(k_0), \\ w_2'(k_0)(f(k_0) - \sigma_1(k_0)) + u(\sigma_1(k_0), k_0) - \alpha r \tilde{u}(\sigma_1(k_0), k_0) = bv_2(k_0) + aw_2(k_0) \end{cases} \tag{9.57}$$

with the boundary conditions

$$\begin{cases} v_2(k_\infty) = \frac{1}{\delta} u(f(k_\infty), k_\infty) + \alpha \tilde{u}(f(k_\infty), k_\infty), \\ w_2(k_\infty) = \frac{1}{\delta} u(f(k_\infty), k_\infty) - \alpha \tilde{u}(f(k_\infty), k_\infty). \end{cases} \tag{9.58}$$

Set $v_3 = v_1 - v_2, w_3 = w_1 - w_2$ and subtract (9.57), (9.58) from (9.53), (9.54) respectively, we get

$$\begin{cases} v_3'(k_0)(f(k_0) - \sigma_1(k_0)) = av_3(k_0) + bw_3(k_0), \\ w_3'(k_0)(f(k_0) - \sigma_1(k_0)) = bv_3(k_0) + aw_3(k_0) \end{cases} \tag{9.59}$$

with the boundary conditions

$$\begin{cases} v_3(k_\infty) = 0, \\ w_3(k_\infty) = 0. \end{cases} \tag{9.60}$$

Obviously, $v_3 = w_3 = 0$ is a solution. We need to show that it is the only one so that $v_1 = v_2$ and $w_1 = w_2$, and this can be verified by Lemma 9.1. Then equation (9.55) becomes

$$v_1(k_0) = \int_0^\infty e^{-\delta t} u(\sigma_1(\mathcal{K}(t; k_0, \sigma_1)), \mathcal{K}(t; k_0, \sigma_1))dt + \alpha r \int_0^\infty e^{-rt} \tilde{u}(\sigma_1(\mathcal{K}(t; k_0, \sigma_1)), \mathcal{K}(t; k_0, \sigma_1))dt, \tag{9.61}$$

which is precisely equation (3.15). Since v_1 satisfies (3.15), then it satisfies (3.19) and (3.20) by Theorem 3.5. ∎

Lemma 9.1. *If (v_3, w_3) is a pair of continuous functions on a neighborhood Ω of k_∞, continuously differentiable for $k \neq k_\infty$, and which solve (9.59) with boundary conditions (9.60) for $k \neq k_\infty$, then $v_3 = w_3 = 0$.*

Proof. First, we omit the subscript of k_0. Set $f(k) - \sigma_1(k) = \varphi(k)$, then $\varphi(k) \to 0$ as $k \to k_\infty$. The system (9.59) can be rewritten as:

$$\begin{pmatrix} \varphi v_3' \\ \varphi w_3' \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} v_3 \\ w_3 \end{pmatrix}.$$

Let

$$\begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} v_3 \\ w_3 \end{pmatrix}, \tag{9.62}$$

then we have

$$\varphi \begin{pmatrix} V' \\ W' \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}^{-1} \begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} \delta & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}. \tag{9.63}$$

We consider the equation $\varphi V' = \delta V$. By calculation, we obtain:

$$V(k) = V(k_0) e^{\int_{k_0}^{k} \frac{\delta}{\varphi(u)} du}. \tag{9.64}$$

No lose of generality, we can assume that $k_0 < k_\infty$.

Let $S = \{k | \varphi(k) = 0, k_0 \le k \le k_\infty\}$, where k_0 is the initial stock, then S is nonempty for $k_\infty \in S$. We consider the following two cases:

(i) $S \backslash \{k_\infty\} = \emptyset$. Then $\varphi(k) \neq 0, \forall k \in [k_0, k_\infty)$. While $\frac{dk}{dt} = \varphi(k)$, then k is monotonic with t. For $k_0 < k_\infty$, we have $\varphi(k) > 0$, for all $k \in [k_0, k_\infty)$. By (9.62), $0 = \sqrt{\frac{1}{2}}(v_3(k_\infty) + w_3(k_\infty)) = V(k_\infty) = V(k_0) e^{\int_{k_0}^{k_\infty} \frac{\delta}{\varphi(u)} du}$, we have $V(k_0) = 0$. Hence, $V(k) \equiv 0$.

(ii) $S \backslash \{k_\infty\} \neq \emptyset$. Then by continuity of φ, $\varphi \equiv 0$ in the closure of S, thus $S = \bar{S}$, i.e. S is a closed set. It follows that $V(k) = \frac{\varphi(k)V'(k)}{\delta} = 0$ in S by (9.63). Denote $k^* = min\{k | k \in S\}$, then

$(k^*, k_\infty)\backslash S$ is an open set, it can be represented as the union of at most countable many disjoint open intervals:

$$[k^*, k_\infty]\backslash S = (k^*, k_\infty)\backslash S = \bigcup_{n=1}^{\infty}(l_n, m_n). \tag{9.65}$$

For any open interval (l_n, m_n), $\varphi(k) \neq 0, \forall k \in (l_n, m_n)$. Let $k_n \in (l_n, m_n)$, we have

$$V(l_n) = V(k_n)e^{\int_{k_n}^{l_n} \frac{\delta}{\varphi(u)}du}, V(m_n) = V(k_n)e^{\int_{k_n}^{m_n} \frac{\delta}{\varphi(u)}du}. \tag{9.66}$$

φ is positive (or negative) in (l_n, m_n). From the facts that $l_n < k_n < m_n$, $V(l_n) = \frac{\varphi(l_n)V'(l_n)}{\delta} = 0$, $V(m_n) = \frac{\varphi(m_n)V'(m_n)}{\delta} = 0$, $\int_{k_n}^{l_n} \frac{\delta}{\varphi(u)}du$ and $\int_{k_n}^{m_n} \frac{\delta}{\varphi(u)}du$ has the opposite sign, so that at least one of $e^{\int_{k_n}^{l_n} \frac{\delta}{\varphi(u)}du}$ and $e^{\int_{k_n}^{m_n} \frac{\delta}{\varphi(u)}du}$ is nonzero, then we must have $V(k_n) = 0$, which further implies $V(k) \equiv 0, \forall k \in (l_n, m_n)$.

During the discussions, we have $V(k) = 0, \forall k \in [k^*, k_\infty]$. Under the unfortunate case $k_0 < k^*$, by $\varphi(k) \neq 0, \forall k \in [k_0, k^*)$, using the similar calculation of case (i), we have $V(k) \equiv 0, \forall k \in [k_0, k^*)$. Then $V(k) \equiv 0$ in $[k_0, k_\infty]$

In any cases, we have $V(k) \equiv 0$. Similarly $W(k) \equiv 0$.Then $v_3 \equiv w_3 \equiv 0$ as required. ∎

E Proof of Lemma 9.3: Existence

We consider the function (v, w) satisfying the system (4.1) with boundary conditions (4.2). From now on, we use variable k instead of k_0 for convenience. By $\sigma(k) = i \circ v'(k) = i(v'(k), k)$, we have

$$\begin{aligned} v'(k)(f(k) - \sigma(k)) + \bar{u}(\sigma(k), k) &= v'(k)(f(k) - i(v'(k), k)) + \bar{u}(i(v'(k), k), k) \\ &= v'(k)f(k) + [\bar{u}(i(v'(k), k), k) - v'(k)i(v'(k), k)]. \end{aligned} \tag{9.67}$$

Let

$$F(y, k) = yf(k) + \bar{u}(i(y, k), k) - yi(y, k) - \bar{u}(f(k), k), \tag{9.68}$$

and we calculate the partial derivatives of F:

$$\begin{aligned} \frac{\partial F(y, k)}{\partial y} &= f(k) + \bar{u}_1'(i(y, k), k)i_1'(y, k) - yi_1'(y, k) - i(y, k) \\ &= f(k) - i(y, k), && (9.69) \\ \frac{\partial^2 F(y, k)}{\partial y^2} &= -i_1'(y, k), && (9.70) \\ \frac{\partial F(y, k)}{\partial k} &= yf'(k) + \bar{u}_2'(i(y, k)) + \bar{u}_1'(i(y, k))i_2'(y, k) - yi_2'(y, k) \\ &\quad -\bar{u}_1'(f(k), k)f'(k) - \bar{u}_2'(f(k), k) \\ &= f'(k)(y - \bar{u}_1'(f(k), k)) + \bar{u}_2'(i(y, k), k) - \bar{u}_2'(f(k), k), && (9.71) \end{aligned}$$

where $i_1'(y,k) = \frac{\partial i(y,k)}{\partial y}, i_2'(y,k) = \frac{\partial i(y,k)}{\partial k}$ and we used (3.18) in Theorem 3.4. Differentiating both side of the first formula of (3.18) with respect to variable y, we have

$$\bar{u}_{11}''(i(y,k),k)i_1'(y,k) = 1. \tag{9.72}$$

Since $\bar{u}$ is a strictly concave function with respect to its first variable, we have, $\bar{u}_{11}''(c,k) < 0$. Then

$$\frac{\partial^2 F(y,k)}{\partial y^2} = -i_1'(y,k) = -\frac{1}{\bar{u}_{11}''(i(y,k),k)} > 0, \tag{9.73}$$

for all y, and thus $F(y,k)$ is a strictly convex function with respect to y. For a fixed k, if $\frac{\partial F(y^*,k)}{\partial y} = 0$, then

$$i(y^*,k) = f(k), \tag{9.74}$$

$F(y^*,k)$ get its minimum value at $y = y*$. The minimum point y^* depends on k, we denote by $y* = y^*(k)$ and then we have

$$y^*(k) = \bar{u}_1'(i(y^*,k),k) = \bar{u}_1'(f(k),k) \tag{9.75}$$

and $F(y^*(k),k) = 0$ by (9.68).

Let

$$\mu(k) = av(k) + bw(k) - \bar{u}(f(k),k). \tag{9.76}$$

We consider $(v(k),\mu(k))$ instead of $(v(k),w(k))$ as the new unknown functions. Then the first equation of the system (4.1) is equivalent to

$$\begin{aligned} F(v'(k),k) &= v'(k)(f(k)-\sigma(k)) + \bar{u}(\sigma(k),k) - \bar{u}(f(k),k) \\ &= av(k) + bw(k) - \bar{u}(f(k),k) \\ &= \mu(k), \end{aligned} \tag{9.77}$$

where the first equality follows from the definition (9.68) of F and the fact $\sigma(k) = i(v'(k),k)$ in Theorem 3.4, the second equality follows from the first equation of (4.1).

Since F is strictly convex with respect to its first variable and $F(y^*(k),k) = 0$, the equation $F(x + y^*(k),k) = \mu(k)$ of variable x has two solutions $x_-(\mu(k),k) < 0 < x_+(\mu(k),k)$ for $\mu(k) > 0$, none for $\mu(k) < 0$ and a single solution $x = 0$ for $\mu(k) = 0$.

From the second equation of (4.1), we have

$$\begin{aligned} w'(k) &= \frac{bv(k) + aw(k) - u(\sigma(k),k) + \alpha r\tilde{u}(\sigma(k),k)}{f(k)-\sigma(k)} \\ &= \left(\frac{av(k) + bw(k) - \bar{u}(\sigma(k),k)}{f(k)-\sigma(k)}\right)\frac{bv(k) + aw(k) - u(\sigma(k),k) + \alpha r\tilde{u}(\sigma(k),k)}{av(k) + bw(k) - \bar{u}(\sigma(k),k)} \\ &= v'(k)\frac{bv(k) + aw(k) - u(\sigma(k),k) + \alpha r\tilde{u}(\sigma(k),k)}{av(k) + bw(k) - \bar{u}(\sigma(k),k)}, \end{aligned} \tag{9.78}$$

where to get the last equality, we have used the first equation of (4.1). Differentiating $\mu(k)$ with respect to k yields

$$
\begin{aligned}
\frac{d\mu}{dk} &= av'(k) + bw'(k) - \bar{u}_1'(f(k),k)f'(k) - \bar{u}_2'(f(k),k) \\
&= av'(k) + bv'(k)\frac{bv(k) + aw(k) - u(\sigma(k),k) + \alpha r\tilde{u}(\sigma(k),k)}{av(k) + bw(k) - \bar{u}(\sigma(k),k)} - \bar{u}_1'(f(k),k)f'(k) - \bar{u}_2'(f(k),k) \\
&= v'(k)\frac{(a^2+b^2)v(k) + 2abw(k) - \delta u(\sigma(k),k) - \alpha r^2\tilde{u}(\sigma(k),k)}{av(k) + bw(k) - \bar{u}(\sigma(k),k)} \\
&\quad -y^*(k)f'(k) - \bar{u}_2'(f(k),k),
\end{aligned}
\tag{9.79}
$$

where we used $a = \frac{\delta+r}{2}, b = \frac{\delta-r}{2}$ and (9.78).

As we just discussed, the necessary and sufficient condition that $F(x + y^*(k), k) = \mu(k)$ has solutions is $\mu(k) \geq 0$, that is

$$
av(k) + bw(k) - \bar{u}(f(k),k) \geq 0. \tag{9.80}
$$

And in this case, we denote these two solutions by x_+ and x_- respectively. Here x_+ and x_- coincide with $\mu(k) = 0$. Denote one of the solutions by $x = x(\mu(k),k)$, from $F(v'(k),k) = \mu(k)$, we conclude that

$$
v'(k) = y^*(k) + x(\mu(k),k). \tag{9.81}
$$

Now we consider the dynamical system on $(v(k), \mu(k))$ given by

$$
\begin{cases}
v'(k) = y^*(k) + x(\mu(k),k), \\
\mu'(k) = (y^*(k) + x(\mu(k),k))\dfrac{(a^2+b^2)v(k) + 2abw(k) - \delta u(\sigma(k),k) - \alpha r^2\tilde{u}(\sigma(k),k)}{av(k) + bw(k) - \bar{u}(\sigma(k),k)} \\
\qquad\qquad - y^*(k)f'(k) - \bar{u}_2'(f(k),k),
\end{cases}
\tag{9.82}
$$

where the second equation on $\mu(k)$ is obtained via replacing $v'(k)$ by the right hand side of (9.81).

From (9.77), $F(x + y^*(k), k) = \mu(k)$. Let $x(k) = x(\mu(k),k)$, $\bar{u}_{12}''(c,k) = \frac{\partial^2 \bar{u}(c,k)}{\partial c \partial k}$ and so on, we have

$$
\mu'(k) = \frac{\partial F}{\partial y}\left(\frac{dx}{dk} + \bar{u}_{11}''(f(k),k)f'(k) + \bar{u}_{12}''(f(k),k)\right) + \frac{\partial F}{\partial k}, \tag{9.83}
$$

and then consider $k = k(x)$ as a function of x, from (9.83), we obtain

$$
\begin{aligned}
\frac{dk}{dx} &= \frac{\frac{\partial F}{\partial y}}{\mu'(k) - \frac{\partial F}{\partial k} - \frac{\partial F}{\partial y}(\bar{u}_{11}''(f(k),k)f'(k) + \bar{u}_{12}''(f(k),k))} \\
&= \frac{(f(k) - i(y^*+x,k))(av + bw - \bar{u}(i(y^*+x,k),k))}{D(x,k,v,w)},
\end{aligned}
\tag{9.84}
$$

where we used (9.69),(9.71) and (9.79), and denote by

$$
\begin{aligned}
D(x,k,v,w) \quad = \quad & (y^*+x)[(a^2+b^2)v(k)+2abw(k)-\delta u(\sigma(k),k)-\alpha r^2\tilde{u}(\sigma(k),k)] \\
& -\Big\{(f(k)-i(y^*+x,k))[\bar{u}''_{11}(f(k),k)f'(k)+\bar{u}''_{12}(f(k),k)] \\
& +(y^*+x)f'(k)+\bar{u}'_2(i(y^*+x,k),k)\Big\}(av+bw-\bar{u}(i(y^*+x,k),k)). \quad (9.85)
\end{aligned}
$$

Further more, by (9.82) and (9.84) we have

$$
\frac{dv}{dx}=\frac{dv}{dk}\frac{dk}{dx}=(y^*+x)\frac{(f(k)-i(y^*+x,k))(av+bw-\bar{u}(i(y^*+x,k),k))}{D(x,k,v,w)}. \tag{9.86}
$$

Now we transform our system to

$$
\begin{cases}
\dfrac{dk}{dx}=\dfrac{(f(k)-i(y^*+x,k))(av+bw-\bar{u}(i(y^*+x,k),k))}{D(x,k,v,w)}, \\
\dfrac{dv}{dx}=\dfrac{(y^*+x)(f(k)-i(y^*+x,k))(av+bw-\bar{u}(i(y^*+x,k),k))}{D(x,k,v,w)}.
\end{cases} \tag{9.87}
$$

We introduce a new variable s such that $\frac{ds}{dx}=\frac{1}{D(x,k,v,w)}$, then the system becomes

$$
\begin{cases}
\dfrac{dx}{ds}=D(x,k,v,w), \\
\dfrac{dk}{ds}=(f(k)-i(y^*+x,k))(av+bw-\bar{u}(i(y^*+x,k),k)), \\
\dfrac{dv}{ds}=(y^*+x)(f(k)-i(y^*+x,k))(av+bw-\bar{u}(i(y^*+x,k),k)).
\end{cases} \tag{9.88}
$$

By (9.76), the definition of $\mu(k)$, we get

$$
w(k)=\frac{\mu(k)+\bar{u}(f(k),k)-av(k)}{b}=\frac{F(y^*(k)+x,k)+\bar{u}(f(k),k)-av(k)}{b}. \tag{9.89}
$$

Plugging it into system (9.88), we have the following system on three independent unknown functions x,k and v:

$$
\begin{cases}
\dfrac{dx}{ds}=\tilde{D}(x,k,v), \\
\dfrac{dk}{ds}=(f(k)-i(y^*+x,k))[F(y^*+x,k)+\bar{u}(f(k),k)-\bar{u}(i(y^*+x,k),k)], \\
\dfrac{dv}{ds}=(y^*+x)(f(k)-i(y^*+x,k))[F(y^*+x,k)+\bar{u}(f(k),k)-\bar{u}(i(y^*+x,k),k)],
\end{cases} \tag{9.90}
$$

where $\tilde{D}(x,k,v)$ is given by

$$\begin{aligned}\tilde{D}(x,k,v) &= D(x,k,v,w)\\ &= (y^*+x)\Big[2aF(y^*+x,k)+(b^2-a^2)v+2a\bar{u}(f(k),k)\\ &\quad -\delta u(i(y^*+x,k),k)-\alpha r^2\tilde{u}(i(y^*+x,k),k)\Big]\\ &\quad -\Big(F(y^*+x,k)+\bar{u}(f(k),k)-\bar{u}(i(y^*+x,k),k)\Big)\Big[(y^*+x)f'(k)+\bar{u}_2'(i(y^*+x,k),k)\\ &\quad +(f(k)-i(y^*+x,k))(\bar{u}_{11}''(f(k),k)f'(k)+\bar{u}_{12}''(f(k),k))\Big],\end{aligned} \tag{9.91}$$

where the twice continuously differentiability of $\tilde{D}(x,k,v)$ needs the three times continuously differentiability of f and the four times continuously differentiability of u, $\tilde{u}$.

If $k=k_\infty$, then

$$\begin{aligned}\mu(k_\infty) &= av(k_\infty)+bw(k_\infty)-\bar{u}(f(k_\infty),k_\infty)\\ &= \frac{\delta+r}{2}(\frac{1}{\delta}u(f(k_\infty),k_\infty)+\alpha\tilde{u}(f(k_\infty),k_\infty))+\frac{\delta-r}{2}(\frac{1}{\delta}u(f(k_\infty),k_\infty)-\alpha\tilde{u}(f(k_\infty),k_\infty))\\ &\quad -(u(f(k_\infty),k_\infty)+\alpha r\tilde{u}(f(k_\infty),k_\infty))\\ &= 0.\end{aligned} \tag{9.92}$$

Then $F(y^*(k_\infty)+x,k_\infty)=\mu(k_\infty)=0$ has only one solution $x=0$. Denote by $v_\infty=v(k_\infty)$. We consider the system near the point $(x,k,v)=(0,k_\infty,v_\infty)$. Denote by $e=(x,k-k_\infty,v-v_\infty)$ and $||\cdot||$ denote the Euclidean norm.

From (9.69), (9.71), (9.74) and (9.75), $\frac{\partial F}{\partial y}|_{(\bar{u}_1'(f(k_\infty),k_\infty),k_\infty)}=\frac{\partial F}{\partial k}|_{(\bar{u}_1'(f(k_\infty),k_\infty),k_\infty)}=0$, and then

$$\frac{\partial F(y^*+x,k)}{\partial x}\Big|_{e=(0,0,0)}=\frac{\partial F(y^*+x,k)}{\partial k}\Big|_{e=(0,0,0)}=F(y^*+x,k)\Big|_{e=(0,0,0)}=0, \tag{9.93}$$

where we consider F as a function of $e=(x,k-k_\infty,v-v_\infty)$, and we have $F(y^*+x,k)=O(||e||^2)$ near $e=0$.

Let $e=(0,0,0)$, we have

$$\begin{aligned}&(b^2-a^2)v+2a\bar{u}(f(k),k)-\delta u(i(y^*+x,k),k)-\alpha r^2\tilde{u}(i(y^*+x,k),k)|_{e=(0,0,0)}\\ =\ &((\frac{\delta-r}{2})^2-(\frac{\delta+r}{2})^2)v_\infty+(\delta+r)\bar{u}(f(k_\infty),k_\infty)-\delta u(i(y^*(k_\infty),k_\infty),k_\infty)\\ &-\alpha r^2\tilde{u}(i(y^*(k_\infty),k_\infty),k_\infty)\\ =\ &-\delta r(\frac{1}{\delta}u(f(k_\infty),k_\infty)+\alpha\tilde{u}(f(k_\infty),k_\infty))+(\delta+r)(u(f(k_\infty),k_\infty)+\alpha r\tilde{u}(f(k_\infty),k_\infty))\\ &-\delta u(f(k_\infty),k_\infty)-\alpha r^2\tilde{u}(f(k_\infty),k_\infty)\\ =\ &0,\end{aligned} \tag{9.94}$$

where we used the first equation of (4.2) and (9.74) in the second equality. Then we consider the following functions

$$\begin{aligned}
&f(k) - i(y^* + x, k),\\
&\bar{u}(f(k), k) - \bar{u}(i(y^* + x, k), k),\\
&y^*(k) + x - \bar{u}_1'(f(k_\infty), k_\infty),\\
&(b^2 - a^2)v + 2a\bar{u}(f(k), k) - \delta u(i(y^* + x, k), k) - \alpha r^2 \tilde{u}(i(y^* + x, k), k),\\
&(y^* + x)f'(k) + \bar{u}_2'(i(y^* + x, k), k) - \bar{u}_1'(f(k_\infty), k_\infty)f'(k_\infty) - \bar{u}_2'(f(k_\infty), k_\infty).
\end{aligned} \tag{9.95}$$

They take the value 0 at the point $(0, k_\infty, v_\infty)$, and then the product of any two of them is $O(||e||^2)$ near the point $(0, k_\infty, v_\infty)$. Then

$$\frac{dk}{ds} = O(||e||^2), \tag{9.96}$$

$$\frac{dv}{ds} = O(||e||^2) \tag{9.97}$$

and

$$\begin{aligned}
\tilde{D}(x, k, v) &= (y^* + x)\Big[(b^2 - a^2)v + 2a\bar{u}(f(k), k) - \delta u(i(y^* + x, k), k) - \alpha r^2 \tilde{u}(i(y^* + x, k), k)\Big]\\
&\quad -\Big(\bar{u}(f(k), k) - \bar{u}(i(y^* + x, k), k)\Big)\Big[(y^* + x)f'(k) + \bar{u}_2'(i(y^* + x, k), k)\Big]\\
&\quad +O(||e||^2)\\
&= \bar{u}_1'(f(k_\infty), k_\infty)\Big[(b^2 - a^2)v + 2a\bar{u}(f(k), k) - \delta u(i(y^* + x, k), k)\\
&\quad -\alpha r^2 \tilde{u}(i(y^* + x, k), k)\Big] - \Big(\bar{u}(f(k), k) - \bar{u}(i(y^* + x, k), k)\Big)\\
&\quad \cdot\Big(\bar{u}_1'(f(k_\infty), k_\infty)f'(k_\infty) + \bar{u}_2'(f(k_\infty), k_\infty)\Big) + O(||e||^2).
\end{aligned} \tag{9.98}$$

Specially we have

$$\tilde{D}(0, k_\infty, v_\infty) = 0. \tag{9.99}$$

Differentiate $\tilde{D}(x, k, v, r)$ with respect to x

$$\begin{aligned}
\left.\frac{\partial \tilde{D}}{\partial x}\right|_{(0,k_\infty,v_\infty)} &= -\bar{u}_1'(f(k_\infty), k_\infty)i_1'(\bar{u}_1'(f(k_\infty), k_\infty), k_\infty)[\delta u_1'(f(k_\infty), k_\infty) + \alpha r^2 \tilde{u}_1'(f(k_\infty), k_\infty)]\\
&\quad +[\bar{u}_1'(f(k_\infty), k_\infty)f'(k_\infty) + \bar{u}_2'(f(k_\infty), k_\infty)]\bar{u}_1'(f(k_\infty), k_\infty)i_1'(\bar{u}_1'(f(k_\infty), k_\infty), k_\infty)\\
&= \bar{u}_1'^2(f(k_\infty), k_\infty)i_1'(\bar{u}_1'(f(k_\infty), k_\infty), k_\infty)\Bigg[f'(k_\infty) -\\
&\quad -\frac{\delta u_1'(f(k_\infty), k_\infty) + \alpha r^2 \tilde{u}_1'(f(k_\infty), k_\infty) - (u_2'(f(k_\infty), k_\infty) + \alpha r \tilde{u}_2'(f(k_\infty), k_\infty))}{u_1'(f(k_\infty), k_\infty) + \alpha r \tilde{u}_1'(f(k_\infty), k_\infty)}\Bigg]\\
&= \bar{u}_1'^2(f(k_\infty), k_\infty)i_1'(\bar{u}_1'(f(k_\infty), k_\infty), k_\infty)[f'(k_\infty) - \bar{g}(k_\infty)],
\end{aligned} \tag{9.100}$$

where in the second equality, we used $\bar{u}'_j = u'_j + \alpha r \tilde{u}'_j,\ j = 1, 2$ and in the last equality, $\bar{g}$ is defined in (4.4).

For convenience, we simply use $i'_{1\infty}, u'_{1\infty}, a_\infty, b_\infty, c_\infty$ to represent $i'_1(u'_1(f(k_\infty), k_\infty), k_\infty, 0)$, $u'_1(f(k_\infty), k_\infty)$, $\frac{\partial \tilde{D}}{\partial k}|_{(0,k_\infty,v_\infty,0)}$, $\frac{\partial \tilde{D}}{\partial x}|_{(0,k_\infty,v_\infty,0)}$ and $\frac{\partial \tilde{D}}{\partial v}|_{(0,k_\infty,v_\infty,0)}$ respectively. Then we have

$$\begin{aligned} \frac{dx}{ds} &= \tilde{D}(x, k, v, r) \\ &= a_\infty x + b_\infty (k - k_\infty) + c_\infty (v - v_\infty) + O(||(e, r)||^2), \end{aligned} \tag{9.101}$$

where by (9.99),

$$\begin{aligned} a_\infty &= \overline{u}'^2_1(f(k_\infty), k_\infty, r) i'_1(\bar{u}'_1(f(k_\infty), k_\infty, r), k_\infty, r)[f'(k_\infty) - \overline{g}(k_\infty, r)]\Big|_{r=0} \\ &= -u'^2_{1\infty} i'_{1\infty}(\overline{g}_0(k_\infty) - f'(k_\infty)). \end{aligned} \tag{9.102}$$

Linearizing the system (9.90) near the point $(0, k_\infty, v_\infty)$ by the previous discussions, we have

$$\frac{d}{ds}\begin{pmatrix} x \\ k - k_\infty \\ v - v_\infty \end{pmatrix} = \begin{pmatrix} -u'^2_{1\infty} i'_{1\infty}(\overline{g}_0(k_\infty) - f'(k_\infty)) & b_\infty & c_\infty \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ k - k_\infty \\ v - v_\infty \end{pmatrix}. \tag{9.103}$$

We denote the square coefficient matrix in the above system by A. By the strictly concavity and increasing of u with respect to the first variable, $u'_{1\infty} = u'_1(f(k_\infty), k_\infty) > 0, u''_1(f(k_\infty), k_\infty) < 0$ and $i'_{1\infty} = i'_1(u'_1(f(k_\infty), k_\infty), k_\infty, 0) = \frac{1}{u''_1(f(k_\infty), k_\infty)} < 0$ by (9.73). And for any k_∞ satisfying (4.10) with $\overline{g}_0(k_\infty) - f'(k_\infty) \neq 0$, we then have

$$-u'^2_{1\infty} i'_{1\infty}(\overline{g}_0(k_\infty) - f'(k_\infty)) \neq 0, \tag{9.104}$$

Let

$$\tilde{x} = x + \frac{b_\infty}{a_\infty}(k - k_\infty) + \frac{c_\infty}{a_\infty}(v - v_\infty), \tag{9.105}$$

then the system (9.103) is transformed into a new system, whose linear part is the linear part of the system (9.90), and has the form:

$$\frac{d}{ds}\begin{pmatrix} \tilde{x} \\ k - k_\infty \\ v - v_\infty \end{pmatrix} = \begin{pmatrix} -u'^2_{1\infty} i'_{1\infty}(\overline{g}_0(k_\infty) - f'(k_\infty)) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ k - k_\infty \\ v - v_\infty \end{pmatrix}. \tag{9.106}$$

Then we use the center manifold theorem (cf. Theorem 1 of [Car1]), there exist an $\epsilon > 0$, and a map $h(k, v)$, defined in a neighborhood $\mathcal{O} = (k_\infty - \epsilon, k_\infty + \epsilon) \times (v_\infty - \epsilon, v_\infty + \epsilon)$ of (k_∞, v_∞) such that

$$h(k_\infty, v_\infty) = 0, \quad \frac{\partial h}{\partial k}(k_\infty, v_\infty) = 0, \quad \frac{\partial h}{\partial v}(k_\infty, v_\infty, 0) = 0 \tag{9.107}$$

and the manifold $\mathcal{M}$ defined by

$$\mathcal{M} = \left\{ \left(h(k,v) - \frac{b_\infty}{a_\infty}(k-k_\infty) - \frac{c_\infty}{a_\infty}(v-v_\infty), k, v \right) \middle| (k,v) \in \mathcal{O} \right\}, \tag{9.108}$$

is invariant under the flow associated to the system (9.90), where we just consider the properties of equations and we do not discuss the reality meaning when r takes non-positive values. The map h and the central manifold $\mathcal{M}$ are of C^2, and $\mathcal{M}$ is three-dimensional and tangent to the critical plane.

If $k = k_\infty$ and $v = v_\infty$, by (9.77) and (9.92), x must equals to 0, then

$$h(k_\infty, v_\infty) = \tilde{x}(k_\infty, v_\infty) = x(k_\infty, v_\infty) = 0, \tag{9.109}$$

where we used (9.105).

We are interested in the solutions which lie on the central manifold $\mathcal{M}$. They can be formed by $x = h(k,v) - \frac{b_\infty}{a_\infty}(k-k_\infty) - \frac{c_\infty}{a_\infty}(v-v_\infty)$ in equation (9.81), we get

$$\begin{cases} \dfrac{dv}{dk} = \bar{u}_1'(f(k),k) + h(k,v) - \dfrac{b_\infty}{a_\infty}(k-k_\infty) - \dfrac{c_\infty}{a_\infty}(v-v_\infty), \\ v(k_\infty) = v_\infty \end{cases} \tag{9.110}$$

which can be viewed as eliminating the variable s from the second and third equations of the system (9.90), and x is found by using the fact that $\mathcal{M}$ is invariant in the flow of (9.90):

$$x(s) = h(k(s), v(s)) - \frac{b_\infty}{a_\infty}(k(s)-k_\infty) - \frac{c_\infty}{a_\infty}(v(s)-v_\infty). \tag{9.111}$$

Since $a_\infty \neq 0$ and the right hand side of the first equation of (9.110) is continuously differentiable in $\mathcal{O}_1 = (k_\infty - \epsilon, k_\infty + \epsilon)$, therefore, is Lipschiz continuous. By $\frac{dv}{dk}|_{k=k_\infty} = \bar{u}_1'(f(k_\infty),k_\infty) \neq 0$ which follows from (9.109) and the first equation of (9.110), the nonconstant solution of this initial-value problem exist in $\mathcal{O}_1$ and we denote by $v(k) = \psi(k)$,where $\psi(k_\infty) = v_\infty$ and $\psi \in C^2(\mathbb{R},\mathbb{R})$ if $h \in C^2(\mathbb{R}^2,\mathbb{R})$. Substituting $v(k) = \psi(k)$ into $x = h(k,v) - \frac{b_\infty}{a_\infty}(k-k_\infty) - \frac{c_\infty}{a_\infty}(v-v_\infty)$ yields

$$x(k) = h(k,\psi(k)) - \frac{b_\infty}{a_\infty}(k-k_\infty) - \frac{c_\infty}{a_\infty}(\psi(k)-v_\infty). \tag{9.112}$$

Finally, $\mu(k) = F(\bar{u}_1'(f(k),k) + x(k), k)$ and $w(k) = \frac{1}{b}(\mu(k) + \bar{u}(f(k),k) - a\psi(k))$ is also C^2, so we have found a C^2 solution of the system (4.1) with boundary conditions (4.2). ∎

F Proof of Lemma 9.3: Convergence

We need the value of $v'(k_\infty)$ and $\sigma'(k_\infty)$, where k_∞ is the equilibrium state of the economy and $\sigma(k) = i \circ v'(k) = i(v'(k),k)$. If the solution exists, let $k \to k_\infty$ in the first equation of (4.1), using the boundary conditions (4.2) and the computations in (9.92), we have

$$v'(k_\infty)f(k_\infty) - v'(k_\infty)i(v'(k_\infty),k_\infty) + \bar{u}(i(v'(k_\infty),k_\infty),k_\infty) = \bar{u}(f(k_\infty),k_\infty), \tag{9.113}$$

thus $F(v'(k_\infty), k_\infty) = 0$ by the definition (9.68) of F. By the strictly convexity of F with respect to its first variable and by the discussions in which leads to (9.75), we have

$$v'(k_\infty) = y^*(k_\infty) = \bar{u}_1'(f(k_\infty), k_\infty). \tag{9.114}$$

To compute $\sigma'(k_\infty)$, we consider (9.41), the integration form of $v'(k)$

$$\begin{aligned} v'(k) &= \int_0^\infty h(t)u_1'(\sigma(\mathcal{K}(\sigma;t,k)), \mathcal{K}(\sigma;t,k))\sigma'(\mathcal{K}(\sigma;t,k))\mathcal{R}(t)dt \\ &\quad + \int_0^\infty h(t)u_2'(\sigma(\mathcal{K}(\sigma;t,k)), \mathcal{K}(\sigma;t,k))\mathcal{R}(t)dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_1'(\sigma(\mathcal{K}(\sigma;t,k)), \mathcal{K}(\sigma;t,k))\sigma'(\mathcal{K}(\sigma;t,k))\mathcal{R}(t)dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_2'(\sigma(\mathcal{K}(\sigma;t,k)), \mathcal{K}(\sigma;t,k))\mathcal{R}(t)dt. \end{aligned} \tag{9.115}$$

Note that $h(t) = e^{-\delta t}$ and $\tilde{h}(t) = e^{-rt}$ exponentially decay. Letting $k \to k_\infty$ and using (9.37) we have

$$\begin{aligned} v'(k_\infty) &= \int_0^\infty h(t)u_1'(f(k_\infty), k_\infty)\sigma'(k_\infty)e^{(f'(k_\infty)-\sigma'(k_\infty))t}dt \\ &\quad + \int_0^\infty h(t)u_2'(f(k_\infty), k_\infty)e^{(f'(k_\infty)-\sigma'(k_\infty))t}dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_1'(f(k_\infty), k_\infty)\sigma'(k_\infty)e^{(f'(k_\infty)-\sigma'(k_\infty))t}dt \\ &\quad + \alpha r \int_0^\infty \tilde{h}(t)\tilde{u}_2'(f(k_\infty), k_\infty)e^{(f'(k_\infty)-\sigma'(k_\infty))t}dt \\ &= \int_0^\infty [u_1'(f(k_\infty), k_\infty)\sigma'(k_\infty) + u_2'(f(k_\infty), k_\infty)]e^{-(\delta - f'(k_\infty)+\sigma'(k_\infty))t}dt \\ &\quad + \alpha r \int_0^\infty [\tilde{u}_1'(f(k_\infty), k_\infty)\sigma'(k_\infty) + \tilde{u}_2'(f(k_\infty), k_\infty)]e^{-(r - f'(k_\infty)+\sigma'(k_\infty))t}dt \\ &= \frac{u_1'(f(k_\infty), k_\infty)\sigma'(k_\infty) + u_2'(f(k_\infty), k_\infty)}{\delta - f'(k_\infty) + \sigma'(k_\infty)} \\ &\quad + \alpha r \frac{\tilde{u}_1'(f(k_\infty), k_\infty)\sigma'(k_\infty) + \tilde{u}_2'(f(k_\infty), k_\infty)}{r - f'(k_\infty) + \sigma'(k_\infty)}, \end{aligned} \tag{9.116}$$

where because $v'(k_\infty) < \infty$ and then we can do the computation of the last equation. Taking $v'(k_\infty) = \bar{u}_1'(f(k_\infty), k_\infty)$ into account, we solute $\sigma'(k_\infty)$ from the fractional equation:

$$\begin{aligned} \sigma'(k_\infty) &= f'(k_\infty) - r\Big[\delta(u_1'(f(k_\infty), k_\infty) + \alpha r \tilde{u}_1'(f(k_\infty), k_\infty)) \\ &\quad -(u_2'(f(k_\infty), k_\infty) + \alpha\delta\tilde{u}_2'(f(k_\infty), k_\infty)) \\ &\quad -f'(k_\infty)(u_1'(f(k_\infty), k_\infty) + \alpha\delta\tilde{u}_1'(f(k_\infty), k_\infty))\Big]/E(k_\infty) \\ &= f'(k_\infty) - r\frac{(u_1'(f(k_\infty), k_\infty) + \alpha\delta\tilde{u}_1'(f(k_\infty), k_\infty))(\underline{g}(k_\infty) - f'(k_\infty))}{(u_1'(f(k_\infty), k_\infty) + \alpha r\tilde{u}_1'(f(k_\infty), k_\infty))(\overline{g}(k_\infty) - f'(k_\infty))}, \end{aligned} \tag{9.117}$$

where

$$\begin{aligned} E(k_\infty) &= \delta u_1'(f(k_\infty),k_\infty) + \alpha r^2 \tilde{u}_1'(f(k_\infty),k_\infty) - (u_2'(f(k_\infty),k_\infty) + \alpha r \tilde{u}_2'(f(k_\infty),k_\infty)) \\ &\quad - f'(k_\infty)(u_1'(f(k_\infty),k_\infty) + \alpha r \tilde{u}_1'(f(k_\infty),k_\infty)) \\ &= (u_1'(f(k_\infty),k_\infty) + \alpha r \tilde{u}_1'(f(k_\infty),k_\infty))(\overline{g}(k_\infty) - f'(k_\infty)) \end{aligned} \tag{9.118}$$

and $\underline{g}$, $\overline{g}$ was defined in (4.3),(4.4).

Linearizing the equation of motion $\frac{dk}{dt} = f(k) - \sigma(k)$ at $k = k_\infty$ yields

$$\frac{dk}{dt} = (f'(k_\infty) - \sigma'(k_\infty))(k - k_\infty). \tag{9.119}$$

That k converges to k_∞ if

$$f'(k_\infty) - \sigma'(k_\infty) < 0. \tag{9.120}$$

Hence by (9.117), the estimate (9.120) is equivalent to

$$r\frac{(u_1'(f(k_\infty),k_\infty) + \alpha\delta \tilde{u}_1'(f(k_\infty),k_\infty))(\underline{g}(k_\infty) - f'(k_\infty))}{(u_1'(f(k_\infty),k_\infty) + \alpha r \tilde{u}_1'(f(k_\infty),k_\infty))(\overline{g}(k_\infty) - f'(k_\infty))} < 0. \tag{9.121}$$

Simplifying it, we obtain

$$\underline{g}(k_\infty) < f'(k_\infty) < \overline{g}(k_\infty). \tag{9.122}$$

For r small enough, the (4.10) guarantees the conditions, thus the convergence is established. ∎

G Proof of Theorem 7.2

Because σ is an admissible strategy, we have

$$\left| \int_0^\infty [k_b(t) - k_\infty] dt \right| < +\infty, \tag{9.123}$$

then we need to prove that σ_ϵ given by (6.2) is also an admissible strategy, i.e., we want to show

$$\left| \int_0^\infty [k_c(t) - k_\infty] dt \right| < +\infty, \tag{9.124}$$

where in the non-degenerate case, $k_c(t)$ converges to the same k_∞ as $k_b(t)$. On the other hand, by (3.3), (3.4), (3.13) and the definition of σ_ϵ, we have $k_b(t) = \mathcal{K}(t-\epsilon; k_b(\epsilon), \sigma)$ and $k_c(t) = \mathcal{K}(t-\epsilon; k_c(\epsilon), \sigma)$ when $t \geq \epsilon$. Then there must exist two positive times t_1, t_2 such that $\mathcal{K}(t_1 - \epsilon; k_b(\epsilon), \sigma) =$

$\mathcal{K}(t_2-\epsilon;k_c(\epsilon),\sigma)$. Therefore, we have

$$\begin{aligned}
\left|\int_{t_2}^{\infty}[k_c(t)-k_\infty]dt\right| &= \left|\int_0^{\infty}[\mathcal{K}(t+t_2-\epsilon;k_c(\epsilon),\sigma)-k_\infty]dt\right| \\
&= \left|\int_0^{\infty}[\mathcal{K}(t;\mathcal{K}(t_2-\epsilon;k_c(\epsilon),\sigma),\sigma)-k_\infty]dt\right| \\
&= \left|\int_0^{\infty}[\mathcal{K}(t;\mathcal{K}(t_1-\epsilon;k_b(\epsilon),\sigma),\sigma)-k_\infty]dt\right| \\
&= \left|\int_0^{\infty}[\mathcal{K}(t+t_1-\epsilon;k_b(\epsilon),\sigma)-k_\infty]dt\right| \\
&= \left|\int_{t_1}^{\infty}[k_b(t)-k_\infty]dt\right| \\
&< +\infty, \qquad (9.125)
\end{aligned}$$

thus (9.124) holds.

Since in the non-degenerate case, the limit capital k_∞ cannot be influenced, then similarly to (9.42), we have

$$\begin{aligned}
P(k_0,\sigma,c) &= \bar{u}(c,k_0)-\bar{u}(\sigma(k_0),k_0) \\
&+(\sigma(k_0)-c)\int_0^{\infty}e^{-\delta t}[u_1'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\sigma'(\mathcal{K}(t;k_0,\sigma)) \\
&+u_2'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))]\mathcal{R}(t)dt \\
&+\alpha(\sigma(k_0)-c)\int_0^{\infty}[\tilde{u}_1'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))\sigma'(\mathcal{K}(t;k_0,\sigma)) \\
&+\tilde{u}_2'(\sigma(\mathcal{K}(t;k_0,\sigma)),\mathcal{K}(t;k_0,\sigma))]\mathcal{R}(t)dt. \qquad (9.126)
\end{aligned}$$

Then we can do the similar steps of Theorem 3.5, Theorem 3.5 and Theorem 4.1 to obtain the theorem. ∎

H Proof of Theorem 7.3

In fact, the prove is a repetition of the proof of Lemma 9.3 in Appendix F and Appendix G. We first show how to obtain the inequalities (7.8).

Similar to Appendix G, we have

$$v'(k_\infty)=\bar{u}_1'(f(k_\infty),k_\infty), \qquad (9.127)$$

and

$$\begin{aligned}
v'(k_\infty) &= \int_0^{\infty}[u_1'(f(k_\infty),k_\infty)\sigma'(k_\infty)+u_2'(f(k_\infty),k_\infty)]e^{-(\delta-f'(k_\infty)+\sigma'(k_\infty))t}dt \\
&+\alpha\int_0^{\infty}[\tilde{u}_1'(f(k_\infty),k_\infty)\sigma'(k_\infty)+\tilde{u}_2'(f(k_\infty),k_\infty)]e^{-(-f'(k_\infty)+\sigma'(k_\infty))t}dt \\
&= \frac{u_1'(f(k_\infty),k_\infty)\sigma'(k_\infty)+u_2'(f(k_\infty),k_\infty)}{\delta-f'(k_\infty)+\sigma'(k_\infty)} \\
&+\alpha\frac{\tilde{u}_1'(f(k_\infty),k_\infty)\sigma'(k_\infty)+\tilde{u}_2'(f(k_\infty),k_\infty)}{-f'(k_\infty)+\sigma'(k_\infty)}. \qquad (9.128)
\end{aligned}$$

From the two above equations, we can compute $\sigma'(k_\infty)$:

$$\sigma'(k_\infty) = f'(k_\infty) - \alpha\delta \frac{\tilde{u}_1'(f(k_\infty), k_\infty)}{u_1'(f(k_\infty), k_\infty) + \alpha\tilde{u}_1'(f(k_\infty), k_\infty)} \cdot \frac{f'(k_\infty) + \frac{\tilde{u}_2'(f(k_\infty),k_\infty)}{\tilde{u}_1'(f(k_\infty),k_\infty)}}{f'(k_\infty) - \frac{\delta u_1'(f(k_\infty),k_\infty) - (u_2'(f(k_\infty),k_\infty) + \alpha\tilde{u}_2'(f(k_\infty),k_\infty))}{u_1'(f(k_\infty),k_\infty) + \alpha\tilde{u}_1'(f(k_\infty),k_\infty)}}. \tag{9.129}$$

Then k converge to k_∞ requires $f'(k_\infty) - \sigma'(k_\infty) < 0$, and hence (7.8).

One more thing we need to do is to show that the strategy $\sigma(k_0) = i(v'(k_0), k_0)$ which we obtained from (7.5) is admissible. And this is directly by $f'(k_\infty) - \sigma'(k_\infty) < 0$, then $k(t)$ exponentially converges to k_∞, and so (7.5) holds. ∎

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