A note on existence and uniqueness for solutions of multidimensional reflected BSDEs

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Abstract

In this note, we provide an innovative and simple approach for proving the existence of a unique solution for multidimensional reflected BSDEs associated to switching problems. Getting rid of a monotonicity assumption on the driver function, this approach simplifies and extends the recent results of Hu & Tang [4] or Hamadene & Zhang [3].

Key words: BSDE with oblique reflections, Switching problems.

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1 Introduction

The theory of Backward Stochastic Differential Equations (BSDEs for short) offers a large number of applications in the field of stochastic control or mathematical finance. Recently, Hu and Tang [4] introduced and studied a new type of BSDE, constrained by oblique reflections, and associated to optimal switching problems. Via heavy arguments, Hamadène and Zhang [3] generalized the form of these multidimensional reflected BSDEs. They allow for the consideration of more general oblique constraints, as well as drivers depending on the global solution of the BSDE. Unfortunately, their framework requires the driver function to be increasing with respect to the components of the global solution of the BSDE. In the context of linear reflections of switching type, we are able to get rid of this limiting monotonicity assumption.

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We provide in this note a new method to prove existence and uniqueness for such type of BSDEs. We follow the classical scheme introduced for e.g. in [2], which consists in proving that a well chosen operator is a contraction for a given norm. This is done via the introduction of a convenient one dimensional dominating BSDE and the use of a standard comparison theorem.
The rest of this note is organized as follows. In Section 2, we present our main result, namely Theorem 2.1. In Section 3, we give its proof which requires several intermediary results.

2 Framework and main result

Let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic basis supporting a q-dimensional Brownian motion W, $q \geq 1$. $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ is the completed filtration generated by the Brownian motion W, and $\mathfrak{P}$ denotes the $\sigma-$algebra on $[0,T] \times \Omega$ generated by $\mathbb{F}-$progressively measurable processes. In the following, we shall omit the dependence on $\omega \in \Omega$ when it is clearly given by the context.

We introduce the following spaces of processes:

- $\mathcal{S}^2$ (resp. $\mathcal{S}^2_c$) is the set of $\mathbb{R}-$valued, adapted and càdlàg[1] (resp. continuous) processes $(Y_t)_{0 \leq t \leq T}$ such that $\|Y\|_{\mathcal{S}^2} := \mathbb{E}\Big[\sup_{t \in [0,T]} |Y_t|^2\Big]^{\frac{1}{2}} < \infty$,
- $\mathcal{H}^2$ is the set of $\mathbb{R}^q-$valued, progressively measurable process $(Z_t)_{0 \leq t \leq T}$ such that $\|Z\|_{\mathcal{H}^2} := \mathbb{E}\Big[\int_0^T |Z_t|^2 \mathrm{d}t\Big]^{\frac{1}{2}} < \infty$,
- $\mathcal{K}^2$ (resp. $\mathcal{K}^2_c$) is the subset of nondecreasing processes $(K_t)_{0 \leq t \leq T} \in \mathcal{S}^2$ (resp. $(K_t)_{0 \leq t \leq T} \in \mathcal{S}^2_c$), starting from $K_0 = 0$.

We are given a matrix valued continuous process $C = (C^{ij})_{1 \leq i,j \leq d}$ such that $C^{ij} \in \mathcal{S}^2_c$, for $i, j \in \mathcal{I}$, where $\mathcal{I} := \{1, \dots, d\}$, $d \geq 2$, and satisfying the following structure condition

$$\begin{cases} (i)\ C^{ii}_{\cdot} = 0\,, & \text{for } i \in \mathcal{I}\ ; \\ (ii)\ \inf_{0 \leq t \leq T} C^{ij}_t > 0\,, & \text{for } i, j \in \mathcal{I}^2\,,\ \text{with } i \neq j\ ; \\ (iii)\ \inf_{0 \leq t \leq T} C^{ij}_t + C^{jl}_t - C^{il}_t > 0, & \text{for } i, j, l \in \mathcal{I}^3,\ \text{with } i \neq j,\ j \neq l\,; \end{cases} \tag{2.1}$$

In "switching problems", the quantity C interprets as a cost of switching. As in [4] or [3] in a more general framework, this assumption makes instantaneous switching irrelevant.

[1] French acronym for right continuous with left limits.

We introduce the family of random closed convex sets $(\mathcal{C}_t)_{0\leq t\leq T}$ associated to C:

$$\mathcal{C}_t := \left\{ y \in \mathbb{R}^d \mid y^i \geq \max_{j\in\mathcal{I}}(y^j - C_t^{ij}) \,, \quad i \in \mathcal{I} \right\} \,, \quad 0 \leq t \leq T \,,$$

and the oblique projection operator $\mathcal{P}$ onto $\mathcal{C}$ defined by

$$\mathcal{P}_t : y \in \mathbb{R}^d \ \mapsto \ \left(\max_{j\in\mathcal{I}} \left\{ y^j - C_t^{ij} \right\} \right)_{i\in\mathcal{I}} \,, \quad 0 \leq t \leq T \,.$$

We are also given a terminal random variable $\xi \in [L^2(\mathcal{F}_T)]^d$ valued in $\mathcal{C}_T$, where $L^2(\mathcal{F}_T)$ is the set of $\mathcal{F}_T$-measurable random variable X satisfying $\mathbb{E}\big[|X|^2\big] < \infty$.

We then consider the following system of reflected BSDEs:
find $(\dot{Y}, \dot{Z}, \dot{K}) \in [\mathcal{S}_c^2 \times \mathcal{H}^2 \times \mathcal{K}_c^2]^d$ such that

$$\begin{cases} \dot{Y}_t^i = \xi^i + \int_t^T f^i(s, \dot{Y}_s, \dot{Z}_s)\mathrm{d}s - \int_t^T \dot{Z}_s^i.\mathrm{d}W_s + \dot{K}_T^i - \dot{K}_t^i \,, \quad 0 \leq t \leq T \,, \\ \dot{Y}_t^i \geq \max_{j\in\mathcal{I}}\{\dot{Y}_t^j - C_t^{ij}\} \,, \quad 0 \leq t \leq T \,, \\ \int_0^T [\dot{Y}_t^i - \max_{j\in\mathcal{I}}\{\dot{Y}_t^j - C_t^{ij}\}]\mathrm{d}\dot{K}_t^i = 0 \,, \ 0 \leq i \leq d \,, \end{cases} \tag{2.2}$$

where $f : \Omega \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{q\times d} \to \mathbb{R}^m$ is $\mathfrak{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{q\times d})-$measurable and satisfies the following Lipschitz property: there exists a constant $L > 0$ such that

$$|f(t, y, z) - f(t, y', z')| \quad \leq \quad L(|y - y'| + |z - z'|),$$

for all $(t, y, y', z, z') \in [0,T] \times [\mathbb{R}^d]^2 \times [\mathbb{R}^{q\times d}]^2$, $\mathbb{P}-$a.s. We also assume that

$$\mathbb{E}\left[\int_0^T |f(t, 0, 0)|^2 dt \right] \quad < \quad \infty \,.$$

The existence and uniqueness of a solution to the system (2.2) has already been derived with the addition of one of the following assumptions:

- (**H**1) The cost process C is constant and, for $i \in \mathcal{I}$, the $i-$th component of the driver function f depends only on y^i, the $i-$th component of y, and z^i, the $i-$th column of z. (see Hu & Tang [4])

- (**H**2) For $i \in \mathcal{I}$, the $i-$th component of the random driver f depends on y and the $i-$th column of z, and is nondecreasing in y^j for $j \neq i$.
(see Hamadène & Zhang [3])

We now introduce the following weaker assumption:

- ($\mathbf{H}f$) For $i \in \mathcal{I}$, the $i-$th component of the random function f depends on y and the $i-$th column of the variable z:
  $f^i(t,y,z) = f^i(t,y,z^i)$, $\quad (t,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^{q\times d}$.

The main contribution of this note is the following result.

Theorem 2.1. *Under* ($\mathbf{H}f$)*, the BSDE* (2.2) *admits a unique solution.*

3 Proof of Theorem 2.1

The proof divides in three steps. First, we slightly generalize the switching representation result presented in Theorem 3.1 of [4], allowing for the consideration of random driver and costs. Second, we introduce a Picard type operator associated to the BSDE (2.2) of interest. Finally, via the introduction of a convenient dominating BSDE and the use of a comparison argument, we prove that this operator is a contraction for a well chosen norm.

3.1 The optimal switching representation property

We first give a key representation property for the solution of (2.2) under the following assumption:

- ($\mathbf{H}3$) For $i \in \mathcal{I}$, the $i-$th component of the random driver f depends only on y^i and the $i-$th column of the variable z:
  $f^i(t,y,z) = f^i(t,y^i,z^i)$, $\quad (i,t,y,z) \in \mathcal{I} \times [0,T] \times \mathbb{R}^d \times \mathbb{R}^{q\times d}$.

Observe that ($\mathbf{H}1$) $\implies$ ($\mathbf{H}3$) $\implies$ ($\mathbf{H}2$). We first provide the existence of a solution to (2.2) under Assumption ($\mathbf{H}3$).

Proposition 3.1. *Under* ($\mathbf{H}3$)*, the BSDE* (2.2) *has at least one solution.*

Proof. Since ($\mathbf{H}3$) is stronger than ($\mathbf{H}2$), the existence of a solution is provided by Theorem 3.2 in [3]. Due to the the particular form of switching oblique reflections and for sake of completeness, we decide to report here a simplified sketch of proof.

We use Picard iteration. Let $(Y^{\cdot,0}, Z^{\cdot,0}) \in [\mathcal{S}^2 \times \mathcal{H}^2]^d$ be the solution to the following BSDE without reflection:

$$Y_t^{i,0} \;=\; \xi^i + \int_t^T f^i(s, Y_s^{i,0}, Z_s^{i,0})\mathrm{d}s - \int_t^T Z_s^{i,0}.\mathrm{d}W_s, \quad 0 \le t \le T, \quad i \in \mathcal{I}\,. \quad (3.1)$$

For $i \in \mathcal{I}$ and $n \geq 1$, define recursively $Y^{i,n}$ as the first component of the unique solution (see Theorem 5.2 in [1]) of the reflected BSDE

$$\begin{cases} Y_t^{i,n} = \xi^i + \int_t^T f^i(s, Y_s^{i,n}, Z_s^{i,n})\mathrm{d}s - \int_t^T Z_s^{i,n}.\mathrm{d}W_s + K_T^{i,n} - K_t^{i,n} , \quad 0 \leq t \leq T , \\ Y_t^{i,n} \geq \max_{j\in\mathcal{I}}\{Y_t^{j,n-1} - C_t^{ij}\} , \quad 0 \leq t \leq T , \\ \int_0^T [Y_t^{i,n} - \max_{j\in\mathcal{I}}\{Y_t^{j,n-1} - C_t^{ij}\}]\mathrm{d}K_t^{i,n} = 0 . \end{cases}$$

For any $i \in \mathcal{I}$, one easily verifies by induction that the sequence $(Y^{i,n})_{n\in\mathbb{N}}$ is nondecreasing and upper bounded by $\bar{Y}$ the first component of the unique solution to

$$\bar{Y}_t = \sum_{i=1}^d |\xi^i| + \int_t^T \sum_{i=1}^d |f^i(s, \bar{Y}_s, \bar{Z}_s)|ds - \int_t^T \bar{Z}_s.\mathrm{d}W_s, \quad 0 \leq t \leq T.$$

For any $i \in \mathcal{I}$, by Peng's monotonic limit Theorem (see Theorem 3.6 in [5]), the limit $\dot{Y}^i$ of $(Y^{i,n})_n$ is a càdlàg process and there exists $(\dot{Z}^i, \dot{K}^i) \in \mathcal{H}^2 \times \mathcal{K}^2$ such that

$$\begin{cases} \dot{Y}_t^i = \xi^i + \int_t^T f^i(s, \dot{Y}_s^i, \dot{Z}_s^i)\mathrm{d}s - \int_t^T \dot{Z}_s^i.\mathrm{d}W_s + \dot{K}_T^i - \dot{K}_t^i , \quad 0 \leq t \leq T , \\ \dot{Y}_t^i \geq \max_{j\in\mathcal{I}}\{\dot{Y}_t^j - C_t^{ij}\} , \quad 0 \leq t \leq T . \end{cases} \tag{3.2}$$

In order to prove that $(\dot{Y}, \dot{Z}, \dot{K})$ is the minimal solution, we then introduce, in the spirit of [6] and for any $i \in \mathcal{I}$, the smallest f^i-supermartingale $\tilde{Y}^i$ with lower barrier $\max_{j\neq i}\{\dot{Y}^j - C^{ij}\}$ defined as the solution of

$$\begin{cases} \tilde{Y}_t^i = \xi^i + \int_t^T f^i(s, \tilde{Y}_s^i, \tilde{Z}_s^i)\mathrm{d}s - \int_t^T \tilde{Z}_s^i.\mathrm{d}W_s + \tilde{K}_T^i - \tilde{K}_t^i , \quad 0 \leq t \leq T , \\ \tilde{Y}_t^i \geq \max_{j\neq i}\{\dot{Y}_t^j - C_t^{ij}\} , \quad 0 \leq t \leq T , \\ \int_0^T [\tilde{Y}_{t^-}^i - \max_{j\neq i}\{\dot{Y}_{t^-}^j - C_t^{ij}\}]\mathrm{d}\tilde{K}_t^i = 0 , \ 0 \leq i \leq d . \end{cases} \tag{3.3}$$

For $i \in \mathcal{I}$, we directly deduce from (3.2) that $\dot{Y}^i \geq \tilde{Y}^i$, and, since $(Y^{\cdot,n})_{n\in\mathbb{N}}$ is increasing, a direct comparison argument leads to $Y^{i,n} \leq \tilde{Y}^i$, for $n \in \mathbb{N}$. Therefore, we get $\dot{Y} = \tilde{Y}$ so that $\dot{Y}$ satisfies (2.2).

It only remains to prove that $\dot{Y}$ is continuous and we suppose on the contrary that $\Delta\dot{Y}_t^{i_0} := \dot{Y}_t^{i_0} - \dot{Y}_{t^-}^{i_0} \neq 0$ for some fixed $(i_0, t) \in [0, T] \times \mathcal{I}$. We deduce from (3.3) that $\Delta\dot{Y}_t^{i_0} = -\Delta K_t^{i_0} < 0$ and $\dot{Y}_{t^-}^{i_0} = \dot{Y}_{t^-}^{i_1} - C_t^{i_0 i_1}$, for some $i_1 \neq i_0$. Then, we have

$$\dot{Y}_{t^-}^{i_1} - C_t^{i_0 i_1} = \dot{Y}_{t^-}^{i_0} > \dot{Y}_t^{i_0} \geq \dot{Y}_t^{i_1} - C_t^{i_0 i_1},$$

which gives $\Delta\dot{Y}_t^{i_1} < 0$. Repeating this argument, we get a finite sequence $(i_k)_{0\leq k\leq n}$ such that $i_n = i_0$ and $\dot{Y}_{t^-}^{i_k} = \dot{Y}_{t^-}^{i_{k+1}} - C_t^{i_k i_{k+1}}$ for any $k < n$. We deduce $\sum_{k=1}^n C^{i_k i_{k-1}} = 0$ which contradicts part (iii) of the structural condition (2.1). □

Under (**H**3), a slight generalization of Theorem 3.1 in [4] allows to represent the process $(\dot{Y}^i)_{i\in\mathcal{I}}$ as the value process of an optimal switching problem on a family of one-dimensional BSDEs. More precisely, we consider the set of admissible[2] strategies $\mathcal{A}$ consisting in the sequence $a = (\theta_k, \alpha_k)_{k\geq 0}$ with

- $(\theta_k)_{k\geq 0}$ a nondecreasing sequence of stopping times valued in $[0,T]$, and such that there exists an integer valued random variable N^a, $\mathcal{F}_T$ measurable and $\theta_{N^a} = T$, $\mathbb{P}$ − a.s.,
- $(\alpha_k)_{k\geq 0}$ a sequence of random variables valued in $\mathcal{I}$ such that α_k is $\mathcal{F}_{\theta_k}$−measurable for all $k \geq 0$,
- the process A^a defined by $A_t^a := \sum_{k\geq 1} C_{\theta_k}^{\alpha_{k-1}\alpha_k} \mathbf{1}_{\theta_k\leq t}$ belonging to $\mathcal{S}^2$.

The state process associated to a strategy $a \in \mathcal{A}$ is denoted $(a_t)_{0\leq t\leq T}$ and defined by

$$a_t \quad = \quad \sum_{k\geq 1} \alpha_{k-1} \mathbf{1}_{(\theta_{k-1},\theta_k]}(t)\,, \quad 0 \leq t \leq T\,.$$

For $(t,i) \in [0,T] \times \mathcal{I}$, we also define $\mathcal{A}_{t,i}$ the subset of admissible strategies restricted to start in state i at time t. For $(t,i) \in [0,T] \times \mathcal{I}$ and $a \in \mathcal{A}_{t,i}$, we consider $(U^a, V^a) \in \mathcal{S}^2 \times \mathcal{H}^2$ the unique solution of the switched BSDE:

$$U_r^a \quad = \quad \xi^{a_T} + \int_r^T f^{a_s}(s, U_s^a, V_s^a)\mathrm{d}s - \int_r^T V_s^a.\mathrm{d}W_s - A_T^a + A_r^a\,, \quad t \leq r \leq T\,.$$

As in Theorem 3.1 in [4], the correspondence between Y and the switched BSDEs is given by the following proposition which provides, as a by-product, the uniqueness of solution to (2.2) under (**H**3).

Proposition 3.2. *Under* (**H**3)*, there exists* $a^* \in \mathcal{A}_{t,i}$ *such that*

$$\dot{Y}_t^i \quad = \quad U_t^{a^*} \quad = \quad \operatorname*{ess\,sup}_{a\in\mathcal{A}_{t,i}} U_t^a\,, \quad (t,i) \in [0,T] \times \mathcal{I}\,.$$

Proof. The proof follows from the exact same reasoning as in the proof of Theorem 3.1 in [4] having in mind that in our framework, the driver f and the costs C are random. This implies that N^{a^*}, the number of switch of the optimal strategy, is only almost surely finite. This is the only difference with [4] where N^{a^*} belongs to $L^2(\mathcal{F}_T)$. □

[2] Our definition of admissible strategies slightly differs from the one given in [4] Definition 3.1. Indeed, the authors assume that N is in $L^2(\mathcal{F}_T)$ rather than $A \in \mathcal{S}^2$. In fact, it is clear that in the context of constant cost process C satisfying (2.1), both definitions are equivalent.

3.2 The contraction operator

We suppose that $(\mathbf{H}f)$ is in force and introduce the operator $\Phi : [\mathcal{H}^2]^d \to [\mathcal{H}^2]^d$, $\Gamma \mapsto Y := \Phi(\Gamma)$, where $(Y, Z, K) \in (\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{K}^2)^d$ is the unique solution of the BSDE

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f^i(s, \Gamma_s, Z_s^i)\mathrm{d}s - \int_t^T Z_s^i.\mathrm{d}W_s + K_T^i - K_t^i \;, \quad 0 \le t \le T \;, \\ Y_t^i \ge \max_{j\in\mathcal{I}}\{Y_t^j - C_t^{ij}\} \;, \quad 0 \le t \le T \;, \\ \int_0^T [Y_t^i - \max_{j\in\mathcal{I}}\{Y_t^j - C_t^{ij}\}]\mathrm{d}K_t^i = 0 \;, \quad i \in \mathcal{I} \;. \end{cases} \tag{3.4}$$

Since $(\mathbf{H}f)$ holds, the random driver $(\omega, t, z) \mapsto [f^i(t, \Gamma_t(\omega), z^i)]_{i\in\mathcal{I}}$ satisfies $(\mathbf{H}3)$, for any $\Gamma \in [\mathcal{H}^2]^d$. Therefore, the existence of a unique solution to (3.4) is given by Proposition 3.1 and Proposition 3.2 and Φ is well defined. Furthermore, observe that Φ is valued in $[\mathcal{S}^2]^d$.
In order to prove that Φ is a contraction on $[\mathcal{H}^2]^d$, we introduce, as in e.g. [2], the norm $\|.\|_{2,\beta}$ defined on $[\mathcal{H}^2]^d$ by

$$\|Y\|_{2,\beta} \;:=\; \Big(\mathbb{E}\Big[\int_0^T e^{\beta t}|Y_t|^2 dt\Big]\Big)^{\frac{1}{2}} \;.$$

We now state the contraction property for Φ, whose proof is postponed to the last section of this note.

Proposition 3.3. *For β large enough, the operator Φ is a contraction on the Banach space $([\mathcal{H}^2]^d, \|.\|_{2,\beta})$, i.e. Φ is $k-$Lipschitz continuous with $k < 1$ for this norm.*

Proof of Theorem 2.1. As a consequence, there exists a unique fixed point in $[\mathcal{H}^2]^d$ for Φ. Since Φ is valued in $[\mathcal{S}^2]^d$, there is a continuous version (still denoted Y) of this fixed point belonging to $[\mathcal{S}^2]^d$, see footnote 5 p. 21 in [2]. Hence, this version Y and the corresponding processes $(Z, K) \in [\mathcal{H}^2 \times \mathcal{K}^2]^d$ are the unique solution of (2.2). □

3.3 Contraction via domination

This section is dedicated to the proof of Proposition 3.3.

Proof of Proposition 3.3. We consider two processes ${}^1\Gamma$ and ${}^2\Gamma$ belonging to $[\mathcal{H}^2]^d$ and denote ${}^1Y := \Phi({}^1\Gamma)$ and ${}^2Y := \Phi({}^2\Gamma)$.

Step 1: *Auxiliary dominating BSDE.* Let us introduce the following BSDE:

$$\begin{cases} \check{Y}_t^i = \xi^i + \int_t^T \check{f}^i(s, \check{Z}_s^i)\mathrm{d}s - \int_t^T \check{Z}_s^i.\mathrm{d}W_s + \check{K}_T^i - \check{K}_t^i \;, \quad 0 \le t \le T \;, \\ \check{Y}_t^i \ge \max_{j\in\mathcal{I}}\{\check{Y}_t^j - C_t^{ij}\} \;, \quad 0 \le t \le T \;, \\ \int_0^T [(\check{Y}_t)^i - \max_{j\in\mathcal{I}}\{\check{Y}_t^j - C_t^{ij}\}]\mathrm{d}\check{K}_t^i = 0 \;, \quad 1 \le i \le d \;, \end{cases} \tag{3.5}$$

where $\check f^i : (t,z^i) \mapsto f^i(t,{}^1\Gamma_t,z^i) \vee f^i(t,{}^2\Gamma_t,z^i)$, for $(i,t,z) \in \mathcal{I}\times[0,T]\times\mathbb{R}^q$. Once again, since (**H**3) holds for $\check f$, there exists a unique solution to (3.5).

Step 2: *Switching representation.* We fix $(t,i) \in [0,T]\times\mathcal{I}$ and, for any $a \in \mathcal{A}_{t,i}$, denote by $(\check U^a, \check V^a)$ and $({}^jU^a, {}^jV^a)$, for $j = 1$ and 2, the respective solutions of the following one-dimensional BSDEs:

$$\check U^a_s = \xi^{a_T}_T + \int_s^T \check f^{a_r}(r,\check V^a_r)\mathrm{d}r - \int_s^T \check V^a_r.\mathrm{d}W_r - A^a_T + A^a_s\ , \quad t \le s \le T,$$

$${}^jU^a_s = \xi^{a_T}_T + \int_s^T f^{a_r}(r,{}^j\Gamma_r,{}^jV^a_r)\mathrm{d}r - \int_t^T {}^jV^a_r.\mathrm{d}W_r - A^a_T + A^a_r\ , \quad t \le s \le T, \quad j = 1,2\ .$$

Since $f(.,{}^1\Gamma_.,.)$, $f(.,{}^2\Gamma_.,.)$ and $\check f$ satisfy (**H**3), we deduce from Proposition 3.2 that

$${}^jY^i_t = \operatorname*{ess\,sup}_{a\in\mathcal{A}_{t,i}} {}^jU^a_t\ , \quad \text{for } j = 1,2\ , \qquad \text{and} \qquad \check Y^i_t = \operatorname*{ess\,sup}_{a\in\mathcal{A}_{t,i}} \check U^a_t =: \check U^{\check a}_t\ , \tag{3.6}$$

where $\check a \in \mathcal{A}_{t,i}$ is the associated optimal strategy given in Proposition 3.2.
Using a comparison argument , we easily check that $\check U^a \ge {}^1U^a \vee {}^2U^a$, for any strategy $a \in \mathcal{A}_{t,i}$. This estimate combined with (3.6) leads to $\check Y^i_t \ge {}^1Y^i_t \vee {}^2Y^i_t$. Since $\check a$ is an admissible strategy for the representation (3.6) of ${}^1Y^i$ and ${}^2Y^i$, we deduce that

$${}^1U^{\check a}_t \le {}^1Y^i_t \le \check U^{\check a}_t \qquad \text{and} \qquad {}^2U^{\check a}_t \le {}^2Y^i_t \le \check U^{\check a}_t\ .$$

This directly leads to

$$|{}^1Y^i_t - {}^2Y^i_t| \le |\check U^{\check a}_t - {}^1U^{\check a}_t| + |\check U^{\check a}_t - {}^2U^{\check a}_t|\ . \tag{3.7}$$

Step 3: *Contraction property.* We first control the first term on the right hand side of (3.7). Denoting $\delta U^{\check a} := \check U^{\check a} - {}^1U^{\check a}$ and $\delta V^{\check a} := \check V^{\check a} - {}^1V^{\check a}$, we apply Ito's formula to the process $e^{\beta\cdot}|\delta U^{\check a}_\cdot|^2$ and compute

$$e^{\beta t}|\delta U^{\check a}_t|^2 + \int_t^T e^{\beta s}|\delta V^{\check a}_s|^2\mathrm{d}s = -\int_t^T \beta e^{\beta s}|\delta U^{\check a}_s|^2\mathrm{d}s - 2\int_t^T e^{\beta s}\delta U^{\check a}_s\delta V^{\check a}_s.\mathrm{d}W_s$$
$$+2\int_t^T e^{\beta s}\delta U^{\check a}_s[\check f^{\check a_s}(s,\check V^{\check a}_s) - f^{\check a_s}(s,{}^1\Gamma_s,{}^1V^{\check a}_s)]\mathrm{d}s\ .$$

Observe that the inequality $|x\vee y - y| \le |x-y|$ combined with the Lipschitz property of f leads to

$$|\check f^{\check a_s}(s,\check V^{\check a}_s) - f^{\check a_s}(s,{}^1\Gamma_s,{}^1V^{\check a}_s)| \le L(|{}^1\Gamma_s - {}^2\Gamma_s| + |\check V^{\check a}_s - {}^1V^{\check a}_s|)\ , \quad t \le s \le T\ .$$

Combining these two estimates with the inequality $2xy \le \frac{1}{\beta}x^2 + \beta y^2$, we get

$$e^{\beta t}|\delta U^{\check a}_t|^2 \le -2\int_t^T \delta U^{\check a}_s\delta V^{\check a}_s.\mathrm{d}W_s + \frac{L}{\beta}\int_t^T e^{\beta s}|{}^1\Gamma_s - {}^2\Gamma_s|^2\mathrm{d}s\ ,$$

for $\beta \geq L$. Taking the expectation in the previous expression and observing that the second term in (3.7) is treated similarly, we deduce

$$\mathbb{E}\Big[e^{\beta t}|^1Y^i_t - {}^2Y^i_t|^2\Big] \leq \frac{2L}{\beta}\mathbb{E}\Big[\int_0^T e^{\beta s}|^1\Gamma_s -^2 \Gamma_s|^2 \mathrm{d}s\Big] \ .$$

Since the last inequality holds true for any (t,i) in $[0,T]\times\mathcal{I}$, we derive

$$||\Phi(^1\Gamma) - \Phi(^2\Gamma)||_{2,\beta} \leq \sqrt{\frac{2LTd}{\beta}}||^1\Gamma -^2 \Gamma||_{2,\beta}.$$

Setting $\beta := \max(L, 8LTd)$, Φ is $\frac{1}{2}$-Lipschitz continuous for the $\|\cdot\|_{2,\beta}$ norm, which concludes the proof of the proposition. □

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