Constrained Backward SDEs with Jumps: Application to Optimal Switching

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Abstract

We study backward stochastic differential equations (BSDEs) with jumps, subject to an additional global constraint involving all the components of the solution. We provide the existence of a unique minimal solution for these so-called constrained BSDEs with jumps via a penalization procedure. These new constrained BSDEs with jumps offer an alternative probabilistic representation for the solution of Brownian multidimensional reflected BSDEs studied in [14] and [12]. Furthermore, they allow for the representation of the value process associated to optimal switching problems, where the switching strategy influences the dynamics of the underlying diffusion.

Key words: Stochastic control, Switching problems, BSDE with jumps, Reflected BSDE.

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1 Introduction

Since its introduction by Pardoux and Peng in [16], Backward Stochastic Differential Equations (BSDEs in short) have been widely studied. In particular, they appear as a very powerful tool to solve partial differential equations (PDEs) and corresponding stochastic optimization problems. Several generalizations of this notion are based on the addition of new constraints on the solution. First, El Karoui et al. [10] study the case where the component Y is forced to stay above a given process, leading to the notion of reflected BSDEs related to optimal stopping and obstacle problems. Motivated by super replication issues under portfolio constraints, Cvitanic et al. [7] consider the case where the component Z is constrained to stay in a fixed convex set. More recently, Kharroubi et al. [15] introduce a constraint on the jump component U of the BSDE, providing a representation of solutions for a class of PDE, called quasi-variational inequalities, arising for e.g. in optimal impulse control problems. The generalization of the results of El Karoui et al. [10] to oblique reflections in a multi-dimensional framework was first given in a very special case (e.g. the generator does not depend on z) by Ramasubramanian [21], who studied a BSDE reflected in an orthant. Then, Hu and Tang [14] followed by Hamadène and Zhang [12] consider general BSDEs with oblique reflections and connect them with systems of variational inequalities and optimal switching problems. Nevertheless, they only consider cases where the switching strategy barely affects the dynamics of the underlying diffusion. Our paper introduces the notion of constrained BSDEs with jumps, which offers in particular a nice and natural probabilistic representation for these types of switching problems. This new notion essentially unifies and extends the notions of constrained BSDE without jumps, BSDE with constrained jumps as well as multidimensional BSDE with oblique reflections.

Let us illustrate our presentation with the example of a switching problem and introduce an underlying diffusion process, whose dynamics are given by

$$X_t^\alpha = X_0 + \int_0^t b_{\alpha_s}(s, X_s^\alpha)ds + \int_0^t \sigma_{\alpha_s}(s, X_s^\alpha)dW_s, \quad 0 \le t \le T, \tag{1.1}$$

where α is a switching control process valued in $\mathcal{I} := \{1, \dots, m\}$. We consider the following switching control problem defined by

$$\sup_\alpha \mathbf{E}\Big[g_{\alpha_T}(X_T^\alpha) + \int_0^T \psi_{\alpha_s}(s, X_s^\alpha)ds + \sum_{0<\tau_k\le T} c_{\alpha_{\tau_k^-},\alpha_{\tau_k}}\Big], \tag{1.2}$$

where $(\tau_k)_k$ denotes the jump times of the control α. As discussed in [6], this type of stochastic control problem is typically encountered by an agent maximizing the production rentability of a given good by switching between m possible modes of production based on different commodities. A switch is penalized by a given cost function c and, since the agent is a large actor on the market, the chosen mode of production influences the dynamics of the corresponding commodities. Whenever the mode of production α does not influence the dynamics of the underlying X, [9] provides a probabilistic representation of the value process, via multidimensional reflected BSDE.As observed by Tang and Yong [24], the value function associated to this problem interprets on $[0, T]$ as the unique viscosity solution of a given coupled system of variational inequalities. The difficulty in the derivation of a BSDE representation for this type of problem is, first, the dependence of the solution in mode $i \in \mathcal{I}$ with respect to the global solution in all possible modes, and second, the dependence on the control of the drift and the volatility of X. As observed in [2], the unique viscosity solution to the corresponding system of variational inequalities interprets as the value function of a well suited stochastic target problem associated to a diffusion with jumps. Using entirely probabilistic arguments, the BSDE representation provided in this paper relies on this type of correspondence. In our approach, we let artificially the strategy jump randomly between the different modes of production. As in [17], this allows to retrieve in the jump component of a one-dimensional backward process, some information regarding the solution in the other modes of production. Indeed, let us introduce a pure jump process $(I_t)_{0\leq t\leq T}$ based on an independent random measure μ and construct the underlying process $(X_t^{I_t})_{0\leq t\leq T}$, whose dynamics are based on the random mode of production I_t at time t, according to equation (1.1). Let consider the following constrained BSDE associated to the two dimensional forward process (I, X^I) (called transmutation-diffusion process in [17]) and defined on $[0,T]$ by:

$$\left\{\begin{array}{rcl} Y_t & = & g_{I_T}(X_T^{I_T}) + \int_t^T \psi_{I_s}(s, X_s^{I_s})ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T\int_{\mathcal{I}} U_s(i)\mu(ds,di)\,, \\ U_t(i) & \geq & c_{i,I_{t-}}, \qquad d\mathbf{P}\otimes dt\otimes\lambda(di)\ \ a.e. \end{array}\right. \tag{1.3}$$

We prove in the last section of this paper that (1.3) has one unique minimal solution which indeed relates directly to the solution of the corresponding switching problem (1.2).

In order to unify our results with the one based on multidimensional reflected BSDE considered in [14] or [12], we extend this approach and introduce the notion of constrained BSDE with jumps whose solution (Y, Z, U, K) satisfies the general dynamics

$$Y_t \;=\; \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds,di), \tag{1.4}$$

a.s., for $0 \leq t \leq T$, as well as the constraint

$$h_i(t, Y_{t^-}, Z_t, U_t(i)) \;\geq\; 0, \qquad d\mathbf{P}\otimes dt\otimes\lambda(di)\ \ a.e.\,, \tag{1.5}$$

where f and h are given random Lipschitz functions, and h is non-increasing in its last variable. Through a penalization argument, we provide in Section 2 the existence of a unique minimal solution to the constrained BSDE with jumps (1.4)-(1.5). This new type of BSDE mainly extends and unifies the existing literature on BSDE in three interconnected directions:

- We generalize the notion of BSDE with constrained jumps considered in [15], letting the driver function f depend on U and considering general constraint function h depending on all the components of the solution.
- We add some jumps in the dynamics of constrained BSDE studied in [20] and let the coefficients depend on the jump component U.
- Via the addition of artificial jumps, a well chosen one-dimensional constrained BSDE with jumps allows to represent the solution of a multidimensional reflected BSDE, in the framework of [12] or [14].

Constrained BSDEs with jumps offer a natural unifying framework to represent these three distinct types of BSDE. We believe that the representation of a multidimensional obliquely reflected BSDE by a one-dimensional constrained BSDE with jumps is also numerically very promising. It offers in particular the opportunity for the extension of the numerical schemes in [3] in order to develop an entirely probabilistic algorithm for the resolution of Markovian switching problems, where the switching strategy modifies the dynamics of the underlying diffusion. Such type of algorithm could also solve high dimensional systems of variational inequalities, which relates directly to multidimensional BSDEs with oblique reflections, see [14] for more details. The algorithm as well as the Feynman Kac representation of general constrained BSDEs with jumps are presented in the separate paper [11].

Similarly to any argument of the paper, the proofs relating constrained BSDEs with jumps and BSDEs with oblique reflections only rely on probabilistic arguments and can be applied in a non-Markovian setting. Nevertheless, the class of possibly non-Markovian reflected BSDE studied in [12] or [14] does not allow for the consideration of switching problems where the dynamics of the underlying diffusion depends in a general manner on the current switching regime. Section 4 of the paper deals with this type of general non Markovian switching problem, typically of the form of (1.2) where the functions g, f and c are possibly random. We relate the value process of the optimal switching problem to a well chosen family of multidimensional reflected BSDEs. We finally link via a penalization procedure this family of reflected BSDEs with only one member of the class of one-dimensional constrained BSDE with jumps. Therefore, constrained BSDEs with jumps offer also a nice probabilistic representation for general switching problems, even in a non-Markovian framework.

We regroup in the Appendix some technical results on BSDE, mainly extensions of existing results, which are not the main focus of the paper but can present some interest in themselves: we provide a comparison and a monotonic limit theorem for constrained BSDEs with jumps, as well as viability and comparison properties for multidimensional constrained BSDEs.

Notations. Throughout this paper we are given a finite terminal time T and a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a d-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$, and a Poisson random measure μ on $\mathbb{R}_+ \times \mathcal{I}$, where $\mathcal{I} = \{1, \dots, m\}$, with intensity measure $\lambda(di)dt$ for some finite measure λ on $\mathcal{I}$ with $\lambda(i) > 0$ for all $i \in \mathcal{I}$. We set $\tilde{\mu}(dt, di) = \mu(dt, di) - \lambda(di)dt$ the compensated measure associated to μ. $\sigma(\mathcal{I})$ denotes the σ-algebra of subsets of $\mathcal{I}$. For $x = (x_1, \dots, x_\ell) \in \mathbb{R}^\ell$ with $\ell \in \mathbb{N}$, we set $|x| = \sqrt{|x_1|^2 + \cdots + |x_\ell|^2}$ the euclidean norm. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ (resp. $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$) the augmentation of the natural filtration generated by W and μ (resp. by W), and by $\mathcal{P}_{\mathbb{G}}$ (resp. $\mathcal{P}_{\mathbb{F}}$, $\mathfrak{P}_{\mathbb{G}}$, $\mathfrak{P}_{\mathbb{F}}$) the σ-algebra of $\mathbb{G}$-predictable (resp. $\mathbb{F}$-predictable $\mathbb{G}$-progressive, $\mathbb{F}$-progressive) subsets of $\Omega \times [0, T]$. We denote by $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$ (resp. $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$) the set of real-valued càd-làg $\mathbb{G}$-adapted (resp. continuous $\mathbb{F}$-adapted) processes $Y = (Y_t)_{0\leq t\leq T}$ such that

$$\|Y\|_{\mathcal{S}^{\mathbf{2}}} \;:=\; \left(\mathbf{E}\Big[\sup_{0\leq t\leq T} |Y_t|^2\Big]\right)^{\frac{1}{2}} \;<\; \infty.$$

$\mathbf{L^p(0,T)}$, $p \geq 1$, is the set of real-valued processes $\phi = (\phi_t)_{0 \leq t \leq T}$ such that

$$\|\phi\|_{\mathbf{L^p(0,T)}} \ := \ \Big(\mathbf{E}\Big[\int_0^T |\phi_t|^p dt\Big]\Big)^{\frac{1}{p}} \ < \ \infty,$$

and $\mathbf{L^p_{\mathbb{F}}(0,T)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(0,T)}$) is the subset of $\mathbf{L^p(0,T)}$ consisting of $\mathfrak{P}_{\mathbb{F}}$-measurable (resp. $\mathfrak{P}_{\mathbb{G}}$-measurable) processes.
$\mathbf{L^p_{\mathbb{F}}(W)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(W)}$), $p \geq 1$, is the set of $\mathbb{R}^d$-valued $\mathcal{P}_{\mathbb{F}}$-measurable (resp. $\mathcal{P}_{\mathbb{G}}$-measurable) processes $Z = (Z_t)_{0 \leq t \leq T} \in \mathbf{L^p_{\mathbb{F}}(0,T)}$ (resp. $\mathbf{L^p_{\mathbb{G}}(0,T)}$) .
$\mathbf{L^p}(\tilde{\mu})$, $p \geq 1$, is the set of $\mathcal{P} \otimes \sigma(\mathcal{I})$-measurable maps $U : \Omega \times [0,T] \times \mathcal{I} \to \mathbb{R}$ such that

$$\|U\|_{\mathbf{L^p}(\tilde{\mu})} \ := \ \bigg(\mathbf{E}\Big[\int_0^T \int_{\mathcal{I}} |U_t(i)|^p \lambda(di) dt\Big]\bigg)^{\frac{1}{p}} \ < \ \infty.$$

$\mathbf{A^2_{\mathbb{F}}}$ (resp. $\mathbf{A^2_{\mathbb{G}}}$) is the closed subset of $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ (resp. $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$) consisting of nondecreasing processes $K = (K_t)_{0 \leq t \leq T}$ with $K_0 = 0$.
Finaly, for $t \in [0,T]$, $\mathcal{T}_t$ denotes the set of $\mathbb{F}$-stopping times τ such that $\tau \in [t,T]$, $\mathbf{P}$-a.s.. For ease of notation, we omit in all the paper the dependence in $\omega \in \Omega$, whenever it is obvious.

2 Constrained Backward SDEs with jumps

This section is devoted to the presentation of constrained Backward SDEs with jumps, generalizing the framework considered in [15] or [20]. Namely:

- We allow the driver function to depend on the jump component of the backward process;
- We extend the class of possible constraint functions by letting them depend on all the components of the solution to the BSDE.

We adapt the arguments developed in [15] in order to derive existence and uniqueness of a minimal solution for this new type of BSDE. No major technical difficulty appears for the derivation of these results and, in order to simplify the readability of the paper, the required extensions for comparison and monotonic limit theorems are reported in the Appendix. From our viewpoint, the nice feature of such constrained BSDE relies in their relation with multidimensional reflected BSDE, developed in the next Section.

2.1 Formulation

A constrained BSDE with jumps is characterized by three objects:

- a terminal condition, i.e. a $\mathcal{G}_T$-measurable random variable ξ,
- a driver function, i.e. a map $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, which is $\mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m)$-measurable,

- a constraint function, i.e. a $\sigma(\mathcal{I}) \otimes \mathfrak{P}_{\mathbb{G}} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R})$-measurable map $h : \mathcal{I} \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that $h_i(\omega, t, y, z, .)$ is non-increasing for all $(i, \omega, t, y, z) \in \mathcal{I} \times \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d$.

Definition 2.1. *A solution to the corresponding constrained BSDE with jumps is a quadruple* $(Y, Z, U, K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds + K_T - K_t - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds, di), \quad (2.1)$$

for $0 \leq t \leq T$ *a.s., as well as the constraint*

$$h_i(t, Y_{t^-}, Z_t, U_t(i)) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(di) \ \ a.e. . \quad (2.2)$$

Furthermore, (Y, Z, U, K) is referred to as the minimal solution to (2.1)-(2.2) whenever, for any other solution $(\check{Y}, \check{Z}, \check{U}, \check{K})$ of (2.1)-(2.2), we have $Y \leq \check{Y}$ a.s. In this case, Y naturally interprets in the terminology of Peng [18] as the smallest supersolution to (2.1)-(2.2).

Remark 2.1. In the case where the driver function f does not depend on U and the constraint function h is of the form $h_i(u + c(t, y, z))$, observe that this BSDE exactly fits in the framework considered in [15]. Similarly, in the Brownian case (i.e. no jump component), this type of BSDEs has been studied in [20]. Therefore, our framework generalizes and unifies those considered in [15] and [20].

In order to derive existence and uniqueness of solution for this BSDE, we require the classical Lipschitz and linear growth conditions on the coefficients, as well as a control on the way the driver function depends on the jump component U of the BSDE. We regroup these conditions in the following assumption.

(H0)

(i) There exists a constant $k > 0$ such that the functions f and h satisfy $\mathbf{P}$-a.s. the uniform Lipschitz property:

$$\begin{aligned} |f(t, y, z, u) - f(t, y', z', u')| &\leq k|(y, z, u) - (y', z', u')| , \\ |h_i(t, y, z, u_i) - h_i(t, y', z', u_i')| &\leq k|(y, z, u_i) - (y', z', u_i')| , \end{aligned}$$

for all $\{i, t, (y, z, u), (y', z', u')\} \in \mathcal{I} \times [0,T] \times [\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m]^2$.

(ii) The coefficients ξ, f and h satisfy the following integrability condition

$$\mathbf{E}|\xi|^2 + \int_0^T \mathbf{E}|f(t, 0, 0, 0)|^2 dt + \sum_{i \in \mathcal{I}} \int_0^T \mathbf{E}|h_i(t, 0, 0, 0)|^2 dt < \infty . \quad (2.3)$$

(iii) There exist two constants $C_1 \geq C_2 > -1$ such that we can find a $\mathcal{P}_{\mathbb{G}} \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^m)$-measurable map $\gamma : \Omega \times [0,T] \times \mathcal{I} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m \to [C_2, C_1]$ satisfying

$$f(t, y, z, u) - f(t, y, z, u') \leq \int_{\mathcal{I}} (u_i - u_i')\gamma_t^{y,z,u,u'}(i)\lambda(di),$$

for all $(i, t, y, z, u, u') \in \mathcal{I} \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times [\mathbb{R}^m]^2$, $\mathbf{P}$-a.s..

Remark 2.2. Under Assumption **(H0)** (i) and (ii), existence and uniqueness of a solution (Y, Z, U, K) to the BSDE (2.1) with $K = 0$ follows from classical results on BSDEs with jumps, see Lemma 2.4 in [23]. In order to add the h-constraint (2.2), one needs as usual to relax the dynamics of Y by adding the non-decreasing process K in (2.1). In mathematical finance, the purpose of this new process K is to increase the super replication price Y of a contingent claim, under additional portfolio constraints. In order to find a minimal solution to the constrained BSDE (2.1)-(2.2), the nondecreasing property of h is crucial for stating comparison principles needed in the penalization approach. The case of upper-bounded jumps constraint, i.e. $h_i(., u) = c_i(.) - u$, corresponds to optimal switching problems and will be considered in Section 4.

Remark 2.3. Part (iii) of Assumption **(H0)** constrains the form of the dependence of the driver f with respect to the jump component of the BSDE. It is inspired by [22] and will ensure comparison results for BSDEs driven by this type of driver, as detailed in Section A.1 of the Appendix.

2.2 Existence, uniqueness and approximation by penalization

In this paragraph, we derive the existence of a unique minimal solution for the constrained BSDE with jumps (2.1)-(2.2). This result requires an extension of Peng's monotonic limit theorem [18] to the case of BSDE with jump, as well as the addition of an increasing component to the comparison results for BSDEs with jumps, derived by Royer [22].

The proof relies on a classical penalization argument and we introduce the following sequence of BSDEs with jumps

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n)ds + n\int_t^T \int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i))\lambda(di)ds \qquad (2.4) \\ &\quad - \int_t^T \langle Z_s^n, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i)\mu(ds, di), \quad 0 \le t \le T,\ n \in \mathbb{N}, \end{aligned}$$

where $h_i^-(.) := \max(-h_i(.), 0)$ is the negative part of the function h_i, $i \in \mathcal{I}$. Under Assumption **(H0)**, the Lipschitz property of the coefficients f and h ensures existence and uniqueness of a solution $(Y^n, Z^n, U^n) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ to (2.4), see Theorem 2.1 in [1].

In order to prove that the sequence $(Y_n)_{n\in\mathbb{N}}$ is convergent, we require a comparison theorem for reflected BSDEs with jumps, that we did not find in the literature. For sake of completeness and in order to simplify the reading of the paper, we report it in Proposition A.1, see Section A.1 in the Appendix. In order to deduce the convergence of the sequence $(Y_n)_{n\in\mathbb{N}}$, we shall require the following additional assumption :

(H1) There exists a quadruple $(\check{Y}, \check{Z}, \check{K}, \check{U}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ solution of (2.1)-(2.2).

This assumption, which may appear restrictive, is rather classical and we present in Section 3 a large class of cases where **(H1)** is satisfied, see also Remark 2.4 below.

With the help of Proposition A.1, we can now state comparison results for the sequence $(Y^n)_{n\in\mathbb{N}}$.

Lemma 2.1. *Under* **(H0)***, the sequence* $(Y^n)_{n\in\mathbb{N}}$ *is nondecreasing, and, for any quadruple* $(\check{Y},\check{Z},\check{U},\check{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying* (2.1)-(2.2)*, we have* $Y^n \leq \check{Y}$ *a.s.,* $n \in \mathbb{N}$*. Under the additional Assumption* **(H1)***, the sequence of processes* $(Y^n)_{n\in\mathbb{N}}$ *converges increasingly and in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0},\mathbf{T})$ *to a process* $Y \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$*.*

Proof. For $n \in \mathbb{N}$, let introduce the Lipschitz map $f^n := f + n\int_{\mathcal{I}} h^- d\lambda$. Since f satisfies **(H0)**(iii) and h is lipschitz and non-increasing, we deduce:

$$f^n(t,y,z,u) - f^n(t,y,z,u') \leq \int_{\mathcal{I}} \{(u_i - u'_i)\gamma_t^{y,z,u,u'}(i) + n(h_i^-(t,y,z,u_i) - h_i^-(t,y,z,u'_i))\}\lambda(di),$$
$$\leq \int_{\mathcal{I}} (u_i - u'_i)(\gamma_t^{y,z,u,u'}(i) + kn\mathbf{1}_{u_i \geq u'_i})\lambda(di), \qquad \mathbf{P}\text{- a.s}\ , \quad n \in \mathbb{N}\ ,$$

for any $(t,y,z,u,u') \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^m$. Thus, for any $n \in \mathbb{N}$, the coefficients f^n and f^{n+1} satisfy **(H0)** as well as $f^n \leq f^{n+1}$. We deduce from a simplified version of Proposition A.1 without the additional increasing process K, that the sequence $(Y^n)_{n\in\mathbb{N}}$ is non-decreasing.

Furthermore, for any quadruple $(\check{Y},\check{Z},\check{U},\check{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying (2.1)-(2.2), we obtain $Y^n \leq \check{Y}$ a.s., $n \in \mathbb{N}$, applying the same Proposition A.1 but with coefficients $f_1 = f_2 = f^n$ and $K^2 = \check{K}$. Therefore, under additional Assumption **(H1)**, the sequence $(Y^n)_{n\in\mathbb{N}}$ is nondecreasing and upper bounded, ensuring its monotonic and in $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0},\mathbf{T})$ convergence. □

We now turn to the convergence of the triplet $(Z^n,U^n,K^n)_{n\in\mathbb{N}}$ where, for any $n \in \mathbb{N}$, the nondecreasing process $K^n \in \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ is defined by

$$K_t^n \;\; = \;\; n\int_0^t \int_{\mathcal{I}} h_i^-(s,Y_s^n,Z_s^n,U_s^n(i))\lambda(di)ds, \quad 0 \leq t \leq T.$$

For this purpose, we shall make use of an extension of Peng's monotonic limit theorem [18] to BSDEs with jumps, provided in Proposition A.2 of the Appendix.

Theorem 2.1. *Under* **(H0)-(H1)***, there exists a unique minimal solution* $(Y,Z,U,K) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *to* (2.1)-(2.2)*, with* K *predictable. Furthermore* Y *is the increasing limit of* $(Y^n)_{n\in\mathbb{N}}$*, and*

$$\|Z^n - Z\|_{\mathbf{L^p(0,T)}} + \|U^n - U\|_{\mathbf{L^p}(\tilde{\mu})} \;\; \longrightarrow \;\; 0, \qquad 1 \leq p < 2\ ,$$

as n *goes to infinity. Moreover* K *is the weak limit of* $(K^n)_{n\in\mathbb{N}}$ *in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0},\mathbf{T})$ *(and hence predictable), and for any* $t \in [0,T]$*,* K_t *is the weak limit of* $(K_t^n)_{n\in\mathbb{N}}$ *in* $\mathbf{L}^{\mathbf{2}}(\mathbf{\Omega},\mathcal{F}_{\mathbf{t}},\mathbf{P})$*.*

Proof. Combining Lemma 2.1 and Proposition A.2, we derive directly the convergence of the sequence $(Y^n,Z^n,U^n,K^n)_{n\in\mathbb{N}}$ to (Y,Z,U,K), solution of (2.1) in the sense precised in the Theorem.

We now prove that (Y, Z, U, K) satisfy (2.2). From the previous convergence result, we derive in particular that $(Y^n, Z^n, U^n)_{n\in\mathbb{N}}$ converges in $\mathbf{L}^{\mathbf{1}}_{\mathbb{G}}(\mathbf{0},\mathbf{T}) \times \mathbf{L}^{\mathbf{1}}_{\mathbb{G}}(\mathbf{0},\mathbf{T}) \times \mathbf{L}^{\mathbf{1}}(\tilde{\mu})$ to (Y, Z, U). Since h is Lipschitz, we have

$$\frac{\mathbf{E}[K_n]}{n} = \mathbf{E}\left[\int_0^T \int_{\mathcal{I}} h_i^-(s, Y_s^n, Z_s^n, U_s^n(i))\lambda(di)ds\right] \to \mathbf{E}\left[\int_0^T \int_{\mathcal{I}} h_i^-(s, Y_s, Z_s, U_s(i))\lambda(di)ds\right] ,$$

as n goes to infinity. Combining this estimate with the uniform bound in $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$ of the sequence $(K^n)_{n\in\mathbb{N}}$ provided in (A.4), see the proof of Proposition A.2, we deduce that the constraint (2.2) is also satisfied.

We finaly prove the uniqueness. From the minimality condition, the uniqueness for the component Y of the solution is obvious. Suppose now that we have two solutions (Y, Z, U, K) and (Y, Z', U', K') with K and K' predictable. Then we have

$$\begin{aligned}\int_0^t [f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z'_s, U'_s)]ds + \int_0^t [Z'_s - Z_s]dW_s \\ + \int_0^t \int_{\mathcal{I}} [U'_s(i) - U_s(i)]\mu(di, ds) + K'_t - K_t \quad = \quad 0\ , \quad 0 \le t \le T. \end{aligned} \tag{2.5}$$

Since μ is a Poisson measure, and hence has unaccessible jumps, we get by taking the predictable projection in (2.5) that

$$\int_0^t [f(s, Y_s, Z_s, U_s) - f(s, Y_s, Z'_s, U'_s)]ds + \int_0^t [Z'_s - Z_s]dW_s + K'_t - K_t \quad = \quad 0\ , \tag{2.6}$$

for $0 \le t \le T$, and

$$\int_0^T \int_{\mathcal{I}} [U'_s(i) - U_s(i)]\mu(di, ds) \quad = \quad 0,$$

which gives $U' = U$. Identifying the finite variation and the Brownian parts in (2.6) we get

$$\int_0^T [Z'_s - Z_s]dW_s \quad = \quad 0,$$

which gives $Z = Z'$. The uniqueness of K follows then from (2.5). □

Remark 2.4. Observe that the purpose of Assumption **(H1)** is simply to ensure an upper bound in $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$ on the sequence of solutions $(Y^n)_{n\in\mathbb{N}}$ to the penalized BSDEs. If such an upper bound already exists, there is a unique minimal solution to (2.1)-(2.2), and therefore **(H1)** is automatically satisfied. Particular cases where Assumption **(H1)** is satisfied are for instance presented in Theorem 3.1 below. In a Markovian setting, sufficient conditions for this assumption are also provided in Remark 3.2 of [11].

3 Link with multi-dimensional reflected Backward SDEs

In this section, we prove that one-dimensional constrained BSDEs with jumps offer a nice alternative for the representation of solutions to multidimensional reflected BSDEs studied in [14] and [12]. This representation has practical implications, since, for example, it opens the door to the numerical resolution of multi-dimensional reflected BSDEs via the approximation of a single one-dimensional constrained BSDE with additional artificial jumps.

The arguments presented here are purely probabilistic and therefore apply in the non Markovian framework considered in [14]. Furthermore, the proofs require precise comparison results for reflected BSDEs based on viability properties that are reported in the Appendix for the convenience of the reader.

Recall that solving a general multidimensional reflected BSDE consists in finding m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^{\mathbf{2}}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^m$ satisfying, for all $i \in \mathcal{I}$,

$$\begin{cases} Y^i_t = \xi^i + \int_t^T \psi_i(s, Y^1_s, \ldots, Y^m_s, Z^i_s)ds - \int_t^T \langle Z^i_s, dW_s\rangle + K^i_T - K^i_t\,, \quad 0 \le t \le T\,, \\ Y^i_t \ge \max_{j\in A_i} h_{i,j}(t, Y^j_t)\,, \quad 0 \le t \le T\,, \\ \int_0^T [Y^i_t - \max_{j\in A_i}\{h_{i,j}(t, Y^j_t)\}]dK^i_t = 0\,, \end{cases} \tag{3.1}$$

where $\psi_i : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathbb{F}$-progressively measurable map, $\xi^i \in \mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P})$, A_i is a nonempty subset of $\mathcal{I} \setminus \{i\}$, and, for any $j \in A_i$, $h_{i,j} : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is a given $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R})$-measurable function. As detailed in Theorem 3.1 and Theorem 4.2 of [12], existence and uniqueness of a solution to (3.1) is ensured by the following assumption:

(H2)

(i) For any $i \in \mathcal{I}$ and $j \in A_i$, we have $\xi^i \ge h_{i,j}(T, \xi^j)$.

(ii) For any $i \in \mathcal{I}$, $\mathbf{E}|\xi^i|^2 + \mathbf{E}\int_0^T \sup_{y\in\mathbb{R}^m} |\psi_i(t, y, 0)|^2 \mathbf{1}_{\{y_i=0\}} dt < +\infty$, and ψ_i is Lipschitz continuous: there exists a constant $k_\psi \ge 0$ such that

$$|\psi_i(t, y, z) - \psi_i(t, y', z')| \le k_\psi(|y - y'| + |z - z'|)\,, \qquad \forall (i, y, z, y', z') \in \mathcal{I} \times [\mathbb{R} \times \mathbb{R}^d]^2\,.$$

(iii) For any $i \in \mathcal{I}$, and $j \ne i$, ψ_i is nondecreasing in its $(j+1)-$th variable i.e. for any $(t, y, y', z) \in \mathcal{I} \times [\mathbb{R}^m]^2 \times \mathbb{R}^d$ such that $y_k = y'_k$ for $k \ne j$ and $y_j \le y'_j$, we have

$$\psi_i(t, y, z) \quad \le \quad \psi_i(t, y', z) \quad \mathbf{P} - a.s.$$

(iv) For any $(i, t, y) \in \mathcal{I} \times [0,T] \times \mathbb{R}$ and $j \in A_i$, $h_{i,j}$ is continuous, $h_{i,j}(t, .)$ is a 1-Lipschitz increasing function satisfying $h_{i,j}(t, y) \le y$, $\mathbf{P}$-a.s. and we have $h_{i,j}(., 0) \in \mathbf{L^2(0, T)}$.

(v) For any $i \in \mathcal{I}$, $\xi^i \ge \max_{j\ne i}\{h_{i,j}(T, \xi^j)\}$.

Remark 3.1. Part (ii) and (iii) of Assumption **(H2)** are classical Lipschitz and monotonicity properties of the driver. Part (iv) ensures a tractable form for the domain of $\mathbb{R}^m$ where $(Y^i)_{i\in\mathcal{I}}$ lies, and (i) implies that the terminal condition is indeed in the domain. Recent results in [4] allow to relax the monotonicity condition (iii) for the case of constraint function h associated to switching problems.

Consider now the following constrained BSDE with jump: find a minimal quadruple $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0},\mathbf{T}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$\begin{aligned}\tilde{Y}_t &= \xi^{I_T} + \int_t^T \psi_{I_{s^-}}(s, \tilde{Y}_s + \tilde{U}_s(1)\mathbf{1}_{I_{s^-}\neq 1}, \ldots, \tilde{Y}_s + \tilde{U}_s(m)\mathbf{1}_{I_{s^-}\neq m}, \tilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t \\ &\qquad - \int_t^T \langle \tilde{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds,di), \quad 0 \leq t \leq T, \ a.s. \qquad (3.2)\end{aligned}$$

with

$$\mathbf{1}_{A_{I_{t^-}}}(i)\Big[\tilde{Y}_{t^-} - h_{I_{t^-},i}(t, \tilde{Y}_{t^-} + \tilde{U}_t(i))\Big] \ \geq \ 0, \qquad d\mathbf{P}\otimes dt \otimes \lambda(di) \ \ a.e.\ , \qquad (3.3)$$

where the process I is a pure jump process defined by

$$I_t \ = \ I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s^-})\mu(ds,di)\,.$$

Remark that, if $\mu = \sum_{n\geq 0}\delta_{(\kappa_n,L_n)}$, the process I is simply the pure jump process which coincides with L_n on each $[\kappa_n,\kappa_{n+1})$. This BSDE enters obviously into the class of constrained BSDEs with jumps of the form (2.1)-(2.2) studied above, with the following correspondence

$$\xi = \xi^{I_T}; f(t,y,z,u) = \psi_{I_{t^-}}(t,(y+u_i\mathbf{1}_{I_{t^-}\neq i})_{i\in\mathcal{I}}, z); h_i(t,y,z,v) = \{y - h_{I_{t^-},i}(t,y+v)\}\mathbf{1}_{i\in A_{I_{t^-}}}.$$

As detailed in the next theorem, **(H2)** implies that **(H0)-(H1)** holds for (3.2)-(3.3), and its minimal solution is directly related to the solution of the multidimensional reflected BSDE (3.1).

Theorem 3.1. *Let Assumption* **(H2)** *hold and* $(Y^i,Z^i,K^i)_{i\in\mathcal{I}}$ *be the unique solution of* (3.1)*. Then* **(H0)**-**(H1)** *are in force for the BSDE* (3.2)-(3.3)*, and the unique minimal solution* $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *to* (3.2)-(3.3) *(whose existence is given by Theorem 2.1) satisfies*

$$\tilde{Y}_t = Y_t^{I_t}\,, \quad \tilde{Z}_t = Z_t^{I_{t^-}}\,, \quad \tilde{U}_t = (Y_t^i - Y_{t^-}^{I_{t^-}})_{i\in\mathcal{I}}\,, \qquad 0 \leq t \leq T\,. \qquad (3.4)$$

Proof. The proof divides in 3 steps. First we prove the existence of a unique minimal solution to (3.2)-(3.3). Then, we introduce a sequence of penalized BSDEs converging to the solution of the multidimensional reflected BSDE (3.1). Finally, we prove that a corresponding sequence of penalized BSDEs with jumps, built via a relation of the form of (3.4), converges indeed to the solution of (3.2)-(3.3).

Step 1: *Existence and uniqueness of a minimal solution to* (3.2)-(3.3)*.*
In order to use Theorem 2.1, we need to verify that Assumptions **(H0)** and **(H1)** are satisfied in this context.

First, parts (i) and (ii) of Assumption **(H0)** are direct consequences of **(H2)**(ii). Fix any $(t,y,z,u,u') \in [0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^m$, and define $v^{(k)} \in \mathbb{R}^m$ by

$$v^{(k)} \ = \ (u'_1,\ldots,u'_{k-1},u_k,\ldots,u_m), \qquad 1 \leq k \leq m+1\,.$$

From the monotonicity assumption **(H2)**(iii) on the Lipschitz function ψ we get

$$\begin{aligned}
f(t,y,z,u)-f(t,y,z,u') &= \sum_{k=1}^{m} \psi_{I_{t-}}(t,(y+v_i^{(k)}\mathbf{1}_{I_{t-}\neq i})_{i\in\mathcal{I}},z)-\psi_{I_{t-}}(t,(y+v_i^{(k+1)}\mathbf{1}_{I_{t-}\neq i})_{i\in\mathcal{I}},z)\\
&\leq k_\psi \sum_{k=1}^{m-1}(u_k-u'_k)\mathbf{1}_{u_k\geq u'_k}\mathbf{1}_{k\neq I_{t-}}\ .
\end{aligned}$$

Taking $\gamma_t^{y,z,u,u'}(i)=\frac{k_\psi}{\lambda(i)}\mathbf{1}_{u_k\geq u'_k}\mathbf{1}_{i\neq I_{t-}}$ (which is well defined, since $\lambda(i)>0$ for any $i\in\mathcal{I}$), we get **(H0)**-(iii).

In order to prove that **(H1)** holds, one needs to verify the existence of a solution to (3.2)-(3.3). One directly computes from (3.1) that the candidate $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K})$ defined in (3.4) satisfies (3.2) as well as (3.3). Indeed, define $N_t:=\mu(\mathcal{I}\times[0,t])$ for $0\leq t\leq T$, the (random) number of stopping times κ_n associated to the random measure μ which satisfy $\kappa_n\in[0,t)$. Then, since Y is a solution to the reflected BSDE (3.1), we have

$$\begin{aligned}
Y_{\kappa_{N_T}}^{L_{N_T}} &= \xi^{L_{N_T}}+\int_{\kappa_{N_T}}^{T}\psi_{L_{N_T}}(s,(Y_s^{L_{N_T}}+U_s(i)\mathbf{1}_{i\neq L_{N_T}})_{i\in\mathcal{I}},Z_s^{L_{N_T}})ds\\
&\quad -\int_{\kappa_{N_T}}^{T}Z_s^{L_{N_t}}dW_s+K_T^{L_{N_T}}-K_{\kappa_{N_T}}^{L_{N_T}}\ .
\end{aligned}$$

Then, still using the equation (3.1) and identifying the jumps at κ_{N_T}, we get at κ_{N_T-1}:

$$\begin{aligned}
Y_{\kappa_{N_T-1}}^{L_{N_T-1}} &= Y_{\kappa_{N_T}}^{L_{N_T}}+\int_{\kappa_{N_T-1}}^{\kappa_{N_T}}\psi_{L_{N_T-1}}(s,(Y_s^{L_{N_T-1}}+U_s(i)\mathbf{1}_{i\neq L_{N_T-1}})_{i\in\mathcal{I}},Z_s^{L_{N_T-1}})ds\\
&\quad -\int_{\kappa_{N_t-1}}^{\kappa_{N_t}}Z_s^{L_{N_T-1}}dW_s+K_{\kappa_{N_T}}^{L_{N_T-1}}-K_{\kappa_{N_T-1}}^{L_{N_T-1}}+(Y_{\kappa_{N_T}}^{L_{N_T-1}}-Y_{\kappa_{N_T}}^{L_{N_T}})\\
&= \xi^{I_T}+\int_{\kappa_{N_T-1}}^{T}\psi_{I_{s-}}(s,Y_s^{I_s}+U_s(i)\mathbf{1}_{i\neq I_{s-}},Z_s^{I_{s-}})ds-\int_{\kappa_{N_T-1}}^{T}Z_s^{I_{s-}}dW_s\\
&\quad -\int_{\kappa_{N_T-1}}^{T}\int_{\mathcal{I}}U_s(i)\mu(di,ds)+K_T^{L_{N_T}}-K_{\kappa_{N_T}}^{L_{N_T}}+K_{\kappa_{N_T}}^{L_{N_T-1}}-K_{\kappa_{N_T-1}}^{L_{N_T-1}}
\end{aligned}$$

Repeating this procedure until time κ_{N_t+1}, we get

$$\begin{aligned}
Y_{\kappa_{N_t}+1}^{L_{N_t+1}} &= \xi^{I_T}+\int_{\kappa_{N_t+1}}^{T}\psi_{I_{s-}}(s,Y_s^{I_s}+U_s(i)\mathbf{1}_{i\neq I_{s-}},Z_s^{I_{s-}})ds-\int_{\kappa_{N_t+1}}^{T}Z_s^{I_{s-}}dW_s\\
&\quad -\int_{\kappa_{N_t+1}}^{T}\int_{\mathcal{I}}U_s(i)\mu(di,ds)+K_T^{L_{N_T}}-K_{\kappa_{N_T}}^{L_{N_T}}+K_{\kappa_{N_T}}^{L_{N_T-1}}-K_{\kappa_{N_T-1}}^{L_{N_T-1}}\\
&\quad +\ldots+K_{\kappa_{N_t+2}}^{L_{N_t+1}}-K_{\kappa_{N_t+1}}^{L_{N_t+1}}\ .
\end{aligned}$$

Combining this last equation with the equation satisfied by $Y^{L_{N_t}}$ between t and κ_{N_t+1} we get the existence of a square integrable increasing process $\tilde{K}$ such that $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K})$ satisfies equation (3.2). From the reflection constraint we directly derive that $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{K})$ satisfies the constraint (3.3).

Therefore **(H0)-(H1)** holds for (3.2)-(3.3) and the existence of a unique minimal solution follows from Theorem 2.1.

Step 2: *Penalization of the multidimensional BSDE* (3.1).
We now introduce the following sequence of multidimensional penalized BSDEs: for $n \in \mathbb{N}$, find m couples $(Y^{i,n}, Z^{i,n})_{i\in\mathcal{I}} \in (\mathcal{S}^{\mathbf{2}}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^m$ satisfying

$$Y^{i,n}_t = \xi^i + \int_t^T \psi^n_i(s, Y^{1,n}_s, \dots, Y^{m,n}_s, Z^n_s)ds - \int_t^T \langle Z^{i,n}_s, dW_s\rangle \ , \ \ 0 \le t \le T, \quad i \in \mathcal{I} \ , \ \ a.s. , \tag{3.5}$$

where the random map ψ^n is defined on $[0,T] \times \mathbb{R}^m \times [\mathbb{R}^d]^m$ by

$$\psi^n_i \ : \ (t, y, z) \mapsto \psi_i(t, y, z_i) + n \sum_{j\in A_i} [y_i - h_{i,j}(t, y_j)]^- \lambda(j) \ , \quad i \in \mathcal{I} \, .$$

For any $n \in \mathbb{N}$, the existence of a unique solution to (3.5) is given in the seminal paper [16] and we prove now that the sequence of solution to these BSDEs converges to the solution of the multidimensional reflected BSDE (3.1).

In order to prove that the sequence $(Y^{i,n})_{n\in\mathbb{N}}$ is nondecreasing and convergent for any $i \in \mathcal{I}$, we shall appeal to a multidimensional comparison theorem for reflected BSDEs presented in Section A.3 of the Appendix. First, since $\psi^i_n \le \psi^i_{n+1}$ for any $i \in \mathcal{I}$ and $n \in \mathbb{N}$, the comparison Theorem 2.1 in [13] implies that the sequence $(Y^{\cdot,n})_{n\in\mathbb{N}}$ is nondecreasing componentwise. Second, we compute from the Lipschitz property of ψ that

$$-2\langle y, \psi^n(t, y', z) - \psi^n(t, y', z')\rangle = -2\langle y, \psi(t, y', z) - \psi(t, y', z')\rangle \le |y|^2 + \sum_{i=1}^m |z_i - z'_i|^2 \ ,$$

$\mathbf{P}$-a.s., for any $\{t, (y, y'), (z, z')\} \in [0,T] \times [\mathbb{R}^+]^m \times \mathbb{R}^m \times [\mathbb{R}^{d\times m}]^2$ and $n \in \mathbb{N}$. Therefore, since $\psi^n(t, Y_t, Z_t) = \psi(t, Y_t, Z_t)$ for $t \in [0,T]$, we deduce from Proposition A.4 in the Appendix that

$$Y^{i,n}_t \le Y^i_t \ , \quad \text{for all} \quad (i, t, n) \in \mathcal{I} \times [0,T] \times \mathbb{N} \, . \tag{3.6}$$

By Peng's monotonic limit theorem, there exist

- $\hat{Y}^1, \dots, \hat{Y}^m$ $\mathbb{F}$-adapted càdlàg processes with $\mathbb{E}[\sup_{0\le t\le T} |\hat{Y}^i_t|^2] < \infty$ for all $i \in \mathcal{I}$,
- $\hat{Z}^1, \dots, \hat{Z}^m \in \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$,
- $\hat{K}^1, \dots, \hat{K}^m$ $\mathbb{F}$-adapted nondecreasing càdlàg processes with $\hat{K}^i_0 = 0$ and $\mathbb{E}[\sup_{0\le t\le T} |\hat{K}^i_T|^2] < \infty$ for all $i \in \mathcal{I}$,

such that $Y^{i,n} \uparrow \hat{Y}^i$ a.e., $Y^{i,n} \to \hat{Y}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, $Z^{i,n} \to \hat{Z}^i$ in $\mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W})$ weakly, $\hat{K}^i_T \to \hat{K}^i_T$ in $\mathbf{L}^{\mathbf{2}}(\mathbf{\Omega}, \mathcal{F}_{\mathbf{T}}, \mathbf{P})$ weakly and

$$\begin{cases} \hat{Y}^i_t = \xi^i + \int_t^T \psi_i(s, \hat{Y}^1_s, \dots, \hat{Y}^m_s, \hat{Z}^i_s)ds - \int_t^T \langle \hat{Z}^i_s, dW_s\rangle + \hat{K}^i_T - \hat{K}^i_t \, , \\ \hat{Y}^i_t \ge \max_{j\in A_i} h_{i,j}(t, \hat{Y}^j_t) \ , \quad 0 \le t \le T \ , \ i \in \mathcal{I} \, . \end{cases} \tag{3.7}$$

Recall that the inequality in (3.7) is satisfied since $\hat{K}^i_T$ is non-negative and converges weakly to $\hat{K}^i_T$ in $\mathbf{L^2(\Omega, \mathcal{F}_T, P)}$ and hence is bounded in $\mathbf{L^1(\Omega, \mathcal{F}_T, P)}$. Consider now the following RBSDEs whose solution exists according to Peng and Xu, see Theorem 2.1 in [19]:

$$\left\{\begin{array}{l} \tilde{Y}^i_t = \xi^i + \int_t^T \psi_i(s, \hat{Y}^1_s, \ldots, \hat{Y}^{i-1}_s, \tilde{Y}^i_s, \hat{Y}^{i+1}_s \ldots, \hat{Y}^m_s, \tilde{Z}^i_s)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad - \int_t^T \langle \tilde{Z}^i_s, dW_s\rangle + \tilde{K}^i_T - \tilde{K}^i_t \,, \\ \tilde{Y}^i_t \geq \max_{j\in A_i} h_{i,j}(t, \hat{Y}^j_t) \,, \quad 0 \leq t \leq T \,, \\ \int_0^T [\tilde{Y}^i_{t^-} - \max_{j\in A_i} h_{i,j}(t, \hat{Y}^j_{t^-})] d\tilde{K}^j_t = 0 \quad i \in \mathcal{I} \,. \end{array}\right. \tag{3.8}$$

We note that (3.7) and (3.8) have the same lower barrier. For any $i \in \mathcal{I}$, since $\tilde{Y}^i$ is the smallest ψ_i-supermartingale with lower barrier $\max_{j\in A_i} h(., \hat{Y}^j_.)$, we have from Theorem 2.1 in [19] that $\tilde{Y}^i \leq \hat{Y}^i$.

On the other hand, we know from **(H2)** (iii) that

$$\psi^n_i(s, \hat{Y}^1_s, \ldots, \hat{Y}^{i-1}_s, y, \hat{Y}^{i+1}_s, \ldots, \hat{Y}^m_s) \quad \geq \quad \psi^n_i(s, Y^{1,n}_s, \ldots, Y^{i-1,n}_s, y, Y^{i+1,n}_s, \ldots, Y^{m,n}_s) \,,$$

for all $(i, s, y, n) \in \mathcal{I} \times [0, T] \times \mathbb{R} \times \mathbb{N}$, $\mathbf{P}$-a.s.. For $i \in \mathcal{I}$, since $\tilde{Y}^i \geq \max_{j\in A_i} h_{i,j}(., Y^j_.)$, combining **(H2)** (iv) and a comparison theorem for one dimensional reflected BSDEs, we get $Y^{i,n} \leq \tilde{Y}^i$ fot any $n \in \mathcal{N}$, and, sending n to infinity, deduce $\hat{Y}^i \leq \tilde{Y}^i$. Therefore $\hat{Y} = \tilde{Y}$ and $(\hat{Y}, \hat{Z}, \hat{K})$ satisfies

$$\left\{\begin{array}{l} \hat{Y}^i_t = \xi^i + \int_t^T \psi_i(s, \hat{Y}_s, \hat{Z}^i_s)ds - \int_t^T \langle \hat{Z}^i_s, dW_s\rangle + \hat{K}^i_T - \hat{K}^i_t \,, \\ \hat{Y}^i_t \geq \max_{j\in A_i} h_{i,j}(t, \hat{Y}^j_t) \,, \quad 0 \leq t \leq T \,, \\ \int_0^T [\hat{Y}^i_{t^-} - \max_{j\in A_i} h_{i,j}(t, \hat{Y}^j_{t^-})] d\hat{K}^j_t = 0 \,, \quad i \in \mathcal{I} \,. \end{array}\right. \tag{3.9}$$

Notice that the minimality condition in (3.9) differs from the expected one in (3.1). Nevertheless, those two coincide whenever $\hat{Y}$ is continuous, property that we derive now.

Suppose on the contrary that $\Delta\hat{Y}^{i_1}_t \neq 0$ for some $(i_1, t) \in \mathcal{I} \times [0, T]$. Then from (3.9) we have $\Delta\hat{Y}^{i_1}_t = -\Delta\hat{K}^{i_1}_t < 0$, which further implies that

$$\hat{Y}^{i_1}_{t^-} = \max_{j\in A_i} h_{i_1,j}(t, \hat{Y}^j_{t^-}) = h_{i_1,i_2}(t, \hat{Y}^{i_2}_{t^-}) \,,$$

for some $i_2 \neq i_1$. Using the constraint satisfied by $\hat{Y}$, we get

$$h_{i_1,i_2}(t, \hat{Y}^{i_2}_{t^-}) = \hat{Y}^{i_1}_{t^-} > \hat{Y}^{i_1}_t \geq \max_{i\in A_{i_1}} h_{i_1,i}(t, \hat{Y}^i_t) \geq h_{i_1,i_2}(t, \hat{Y}^{i_2}_t).$$

Thus $\Delta\hat{Y}^{i_2}_t < 0$. Repeating this argument we get a finite cyclic sequence $(i_k)_{1\leq k\leq N}$ such that $i_N = i_i$ and

$$\hat{Y}^{i_{k-1}}_{t^-} \quad = \quad h_{i_{k-1},i_k}(t, \hat{Y}^{i_k}_{t^-}) \,, \quad 2 \leq k \leq N \,,$$

which contradicts **(H2)** (iv).

Step 3: *Link between solutions of BSDE* (3.1) *and BSDE* (3.2)-(3.3).
For $n \in \mathbb{N}$, define the process $(Y^{I,n}, Z^{I,n}, U^{I,n}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu})$ by

$$Y^{I,n}_t := Y^{I_t,n}_t \,, \quad Z^{I,n}_t := Z^{I_{t^-},n}_t \quad \text{and} \quad \tilde{U}^{I,n}_t := (Y^{i,n}_t - Y^{I,n}_{t^-})_{i\in\mathcal{I}} \,, \quad 0 \leq t \leq T. \tag{3.10}$$

In order to obtain (3.4), it remains to prove that $(Y^{I,n}, Z^{I,n}, U^{I,n})$ converges to $(\tilde{Y}, \tilde{Z}, \tilde{U})$.

Writing (3.5) between each successive stopping times associated to the random measure μ, we easily check that $(Y^{I,n}, Z^{I,n}, U^{I,n})$ is the unique solution of the following penalized BSDE

$$\begin{aligned} Y_t^{I,n} &= \xi^{I_T} + \int_t^T \psi_{I_{s^-}}(s, Y_s^{I,n} + U_s^{I,n}(1)\mathbf{1}_{I_{s^-}\neq 1}, \dots, Y_s^{I,n} + U_s^{I,n}(m)\mathbf{1}_{I_{s^-}\neq m}, Z_s^{I,n})ds \\ &- \int_t^T \langle Z_s^{I,n}, dW_s\rangle + n\int_t^T\int_{\mathcal{I}} h_i^-(s, Y_{s^-}^{I,n}, Z_s^{I,n}, U_s^{I,n}(i))\lambda(di)ds + \int_t^T\int_{\mathcal{I}} U_s^{I,n}(i)\mu(ds,di)\,, \end{aligned}$$

for $0 \le t \le T$. Since (3.6) holds, the sequence $(Y^{I,n})_{n\in\mathbb{N}}$ is bounded in $\mathcal{S}^2_{\mathbb{G}}$ and, using Theorem 2.1, we get

$$\|Y^{I,n} - \tilde{Y}\|_{\mathbf{L^2(0,T)}} + \|Z^{I,n} - \tilde{Z}\|_{\mathbf{L^p(0,T)}} + \|U^{I,n} - \tilde{U}\|_{\mathbf{L^p}(\tilde{\mu})} \longrightarrow 0\,, \quad p<2, \tag{3.11}$$

where we recall that $(\tilde{Y}, \tilde{Z}, \tilde{U})$ is the minimal solution to (3.2)-(3.3). Combining this result with (3.10) and Step 2. concludes the proof. □

Observe that the previous result offers an alternative BSDE representation for solutions of multidimensional reflected BSDEs. In particular, they allow for a new probabilistic representation of non-markovian switching problems presented in [12] or [14]. Nevertheless, BSDE with oblique reflections do not allow in general to represent solutions of optimal switching problems, where the switching strategy influences the dynamics of the underlying diffusion. The next Section is dedicated to the derivation of a probabilistic representation via constrained BSDE with jumps for this particular type of problems. Since all the results rely on probabilistic arguments, we choose to present them in a non-markovian framework.

4 BSDE representation for general switching problem

This section is devoted to the interpretation of non-Markovian switching problems in terms of solutions to BSDEs with constrained jumps. In particular, we consider useful practical cases where the current switching regime influences the dynamics of the underlying diffusion. For example, an electricity producer, who is also a large investor on the commodity market faces this type of control problem. To our knowledge, no BSDE representation has yet been established in this general framework. We first extend the results of [14] and relate the solution of a general non-Markovian switching problem with a well chosen family of multidimensional BSDE with oblique reflections. We finally link this family of BSDE with a single one-dimensional constrained BSDE with jumps leading to the announced representation property.

4.1 Non-Markovian optimal switching

Given the set $\mathcal{I} = \{1, \dots, m\}$ and the maturity T, an impulse strategy α consists in a sequence $\alpha := (\tau_k, \zeta_k)_{k\ge1}$, where $(\tau_k)_{k\ge1}$ is an increasing sequence of $\mathbb{F}$-stoppping times smaller than T, and ζ_i are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathcal{I}$. To a strategy

$\alpha = (\tau_k, \zeta_k)_{k\geq 1}$ and an initial regime i_0, we naturally associate the state process $(\alpha_t)_{t\leq T}$ defined by

$$\alpha_t \quad := \quad \sum_{k\geq 0} \zeta_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t) \,, \quad 0 \leq t \leq T \,,$$

with $\tau_0 = 0$ and $\zeta_0 = i_0$. We denote by $\mathcal{A}$ the set of admissible strategies and $\mathcal{A}_{t,i}$ the subset of strategies starting from state $i \in \mathcal{I}$ at time $t \in [0, T]$. Given a strategy $\alpha \in \mathcal{A}$ and an initial condition $(i_0, X_0) \in \mathcal{I} \times \mathbb{R}^d$, we define the controlled process X^α by

$$X_t^\alpha = X_0 + \int_0^t b_{\alpha_s}(s, X_s^\alpha) ds + \int_0^t \sigma_{\alpha_s}(s, X_s^\alpha) dW_s \,, \tag{4.1}$$

and we consider the total profit at horizon T defined by

$$J(\alpha) \quad := \quad \mathbf{E}\left[g_{\alpha_T}(X_T^\alpha) + \int_0^T \psi_{\alpha_s}(s, X_s^\alpha) ds - \sum_{0<\tau_k\leq T} c_{\zeta_{k-1},\zeta_k}(\tau_k) \right] \,, \tag{4.2}$$

where $(b, \sigma, \psi) : \ \Omega \times [0, T] \times \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^{d\times d} \times \mathbb{R}$ are $\mathfrak{P}_{\mathbb{F}} \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions, $g : \ \Omega \times \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{F}_T \otimes \sigma(\mathcal{I}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and $c : \ \Omega \times [0, T] \times \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ is a $\mathfrak{P}_{\mathbb{F}} \otimes \sigma(\mathcal{I}) \otimes \sigma(\mathcal{I})$-measurable function.

Given the starting initial production mode i_0, the switching problem consists in finding a strategy $\alpha^* \in \mathcal{A}_{0,i_0}$ such that

$$J(\alpha^*) = \sup_{\alpha\in\mathcal{A}_{0,i_0}} J(\alpha) \,. \tag{4.3}$$

Such a strategy is called optimal and we shall work under the following assumption.

(H3)

(i) b and σ satisfy the Lipschitz property: there exists a constant k such that $\mathbf{P}-$a.s.

$$|b_i(\omega, t, x) - b_i(\omega, t, x')| + |\sigma_i(\omega, t, x) - \sigma_i(\omega, t, x')| \quad \leq \quad k\, |x - x'| \,, \quad i \in \mathcal{I} \,,$$

for all $(\omega, t, x, x') \in \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

(ii) The terminal condition g satisfies the following structural condition:

$$g_i(\omega, x) \geq \max_{j\in\mathcal{I}} \{g_j(\omega, x) - c_{i,j}(\omega, T)\} \,, \quad \text{for all } (i, \omega, x) \in \mathcal{I} \times \Omega \times \mathbb{R}^d \,.$$

(iii) The functions $|g|$ and $|\psi|$ are uniformly upper-bounded by the constants $\bar{g}$ and $\bar{\psi}$.

(iv) The cost function c is lower-bounded, i.e. there exists a constant $\bar{c} > 0$ such that

$$c_{i,j}(.) \geq \bar{c} \,, \qquad \text{for all } \ (i, j) \in \mathcal{I} \times \mathcal{I} \quad \text{such that} \quad j \neq i \,.$$

Furthermore $c_{i,j} \in \mathcal{S}^2_{\mathbb{F}}$ for all $i, j \in \mathcal{I}$, and we have

$$c_{i,l} \ < \ c_{i,j} + c_{j,l} \,, \quad \text{for all} \quad (i, j, l) \in [\mathcal{I}]^3 \quad \text{such that} \quad j \neq i \,,\ j \neq l \,.$$

Remark 4.1. Part (i) of Assumption **(H3)** provides the existence of a unique solution to (4.1). Part (ii) ensures the non-optimality of a switching at maturity, (iv) makes indirect switching strategy irrelevant and (iii)-(iv) ensures that the problem is well posed.

Let define the set of finite strategies $\mathcal{D} := \cup_{i\in\mathcal{I}}\mathcal{D}_{0,i}$, with

$$\mathcal{D}_{0,i} \quad := \quad \{\alpha = (\tau_k, \zeta_k)_{k\geq 1} \in \mathcal{A}_{0,i} \mid \mathbf{P}(\tau_k < T, \ \ \forall k \geq 1) = 0\} \ , \quad i \in \mathcal{I} \ .$$

Let first observe the following property:

Proposition 4.1. *Under **(H3)**, the supremum of J over $\mathcal{A}_{0,i}$ coincides with the one of J over $\mathcal{D}_{0,i}$, that is*

$$\sup_{\alpha\in\mathcal{A}_{0,i}} J(\alpha) \quad = \quad \sup_{\alpha\in\mathcal{D}_{0,i}} J(\alpha) \,, \qquad i \in \mathcal{I} \,. \tag{4.4}$$

Proof. Fix $i \in \mathcal{I}$ and consider a strategy $\alpha = (\tau_k, \zeta_k)_{k\geq 0} \in \mathcal{A}_{0,i}$. Suppose that $\alpha \notin \mathcal{D}_{0,i}$ and introduce $B := \{\omega \in \Omega \mid \tau_n(\omega) < T, \ \forall n \in \mathbb{N}^*\}$ so that $\mathbf{P}(B) > 0$. We derive from **(H3)** (iii) and (iv) that

$$J(\alpha) \quad \leq \quad \bar{g} + T\bar{\psi} - \mathbf{E}\left[\mathbf{1}_B \sum_{0<\tau_k\leq T} \bar{c}\right] \ = \ -\infty \,,$$

and directly deduce (4.4). □

Remark 4.2. Using a change a probability argument, Hu and Tang [14] provide a BSDE representation of the solution to (4.3) in terms of reflected multidimensional BSDE. Unfortunately their representation restricts to particular cases, where the volatility function σ does not depend on the current switching mode I. As detailed below, the use of constrained BSDEs with jumps allows to get rid of this limiting assumption.

4.2 Family of reflected BSDEs and optimal switching

In the spirit of [8], we consider in this section a family of reflected BSDE. Any element of the family will be characterized by a couple (ν, η) with ν a stopping time valued in $[0, T]$ and η an $\mathcal{F}_\nu$-measurable random variable taking values in $\mathbb{R}^d$. The set of such couple (ν, η) will be denoted $\mathcal{K}$.

For any parameter $(\nu, \eta) \in \mathcal{K}$, we consider the following reflected BSDE

$$\begin{cases} (Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}} \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^2_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^2_{\mathbb{F}})^m, \\ Y^{\nu,i,\eta}_t = g_i(X^{\nu,i,\eta}_T) + \int_{t\wedge\nu}^T \psi_i(s, X^{\nu,i,\eta}_s)\mathbf{1}_{s\geq\nu}ds - \int_{t\wedge\nu}^T \langle Z^{\nu,i,\eta}_s, dW_s\rangle + K^{\nu,i,\eta}_T - K^{\nu,i,\eta}_t, \\ Y^{\nu,i,\eta}_t \geq \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t - c_{i,j}(t)\}, \qquad 0 \leq t \leq T \ , \\ \int_0^T [Y^{\nu,i,\eta}_t - \max_{j\in\mathcal{I}}\{Y^{\nu,j,\eta}_t - c_{i,j}(t)\}]dK^{\nu,i,\eta}_t \ = \ 0, \end{cases} \tag{4.5}$$

where $X^{\nu,i,\eta}$ is the diffusion defined by

$$X^{\nu,i,\eta}_t \quad = \quad \eta\mathbf{1}_{t\geq\nu} + \int_\nu^t b_i(s, X^{\nu,i,\eta}_s)ds + \int_\nu^t \sigma_i(s, X^{\nu,i,\eta}_s)dW_s \,, \qquad \forall t \geq 0. \tag{4.6}$$

Under **(H3)**, Theorem 4.2 in [12] provides the existence of a unique solution to (4.5), for any parameter $(\nu, \eta) \in \mathcal{K}$, and we denote by $\mathcal{O}^{\nu,\cdot,\eta}$ the corresponding frontier for the domain of $Y^{\nu,\cdot,\eta}$ defined by

$$\mathcal{O}_t^{\nu,i,\eta} \; := \; \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta} - c_{i,j}(t)\}\,, \qquad i \in \mathcal{I}\,,\;\; t \leq T. \tag{4.7}$$

We aim at relating the solutions of this class of reflected BSDEs to the solution of the optimal non Markovian switching problem presented in (4.2)-(4.3). The next proposition provides a stability property, a Snell envelope representation and a global estimate on the family of processes $(Y^{\nu,\cdot,\eta})_{(\nu,\eta)\in\mathcal{K}}$.

Proposition 4.2. *If **(H3)** is in force, the following holds.*

(i) For any η, ν and ν' such that $(\eta, \nu) \in \mathcal{K}$, $(\eta, \nu') \in \mathcal{K}$ and $\nu \leq \nu'$, we have

$$Y_t^{\nu,i,\eta} \;\; = \;\; Y_t^{\nu',i,X_{\nu'}^{\nu,i,\eta}}\,, \qquad \mathbf{P} - a.s.\,, \qquad t \geq \nu'\,, \qquad i \in \mathcal{I}\,.$$

(ii) For all $i \in \mathcal{I}$, $(\eta, \nu) \in \mathcal{K}$ and $t \geq \nu$, we have the following representation

$$Y_t^{\nu,i,\eta} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbf{E}\left[\int_{t\wedge\nu}^{\tau} \psi_i(s, X_s^{\nu,i,\eta})ds + \mathcal{O}_\tau^{\nu,i,\eta}\mathbf{1}_{\tau<T} + g_i(X_T^{\nu,i,\eta})\mathbf{1}_{\tau=T}\right|\left.\mathcal{F}_t\right]\,. \tag{4.8}$$

(iii) There exists a constant $\bar{y}$ such that

$$\sup_{(t,i,\nu,\eta)\in[0,T]\times\mathcal{I}\times\mathcal{K}} |Y_t^{\nu,i,\eta}| \;\; \leq \;\; \bar{y}\,, \quad \mathbf{P} - a.s. \tag{4.9}$$

Proof. We prove each assertion separately.
(i) Fix $i \in \mathcal{I}$ and η, ν, ν' as required. Notice first that $X^{\nu,i,\eta}$ and $X^{\nu',i,X_{\nu'}^{\nu,\zeta,\eta}}$ solve the same SDE on $[\nu', T]$, namely

$$X_{\nu'} = X_{\nu'}^{\nu,i,\eta} \quad \text{ and } \quad dX_t = b_i(t, X_t)dt + \sigma_i(t, X_t)dW_t\,, \quad \text{ for } t \geq \nu'. \tag{4.10}$$

Under **(H3)** (i), equation (4.10) admits a unique solution and we have $X^{\nu',i,X_{\nu'}^{\nu,i,\eta}} = X^{\nu,i,\eta}$ on $[\nu', T]$. We deduce that $(Y^{\nu',i,X_{\nu'}^{\nu,i,\eta}}, Z^{\nu',i,X_{\nu'}^{\nu,i,\eta}}, K^{\nu',i,X_{\nu'}^{\nu,i,\eta}})_{i\in\mathcal{I}}$ satisfies the same BSDE as $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}}$ on $[\nu', T]$. Under **(H3)**, Theorem 4.2 in [12] provides uniqueness of solution to this BSDE and concludes the argumentation.

(ii) Fix $i \in \mathcal{I}$ and $(\eta, \nu) \in \mathcal{K}$. Regarding of (4.5), $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})$ interprets as the solution of a one-dimensional reflected BSDE with single barrier $\mathcal{O}^{\nu,i,\eta}$. We deduce from Proposition 2.3 in [10] that $Y^{\nu,i,\eta}$ admits the Snell envelope representation (4.8).

(iii) Fix $(\nu, \eta) \in \mathcal{K}$. We know from the proof of Theorem 2.1 in [12] that $(Y^{\nu,\cdot,\eta,n}, Z^{\nu,\cdot,\eta,n}, K^{\nu,\cdot,\eta,n})_{n\in\mathbb{N}}$ converges in $\mathcal{S}^\mathbf{2}_\mathbb{F}\times\mathbf{L}^\mathbf{2}_\mathbb{F}(\mathbf{W})\times\mathcal{S}^\mathbf{2}_\mathbb{F}$ to $(Y^{\nu,\cdot,\eta}, Z^{\nu,\cdot,\eta}, K^{\nu,\cdot,\eta})$, where the sequence $(Y^{\nu,\cdot,\eta,n}, Z^{\nu,\cdot,\eta,n}, K^{\nu,\cdot,\eta,n})_{n\in\mathbb{N}}$ is defined recursively by

$$Y_t^{\nu,i,\eta,0} \;\; = \;\; g_i(X_T^{\nu,\eta}) + \int_{t\wedge\nu}^{T} \psi_i(s, X_s^{\nu,i,\eta})ds - \int_{t\wedge\nu}^{T} \langle Z_s^{\nu,i,\eta,0}, dW_s\rangle \quad \text{ and } \quad K_t^{\nu,i,\eta,0} \;\; = \;\; 0\,, \;\; i \in \mathcal{I}\,,$$

and, for $n \geq 1$, by

$$\left\{\begin{array}{l} Y_t^{\nu,i,\eta,n} = g_i(X_T^{\nu,i,\eta}) + \int_{t\wedge\nu}^T \psi_i(s, X_s^{\nu,i,\zeta})ds - \int_{t\wedge\nu}^T \langle Z_t^{\nu,i,\eta,n}, dW_s + \rangle K_T^{\nu,i,\eta,n} - K_t^{\nu,i,\eta,n}, \\ Y_t^{\nu,i,\eta,n} \geq \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta,n-1} - c_{i,j}(t)\}, \quad 0 \leq t \leq T\,, \\ \int_0^T [Y_t^{\nu,i,\eta,n} - \max_{j\in\mathcal{I}}\{Y_t^{\nu,j,\eta,n-1} - c_{i,j}(t)\}]dK_t^{\nu,i,\eta,n} \;=\; 0\,, \end{array}\right. \tag{4.11}$$

for $i \in \mathcal{I}$. In order to derive (4.9), it thus suffices to prove by induction on n that

$$|Y_t^{\nu,i,\eta,n}| \leq (T - t + 1)\max\{\bar{\psi}, \bar{g}\}\,, \quad \mathbf{P} - a.s.\,, \quad i \in \mathcal{I}\,,\ \ 0 \leq t \leq T\,,\ \ n \in \mathbb{N}\,.$$

First, rewriting $Y^{\nu,\cdot,\eta,0}$ as a conditional expectation, we directly get

$$|Y_t^{\nu,i,\eta,0}| \leq (T - t)\bar{\psi} + \bar{g} \leq (T - t + 1)\max\{\bar{\psi}, \bar{g}\}\,, \qquad 0 \leq t \leq T\,,\ \ i \in \mathcal{I}\,.$$

Fix $n \in \mathbb{N}$ and suppose the result is true for $Y^{\nu,\cdot,\eta,n}$. For $i \in \mathcal{I}$, using the representation of $Y^{\nu,i,\eta,n+1}$ as a Snell envelope given by Proposition 2.3 in [10], we derive

$$Y_t^{\nu,i,\eta,n+1} \;=\; \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbf{E}\left[\int_t^\tau \psi_i(s, X_s^{\nu,i,\eta})\mathbf{1}_{s\geq\nu}ds + \mathcal{O}_\tau^{\nu,i,\eta,n+1}\mathbf{1}_{\tau<T} + g_i(X_T^{\nu,i,\eta})\mathbf{1}_{\tau=T}\right|\mathcal{F}_t\bigg]\,,$$

where $\mathcal{O}_s^{\nu,i,\eta,n+1} := \max_{j\in\mathcal{I}}\{Y_s^{\nu,j,\eta,n} - c_{i,j}(s)\}$, for $s \leq T$. Combining this representation with Assumption **(H3)**(iii)-(iv) as well as the recursive estimate, we get

$$\begin{aligned} Y_t^{\nu,i,\eta,n+1} \;&\leq\; \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbf{E}\left[(\tau - t)\bar{\psi} + (T - \tau + 1)\max\{\bar{g}, \bar{\psi}\}\mathbf{1}_{\tau<T} + \bar{g}\mathbf{1}_{\tau=T}\,\middle|\,\mathcal{F}_t\right] \\ &\leq\; (T - t + 1)\max\{\bar{\psi}, \bar{g}\}\,, \qquad 0 \leq t \leq T\,,\ \ i \in \mathcal{I}\,. \end{aligned}$$

By induction and arbitrariness of $(\nu, \eta) \in \mathcal{K}$, we deduce (4.9). □

For any $(\nu, \eta) \in \mathcal{K}$ and any $\mathcal{I}$-valued random variable ζ, we naturally introduce the processes $Y^{\nu,\zeta,\eta}$ and $\mathcal{O}^{\nu,\zeta,\eta}$ defined by

$$Y_t^{\nu,\zeta,\eta} \;:=\; \sum_{i\in\mathcal{I}} Y_t^{\nu,i,\eta}\mathbf{1}_{\zeta=i} \quad \text{and} \quad \mathcal{O}_t^{\nu,\zeta,\eta} \;:=\; \sum_{i\in\mathcal{I}} \mathcal{O}_t^{\nu,i,\eta}\mathbf{1}_{\zeta=i}. \tag{4.12}$$

We are now able to state a representation of the optimal solution to the switching problem (4.2) using the family of multidimensional reflected BSDEs $(Y^{\nu,\cdot,\eta}, Z^{\nu,\cdot,\eta}, K^{\nu,\cdot,\eta})_{(\nu,\eta)\in\mathcal{K}}$ given by (4.5).

Proposition 4.3. *Let $\alpha^* = (\tau_n^*, \zeta_n^*)_{n\geq0}$ be the strategy given by $(\tau_0^*, \zeta_0^*) = (0, i_0)$ and defined recursively for $n \geq 1$ by*

$$\tau_n^* \quad := \quad \inf\left\{s \in [\tau_{n-1}^*, T]\ ;\ Y_s^{\tau_{n-1}^*,\zeta_{n-1}^*,X_{\tau_{n-1}^*}^*} = \mathcal{O}_s^{\tau_{n-1}^*,\zeta_{n-1}^*,X_{\tau_{n-1}^*}^*}\right\}\,, \tag{4.13}$$

$$\zeta_n^* \quad is \quad s.t. \quad \mathcal{O}_{\tau_n^*}^{\tau_{n-1}^*,\zeta_{n-1}^*,X_{\tau_{n-1}^*}^*} \;=\; Y_{\tau_n^*}^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*} - c_{\zeta_{n-1}^*,\zeta_n^*}(\tau_n^*)\,, \tag{4.14}$$

with X^* the diffusion defined by

$$X_t^* = \quad x_0 + \sum_{n\geq1}\int_{\tau_{n-1}^*}^{\tau_n^*} b_{\zeta_{n-1}^*}(s, X_s^*)\mathbf{1}_{s\leq t}ds + \sum_{n\geq1}\int_{\tau_{n-1}^*}^{\tau_n^*} \sigma_{\zeta_{n-1}^*}(s, X_s^*)\mathbf{1}_{s\leq t}dW_s, \quad t \geq 0. \tag{4.15}$$

Under ***(H3)****, the strategy* α^* *is optimal for the switching problem* (4.2) *and we have*

$$Y_0^{0,i_0,x_0} \;=\; J(\alpha^*). \tag{4.16}$$

Proof. The proof is performed in two steps.

Step 1. *The strategy* $\alpha^* \in \mathcal{D}_{0,i_0}$ *and satisfies* $Y_0^{0,i_0,x_0} = J(\alpha^*)$.
The representation (4.8) in Proposition 4.2 rewrites

$$Y_0^{0,i_0,x_0} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0} \mathbf{E}\left[\int_0^{\tau} \psi_{i_0}(s, X_s^{0,i_0,x_0})ds + \mathcal{O}_{\tau}^{0,i_0,x_0}\mathbf{1}_{\tau<T} + g_{i_0}(X_T^{0,i_0,x_0})\mathbf{1}_{\tau=T}\right] . \tag{4.17}$$

Since the boundary $\mathcal{O}^{0,i_0,x_0}$ is continuous, the stopping time τ_1^* is optimal for (4.17) (see the proof of Proposition 2.3 in [10]) and we get

$$\begin{aligned}
Y_0^{0,i_0,x_0} &= \mathbf{E}\left[\int_0^{\tau_1^*} \psi_{i_0}(s, X_s^{0,i_0,x_0})ds + \mathcal{O}_{\tau_1^*}^{0,i_0,x_0}\mathbf{1}_{\tau_1^*<T} + g_{i_0}(X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1^*=T}\right]\\
&= \mathbf{E}\left[\int_0^{\tau_1^*} \psi_{\zeta_0^*}(s, X_s^*)ds + \left(Y_{\tau_1^*}^{\tau_1^*,\zeta_1^*,X_{\tau_1^*}^*} - c_{\zeta_0^*,\zeta_1^*}(\tau_1^*)\right)\mathbf{1}_{\tau_1^*<T} + g_{\zeta_0^*}(X_T^*)\,\mathbf{1}_{\tau_1^*=T}\right] ,
\end{aligned}$$

where the last equality follows from the definitions of ζ_1^* and X^* as well as Part (i) of Proposition 4.2. When $\tau_1^* < T$, using the Snell envelope representation of $Y_{\tau_1^*}^{\tau_1^*,\zeta_1^*,X_{\tau_1^*}^*}$ given by (4.2), we deduce recursively that

$$\begin{aligned}
Y_0^{0,i_0,x_0} = \mathbf{E}\Bigg[&\sum_{k=1}^{n}\int_{\tau_{k-1}^*}^{\tau_k^*} \psi_{\zeta_k^*}(s, X_s^*)ds + Y_{\tau_n^*}^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*}\mathbf{1}_{\tau_n^*<T} \\
&-\sum_{k=1}^{n} c_{\zeta_{k-1}^*,\zeta_k^*}(\tau_k^*)\mathbf{1}_{\tau_k^*<T} + \sum_{k=1}^{n} g_{\zeta_{k-1}^*}(X_T^*)\mathbf{1}_{\tau_{k-1}^*<\tau_k^*=T}\Bigg] , \quad n\in\mathbb{N}^* .
\end{aligned} \tag{4.18}$$

We now prove $\alpha^* \in \mathcal{D}_{0,i_0}$ and assume on the contrary that $p := \mathbf{P}(\tau_n^* < T,\ \forall n \in \mathbb{N}) > 0$. Combining **(H3)**(iii)-(iv), (4.9) and (4.18), we derive

$$\begin{aligned}
Y_0^{0,i_0,x_0} &\leq \bar{\psi}T + \mathbf{E}\left[\sup_{s\leq T}\left|Y_s^{\tau_n^*,\zeta_n^*,X_{\tau_n^*}^*}\right|\right] - n\bar{c}\,\mathbf{P}(\tau_k^* < T\ ,\ \forall k \geq 0) + \bar{g}\\
&\leq \bar{\psi}T + \bar{y} - n\bar{c}p + \bar{g}\,, \qquad n\in\mathbb{N}^* .
\end{aligned}$$

Sending n to infinity in the previous expression leads to $Y_0^{0,i_0,x_0} = -\infty$ which contradicts $Y^{0,i_0,x_0} \in \mathcal{S}_{\mathbb{F}}^2$. Therefore $\mathbf{P}(\tau_k^* < T\ ,\ \forall k \geq 0) = 0$ and $\alpha^* \in \mathcal{D}_{0,i_0}$. Finally, taking the limit as $n \to \infty$ in (4.18) leads to (4.16).

Step 2. *The strategy* α^* *is optimal.*
According to Proposition 4.1, it suffices to consider finite strategies and we pick any $\alpha = (\tau_n, \zeta_n)_{n\geq 0} \in \mathcal{D}_{0,i_0}$. Since τ_1^* is optimal, we deduce from parts (i) and (ii) of Proposition

4.2 that

$$\begin{aligned} Y_0^{0,i_0,x_0} &\geq \mathbf{E}\left[\int_0^{\tau_1}\psi_{i_0}(s,X_s^{0,i_0,x_0})ds+\mathcal{O}_{\tau_1}^{0,i_0,x_0}\mathbf{1}_{\tau_1<T}+g_{i_0}(X_T^{0,i_0,x_0})\mathbf{1}_{\tau_1=T}\right] \\ &\geq \mathbf{E}\left[\int_0^{\tau_1}\psi_{\zeta_0}(s,X_s^{\alpha})ds+\left(Y_{\tau_1}^{\tau_1,\zeta_1,X_{\tau_1}^{\alpha}}-c_{\zeta_0,\zeta_1}(\tau_1)\right)\mathbf{1}_{\tau_1<T}+g_{i_0}(X_T^{\alpha})\mathbf{1}_{\tau_1=T}\right]. \end{aligned}$$

Proceeding exactly as in step 1, an induction argument leads to

$$\begin{aligned} Y_0^{0,i_0,x_0} &\geq \mathbf{E}\Bigg[\int_0^{\tau_n}\psi_{\alpha_s}(X_s^{\alpha})ds+Y_{\tau_n}^{\tau_n,\zeta_n,X_{\tau_n}^{\alpha}}\mathbf{1}_{\tau_n<T} \\ &\qquad -\sum_{k=1}^{n}c_{\zeta_{k-1},\zeta_k}(\tau_k)\mathbf{1}_{\tau_k<T}+\sum_{k=1}^{n}g_{\zeta_{k-1}^*}(X_T^{\alpha})\mathbf{1}_{\tau_{k-1}^*<\tau_k^*=T}\Bigg]\,, \quad n\in\mathbb{N}^*\,. \end{aligned}$$

Sending n to infinity, since the strategy α is finite and **(H3)**(ii) is satisfied, we get

$$Y_0^{0,i_0,x_0} \;\geq\; \mathbf{E}\left[\int_0^{T}\psi_{\alpha_s}(X_s^{\alpha})ds-\sum_{k\geq 1}c_{\zeta_{k-1},\zeta_k}(\tau_k)\mathbf{1}_{\tau_k<T}+g_{\alpha_T}(X_T^{\alpha})\right] \;=\; J(\alpha)\,.$$

The arbitrariness of $\alpha\in\mathcal{D}_{0,i_0}$ concludes the proof. □

4.3 Constrained BSDE with jumps and optimal switching

In the previous paragraph, we observed that a large family of multidimensional reflected BSDE is necessary to represent the solution of a switching problem, whenever the switching strategy modifies the dynamics of the underlying diffusion. We prove here that a unique one-dimensional constrained BSDE with jumps allows for the probabilistic representation of the solution to the non-Markovian switching problem (4.3) of this type.

Consider the constrained BSDE with jumps: find a quadruple $(\bar{Y},\bar{Z},\bar{U},\bar{K})\in\mathcal{S}^{\mathbf{2}}_{\mathbb{G}}\times\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W})\times\mathbf{L}^{\mathbf{2}}(\tilde{\mu})\times\mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ satisfying

$$\bar{Y}_t \;=\; g_{I_T}(X_T^I)+\int_t^T\psi_{I_s}(s,X_s^I)ds+\bar{K}_T-\bar{K}_t-\int_t^T\langle\bar{Z}_s,dW_s\rangle-\int_t^T\int_{\mathcal{I}}\bar{U}_s(i)\mu(ds,di)\,, \tag{4.19}$$

for $0\leq t\leq T$, together with the constraint

$$\bar{U}_t(i) \;\leq\; c_{I_{t^-},i}(t)\,,\qquad d\mathbf{P}\otimes dt\otimes\lambda(di)\;\; a.e., \tag{4.20}$$

where the process (I,X^I) is defined on $[0,T]$ as the unique solution of

$$\begin{aligned} I_t &= i_0+\int_0^t\int_{\mathcal{I}}(i-I_{t^-})\mu(dt,di), \\ X_t^I &= x_0+\int_0^t b_{I_s}(s,X_s^I)ds+\int_0^t\sigma_{I_s}(s,X_s^I)dWs\,. \end{aligned}$$

The representation of the solution to the optimal switching problem (4.3) is obtained via the following link between the minimal solution of the constrained BSDE with jumps (4.19)-(4.20) and the family of $(Y^{\nu,.,\eta},Z^{\nu,.,\eta},K^{\nu,.,\eta})_{(\nu,\eta)\in\mathcal{K}}$ given by (4.5).

Theorem 4.1. *Under **(H3)**, there is a unique minimal solution* $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K}) \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *to* (4.19)-(4.20) *and we have*

$$\bar{Y}_t = Y_t^{t,I_t,X_t^I} \;, \quad \bar{Z}_t = Z_t^{t,I_{t^-},X_t^I} \quad \text{and} \quad \bar{U}_t(i) = Y_t^{t,i,X_t^I} - Y_{t^-}^{t,I_{t^-},X_t^I} \;, \tag{4.21}$$

for $0 \leq t \leq T$. *In particular, we deduce*

$$\bar{Y}_0 \;=\; J(\alpha^*) \;=\; \sup_{\alpha \in \mathcal{A}_{0,i_0}} J(\alpha) \;.$$

Proof. First observe that, under **(H3)**, the coefficients of (4.19)-(4.20) satisfy **(H2)**. For any $(\nu, \eta) \in \mathcal{K}$ and $n \in \mathbb{N}$, let define the sequence of processes $(\tilde{Y}^{\nu,i,\eta,n}, \tilde{Z}^{\nu,i,\eta,n}, \tilde{K}^{\nu,i,\eta,n})_{i\in\mathcal{I}} \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^{\mathcal{I}}$ as the solution of the penalized BSDE

$$\begin{aligned} Y_t^{\nu,i,\eta,n} \;&=\; g_i(X_T^{\nu,i,\eta}) + \int_t^T \psi_i(s, X_s^{\nu,i,\eta})ds - \int_t^T \langle \tilde{Z}_s^{\nu,i,\eta,n}, dWs \rangle \\ &\quad + n \int_t^T \Big\{ \sum_{j\in\mathcal{I}} [\tilde{Y}_s^{\nu,j,\eta,n} - c_{i,j}(s) - \tilde{Y}_s^{\nu,i,\eta,n}]^- \lambda(j) \Big\} ds \;, \quad i \in \mathcal{I}\,, \;\; 0 \leq t \leq T\,. \end{aligned}$$

Following the lines of the proof of Theorem 2.1 in [14], we know that each of these BSDEs admits a unique solution and that the sequence $(Y^{\nu,.,\eta,n}, Z^{\nu,.,\eta,n}, K^{\nu,.,\eta,n})_{n\in\mathbb{N}}$ converges in $\mathcal{S}^{\mathbf{2}}_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathcal{S}^{\mathbf{2}}_{\mathbb{F}}$ to $(Y^{\nu,i,\eta}, Z^{\nu,i,\eta}, K^{\nu,i,\eta})_{i\in\mathcal{I}}$ as n goes to ∞, for each $(\nu, \eta) \in \mathcal{K}$. Similarly, one easily checks that the triplet $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ defined by

$$\bar{Y}_t^n := \bar{Y}_t^{t,I_t,X_t^I,n}, \quad \bar{Z}_t^n := \bar{Z}_t^{t,I_{t^-},X_t^I,n} \quad \text{and} \quad \bar{U}_t^n(i) := \bar{Y}_t^{t,i,X_t^I,n} - \bar{Y}_{t^-}^{t,I_{t^-},X_t^I,n}$$

is solution to the penalized BSDE associated to (4.19)-(4.20), namely

$$\begin{aligned} Y_t \;&=\; g_{I_T}(X_T^I) + \int_t^T \psi_{I_s}(s, X_s^I)ds + n \int_t^T \int_{\mathcal{I}} [U_s(i) + c_{I_{s^-},i}(s)]^- \lambda(di) ds \\ &\quad - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di), \quad 0 \leq t \leq T.\,. \end{aligned}$$

From Proposition 4.2 (iii), we check that the monotone sequence $(\bar{Y}^n)_n$ is bounded almost surely, and we derive from Remark 2.4 that Assumption **(H1)** holds for (4.19)-(4.20). Therefore there exists a unique minimal solution $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ to (4.19)-(4.20) and we have a stronger convergence that the one obtained before :

$$\|\bar{Y} - \bar{Y}^n\|_{\mathcal{S}^{\mathbf{2}}} + \|\bar{Z} - \bar{Z}^n\|_{\mathbf{L^2(0,T)}} + \|\bar{U} - \bar{U}^n\|_{\mathbf{L^2}(\tilde{\mu})} \;\longrightarrow\; 0\;,$$

as n goes to infinity. □

Remark 4.3. An optimal strategy for the switching problem (4.3) can also be described via the constrained BSDE with jumps (4.19)-(4.20). Indeed, using the definition of $(\tau_n^*, \zeta_n^*)_{n\geq 0}$ in Proposition 4.3 and the identification (4.21), we get $(\tau_0^*, \zeta_0^*) = (0, i_0)$ and, for any $n \in \mathbb{N}$,

$$\begin{aligned} \tau_{n+1}^* \quad &= \quad T \wedge \inf \Big\{ t \geq \tau_n^* \;;\; \max_{j\in\mathcal{I}} \mathbf{E}\Big[U_t(j) - c_{\zeta_n^*,j}(t) \Big| I_s = \zeta_n^* \;\; \forall s \geq \tau_n^* \Big] = 0 \Big\}\;, \\ \zeta_{n+1}^* \quad &\text{is} \quad \text{s.t.} \;\; \mathbf{E}\Big[U_{\tau_{n+1}^*}(\zeta_{n+1}^*) - c_{\zeta_n^*,\zeta_{n+1}^*}(\tau_{n+1}^*) \Big| I_s = \zeta_n^* \;\; \forall s \geq \tau_n^* \Big] \;=\; 0\,. \end{aligned}$$

A Appendix

A.1 A comparison theorem for reflected BSDEs with jumps

We derive here a general comparison theorem for reflected BSDEs with jumps. This extends the results of Theorem 2.5 in [22] obtained in the non-reflected case.

Proposition A.1. *Let* $f_1, f_2 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ *two generators satisfying Assumption* ***(H0)*** *and* $\xi_1, \xi_2 \in \mathbf{L}^2(\Omega, \mathcal{G}_T, \mathbf{P})$*. Let* $(Y^1, Z^1, U^1) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu})$ *satisfying on* $[0,T]$

$$Y^1_t \;=\; \xi^1 + \int_t^T f_1(s, Y^1_s, Z^1_s, U^1_s)ds - \int_t^T \langle Z^1_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^1_s(i)\mu(ds, di)\ , \quad \text{(A.1)}$$

and $(Y^2, Z^2, U^2, K^2) \in \mathcal{S}^2_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying on* $[0,T]$

$$Y^2_t = \;\xi^2 + \int_t^T f_2(s, Y^2_s, Z^2_s, U^2_s)ds - \int_t^T \langle Z^2_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U^2_s(i)\mu(ds, di) + K^2_T - K^2_t\ . \quad \text{(A.2)}$$

If $\xi^1 \leq \xi^2$ *and* $f_1(t, Y^1_t, Z^1_t, U^1_t) \leq f_2(t, Y^1_t, Z^1_t, U^1_t)$ *for all* $t \in [0,T]$*, then we have*

$$Y^1_t \;\leq\; Y^2_t\ , \qquad 0 \leq t \leq T\ .$$

Proof. Let us denote $\bar{Y} := Y^2 - Y^1$, $\bar{Z} := Z^2 - Z^1$, $\bar{U} := U^2 - U^1$, $\bar{f} = f_2(., Y^2, Z^2, U^2) - f_1(., Y^1, Z^1, U^1)$ and $\bar{\xi} = \xi^2 - \xi^1$ so that

$$\bar{Y}_t = \;\bar{\xi} + \int_t^T \bar{f}_s ds - \int_t^T \langle \bar{Z}_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i)\mu(ds, de) + K^2_T - K^2_t, \;\; 0 \leq t \leq T,\ a.s. \quad \text{(A.3)}$$

Let now define the process a by

$$a_t \;=\; \frac{f_2(t, Y^2_t, Z^2_t, U^2_t) - f_2(t, Y^1_t, Z^2_t, U^2_t)}{\bar{Y}_t}\mathbf{1}_{\{\bar{Y}_t \neq 0\}}\ , \quad 0 \leq t \leq T\ ,$$

and b the $\mathbb{R}^d$-valued process defined component by component by

$$b^k_t \;=\; \frac{f_2(t, Y^1_t, Z^{(k-1)}_t, U^2_t) - f_2(t, Y^1_t, Z^{(k)}_t, U^2_t)}{V^k_t}\mathbf{1}_{\{V^k_t \neq 0\}}\ , \;\; k = 1, \ldots, d\ , \quad 0 \leq t \leq T\ ,$$

where $Z^{(k)}_t$ is the $\mathbb{R}^d$-valued random vector whose k first components are those of Z^1 and whose $(d-k)$ lasts are those of Z^2, and V^k_t is the k-th component of $Z^{(k-1)}_t - Z^{(k)}_t$.

Notice that the processes a, b are $\mathbf{P}$-$a.s.$ bounded since f_2 is Lipschitz continuous. Observe also that the process $\bar{K}$ defined on $[0,T]$ by

$$\bar{K}_t \;:=\; K^2_t - \int_0^t \int_{\mathcal{I}} {}^2\gamma_s^{Y^1_{s^-}, Z^1_s, U^1_s, U^2_s}\bar{U}_s(i)\lambda(di)ds + \int_0^t (f_2(s, Y^1_s, Z^1_s, U^2_s) - f_1(s, Y^1_s, Z^1_s, U^1_s))ds$$

is a non-decreasing process since f_2 satisfies **(H0)** (iii) with associated bounded process ${}^2\gamma$, and $f_1(t, Y_t^1, Z_t^1, U_t^1) \leq f_2(t, Y_t^1, Z_t^1, U_t^1)$, for all $t \in [0, T]$. With these notations, we rewrite (A.3) as:

$$\begin{aligned}\bar{Y}_t &= \bar{\xi} + \int_t^T \left(a_s \bar{Y}_s + \langle b_s, \bar{Z}_s \rangle + \int_{\mathcal{I}} {}^2\gamma_s^{Y^1_{s^-}, Z^1_s, U^1_s, U^2_s}(i) \bar{U}_s(i) \lambda(di) \right) ds \\ &\qquad - \int_t^T \langle \bar{Z}_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} \bar{U}_s(i) \mu(ds, di) + \bar{K}_T - \bar{K}_t\,.\end{aligned}$$

Consider now the positive process Γ solution of the s.d.e.:

$$d\Gamma_t = \Gamma_{t^-} \left(a_t dt + \langle b_t, dW_t \rangle + \int_{\mathcal{I}} {}^2\gamma_t^{Y^1_{t^-}, Z^1_t, U^1_t, U^2_t}(i) \mu(dt, di) \right), \quad \Gamma_0 = 1.$$

Notice that Γ lies in $\mathcal{S}^2_{\mathbb{G}}$ since a, b and γ are bounded, and Γ is positive since ${}^2\gamma > -1$. A direct application of Itô's formula leads to

$$d(\Gamma \bar{Y})_t = \langle \Gamma_{t^-} \bar{Z}_t + \bar{Y}_{t^-} \Gamma_{t^-} b_t, dW_t \rangle + \Gamma_{t^-} \int_{\mathcal{I}} {}^2\gamma_t^{Y^1_{t^-}, Z^1_t, U^1_t, U^2_t}(i) \bar{U}_t(i) \tilde{\mu}(ds, di) - \Gamma_{t^-} d\bar{K}_t\,,$$

recall that $\tilde{\mu}$ is the compensated measure associated to μ. Hence, the process $\Gamma \bar{Y}$ is a supermartingale since $\Gamma > 0$. Hence

$$\Gamma_t \bar{Y}_t \geq \mathbb{E}\left[\Gamma_T \bar{Y}_T \middle| \mathcal{G}_t \right] = \mathbb{E}\left[\Gamma_T \bar{\xi} \middle| \mathcal{G}_t \right] \geq 0\,, \quad 0 \leq t \leq T\,,$$

leading to $\bar{Y} \geq 0$. □

A.2 Monotonic limit theorem for BSDE with jumps

This paragraph is devoted to the extension of Peng's monotonic limit theorem to the framework of BSDEs driven by a Brownian motion and a Poisson random measure. In the particular case where the driver f does not depend on the jump component U, this extension can be obtained combining several results derived in Section 3 of [15]. For sake of completeness, we provide here a sketch of the proof, trying to insist on the main arguments of the proof, and more precisely on the predictability of the processes $(K^n)_{n\in\mathbb{N}}$.

Proposition A.2. *Let* $(Y^n, Z^n, U^n, K^n)_n$ *be a sequence in* $\mathcal{S}^{\mathbf{2}}_{\mathbb{G}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *satisfying*

$$Y_t^n = Y_T^n + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n) ds - \int_t^T \langle Z_s^n, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s^n(i) \mu(ds, di) + K_T^n - K_t^n,$$

for all $t \in [0, T]$. *If* $(Y^n)_n$ *converges increasingly to* Y *with* $\mathbf{E}[\sup_{t\in[0,T]} |Y_t|^2] < \infty$ *and each* K^n *is predictable, then* $Y \in \mathcal{S}^{\mathbf{2}}_{\mathbb{G}}$ *(up to a modification) and there exists* $(Z, U, K) \in \mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{G}}$ *with* K *predictable, such that*

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T \langle Z_s, dW_s \rangle - \int_t^T \int_{\mathcal{I}} U_s(i) \mu(ds, di) + K_T - K_t,$$

for all $t \in [0, T]$. *Moreover* (Z, U) *is the weak (resp. strong) limit of the sequence* $(Z^n, U^n)_n$ *in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{2}}(\tilde{\mu})$ *(resp.* $\mathbf{L}^{\mathbf{p}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{p}}(\tilde{\mu})$, *for* $p < 2$*). Finally,* K *is the weak limit of* $(K^n)_n$ *in* $\mathbf{L}^{\mathbf{2}}_{\mathbb{G}}(\mathbf{0}, \mathbf{T})$ *and, for any* $t \in [0, T]$, K_t *is the weak limit of* $(K_t^n)_n$ *in* $\mathbf{L}^{\mathbf{2}}(\boldsymbol{\Omega}, \mathcal{F}_{\mathbf{t}}, \mathbf{P})$.

Proof. The proof of Proposition A.2 is an adaptation of the proof of Theorem 2.4 in [18]. The main assumption which allows to extend the arguments of [18] is the predictably of each process K^n, $n \in \mathbb{N}$. We recall the main steps of the proof and explain how the predictability assumption provides the result.

1. *Uniform estimate.* Applying Itô's formula to $|Y^n|^2$ and using standard arguments (BDG inequality and Gronwall's Lemma) we get the existence of a constant $C > 0$ such that

$$\|Y^n\|_{\mathcal{S}^2_{\mathbb{G}}} + \|Z^n\|_{\mathbf{L}^2_{\mathbb{G}}(\mathbf{W})} + \|U^n\|_{\mathbf{L}^2(\tilde{\mu})} + \|K^n\|_{\mathcal{S}^2_{\mathbb{G}}} \;\leq\; C\,, \qquad n \in \mathbb{N}\,. \tag{A.4}$$

2. *Weak convergence.* Using the previous uniform estimate and the Hilbert structure of $\mathbf{L}^2_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0},\mathbf{T}) \times \mathbf{L}^2_{\mathbb{G}}(\mathbf{0},\mathbf{T})$, we deduce the existence of a subsequence of $(Z^n, U^n, K^n, f(.,Y^n,Z^n,U^n))_n$, which converges weakly to some (Z,U,K,F). Identifying the limits of $(Y^n)_n$ and $(Z^n, U^n, K^n, f(.,Y^n,Z^n,U^n))_n$, we get

$$Y_t \;=\; Y_T + \int_t^T F_s ds - \int_t^T \langle Z_s, dW_s\rangle - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds,di) + K_T - K_t, \quad 0 \leq t \leq T. \tag{A.5}$$

The predictability of the process K comes from the predictability of each K^n and the completeness of $\mathbf{L}^2_{\mathbb{G}}(\mathbf{W})$.

3. *Properties of the process K.* We first observe from Lemma 2.2 in [18] that the process K admits a càdlàg modification. We then establish that the contribution of the jumps of K is mainly concentrated within a finite number of intervals with sufficiently small total length.

As in Lemma 2.3 in [18], for any $\delta, \epsilon > 0$, there exists a finite number of pairs of stopping times $(\sigma_k, \tau_k)_{0\leq k\leq N}$ with $0 < \sigma_k \leq \tau_k \leq T$ such that

(i) $(\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset$ for $j \neq k$;

(ii) $\mathbf{E}\sum_{k=0}^{N}(\tau_k - \sigma_k) \geq T - \varepsilon$;

(iii) $\mathbf{E}\sum_{k=0}^{N}\sum_{\sigma_k<t\leq\tau_k}|\Delta K_t|^2 \leq \delta$.

This result is derived with similar arguments as in [18], relying only on the right continuity of the filtration and the predictability of the process K. More precisely, its proof is based on Lemma A.1 in [18] and the fact that, since K is predictable, its jump times are predictable stopping times and hence could be announced i.e. if τ is a jump time of K then there exist a sequence of stopping times $(\tau_k)_k$ with $\tau_k < \tau$ for each k and $\tau_k \uparrow \tau$ as k goes to infinity. Then combining these two results we end this step as in [18].

4. *Strong convergence.* Following the arguments of the proof of Theorem 3.1 in [15], we deduce from the results on contribution of jumps of K the convergence of $(Z^n, U^n)_n$ to (Z,U) in $dt \otimes d\mathbf{P}-$measure. Together with the uniform estimate (A.4), this leads to the strong convergence of $(Z^n, U^n)_n$ to (Z,U) in $\mathbf{L}^{\mathbf{p}}_{\mathbb{G}}(\mathbf{W}) \times \mathbf{L}^{\mathbf{p}}(\tilde{\mu})$, $p < 2$. Combining the Lipschitz property of f with (A.5), we conclude that (Y,Z,U,K) satisfies

$$\begin{aligned} Y_t &= Y_T + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T \langle Z_s, dW_s\rangle \\ &\qquad - \int_t^T \int_{\mathcal{I}} U_s(i)\mu(ds,di) + K_T - K_t, \quad 0 \le t \le T. \end{aligned}$$

□

A.3 Viability and comparison property for multi-dimensional BSDEs

We generalize in this paragraph some viability and comparison properties for multidimensional BSDEs in a closed convex cone $\mathcal{C}$ of $\mathbb{R}^{2m}$, whenever we add some reflections on the Y-component of the BSDE. The two following propositions are respectively extensions of Theorem 2.5 in [5] and a simplifying extension of Theorem 2.1 in [13]. Their derivations do not present major difficulty and we choose to detail them for sake of completeness.

Let $(Y,Z) \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^{2m}$ satisfying

$$Y_t = Y_T + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T \langle Z_s, dW_s\rangle + K_T - K_t\ , \quad 0 \le t \le T\ , \qquad \text{(A.6)}$$

where $F : \Omega \times [0,T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d} \to \mathbb{R}^{2m}$ is a progressively measurable function satisfying **(H2)** (ii) and K is an $\mathbb{R}^{2m}$-valued finite variation process such that

$$K_t = \int_0^t k_s d|K|_s\ ,$$

with $k_t \in \mathcal{C}$ and $|K|_s$ the variation of K on $[0,s]$. We denote by $d_{\mathcal{C}}$ the distance to $\mathcal{C}$, i.e. $d_{\mathcal{C}} : x \mapsto \min_{y\in\mathcal{C}} |x-y|$, and introduce $\Pi_{\mathcal{C}}$ the projection operator onto $\mathcal{C}$.

Proposition A.3. *Suppose $Y_T \in \mathcal{C}$ and there exists a constant C^0 such that F satisfies*

$$4\langle y - \Pi_{\mathcal{C}}(y), F(t,y,z)\rangle \le \langle D^2|d_{\mathcal{C}}|^2(y)z, z\rangle + 2C^0|d_{\mathcal{C}}|^2(y) \qquad \mathbf{P}-a.s.\ , \qquad \text{(A.7)}$$

for any $(t,y,z) \in [0,T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d}$ such that $|d_{\mathcal{C}}|^2$ is twice differentiable at the point y. Then, we have

$$Y_t \in \mathcal{C}\,, \qquad 0 \le t \le T\ , \qquad \mathbf{P}-a.s.$$

Proof. The proof presented here is an adaptation of the one of Theorem 2.5 in [5], allowing to tackle the additional difficulty due to the dK term in the dynamics of Y.
Let $\eta \in C^\infty(\mathbb{R}^{2m})$ be a non-negative function, with support in the unit ball, such that $\int_{\mathbb{R}^{2m}} \eta(x)dx = 1$. For $\delta > 0$ and $x \in \mathbb{R}^{2m}$, we define

$$\eta_\delta(x) := \frac{1}{\delta^{2m}}\eta\Big(\frac{x}{\delta}\Big) \qquad \text{and} \qquad \phi_\delta(x) := \int_{\mathbb{R}^{2m}} |d_{\mathcal{C}}(x-y)|^2\eta_\delta(y)dy\ .$$

Via direct computation, one can verify that $\phi_\delta \in C^\infty(\mathbb{R}^{2m})$ and

$$\begin{cases} 0 \ \le \ \phi_\delta(x) \ \le \ (d_{\mathcal{C}}(x)+\delta)^2 \ , \\ D\phi_\delta(x) \ = \ \int_{\mathbb{R}^{2m}} D|d_{\mathcal{C}}(y)|^2 \eta_\delta(x-y)dy \ \text{ and } |D\phi_\delta(x)| \ \le \ 2(d_{\mathcal{C}}(x)+\delta) \ , \\ D^2\phi_\delta(x) \ = \ \int_{\mathbb{R}^{2m}} D^2|d_{\mathcal{C}}(y)|^2 \eta_\delta(x-y)dy \text{ and } \ 0 \ \le \ |D^2\phi_\delta(x)| \ \le \ 2I_{2m} \ , \end{cases} \tag{A.8}$$

for any $x \in \mathbb{R}^{2m}$. An application of Itô's formula to $\phi_\delta(Y)$, combined with these estimates and $d_{\mathcal{C}}(Y_T) = 0$, leads to

$$\begin{aligned} \mathbf{E}\phi_\delta(Y_t) \ &= \ \mathbf{E}\phi_\delta(Y_T) + \mathbf{E}\int_t^T \langle D\phi_\delta(Y_s), F(s,Y_s,Z_s)\rangle ds - \frac{1}{2}\mathbf{E}\int_t^T \langle D^2\phi_\delta(Y_s)Z_s, Z_s\rangle ds \\ &\quad +\mathbf{E}\int_t^T \langle D\phi_\delta(Y_s), k_s\rangle d|K|_s \\ &\le \ \delta^2 + \mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} \Big[\langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)\rangle - \frac{1}{2}\langle D^2|d_{\mathcal{C}}(y)|^2 Z_s, Z_s\rangle\Big]\eta_\delta(Y_s-y)dyds \\ &\quad -\mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} \langle D|d_{\mathcal{C}}(y)|^2, F(s,y,Z_s)-F(s,Y_s,Z_s)\rangle \eta_\delta(Y_s-y)dyds \\ &\quad +\mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} \langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle \eta_\delta(Y_s-y)dyd|K|_s\,, \qquad 0 \le t \le T\ . \end{aligned} \tag{A.9}$$

Since k is valued in the closed convex cone $\mathcal{C}$, we observe that

$$\langle D|d_{\mathcal{C}}(y)|^2, k_s\rangle \ \le \ 0\ , \qquad 0 \le s \le T\ , \quad y \in \mathbb{R}^{2m}\ .$$

Then, plugging this expression, (A.7) and inequality $2d_c(.) \le 1 + d_c(.)^2$ in (A.9), we get

$$\begin{aligned} \mathbf{E}\phi_\delta(Y_t) &\le \delta^2 + C^0\mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} |d_{\mathcal{C}}(y)|^2\eta_\delta(y-Y_s)dyds \\ &\quad +2\mathbf{E}\int_t^T\int_{\mathbb{R}^{2m}} d_{\mathcal{C}}(y)\eta_\delta(Y_s-y) \max_{y'\colon |y'-Y_s|\le\delta} |F(s,y',Z_s)-F(s,Y_s,Z_s)|dyds \\ &\le \delta^2 + C^0\int_t^T \mathbf{E}\phi_\delta(Y_s)ds + \mathbf{E}\int_t^T (1+\phi_\delta(Y_s)) \max_{y'\colon |y'-Y_s|\le\delta} |F(s,y',Z_s)-F(s,Y_s,Z_s)|ds\,, \end{aligned}$$

for any $t \in [0,T]$. Using the uniform Lipschitz property of F, we deduce

$$\mathbf{E}\phi_\delta(Y_t) \ \le \ C\left\{\delta^2 + \delta + \int_t^T \mathbf{E}\phi_\delta(Y_s)ds\right\}\ , \quad 0 \le t \le T\ ,\ \delta > 0\,,$$

and Gronwall's lemma leads to

$$\mathbf{E}\phi_\delta(Y_t) \ \le \ C(\delta^2+\delta), \quad 0 \le t \le T\ ,\ \delta > 0\,.$$

Finally, from Fatou's Lemma, we have

$$\mathbf{E}|d_{\mathcal{C}}(Y_t)|^2 \ \le \ \liminf_{\delta\to 0} \mathbf{E}\phi_\delta(Y_t) \ = \ 0\ , \quad 0 \le t \le T\ ,$$

which concludes the proof. □

We now turn to the obtention of a multidimensional comparison result for BSDEs, whenever the dominating BSDE suffers additional reflections. This generalizes and also simplifies the results of Theorem 2.1 in [13].

Consider $(Y^1, Z^1, K^1) \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}) \times \mathbf{A}^{\mathbf{2}}_{\mathbb{F}})^m$ satisfying

$$Y^1_t \;=\; Y^1_T + \int_t^T F_1(s, Y^1_s, Z^1_s)ds - \int_t^T \langle Z^1_s, dW_s\rangle + K^1_T - K^1_t\;, \quad 0 \le t \le T\;,$$

and $(Y^2, Z^2) \in (\mathcal{S}^2_{\mathbb{F}} \times \mathbf{L}^{\mathbf{2}}_{\mathbb{F}}(\mathbf{W}))^m$ satisfying

$$Y^2_t \;=\; Y^2_T + \int_t^T F_2(s, Y^2_s, Z^2_s)ds - \int_t^T \langle Z^2_s, dW_s\rangle\;, \quad 0 \le t \le T\;,$$

where F_1 and F_2 are two driver functions satisfying **(H2)** (ii).

Proposition A.4. *Suppose $Y^1_T \ge Y^2_T$ and the existence of a constant C^1 such that we have*

$$-2\langle y, F_1(t, y', z) - F_2(t, y', z')\rangle \;\le\; C^1|y|^2 + \sum_{i=1}^m |z_i - z'_i|^2 \;\mathbf{P}-a.s.\;, \qquad \text{(A.10)}$$

for any $(t, y, y', z, z') \in [0, T] \times (R^+)^m \times \mathbb{R}^m \times [\mathbb{R}^{m\times d}]^2$. Then $Y^1_t \ge Y^2_t$, for all $t \in [0, T]$.

Proof. The process $(Y^1 - Y^2, Y^2)$ is valued in $\mathbb{R}^{2m}$ and solution of a BSDE of the form (A.6) associated to the driver

$$F \;:\; (t, (y, y'), (z, z')) \;\;\mapsto\;\; (F_1(t, y + y', z + z') - F_2(t, y', z'), F_2(t, y', z'))\;,$$

for any $\{t, (y, y'), (z, z')\} \in [0, T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d}$. Introducing the closed convex cone $\mathcal{C} := (\mathbb{R}^+)^m \times \mathbb{R}^m$ of $\mathbb{R}^{2m}$, we see that $d_{\mathcal{C}}(y, y') = |y^-|$ for $(y, y') \in \mathbb{R}^{2m}$. Therefore, we deduce from the Lipschitz property of F_1 and (A.10) that

$$\begin{aligned}
& 4\langle (y, y') - \Pi_{\mathcal{C}}(y, y'), F(t, (y, y'), (z, z'))\rangle \\
=\; & 4\langle -y^-, F_1(t, y + y', z + z') - F_1(t, y', z + z')\rangle + 4\langle -y^-, F_1(t, y', z + z') - F_2(t, y', z')\rangle \\
\le\; & 4k|y^-|^2 + 2\sum_{i=1}^m \mathbf{1}_{y_i<0}|z_i|^2 + 2C^1|y^-|^2 \\
=\; & \langle D^2|d_{\mathcal{C}}|^2(y, y')(z, z'), (z, z')\rangle + (2C^1 + 4k)|d_{\mathcal{C}}|^2(y, y') \qquad \mathbf{P}-a.s.\,,
\end{aligned}$$

for any $\{t, (y, y'), (z, z')\} \in [0, T] \times \mathbb{R}^{2m} \times \mathbb{R}^{2m\times d}$. Applying Proposition A.3 with $C^0 = C^1 + 2k$, we deduce that the process $(Y^1 - Y^2, Y^2)$ is valued in $\mathcal{C}$ and complete the proof. □

References

[1] Barles G., Buckdahn R. and E. Pardoux (1997), Backward stochastic differential equations and integral-partial differential equations, *Stochastics and Stochastics Reports*, **60**, 57-83.

[2] Bouchard B. (2009), A stochastic target formulation for optimal switching problems in finite horizon, *Stochastics*, **81 (2)**, 171-197.

[3] Bouchard B. and R. Elie (2008), Discrete-time approximation of decoupled forward-backward SDE with jumps, *Stochastic Processes and their Applications*, **118**, 53-75.

[4] Chassagneux J.F., R. Elie and I. Kharroubi (2010), A note on existence and uniqueness of multidimensional reflected BSDEs, *Preprint*.

[5] Buckdahn R., M. Quincampoix and A. Rascanu (2000), Viability property for a backward stochastic differential equation and applications to partial differential equations, *Probab. Theory. Relat. Fields*, **116**, 485-504.

[6] Carmona R. and M. Ludkovski (2008), Pricing Asset Scheduling Flexibility Using Optimal Switching, *Preprint. Applied Mathematical Finance*, **15**(6), 405-447.

[7] Cvitanic J., Karatzas I. and M. Soner (1998), Backward stochastic differential equations with constraints on the gain-process, *Annals of Probability*, **26**, 1522-1551.

[8] Djehiche B., Hamadène S. and I. Hdhiri (2008), Stochastic Impulse Control of Non-Markovian Processes. *Preprint*.

[9] Djehiche B., Hamadène S. and A. Popier (2009), The finite horizon optimal multiple switching problem. *SIAM Journal fo Control and Optimization*, 48(4), 2751-2770.

[10] El Karoui N., Kapoudjian C., Pardoux E., Peng S. and M.C. Quenez (1997), Reflected solutions of Backward SDE's, and related obstacle problems for PDEs, *Annals of Probability*, **25**, 702-737.

[11] Elie R. and I. Kharroubi (2009), Probabilistic representation and approximation for coupled systems of variational inequalities, *Preprint*.

[12] Hamadène S. and J. Zhang (2008), The Starting and Stopping Problem under Knightian Uncertainty and Related Systems of Reflected BSDEs. To appear in *Stochastic Processes and Applications*.

[13] Hu Y. and S. Peng (2006), On comparison theorem for multi-dimentional BSDEs, *C. R. Acad. Sci. Paris*, **343**, 135-140.

[14] Hu Y. and S. Tang (2010), Multi-dimensional BSDE with oblique Reflection and optimal switching, *Prob. Theory and Related Fields*, **147**(1-2) 89-121.

[15] Kharroubi I., Ma J., Pham H. and J. Zhang (2008), Backward SDEs with constrained jumps and Quasi-Variational Inequalities, *Annals of Probability*, **38**(2) 794-840.

[16] Pardoux E. and S. Peng (1990), Adapted solution of a backward stochastic differential equation, *Systems and Control Letters*, **14**, 55-61.

[17] PARDOUX E., F. PRADEILLES AND Z. RAO (1997), Probabilistic interpretation of a system of semi-linear parabolic partial differential equations, *Ann. Inst. Henri Poincaré*, section B, **33** (4), 467-490.

[18] PENG S. (1999), Monotonic limit theory of BSDE and nonlinear decomposition theorem of Doob-Meyer's type, *Prob. Theory and Related Fields*, **113**, 473-499.

[19] PENG S. AND M. XU (2005), The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles, *Ann. I.H.P.* **41**, 605-630.

[20] PENG S. AND M. XU (2007), Constrained BSDE and viscosity solutions of variation inequalities, *Preprint.*

[21] RAMASUBRAMANIAN S. (2002), Reflected backward stochastic differential equations in an orthant, *Proc. indian Acad. Sci.,* **112**(2), 347-360.

[22] ROYER M. (2006), Backward stochastic differential equations with jumps and related nonlinear expectations, *Stochastic Processes and their Applications*, **116**, 1358-1376.

[23] TANG S. AND X. LI (1994), Necessary conditions for optimal control of stochastic systems with jumps, *SIAM J. Control and Optimization*, **32**, 1447-1475.

[24] TANG S. AND J. YONG (1993), Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach, *Stochastics and Stochastic Reports*, **45**, 145-176.