# $L^1$-error estimates for numerical approximations of Hamilton-Jacobi-Bellman equations in dimension 1

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Abstract. The goal of this paper is to study some numerical approximations of particular Hamilton-Jacobi-Bellman equations in dimension 1 and with possibly discontinuous initial data. We investigate two anti-diffusive numerical schemes, the first one is based on the Ultra-Bee scheme and the second one is based on the Fast Marching Method. We prove the convergence and derive $L^1$-error estimates for both schemes. We also provide numerical examples to validate their accuracy in solving smooth and discontinuous solutions.

**Keywords:** Hamilton-Jacobi-Bellman equations, Fast Marching Method, Ultra-Bee scheme, $L^1$-error estimate, anti-diffusive scheme.

**Mathematics Subject Classification:** 49L99, 65M15.

## 1 Introduction

This paper discuss two explicit numerical approximations of the following one-space dimensional Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} \vartheta_t + \max\left(f_+(x)\vartheta_x, f_-(x)\vartheta_x\right) = 0 & \text{in} \quad \mathbb{R}\times(0,T) \\ \vartheta(\cdot,0) = v_0 & \text{in} \quad \mathbb{R} \end{cases} \tag{1.1}$$

where $v_0 \in L^\infty(\mathbb{R})$. In particular, $v_0$ can be discontinuous.

In optimal control theory, the solution $\vartheta$ of equation (1.1) corresponds to the value function of an optimization problem [3, 2]. It often happens that this function, as well as the "final" cost $v_0$, is discontinuous (for instance for target or Rendez-Vous problems). The discontinuities of $\vartheta$ will represent, for example, the interface between the domain of admissible trajectories and the one of prohibited trajectories and then it is very important to localize the discontinuities. This is the reason why, in the discontinuous case, the classical monotone schemes for HJB equations ([8, 11, 1, 13]) are no more adapted. Indeed, if we attempt to use these schemes, we observe an increasing numerical diffusion around discontinuities, and this is due to the fact that monotone schemes use at some level finite differences and/or interpolation technics.

In this work, we investigate two different schemes to solve (1.1) for discontinuous initial data. The first one is a Fast Marching Method type scheme. This method, introduced by Sethian [14], is a very efficient scheme to solve numerically the eikonal equation for given positive velocity $c(x)$. This scheme has been improved by Carlini et al in [7, 12] where the case of changing sign velocity is considered.

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Recall that the FMM method was built [14, 12] to deal with eikonal equations with initial condition taking values in $\{0,1\}$, this schemes allows to concentrate numerical effort only in a *Narrow band* around the interface separating the zone of 0-value from the zone of 1-value.

Here we consider the case where the initial condition $v_0$ is any bounded lower semi continuous (l.s.c) function. We first define a level-set approximation $w_0$ of $v_0$ in the following form: given $p \geq 1$ and given $(h_k)_{k=1,\cdots,p} \subset \mathbb{R}^*_+$, we set

$$w_0(x) = \sum_{k=1}^{p} h_k \; w_{0,k}(x), \quad \text{ with } w_{0,k}(x) := \begin{cases} 1 & \text{if } v_0(x) > \displaystyle\sum_{i=1}^{k} h_i \\ 0 & \text{otherwise.} \end{cases}$$

Now for each level $k = 1, \cdots, p$, the function $w_{0,k}$ takes values only in $\{0,1\}$. Therefore, we propose an algorithm based on the FMM which make evolve each level-set function $w_{0,k}$, for $k = 1, \cdots, p$. Hence, we obtain for every $k$ an approximation $\vartheta^\rho_k$ (with $\rho := (\Delta x, (h_k)_k)$) of the solution of (1.1) associated to the initial condition $w_{0,k}$. Thanks to a comparison principle, we prove that an approximation of the solution $\vartheta$ of (1.1) is obtained by:

$$\vartheta^\rho(t,x) = \sum_{k=1}^{p} h_k \; \vartheta^\rho_k(t,x).$$

We derive an $L^1$-error estimate (in finite time $t$) in order of $\Delta x + h$, where $\Delta x$ is the mesh step size and $h = \max_{i=1,\cdots,p} h_i$.

Let us mention that Y. Brenier [6] has used a similar level decomposition in the case of conservation laws.

The second scheme is a modified version of the Ultra-Bee scheme for HJB equations proposed in [5] and which convergence has been proved in [4]. Let us mention that this scheme was first studied in [9, 10] for linear advection equations with constant velocity. In this case, it is proved that the scheme is exact whenever the initial function takes values in $\{0,1\}$ and the discontinuities are separated by $3\Delta x$ ($\Delta x$ being the mesh step size). The scheme keeps nice anti-diffusive properties when we deal with advection or HJB equations with changing sign velocities and initial condition taking only values 0 and 1.

A generalisation of the Ultrabee scheme is proposed in [4] for HJB equation with l.s.c bounded initial condition $v_0$. This generalisation use additional steps of *truncation and prediction* when two discontinuities get close (closer than $3\Delta x$).

In this paper, we use the level-set decomposition of $v_0$ (as explained above). We are lead back to an HJB equation in the form of (1.1) with initial condition $w_{0,k}$ which takes only values 0 and 1. The evolution of each level-set function can be accurately approximated by Ultrabee scheme, and the resulting approximation of the solution $\vartheta$ of (1.1) is very satisfactory. The Ultrabee scheme combined with level-set decomposition has almost the same $L^1$-error bound than the Ultrabee scheme studied in [4], but numerically it seems that the method proposed in this paper give more accurate results (see Section 5, for a numerical comparison).

The paper is organized as follow. In Section 2 we present our main results: a scheme based on the Fast Marching Method (FMM), an Ultra-Bee scheme (UB), and main convergence results for both schemes in an $L^1$-error approximation bound. Section 3 is devoted to some preliminary results. Next Section deals with the convergence proof for the FMM. Numerical simulations are finally presented in Section 5. Some technical proofs are postponed to the Appendix.

## 2 Main results

In this section, we present the convergence results for the FMM scheme and for the Ultra-Bee scheme.

Throughout the paper, we shall use the following assumptions on the dynamics:

**(H1)** $f_+$ and $f_-$ are $L$-Lipschitz continuous.

**(H2)** $\exists \varepsilon > 0 \; \forall x \in \mathbb{R}, \; f_-(x) + \varepsilon \leq f_+(x)$.

**Remark 2.1.** *This last assumption will allow us to compare the velocities $f_+$ and $f_-$ on different nearby points. It can be replaced by*

**(H2')** $f_-(x) \leq f_+(x), \; \forall x \in \mathbb{R}$, *and $f_+$ and $f_-$ are non-decreasing functions on $\mathbb{R}$.*

*or by*

**(H2")** $f_-(x) \leq 0 \leq f_+(x), \; \forall x \in \mathbb{R}$.

On the initial condition $v_0$, we assume that

**(H3)** $v_0 \in L^\infty(\mathbb{R})$, $v_0$ is lower semi-continuous, and has a finite number of extrema, in the following sense:

$$\left\{\begin{array}{l} \text{There exist real numbers } A_1, \ldots, A_{q+1} \text{ and } B_1, \ldots, B_q \text{ with} \\ \qquad A_1 = -\infty \leq B_1 < A_2 < \cdots < B_q \leq A_{q+1} = +\infty, \\ \text{(with possibly } B_1 = -\infty \text{ or } B_q = +\infty\text{), such that } v_0 \nearrow \text{ on each } [A_i, B_i[, \\ v_0 \searrow \text{ on each } ]B_i, A_{i+1}], \text{ and } v_0(B_i) = \min(v_0(B_i^-), v_0(B_i^+)). \end{array}\right.$$

($A_i$ are local minima of $v_0$, and $B_i$ are local maxima of $v_0$).
We also assume that $v_0 \geq 0$ and is compactly supported: $\exists \alpha, \beta \geq 0$,

$$\operatorname{supp}(v_0) \subset [\alpha, \beta]. \tag{2.2}$$

We finally assume that $v_0$ is locally Lipschitz continuous in a neighbourhood of its local minima:

**(H4)** $\exists \delta_0 > 0, \; \forall i = 2, \ldots q, \; v_0$ is Lipschitz continuous in $[A_i - \delta_0, A_i + \delta_0]$

Let us recall the definition of total variation of a real-valued function.

**Definition 2.1.** *Let $w$ be a real-valued function. the total variation of $w$ is defined by:*

$$TV(w) := \sup \left\{ \sum_{j=1,\ldots,k} |w(y_{j+1}) - w(y_j)|; \; k \in \mathbb{N}^*, \textit{and } (y_j)_{1 \leq j \leq k+1} \textit{ non-decreasing sequence} \right\}.$$

## 2.1 Level-set decomposition

Let us consider steps $(h_k)_{k=1,\ldots,p}$ such that $h_k > 0$ and $\displaystyle\sum_{k=1}^{p} h_k > ||v_0||_\infty$, and let

$$\bar{h}_k := \sum_{i=1}^{k} h_i, \text{ for } k \geq 1, \quad \text{and} \quad h := \sup_{1\leq k\leq p} h_k. \tag{2.3}$$

Let $\Delta x > 0$ be a step size of a spatial grid, and let $x_j := j\ \Delta x$ denote a uniform mesh, with $j \in \mathbb{Z}$. Consider also

$$x_{j+\frac{1}{2}} := (j + \frac{1}{2})\,\Delta x, \quad \text{ and } \quad I_j := ]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[.$$

We define $w_0$, a level set decomposition of $v_0$, and the function $w_{0,k}$ of level $k$, by:

$$w_{0,k}(x_i) := 1_{\left\{\bar{h}_k < v_0\right\}}(x_i) = \begin{cases} 1 & \text{if } \bar{h}_k < v_0(x_i) \\ 0 & \text{otherwise,} \end{cases} \tag{2.4a}$$

$$w_{0,k}(x) = w_{0,k}(x_i) \quad \text{for } x \in I_i. \tag{2.4b}$$

We also set

$$w_0(x) := \sum_{k=1,\ldots,p} h_k\, w_{0,k}(x), \quad x \in \mathbb{R}. \tag{2.4c}$$

**Remark 2.2.** *Definition* (2.4) *clearly implies that* $w_k(t,x) \in \{0,1\}$*. Moreover, if* $1 \leq k_1 \leq k_2 \leq p$*, then from the comparison principle [3] and the fact that* $w_{0,k_1}(x) \leq w_{0,k_2}(x)$*,* $\forall x \in \mathbb{R}$*, we obtain* $w_{k_1}(t,x) \leq w_{k_2}(t,x)$*,* $\forall t \geq 0$ *and* $x \in \mathbb{R}$*.*

Now, the idea is to propose two algorithms to compute numerically the approximation $\vartheta_k^\rho$ (where $\rho = (\Delta x, h)$ represents the space discretization) of the solution $w_k$ of (1.1) with initial data $w_{0,k}$. The first scheme is based on the Fast Marching Method (FMM) and the second one is the Ultra-Bee scheme (UB). As soon as we have computed the numerical solution $\vartheta_k^\rho$, a natural approximation of the solution $\vartheta$ of (1.1) is simply given by $\vartheta^\rho = \sum_{k=1}^{p} h_k \vartheta_k^\rho$ (see Proposition 3.3). We now describe in details the two algorithms as well as the convergence result we obtain.

First, we give an error approximation estimate between $v_0$ and its projection $w_0$. The proof is left to the reader.

**Lemma 2.2. (Error at initial time)**
*We have the following estimate*

$$\|w_0 - v_0\|_{L^1(\mathbb{R})} \leq (\beta - \alpha)h + TV(v_0)\Delta x, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

Next, we compare the evolution of the viscosity solutions of (1.1) associated to initial data $v_0$ and $w_0$. The proof of the following proposition is postponed to Appendix A.

**Proposition 2.3.** *Assume (H1)-(H4) with* $\Delta x \leq \delta_0$*. Let* $w_k$ *(resp.* $w$*) be the viscosity solution of* (1.1) *with initial data* $w_{0,k}$ *(resp.* $w_0$*). Then*

$$(i) \qquad w(t,x) = \sum_{k=1,\ldots,p} h_k w_k(t,x), \quad \forall t > 0,\ x \in \mathbb{R},$$

$$(ii) \qquad \|w(t,.) - \vartheta(t,.)\|_{L^1(\mathbb{R})} \leq e^{Lt}\,(\beta - \alpha + M_0\, t)\, h + \left(e^{Lt} TV(v_0) + M_1\right) \Delta x.$$

*where*

$$M_0 = M_0(v_0, f) := \sum_{i=2,\dots,q} (|f_+(A_i)| + |f_-(A_i)|) \quad \textit{and} \quad M_1 = \max_{j, \exists i,\ A_i \in \bar{I}_j} \|v_0'\|_{L^\infty(I_j)}$$

*are constant.*

**Remark 2.3.** *In this proposition indeed only $f_-(x) \le f_+(x)$ is needed, not (H2).*

## 2.2 Fast Marching Method

The idea is to make evolve each level set $w_{0,k}$ using an adaptation of the Generalized Fast Marching Method introduced in [7] (see also [12]).

### 2.2.1 Notations and algorithm

Let $\Delta x > 0$ be a mesh step size of a uniform grid. For $k = 1, \cdots, p$ and $i \in \mathbb{Z}$, we consider:

$$\theta_i^{0,k} := 2w_{0,k}(x_i) - 1 = \begin{cases} \ \ 1 & \text{if} \quad v_0(x_i) > \bar{h}_k \\ -1 & \text{if} \quad v_0(x_i) \le \bar{h}_k \end{cases} \tag{2.6}$$

(the introduction of $\theta$ is just useful to formulate the algorithm in a simple way and in particular to have some symmetry properties of the algorithm).

As in [7], we also define approximated piece-wise constant velocity functions $\widehat{f}_+$ and $\widehat{f}_-$, for $x \in ]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[$ and for $j \in \mathbb{Z}$:

$$\widehat{f}_\alpha(x) := \begin{cases} 0 & \text{if } \exists\, i \in \{j \pm 1\} \text{ s.t.} f_\alpha(x_i) f_\alpha(x_j) \le 0 \text{ and } |f_\alpha(x_j)| \le |f_\alpha(x_i)|, \\ f_\alpha(x_j) & \text{otherwise.} \end{cases} \tag{2.7}$$

This "regularization" allows us to introduce a numerical band of zero to separate the region where the velocity is positive from the one where the velocity is negative. This separation is needed to avoid the duplication of the front (see [7]).

Let us remark that, from (1.1), when $\theta_i^{0,k} = -1 = -\theta_{i+1}^{0,k}$, then the discontinuity evolves with the velocity $f_+$ (to the right if $f_+ > 0$ and to the left if $f_+ < 0$), while when $\theta_i^{0,k} = 1 = -\theta_{i+1}^{0,k}$, the discontinuity evolves with the velocity $f_-$ (see Figure 1).

We now define, for each control $\alpha \in \{-, +\}$, the stencil of grid points useful to compute the value at point $x_i$, for $i \in \mathbb{Z}$

$$\mathcal{U}_\alpha^{n,k}(i) := \begin{cases} i+1 & \text{if } \theta_i^{n,k} = -\theta_{i+1}^{n,k} = -\alpha 1 \text{ and } \widehat{f}_\alpha(x_i) < 0 \\ \\ i-1 & \text{if } \theta_{i-1}^{n,k} = -\theta_i^{n,k} = -\alpha 1 \text{ and } \widehat{f}_\alpha(x_i) > 0 \\ \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathcal{U}_\alpha^{n,k} = \bigcup_i \mathcal{U}_\alpha^{n,k}(i).$$

The set $\mathcal{U}^{n,k}$ will play the role of the *frozen* points of the classical Fast Marching Method. We point out that the set $\mathcal{U}_\alpha^{n,k}(i)$ is either empty or a singleton. We also define a set of Narrow Bands by:

$$\mathrm{NB}_\alpha^{n,k} := \left\{ i,\ \mathcal{U}_\alpha^{n,k}(i) \ne \emptyset \right\}, \quad \mathrm{NB}^{n,k} := \mathrm{NB}_+^{n,k} \cup \mathrm{NB}_-^{n,k} \quad \text{and} \quad \mathrm{NB}^n := \bigcup_{k=1,..,p} \mathrm{NB}^{n,k}.$$

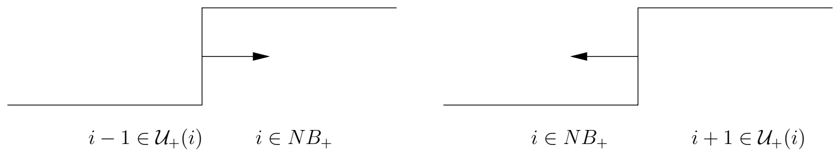



Figure 1: Representation of the useful points and of the Narrow Band for the control $\alpha = +$. Left: $f_+(x_i) > 0$ and the discontinuity moves to the right. Right: $f_+(x_i) < 0$ and the discontinuity moves to the left.

We describe now our FMM for Hamilton-Jacobi-Bellman equation (1.1). This is an adaptation of the one proposed in Carlini *et al.* [7, 12]. In order to track correctly the evolution we need to introduce a discrete function $\tau_{i,\alpha}^{n,k} \in \mathbb{R}^+$, defined only for the points $i \in \mathcal{U}_\alpha^{n,k}$, to represent the approximated physical time for the front propagation at the nodes $i$ for the level set $k$, the control $\alpha$ and at the $n$-th iteration of the algorithm (FMM).

The idea of the algorithm is then very simple. For each point $i$ of the Narrow Band $NB$, we compute a tentative value $\tilde{\tau}_i$ of the arrival time of the front, using the time of the useful points. We then find the minimum of the $\tilde{\tau}_i$ and we accept the nodes that realize the minimum (i.e. we change the value of the $\theta$) and we iterate the process. Let us now give our algorithm in details.

**Initialization:** for $n = 0$, initialize the field $\theta^{0,k}$ as in (2.6), and set

$$\tau_{i,\alpha}^{0,k} := \begin{cases} 0 & \text{if } i \in \mathcal{U}_\alpha^{0,k}, \\ +\infty & \text{otherwise,} \end{cases} \qquad \text{for } k = 1, \cdots, p \ \text{ and } \alpha = \pm.$$

**Loop:** for $n \geq 1$,

1. Compute $\tilde{\tau}^{n-1,k}$ on $\mathrm{NB}^{n-1,k}$ as follows: for $\alpha = \pm$, define

$$\tilde{\tau}_{i,\alpha}^{n-1,k} := \begin{cases} \tau_{\bar{\imath},\alpha}^{n-1,k} + \dfrac{\Delta x}{|\widehat{f}_\alpha(x_i)|} & \text{if } i \in \mathrm{NB}_\alpha^{n-1,k}, \\ \\ +\infty & \text{otherwise,} \end{cases}$$

   where $\bar{\imath} \in \mathcal{U}_\alpha^{n-1,k}(i)$, and set

$$\tilde{\tau}_i^{n-1,k} := \min_{\alpha \in \{+,-\}} \tilde{\tau}_{i,\alpha}^{n-1,k}.$$

2. Set $t_n := \min \left\{ \tilde{\tau}_i^{n-1,k},\ i \in \mathrm{NB}^{n-1,k},\ k \in \{1, ..., p\} \right\}$.
3. Define the new accepted point

$$\mathrm{NA}^{n,k} = \{ i \in \mathrm{NB}^{n-1,k},\ \tilde{\tau}_i^{n-1,k} = t_n \}.$$

4. Update the values of $\theta^{n,k}$:

$$\theta_i^{n,k} = \begin{cases} -\theta_i^{n-1,k} & \text{if } i \in \mathrm{NA}^{n,k} \\ \theta_i^{n-1,k} & \text{otherwise} \end{cases}$$

5. Reinitialize $\tau_{i,\alpha}^{n,k}$ on $\mathcal{U}_\alpha^{n,k}$:

$$\tau_{i,\alpha}^{n,k} = \begin{cases} \min(t_n, \tau_{i,\alpha}^{n-1,k}) & \text{if } i \in \mathcal{U}_\alpha^{n,k} \\ +\infty & \text{otherwise} \end{cases}$$

6. If $t_n \geq T$ then stop. Else, set $n := n+1$.

**Remark 2.4.** *Let us remark that in our algorithm, the minimum time $t_n$ is taken on all the level sets. This allows us in particular to have a comparison principle between the level sets (see Corollary 2.5).*

### 2.2.2 Main results for the FMM scheme

We extend the function $\theta^{n,k}$ in the following way

$$\theta^{\rho,k}(t,x) := \theta_i^{n,k} \quad \text{if } x \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} + \Delta x) \text{ and } t \in [t_n, t_{n+1}) \tag{2.8}$$

where $\rho$ denotes $(\Delta x, h)$. Hence, we define a function $\vartheta^\rho$ by:

$$\vartheta^\rho(t,x) := \sum_{k=1}^{p} \left( \frac{\theta^{\rho,k}(t,x)+1}{2} \right) h_k. \tag{2.9}$$

First we shall check that $\vartheta^\rho$ well define a numerical approximation of the solution $\vartheta$ of (1.1). This claim is a consequence of a comparison principles for the numerical level-set functions $(\theta^{\rho,k})$.

**Theorem 2.4. (Discrete comparison principle)**
*Let $1 < k_1 < k_2 \leq p$. For all $n \in \mathbb{N}$ and all $i \in \mathbb{Z}$, we have either*

$$\theta_i^{n,k_1} > \theta_i^{n,k_2}$$

*or*

$$\theta_i^{n,k_1} = \theta_i^{n,k_2} =: \sigma_i = \pm 1$$

*and if $i \in \mathcal{U}_\alpha^{n,k_1} \cap \mathcal{U}_\alpha^{n,k_2}$, then*

$$\begin{cases} \tau_{i,\alpha}^{n,k_1} \leq \tau_{i,\alpha}^{n,k_2} & \text{if} \quad \sigma_i = +1 \\ \tau_{i,\alpha}^{n,k_1} \geq \tau_{i,\alpha}^{n,k_2} & \text{if} \quad \sigma_i = -1 \end{cases}$$

The proof of this theorem is technical and is given in Appendix B. A first straightforward consequence of this discrete comparison principle can be formulated as follows:

**Corollary 2.5. (Comparison principle for the level-set functions)**
*Let $1 \leq k_1 < k_2 \leq p$. Then*

$$\theta^{\rho,k_2} \leq \theta^{\rho,k_1}.$$

Now, we can give the statement of the main result of this section. (The proof will be done in Section 4.)

**Theorem 2.6. (Convergence of the FMM scheme)**
*Assume (H1)-(H4), and let $\rho = (\Delta x, h)$ with $\Delta x < \min(\frac{\varepsilon}{L}, \delta_0)$ and $h$ as in (2.3). The numerical*

*solution $\vartheta^\rho$ given by the FMM scheme, defined as in (2.9), converges to the viscosity solution $\vartheta$ of (1.1), and for $t \geq 0$, the following error estimate holds:*

$$\|\vartheta^\rho(t,.) - \vartheta(t,.)\|_{L^1(\mathbb{R})} \leq \left( \left( \frac{5}{2} e^{Lt} + 3Lte^{Lt} \right) TV(v_0) + M_1 \right) \Delta x + e^{Lt}(\beta - \alpha + 2M_0\, t)h, \quad (2.10)$$

*where*

$$M_0 = M_0(v_0, f) := \sum_{i=2,\ldots,q} (|f_+(A_i)| + |f_-(A_i)|) \quad \text{and} \quad M_1 = \max_{j, \exists i,\, A_i \in \bar{I}_j} \|v_0'\|_{L^\infty(I_j)}$$

*are constant.*

**Remark 2.5.** *Furthermore if $h$ is chosen to be of the order of $\Delta x$ (for instance using $h_k \equiv h := \Delta x$, $\forall k$), we deduce a global estimate of order $\Delta x$ in $L^1$-error.*

**Remark 2.6.** *In the level set decomposition, we can choose $(h_j)_j$ such that $v_0(A_i) = w_0(A_i)$ for $i = 2, \ldots, q$. In this case, assumption (H4) is not needed (see the proof of Proposition 2.3 in the appendix).*

**Remark 2.7.** *When the velocities $f_+$ and $f_-$ depend on time, it is possible to adapt the algorithm as in [12] and to obtain the comparison principle and the convergence result (in the same way as [7, 12]). Nevertheless, we are not able, in this case, to prove the $L^1$ error estimate.*

## 2.3 Ultra-Bee scheme

### 2.3.1 Algorithm (UB)

Let $\Delta t > 0$ be a constant time step, and $t_n := n\Delta t$ for $n \geq 0$. Let us notice that in the FMM approach, each iteration takes into account the evolution of all levels-set functions $w_k$. On the contrary, the UB scheme should be performed starting from each $w_{0,k}$ independently of the others.

This scheme aims to compute, for every $k = 1, \cdots, p$, a numerical approximation of the averages $\overline{w}_j^{n,k} := \frac{1}{\Delta x} \int_{I_j} w_k(t_n, x)\, dx$, for $j \in \mathbb{Z}$. Since the function $w_k(t_n, \cdot)$ takes only values in $\{0, 1\}$, their averages $\overline{w}_j^{n,k}$ contain the information of the discontinuities localization. The UB scheme gives an accurate approximation of $(\overline{w}_j^{n,k})_j$ as long as the discontinuities are separated by more than $2\Delta x$. Otherwise, when two discontinuities are sufficiently close, a truncation step is made in order to avoid numerical diffusion around these discontinuities. The scheme takes the following form.

$$\frac{V_j^{n+1,1} - V_j^n}{\Delta t} + \max_{\alpha=\pm} \left( f_\alpha(x_j) \frac{V^{n,L}_{j+\frac{1}{2},\alpha} - V^{n,R}_{j-\frac{1}{2},\alpha}}{\Delta x} \right) = 0, \quad (2.11a)$$

$$V_j^{n+1} = [\mathrm{Trunc}(V^{n+1,1})]_j, \quad (2.11b)$$

with the initialization:

$$V_j^0 := \frac{1}{\Delta x} \int_{I_j} w_{0,k}(x) dx. \quad (2.12)$$

Here $V^{n,L}_{j+\frac{1}{2},\alpha}$ and $V^{n,R}_{j+\frac{1}{2},\alpha}$ are numerical *fluxes* that will be defined below, while Trunc denotes a truncation operator that will also be made precise below.

We first set, for $j \in \mathbb{Z}$ and $\alpha = \pm$,

$$\nu_{j,\alpha} := \frac{\Delta t}{\Delta x} f_\alpha(x_j),$$

the "local CFL" number. We assume that

$$|\nu_{j,\alpha}| \leq 1, \quad \forall j \in \mathbb{Z}, \;\text{ and for } \alpha = \pm. \tag{2.13}$$

We also consider

$$\text{if } \nu_{j,\alpha} > 0, \quad \begin{cases} b^+_{j,\alpha} := \max(V^n_j, V^n_{j-1}) + \dfrac{1}{\nu_{j,\alpha}}(V^n_j - \max(V^n_j, V^n_{j-1})), \\ B^+_{j,\alpha} := \min(V^n_j, V^n_{j-1}) + \dfrac{1}{\nu_{j,\alpha}}(V^n_j - \min(V^n_j, V^n_{j-1})), \end{cases} \tag{2.14a}$$

$$\text{if } \nu_{j,\alpha} < 0, \quad \begin{cases} b^-_{j,\alpha} := \max(V^n_j, V^n_{j+1}) + \dfrac{1}{|\nu_{j,\alpha}|}(V^n_j - \max(V^n_j, V^n_{j+1})), \\ B^-_{j,\alpha} := \min(V^n_j, V^n_{j+1}) + \dfrac{1}{|\nu_{j,\alpha}|}(V^n_j - \min(V^n_j, V^n_{j+1})). \end{cases} \tag{2.14b}$$

Under condition (2.13), these numbers satisfy $b^+_{j,\alpha} \leq B^+_{j,\alpha}$, $b^-_{j,\alpha} \leq B^-_{j,\alpha}$, and correspond to flux limiters that ensure stability properties.

Now, we define the UB scheme as follows (see [5, 4]).

**UB Algorithm:** For each level-set function $w_{0,k}$, $k = 1, \cdots, p$, we consider the following evolution algorithm.

**Initialization:** We compute the initial averages $(V^0_j)_{j\in\mathbb{Z}}$ as in (2.12).

**Loop:** For $n \geq 0$, We compute $V^{n+1} = (V^{n+1}_j)_{j\in\mathbb{Z}}$ by:

*A) Evolution.* Define "fluxes" $V^n_{j+\frac{1}{2},\alpha}$ for $\alpha \in \{-,+\}$ as follows:

- If $\nu_{j,\alpha} \geq 0$, set

$$V^{n,L}_{j+\frac{1}{2},\alpha} := \begin{cases} \min(\max(V^n_{j+1}, b^+_{j,\alpha}), B^+_{j,\alpha}) & \text{if } \nu_{j,\alpha} > 0 \\ V^n_{j+1} & \text{if } \nu_{j,\alpha} = 0 \text{ and } V^n_j \neq V^n_{j-1} \\ V^n_j & \text{if } \nu_{j,\alpha} = 0 \text{ and } V^n_j = V^n_{j-1}, \end{cases}$$

- If $\nu_{j,\alpha} \leq 0$, set

$$V^{n,R}_{j-\frac{1}{2},\alpha} := \begin{cases} \min(\max(V^n_{j-1}, b^-_{j,\alpha}), B^-_{j,\alpha}) & \text{if } \nu_j < 0 \\ V^n_{j-1} & \text{if } \nu_{j,\alpha} = 0 \text{ and } V^n_j \neq V^n_{j+1} \\ V^n_j & \text{if } \nu_{j,\alpha} = 0 \text{ and } V^n_j = V^n_{j+1}, \end{cases}$$

  (where $b^+_{j,\alpha}$, $b^-_{j,\alpha}$, $B^+_{j,\alpha}$ and $B^-_{j,\alpha}$ are defined by (2.14a)-(2.14b)).
- If $\nu_{j,\alpha} \geq 0$ and $\nu_{j+1,\alpha} > 0$, set $V^{n,R}_{j+\frac{1}{2},\alpha} := V^{n,L}_{j+\frac{1}{2},\alpha}$.
- If $\nu_{j+1,\alpha} \leq 0$ and $\nu_{j,\alpha} < 0$, set $V^{n,L}_{j+\frac{1}{2},\alpha} := V^{n,R}_{j+\frac{1}{2},\alpha}$.
- If $\nu_{j,\alpha} < 0$ and $\nu_{j+1,\alpha} > 0$, then set

$$V^{n,R}_{j+\frac{1}{2},\alpha} := \begin{cases} V^n_{j+1} & \text{if } V^n_{j+1} = V^n_{j+2} \\ V^n_j & \text{otherwise} \end{cases} \quad \text{and } V^{n,L}_{j+\frac{1}{2},\alpha} := \begin{cases} V^n_j & \text{if } V^n_j = V^n_{j-1} \\ V^n_{j+1} & \text{otherwise.} \end{cases} \tag{2.15}$$

Set $V_j^{n+1,1} := \min_{\{\alpha=\pm\}}\left(V_j^n - \nu_{j,\alpha}\left(V^{n,L}_{j+\frac{1}{2},\alpha} - V^{n,R}_{j-\frac{1}{2},\alpha}\right)\right)$.

*B) Truncation:* Set $V^{n+1} := \mathrm{Trunc}(V^{n+1,1})$ as follows.

- For all indexes $j$ such that

$$\begin{cases} \max(V_{j-1}^{n+1,1}, V_{j+1}^{n+1,1}) < 1, \text{ and } V_j^{n+1,1} = 1 \\ \quad \text{or} \\ 0 < \max(V_j^{n+1,1}, V_{j+1}^{n+1,1}) < 1, \text{ and } V_{j-1}^{n+1,1} = V_{j+2}^{n+1,1} = 0, \end{cases}$$

set

$$V_{j-1}^{n+1} = V_j^{n+1} = V_{j+1}^{n+1} := 0.$$

- Otherwise set $V_j^{n+1} := V_j^{n+1,1}$.

For every $k = 1, \cdots, p$, we associate to the scheme values $(V_j^n)_j$, the l.s.c. step function $\vartheta_k^\rho$ defined for every $t \geq 0$, $x \in \mathbb{R}$ by

$$\vartheta_k^\rho(t,x) := \begin{cases} V_j^n & \text{if } x \in ]x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}[, t \in [t_n, t_{n+1}[, \\ \min(V_j^n, V_{j+1}^n) & \text{if } x = x_{j+\frac{1}{2}}, t \in [t_n, t_{n+1}[. \end{cases} \tag{2.16}$$

The UB scheme approximation of the solution $\vartheta$ of (1.1) is finally determined by

$$\vartheta^\rho(t,x) := \sum_{k=1}^{p} h_k\, \vartheta_k^\rho(t,x). \tag{2.17}$$

**Remark 2.8.** *A general version of the Ultra-bee scheme is given in [4], for any l.s.c. initial condition in $L^1_{loc}(\mathbb{R})$. Here, the algorithm (and specially the truncation step) is specified to the case of an initial condition taking values only in $\{0,1\}$. Also, in the algorithm of [4] there is a prediction step that is unnecessary in our context.*

Several stability and convergence properties of this scheme can be found in [5, 4].

### 2.3.2 Convergence Result for the Ultra-Bee scheme

For every $k = 1, \cdots, p$, the function $\vartheta_k^\rho$ (given by (2.16)) corresponds to a step wise function of level $k$. We have the following $L^1$ error estimate between the solution $\vartheta$ of (1.1) and its numerical approximation $\vartheta^\rho$ given by the Ultra-Bee scheme. (Proof is postponed to the end of Section 3.)

**Theorem 2.7. (Error estimate)**
*Assume (H1)-(H4). We assume that $\Delta x \leq \min(\frac{\varepsilon}{2L}, \delta_0)$. Let $\vartheta^\rho$ be as in (2.17) defined by the algorithm UB. Then for all $t_n \geq 0$, we have the following estimate:*

$$\|\vartheta^\rho(t_n,.) - \vartheta(t_n,.)\|_{L^1(\mathbb{R})} \leq \left((1 + e^{Lt_n} + Lt_n e^{Lt_n})TV(v_0) + M_1\right)\Delta x + e^{Lt_n}(\beta - \alpha + 2M_0\, t_n)h,$$

*where $M_0$ and $M_1$ are the same constant as in Theorem 2.6.*

**Proof of Theorem 2.7:** From [4, Theorem 4], for every $k = 1, \cdots, p$ we have

$$\|\vartheta_k^\rho(t_n, .) - w_k(t_n, .)\|_{L^1(\mathbb{R})} \leq (Lt_n e^{Lt_n} + 1) TV(w_{0,k}) \Delta x.$$

Summing for $k = 1, \ldots, p$, we obtain

$$\|\vartheta^\rho(t_n, .) - w(t_n, .)\|_{L^1(\mathbb{R})} \leq (Lt_n e^{Lt_n} + 1) TV(v_0) \Delta x$$

where we have used that $\sum_{k=1}^{p} h_k TV(w_{0,k}) = TV(w_0) \leq TV(v_0)$ (see for instance [4, Lemma B.3]). Together with Proposition 2.3, we get the desired estimate. $\square$

### 2.4 Some remarks on the schemes

It is important to note that the FMM scheme does not use any time grid and the time-step is adapted, at each iteration, according to the displacement of the fronts. The UB scheme needs a time grid with a mesh satisfying the CFL condition. The later is essential to ensure the stability and the convergence of the scheme.

On the other hand, as we already pointed out, each iteration of the FMM takes into account the evolution of all the level-sets. While the UB scheme makes evolve each level-set independently of the others, hence it is possible to parallelize the computations.

Also let us note that the two schemes allow to concentrate calculations only around the fronts. Indeed, computation in the FMM are carried out only in the narrow-band, and the UB scheme requires calculations only around discontinuities, thanks to its nice antidiffusive property.

Therefore, the two schemes (with the level-set decomposition) can be efficiently implemented in order to have a cpu time computation comparable to classical schemes.

## 3 Preliminary results

From now on, consider an initial condition $v_0$ satisfying (H3), $\vartheta$ solution of (1.1) with the initial condition $v_0$, and the level set decomposition $w_0$ of $v_0$ ($w_0$ defined as in (2.4)).

**Mesh approximation.** First, we define exact and approximated characteristics that will be useful throughout the paper. As the dynamics $f_-$ and $f_+$ are Lipschitz continuous, then for any $a \in \mathbb{R}$ we can define characteristics $X_{a,+}$ and $X_{a,-}$ as the solutions of the following Cauchy problems:

$$\begin{cases} \dot{X}_{a,+}(t) = f_+(X_{a,+}(t)), \\ X_{a,+}(0) = a, \end{cases} \quad \text{and} \quad \begin{cases} \dot{X}_{a,-}(t) = f_-(X_{a,-}(t)). \\ X_{a,-}(0) = a. \end{cases} \tag{3.18}$$

In general, the differential equation

$$\dot{\chi}_a(t) = \widehat{f}_+(\chi_a(t)), \ a.e. \ t \geq 0, \quad \chi_a(0) = a, \tag{3.19}$$

may have more than one absolutely continuous solution. The non-uniqueness comes from the behaviour on boundary points ($x_{j+\frac{1}{2}}$) in the case when the velocity vanishes (or changes sign). Throughout this paper, we shall denote by $X_{a,+}^S$ the function defined by:

$$X_{a,+}^S \text{ is an absolutely continuous solution of (3.19), and} \tag{3.20}$$

$$\text{if} \left( \exists t^* \geq 0, \ \exists j \in \mathbb{Z} \text{ s.t.} \begin{cases} X_{a,+}^S(t^*) = x_{j+\frac{1}{2}}, \\ f_+(x_j) f_+(x_{j+1}) \leq 0 \end{cases} \right) \text{ then } X_{a,+}^S(t) = x_{j+\frac{1}{2}} \ \forall t \geq t^*.$$

We have uniqueness of such solution (see [4, Appendix A]). We construct $X^S_{a,-}$ in a similar way. By using the same arguments as in [4, Lemma 2.1 and Lemma 5.3], we get:

**Lemma 3.1.** *Assume that (H1) and (H2) hold. Let $a, b$ be in $\mathbb{R}$. The following assertions are satisfied:*
*(i) Let $s \le t$ and assume that $X^S_{b,-}(\theta) \ge X^S_{a,+}(\theta)$, for every $\theta \in [s,t]$. Then*

$$|X^S_{b,-}(t) - X^S_{a,+}(t)| + 2\Delta x \le e^{L(t-s)}(|X^S_{b,-}(s) - X^S_{a,+}(s)| + 2\Delta x).$$

*(ii) If $a \ge b + \Delta x$ and $\Delta x \le \frac{\varepsilon}{L}$, then $X^S_{a,+}(t) \ge X^S_{b,-}(t) + \Delta x$ for every $t \ge 0$.*

Also, we have the following representation of the solutions $w_k$, $k = 1, \cdots, p$.

**Lemma 3.2. ([4, Lemma 2.2])**
*Assume that (H1)-(H2) hold. Then the unique viscosity solution of* (1.1) *with initial condition $w_{0,k}$ is given by:*

$$w_k(t,x) = \min_{y \in [X_{x,+}(-t), X_{x,-}(-t)]} w_{0,k}(y), \quad \forall t > 0,\ x \in \mathbb{R}. \tag{3.21}$$

We also consider the function $w^S_k$ which is defined in an analogous way as in (3.21), but with the approximated characteristics $X^S_{x,+}, X^S_{x,-}$ instead of $X_{x,+}$ and $X_{x,-}$:

$$w^S_k(t,x) := \min_{y \in [X^S_{x,+}(-t), X^S_{x,-}(-t)]} w_{0,k}(y), \quad \forall t > 0,\ x \in \mathbb{R}, \tag{3.22}$$

The approximate function $w^S_k$ will play an important role. Indeed, the two studied schemes give approximation of the function $w^S_k$. By using the same arguments as in [4, Proposition 1], we have the following $L^1$-error estimate:

$$\|w^S_k(t,.) - w_k(t,.)\|_{L^1(\mathbb{R})} \le 3Lte^{Lt}\, TV(w_{0,k})\, \Delta x. \tag{3.23}$$

**Proposition 3.3. (Reconstruction of global approximation)**
*Assume (H1)-(H4). Let $\rho := (\Delta x, h)$ with $\Delta x \le \delta_0$ and $h$ as in (2.3). Assume that we have constructed, for every $k = 1, \cdots, p$, an approximation $\vartheta^\rho_k$ of $w^S_k$ such that*

$$\|\vartheta^\rho_k(t,.) - w^S_k(t,.)\|_{L^1(\mathbb{R})} \le C_t\, TV(w_{0,k})\, \Delta x, \tag{3.24}$$

*for some constant $C_t \ge 0$. Define a global approximation by*

$$\vartheta^\rho := \sum_{k=1,\ldots,p} h_k \vartheta^\rho_k.$$

*Then we have the estimate*

$$\|\vartheta^\rho(t,.) - \vartheta(t,.)\|_{L^1(\mathbb{R})} \le \big((C_t + e^{Lt} + 3Lte^{Lt})TV(v_0) + M_1\big)\, \Delta x + e^{Lt}(\beta - \alpha + 2M_0\, t)h, \tag{3.25}$$

*where $M_0$ and $M_1$ are defined as in Theorem 2.6.*

**Proof:** Set $w^S(t,.) := \sum_{k=1,\ldots,p} h_k w^S_k(t,.)$. By summing the estimate (3.24) for $k = 1, \ldots, p$, we obtain

$$\|\vartheta^\rho(t,.) - w^S(t,.)\|_{L^1(\mathbb{R})} \quad \le \quad C_t \sum_{k=1}^{p} h_k TV(w_{0,k})\Delta x \le C_t TV(v_0)\Delta x,$$

where we have used that $\sum_{k=1}^{p} h_k TV(w_{0,k}) = TV(w_0) \leq TV(v_0)$ (see for instance [4, Lemma B.3]). On the other hand, by using (3.23), we obtain

$$\|w^S(t,.) - w(t,.)\|_{L^1(\mathbb{R})} \leq 3Lte^{Lt}\, TV(v_0)\, \Delta x.$$

We conclude the proof by combining Proposition 2.3 and the previous bounds. □

# 4 Fast Marching Method

## 4.1 General remarks on the algorithm

**Proposition 4.1.** *The following properties hold:*

*(i) For all* $i \in \mathcal{U}_\alpha^{n,k}$, *we have*

$$\tau_{i,\alpha}^{n,k} \leq t_n$$

*(ii) If* $i \in \mathrm{NA}^{n,k}$, *then*

$$\tau_{i,\alpha}^{n,k} = \begin{cases} t_n & \text{if } i \in \mathcal{U}_\alpha^{n,k} \\ +\infty & \text{otherwise} \end{cases}$$

*(iii) If* $i \in \mathcal{U}_\alpha^{n-1,k} \cap \mathcal{U}_\alpha^{n,k}$, *then*

$$\tau_{i,\alpha}^{n,k} = \tau_{i,\alpha}^{n-1,k}.$$

*(iv) If* $i \in \mathcal{U}_\alpha^{n,k} \backslash \mathcal{U}_\alpha^{n-1,k}$, *then*

$$\tau_{i,\alpha}^{n,k} = t_n.$$

**Proof** $(i)$ This is a straightforward consequence of Step 5 of the algorithm.

$(ii)$ By Step 5 of the algorithm, we just have to prove that if $i \in \mathrm{NA}^{n,k} \cap \mathcal{U}_\alpha^{n,k}$, then $\tau_\alpha^{n-1,k} = +\infty$. Let us consider the case when $\alpha = +$ and $i \in \mathcal{U}_+^{n,k}(i+1)$ (the other cases being similar). Thus, $\widehat{f}_+(x_{i+1}) > 0$, $\theta_i^{n,k} = -1$ and $\theta_{i+1}^{n,k} = 1$. Also, since $i \in NA^n$, then we have:

$$\theta_i^{n-1,k} = 1, \quad \text{and} \quad i \in NB^{n-1,k}. \tag{4.26}$$

Assume that $\tau_+^{n-1,k} < +\infty$. This implies in particular that $i \in \mathcal{U}_+^{n-1,k}$. We claim that: $i \in \mathrm{NB}_-^{n-1,k}$, and $i+1 \in \mathrm{NA}^n$.

Indeed, (4.26) implies that $i \in \mathcal{U}_+^{n-1,k}(i-1)$ and $\widehat{f}_+(x_{i-1}) < 0$ which gives that $i \notin \mathrm{NB}_+^{n-1,k}$. Since $i \in NB^{n-1,k}$, we get that $i \in NB_-^{n-1,k}$. Moreover, the fact that $\theta_i^{n-1,k} = 1$ and $i \in \mathrm{NB}_-^{n-1,k}$ implies that $i+1 \in \mathcal{U}_-^{n-1,k}(i)$. It yields that

$$\widehat{f}_-(x_i) < 0 \quad \text{and} \quad \theta_{i+1}^{n-1,k} = -1.$$

Since $\theta_{i+1}^{n-1,k} = -1 = -\theta_{i+1}^{n,k}$, we deduce that $i+1 \in \mathrm{NA}^n$.

On the other hand, the fact that $\theta_{i+1}^{n-1,k} = -1$ and $\widehat{f}_-(x_i)$ implies that $i+1 \notin \mathrm{NB}_-^{n-1}$. But $i+1 \in \mathrm{NA}^n$ and thus $i+1 \in \mathrm{NB}_+^{n-1,k}$. This implies that $\widehat{f}_+(x_{i+1}) < 0$, which leads to a contradiction.

$(iii)$ If $i \in \mathcal{U}_\alpha^{n-1,k} \cap \mathcal{U}_\alpha^{n,k}$, then by Step 5 of the algorithm, we have

$$\tau_{i,\alpha}^{n,k} = \min(\tau_{i,\alpha}^{n-1,k}, t_n) = \tau_{i,\alpha}^{n-1,k}.$$

$(iv)$ If $i \in \mathcal{U}_\alpha^{n,k} \backslash \mathcal{U}_\alpha^{n-1,k}$, then by Step 5 of the algorithm, we have

$$\tau_{i,\alpha}^{n,k} = \min(\tau_{i,\alpha}^{n-1,k}, t_n) = t_n$$

where we have used that $\tau_{i,\alpha}^{n-1,k} = +\infty$ since $i \notin \mathcal{U}_\alpha^{n-1,k}$. $\square$

## 4.2 Proof of Theorem 2.6

In view of Proposition 3.3 we mainly have to deal with the case $p = 1$ (one level set).

For simplicity of notation we denote $\theta^n \equiv \theta^{n,1}$. Let $(a_i)_{i=1,\dots,d}$, $(b_i)_{i=1,\dots,d} \subset (\mathbb{Z} + \frac{1}{2})\Delta x$ be such that

$$a_1 < b_1 < a_2 < b_2 < \dots < a_d < b_d.$$

We consider an initial data of the form

$$w_0(x) := \sum_{i=1}^{d} 1_{]a_i,b_i[}(x). \tag{4.27}$$

We have the following convergence result for one level set which proof is given in the following subsection:

**Proposition 4.2. (Convergence of the FMM scheme for one level)**
*Assume (H1)-(H2) and $p = 1$. Let $\rho = \Delta x \le \frac{\varepsilon}{L}$. We have the following error estimate between the numerical solution $\vartheta^\rho$ given by the FMM scheme, defined as in* (2.9)*, and the approximate solution $w^S$ defined in* (3.22)

$$\|\vartheta^\rho(t,.) - w^S(t,.)\|_{L^1(\mathbb{R})} \le \frac{3}{2} e^{Lt} TV(v_0)\Delta x. \tag{4.28}$$

The proof of this proposition is given in the following subsection.

**Proof of Theorem 2.6**
Theorem 2.6 is a straightforward consequence of Propositions 3.3 and 4.2.

$\square$

## 4.3 Proof of Proposition 4.2

Before to give the proof of Proposition 4.2, we have to give some definitions and preliminary results.

**Notations.** We define the numerical discontinuities in the following way: Let $a \in (\mathbb{Z} + \frac{1}{2})\Delta x$ and $i_a^0 \in \mathbb{Z}$ such that $a = x_{i_a^0 - \frac{1}{2}}$. Assume that $\theta_{i_a^0}^0 = \alpha 1 = -\theta_{i_a^0 - 1}^0$ for some $\alpha \in \{+,-\}$. In particular, the numerical discontinuity starting from $a$ will move with the velocity $\widehat{f}_\alpha$. We denote by $(t_a^{n,\Delta})_n$ the sequence of time at which the numerical discontinuity starting from $a$ moves and by $X_{a,\alpha}^{S,\Delta}(t_a^{n,\Delta})$ the position of the discontinuity at time $t_a^{n,\Delta}$. More precisely, we define:

$$t_a^{0,\Delta} = 0 \quad \text{and} \quad X_{a,\alpha}^{S,\Delta} = a = x_{i_a^0 - \frac{1}{2}}$$

and for $n \geq 1$,

$$\begin{cases} t_a^{n,\Delta} = \inf\left\{t_m > t_a^{n-1,\Delta} \text{ s.t. } i_a^{n-1} \in \mathrm{NA}^m \text{ or } i_a^{n-1} - 1 \in \mathrm{NA}^m\right\}, \\ \\ i_a^n = \begin{cases} i_a^{n-1} + 1 & \text{if } \widehat{f}_\alpha(x_{i_a^{n-1}}) > 0 \text{ and } i_a^{n-1} \in \mathrm{NA}^m, \\ i_a^{n-1} - 1 & \text{if } \widehat{f}_\alpha(x_{i_a^{n-1}} - \Delta x) < 0 \text{ and } i_a^{n-1} - 1 \in \mathrm{NA}^m, \\ i_a^{n-1} & \text{otherwise} \end{cases} \\ \\ X_{a,\alpha}^{S,\Delta}(t_a^{n,\Delta}) = x_{i_a^n - \frac{1}{2}}. \end{cases} \tag{4.29}$$

We now define the extinction time $T_a^\Delta$ of the numerical discontinuity starting from $a$ by

$$T_a^\Delta = \inf\{t_a^{n,\Delta},\ \theta_{i_a^n}^m = \theta_{i_a^n - 1}^m \text{ for } m \text{ such that } t_m = t_a^{n,\Delta}\}$$

**Remark 4.1.** *In the definition of $i_a^n$ (4.29), we have $i_a^n = i_a^{n-1}$ only if the discontinuity $X_{a,\alpha}^{S,\Delta}$ do not move and disappear.*

**Lemma 4.3. (Profile of the discontinuity)**
*Let $i_a^0 \in \mathbb{N}$ be such that $\theta_{i_a^0 - 1}^0 = \alpha 1 = -\theta_{i_a^0}^0$ with $\alpha \in \{+,-\}$. Let $n \in \mathbb{N}$ be such that $t_a^{n,\Delta} < T_a^\Delta$. Then, for all $m \in \mathbb{N}$ such that $t_a^{n,\Delta} \leq t_m < t_a^{n+1,\Delta}$, we have*

$$\theta_{i_a^n}^m = -\theta_{i_a^n - 1}^m = \alpha 1.$$

This lemma claims in fact that the numerical discontinuity starting from $a$ and evolving with the velocity $f_\alpha$ is located at a discontinuity of $\theta^m$ and always keep the same profile.
**Proof.** Let us assume that $\alpha = +$ (the case $\alpha = -$ being similar). By recursion, let us assume that

$$\theta_{i_a^{n-1}}^l = -\theta_{i_a^{n-1} - 1}^l = \alpha 1$$

for all $l \in \mathbb{N}$ such that $t_a^{n-1,\Delta} \leq t_l < t_a^{n,\Delta}$.

Let us define $m$ such that $t_m = t_a^{n,\Delta}$. We have

$$i_a^{n-1} \in \mathrm{NA}^m \quad \text{or} \quad i_a^{n-1} - 1 \in \mathrm{NA}^m.$$

**Step 1:** $i_a^n \neq i_a^{n-1}$
By contradiction, assume that $i_a^n = i_a^{n-1}$. Let us assume that $i_a^{n-1} \in \mathrm{NA}^m$ (the other case being similar). This implies in particular that $\widehat{f}_+(x_{i_a^{n-1}}) \leq 0$. Moreover, we have $\theta_{i_a^{n-1}}^{m-1} = 1 = -\theta_{i_a^{n-1}-1}^{m-1}$. This implies that $i_a^{n-1} \notin \mathrm{NB}_+^{m-1}$. But $i_a^{n-1} \in \mathrm{NA}^m$ and so $i_a^{m-1} \in \mathrm{NB}_-^{m-1}$ which implies that

$$\widehat{f}_-(x_{i_a^{n-1}}) < 0. \tag{4.30}$$

Since $i_a^{n-1} \in \mathrm{NA}^m$, we also deduce that $\theta_{i_a^n}^m = \theta_{i_a^{n-1}}^m = -1$. But $t_a^{n,\Delta} < T_a^\Delta$ and so $\theta_{i_a^{n-1}-1}^m = \theta_{i_a^n - 1}^m = 1$. This implies that $i_a^{n-1} - 1 \in \mathrm{NA}^m$. Since $i_a^n = i_a^{n-1}$, we then deduce that $\widehat{f}_+(x_{i_a^{n-1}-1}) \geq 0$ and so $i_a^{n-1} - 1 \notin \mathrm{NB}_+^{m-1}$. This implies that $i_a^{n-1} - 1 \in \mathrm{NB}_-^{m-1}$ and so $\widehat{f}_-(x_{i_a^{n-1}-1}) > 0$. This contradicts (4.30).

**Step 2:** $\theta_{i_a^n}^m = -\theta_{i_a^n - 1}^m = 1$
Let us assume that $i_a^n = i_a^{n-1} + 1$ (the case $i_a^n = i_a^{n-1} - 1$ being similar). This implies that

$i_a^{n-1} \in \mathrm{NA}^m$. But $\theta^{m-1}_{i_a^{n-1}} = 1$ and so $\theta^m_{i_a^n-1} = \theta^m_{i_a^{n-1}} = -1$. Moreover, since $t_a^{n,\Delta} < T_a^\Delta$, we deduce that

$$\theta^m_{i_a^n} = -\theta^m_{i_a^n-1} = 1.$$

**Step 3:** conclusion
By definition of $t_a^{n+1,\Delta}$, for all $l$ such that $t_a^{n,\Delta} \le t_l < t_a^{n+1,\Delta}$, we have $i_a^n \notin \mathrm{NA}^l$ and $i_a^n - 1 \notin \mathrm{NA}^l$ and so

$$\theta^l_{i_a^n} = \theta^m_{i_a^n} \quad \text{and} \quad \theta^l_{i_a^n-1} = \theta^m_{i_a^n-1}.$$

By Step 2, we deduce the result.

□

**Lemma 4.4.** **($\tau_{i_a^n,\alpha}$ is the time when the discontinuity $a$ reached the point $x_{i_a^n-\frac{1}{2}}$)**
*Let $i_a^0 \in \mathbb{N}$ be such that $\theta^0_{i_a^0} = \alpha 1 = -\theta^0_{i_a^0-1}$ with $\alpha \in \{+,-\}$. Let $n \in \mathbb{N}$. Assume that $t_a^{n,\Delta} < T_a^\Delta$. Then, for all $m \in \mathbb{N}$ such that $t_a^{n,\Delta} \le t_m < t_a^{n+1,\Delta}$, we have*

$$\begin{cases} \tau^m_{i_a^n,\alpha} = t_a^{n,\Delta} & \text{if} \quad \widehat{f}_\alpha(x_{i_a^n} - \Delta x) < 0 \\ \tau^m_{i_a^n-1,\alpha} = t_a^{n,\Delta} & \text{if} \quad \widehat{f}_\alpha(x_{i_a^n}) > 0 \end{cases}$$

**Remark 4.2.** *If $\widehat{f}_\alpha(x_{i_\alpha^n} - \Delta x) < 0$, since by Lemma 4.3 we have $\theta^m_{i_a^n} = -\theta^m_{i_a^n-1}$, we deduce that $i_a^n \in \mathcal{U}^m_\alpha(i_a^n-1)$ and so $\tau^m_{i_a^n,\alpha}$ is well defined. In the same way, if $\widehat{f}_\alpha(x_{i_a^n}) > 0$, then $i_a^n - 1 \in \mathcal{U}^m_\alpha(i_a^n)$ and $\tau^m_{i_a^n-1,\alpha}$ is well defined.*

**Proof of Lemma 4.4**
We assume that $\alpha = +$ and that $\widehat{f}_\alpha(x_{i_a^n} - \Delta x) < 0$ (the others cases being similes). Let $m^*$ be such that $t_{m^*} = t_a^{n,\Delta}$. The proof is decomposed into two steps:

**Step 1** $\tau^{m^*}_{i_a^n,+} = t_a^{n,\Delta}$.
If $n = 0$, then $m^* = 0$ and $\tau^0_{i_a^n} = 0 = t_0 = t_a^{n,\Delta}$.
Let us treat the case $n \ge 1$. We claim that $i_a^n = i_a^{n-1} - 1$. Indeed, if $i_a^n = i_a^{n-1} + 1$, then, by (4.29), we have $\widehat{f}_+(x_{i_a^n} - \Delta x) = \widehat{f}_+(x_{i_a^{n-1}}) > 0$, which is absurd. By (4.29), we then deduce that $i_a^n = i_a^{n-1} - 1 \in \mathrm{NA}^{m^*}$. This implies that

$$\tau^{m^*}_{i_a^n,+} = t_{m^*} = t_a^{n,\Delta}.$$

**Step 2** $\tau^m_{i_a^n,+} = t_a^{n,\Delta}$ for all $m$ such that $t_a^{n,\Delta} < t_m < t_a^{n+1,\Delta}$.
By Lemma 4.3, for all $m$ such that $t_a^{n,\Delta} < t_m < t_a^{n+1,\Delta}$, we have $\theta^m_{i_a^n} = -\theta^m_{i_a^n-1} = 1$. Since $\widehat{f}_+(x_{i_a^n-1}) < 0$, this implies that $i_a^n \in \mathcal{U}^m_+(i_a^n - 1)$ for all $m$ such that $t_a^{n,\Delta} < t_m < t_a^{n+1,\Delta}$. By Proposition 4.1 (iii), we then get that

$$\tau^m_{i_a^n,+} = \tau^{m^*}_{i_a^n,+} = t_a^{n,\Delta}.$$

This ends the proof of the lemma. □

For $a \in (\mathbb{Z} + \frac{1}{2})\Delta x$, we also define the time when the discontinuity $X^S_{a,+}$ changes of mesh by

$$\begin{cases} t_a^0 = 0 \\ \forall n \ge 0,\ t_a^{n+1} = \inf\{t > t_a^n, |X^S_{a,+}(t) - X^S_{a,+}(t_a^n)| = \Delta x\}. \end{cases}$$

For $b \in (\mathbb{Z}+\frac{1}{2})\Delta x$, we define the time when the discontinuity $X^S_{b,-}$ changes of mesh in the same way. We define the extinction time of the discontinuity $a_i$ and $b_i$ by

$$T_{a_i} = T_{b_i} := \inf\{t, X^S_{a_i,+}(t) = X^S_{b_i,-}(t)\}.$$

The following proposition claims that the FMM scheme computes exactly the position of the discontinuity before it disappear.

**Proposition 4.5. (Exact computation of the discontinuity before the meeting of discontinuity)**
*Let $a \in (\mathbb{Z}+\frac{1}{2})\Delta x$ and $i^0_a \in \mathbb{N}$ be such that $a = x_{i^0_a - \frac{1}{2}}$. We assume that $\theta^0_{i^0_a} = \alpha 1 = -\theta^0_{i^0_a - 1}$ with $\alpha \in \{+,-\}$. For all $n \in \mathbb{N}$, if $t^{n,\Delta}_a < T^\Delta_a$, then*

$$\begin{cases} t^n_a = t^{n,\Delta}_a \\ X^S_{a,\alpha}(t^n_a) = X^{S,\Delta}_{a,\alpha}(t^{n,\Delta}_a). \end{cases}$$

**Proof of Proposition 4.5**
Let us assume that $\alpha = +$, the case $\alpha = -$ being similar. By recurrence, assume that

$$\begin{cases} t^{n-1}_a = t^{n-1,\Delta}_a \\ X^S_{a,+}(t^{n-1}_a) = X^{S,\Delta}_{a,+}(t^{n-1,\Delta}_a). \end{cases}$$

Since $t^{n,\Delta}_a < \infty$, we have either

$$\widehat{f}_+(X^S_{a,+}(t^{n-1}_a) - \Delta x) < 0 \quad \text{or} \quad \widehat{f}_+(X^S_{a,+}(t^{n-1}_a)) > 0.$$

Let us assume that $\widehat{f}_+(X^S_{a,+}(t^{n-1}_a) - \Delta x) < 0$ (the other case being similar). This implies that the discontinuities $X^S_{a,+}(t^{n-1}_a)$ and $X^{S,\Delta}_{a,+}(t^{n-1}_a)$ will move to the left. For the approached discontinuity, we have

$$t^n_a = t^{n-1}_a + \frac{\Delta x}{|\widehat{f}^+(X^S_{a,+}(t^{n-1}_a) - \Delta x)|} \quad \text{and} \quad X^S_{a,+}(t^n_a) = X^S_{a,+}(t^{n-1}_a) - \Delta x.$$

We now turn to the numerical discontinuity. We define $m$ such that $t_m = t^{n,\Delta}_a$. In particular, since $\widehat{f}_+(x_{i^{n-1}_a - 1}) < 0$, we have $i^n_a = i^{n-1}_a - 1 \in \mathrm{NA}^m$.

Let us now compute $t_m$. For simplicity of notation, let us denote $i^{n-1}_a - 1$ by $i$. By the algorithm, we have

$$\tilde{\tau}^{m-1}_{i,+} = \tau^{m-1}_{i^n_a,+} + \frac{\Delta x}{|\widehat{f}_+(x_i)|}.$$

We now claim that $i \notin \mathrm{NB}^{n-1}_-$. By contradiction, assume that $i \in \mathrm{NB}^{n-1}_-$. By Lemma 4.3, we have $\theta^{m-1}_i = -1$. This implies that $\theta^{m-1}_{i-1} = 1$ and $\widehat{f}_-(x_i) > 0$. This gives in particular that $i-1 \notin \mathrm{NB}^{m-1}_-$. Moreover, since $\widehat{f}_+(x_{i-1}) \leq 0$, we also have deduce that $i-1 \notin \mathrm{NB}^{m-1}_+$ and so $i-1 \notin \mathrm{NA}^m$ which gives

$$\theta^m_{i-1} = \theta^{m-1}_{i-1} = 1.$$

Since $i \in \mathrm{NA}^m$, we also deduce that $\theta^m_i = 1$. This contradicts the fact that $t^{n,\Delta}_a < T^\Delta_a$ and proves that $i \notin \mathrm{NB}^{n-1}_-$.

We then deduce that $\tilde{\tau}_{i,-}^{n-1} = +\infty$ and so

$$t_m = \tilde{\tau}_i^{m-1} = \tilde{\tau}_{i,+}^{m-1}.$$

By Lemma 4.4, we have $\tau_{i_a^n,+}^{m-1} = t_a^{n-1,\Delta}$. So we recover that $t_a^{n,\Delta} = t_m = t_a^n$.

Moreover, we have $i_a^n = i_a^{n-1} - 1$ and

$$X_a^{S,\Delta}(t_a^{n,\Delta}) = x_{i_a^n - \frac{1}{2}} = x_{i_a^{n-1} - \frac{1}{2}} - \Delta x = X_{a,+}^{S,\Delta}(t_a^{n-1,\Delta}) - \Delta x = X_{a,+}^{S}(t_a^{n-1}) - \Delta x = X_{a,+}^{S}(t_a^n).$$

This ends the proof of the proposition. $\square$

The following proposition claims that the discontinuities $X_{a_i,+}^{S,\Delta}(t)$ and $X_{b_{i-1},-}^{S,\Delta}(t)$ cannot meet. This essentially comes from our assumption (H2).

**Proposition 4.6. (No meeting for minima)**
*Assume (H2) and $\Delta x \leq \frac{\varepsilon}{L}$. Let $(a_i)_{i=1,...,d}$, $(b_i)_{i=1,...,d} \subset (\mathbb{Z} + \frac{1}{2})\Delta x$ be such that*

$$a_1 < b_1 < a_2 < b_2 < ... < a_d < b_d.$$

*Assume that*

$$\theta_i^0 = \begin{cases} 1 & \text{if } \exists\, j \text{ s.t } \quad a_j < x_i < b_j \\ -1 & \text{if } \exists\, j \text{ s.t } \quad b_j < x_i < a_{j+1} \end{cases}$$

*Then, for all $t \geq 0$, $i \in \{1, ..., d-1\}$, we have*

$$X_{a_i,+}^{S,\Delta}(t) - X_{b_{i-1},-}^{S,\Delta}(t) \geq \Delta x.$$

**Proof of Proposition 4.6**
By contradiction, let us define

$$t^* = \inf\{t \geq 0,\ \exists i \in \{1, ..., d\} \text{ s.t. } X_{a_i,+}^{S,\Delta}(t) - X_{b_{i-1},-}^{S,\Delta}(t) < \Delta x\}.$$

We denote by $\bar{\imath}$ the index such that $X_{a_{\bar{\imath}},+}^{S,\Delta}(t^*) - X_{b_{\bar{\imath}-1},-}^{S,\Delta}(t^*) < \Delta x$ and by $a = a_{\bar{\imath}}$, $b = b_{\bar{\imath}-1}$. Let us define $n$ and $m$ such that

$$\begin{cases} t_a^{n+1,\Delta} \leq t^* < t_a^{n+2,\Delta} \\ t_b^{m+1,\Delta} \leq t^* < t_b^{m+2,\Delta} \end{cases}$$

In particular, we have either $t^* = t_a^{n+1,\Delta}$ or $t^* = t_b^{m+1,\Delta}$. Finally we define $i \in \mathbb{Z}$ such that $x_i = X_{a,+}^{S,\Delta}(t^*) + \frac{1}{2}$. The proof is decomposed into three cases:

**Case 1:** $t_a^{n+1,\Delta} = t_b^{m+1,\Delta} = t^*$
In this case, the two discontinuities have moved at time $t^*$ and we have (since we have $X_{a,+}^{S,\Delta}(t_a^{n,\Delta}) - X_{b,-}^{S,\Delta}(t_b^{m,\Delta}) \geq \Delta x$)

$$i_a^{n+1} = i_b^m = i_b^{m+1} - 1 = i_a^n - 1 \quad \text{or} \quad i_b^m + 1 = i_b^{m+1} = i_a^{n+1} = i_a^n - 1.$$

In the first case, we then have $\widehat{f}_-(x_i) = \widehat{f}_-(x_{i_b^m}) > 0$ and $\widehat{f}_+(x_i) = \widehat{f}_+(x_{i_a^n} - \Delta x) < 0$ which contradicts the fact that $\widehat{f}_+ \geq \widehat{f}_-$.

In the other case, we have

$$0 < \widehat{f}_-(x_{i_b^m}) = \widehat{f}_-(x_{i-1}) = f_-(x_{i-1}) \leq f_+(x_{i-1}) - \varepsilon \leq f_+(x_i) = \widehat{f}_+(x_i) = \widehat{f}_+(x_{i_a^n}) < 0$$

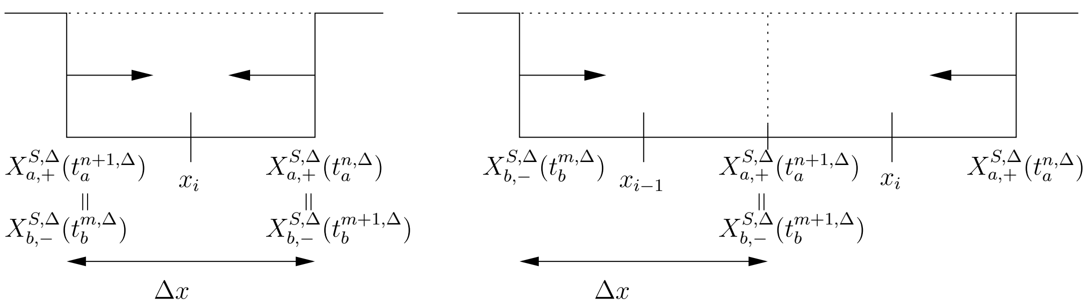



Figure 2: Representation of the discontinuity before the meeting.

where we have used assumption (H2) and the fact that $\widehat{f_\alpha}(x_j) \neq 0$ implies that $\widehat{f_\alpha}(x_j) = f_\alpha(x_j)$. This is absurd.

**Case 2:** $t_b^{m+1,\Delta} < t_a^{n+1,\Delta} = t^*$
In this case, only the discontinuity $X^{S,\Delta}_{a,+}$ has moved at time $t^*$ and we have (since $X^{S,\Delta}_{a,+}(t^{n,\Delta}_a) - X^{S,\Delta}_{b,-}(t^{m,\Delta}_b) \geq \Delta x$)

$$i_b^{m+1} = i_a^{n+1} = i_a^n - 1.$$

We then deduce that $f_+(x_i) = \widehat{f}_+(x_i) = \widehat{f}_+(x_{i_a^{n+1}}) < 0$ and so $\widehat{f}_-(x_i) < 0$. We define $k$ such that

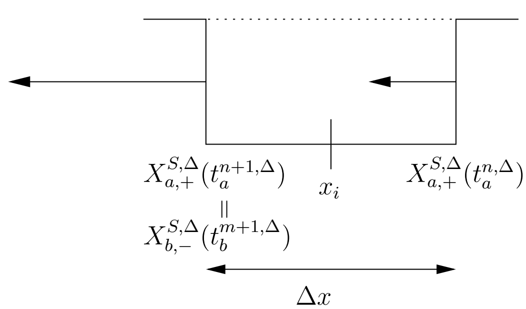



Figure 3: Representation of the discontinuity before the meeting.

$t_k = t_a^{n+1,\Delta} = t^*$.

**Step 1:** Ordering $\tau^{k-1}_{i,-} \leq \tau^{k-1}_{i+1,+}$.
By Lemma 4.4, since $\widehat{f}_+(x_i) = \widehat{f}_+(x_{i_a^{n+1}}) = \widehat{f}_+(x_{i_a^n - 1}) < 0$, we get

$$\tau^{k-1}_{i+1,+} = \tau^{k-1}_{i_a^n,+} = t_a^{n,\Delta}.$$

Moreover, using assumption (H2), we deduce that

$$f_-(x_{i_b^{m+1}-1}) = f_-(x_{i-1}) \leq f_+(x_{i-1}) - \varepsilon \leq f_+(x_i) < 0.$$

We now claim that $\widehat{f}_-(x_{i-1}) = f_-(x_{i-1})$. By contradiction, if it is not the case then (since $f_-(x_{i-1}) < 0$ and $f_-(x_i) < 0$)

$$f_-(x_{i-2}) > 0 \quad \text{and} \quad |f_-(x_{i-1})| \leq |f_-(x_{i-2})|.$$

But, by assumption (H2), we also have

$$f_-(x_{i-1}) \leq f_+(x_i) < 0 \quad \text{and} \quad 0 < f_-(x_{i-2}) \leq f_+(x_{i-1})$$

and so

$$|f_+(x_{i-1})| \geq |f_-(x_{i-2})| \geq |f_-(x_{i-1})| \geq |f_+(x_i)|.$$

This implies that $\widehat{f}_+(x_i) = 0$, which is absurd. This implies that $\widehat{f}_-(x_{i-1}) = f_-(x_{i-1}) < 0$. By Lemma 4.4, we then get

$$\tau_{i,-}^{k-1} = \tau_{i_b^{m+1},-}^{k-1} = t_b^{m+1,\Delta}.$$

To get the result, we then have to prove that $t_b^{m+1,\Delta} \leq t_a^{n,\Delta}$. By contradiction, assume that $t_b^{m+1,\Delta} > t_a^{n,\Delta}$ . We then have

$$t_a^{n,\Delta} < t_b^{m+1,\Delta} < t_a^{n+1,\Delta} < t_b^{m+2,\Delta}.$$

Moreover, at time $\bar{t} = \max(t_b^{m,\Delta}, t_a^{n,\Delta}) < t^*$, we have

$$X_{a,+}^{S,\Delta}(\bar{t}) = X_{a,+}^{S,\Delta}(t_a^{n,\Delta}) = x_{i_a^n - \frac{1}{2}} = x_{i+\frac{1}{2}}, \quad X_{b,+}^{S,\Delta}(\bar{t}) = X_{a,+}^{S,\Delta}(t_b^{m,\Delta}) = x_{i_b^m - \frac{1}{2}} = x_{i+\frac{1}{2}}$$

which contradicts the definition of $t^*$.

**Step 2: contradiction**
We have

$$\begin{aligned} t_a^{n+1,\Delta} = t_k = \tilde{\tau}_{i,+}^{k-1} = &\tau_{i+1,+}^{k-1} + \frac{\Delta x}{|\widehat{f}_+(x_i)|} \geq \tau_{i,-}^{k-1} + \frac{\Delta x}{|\widehat{f}_+(x_{i-1})|} \\ \geq &\tau_{i,-}^{k-1} + \frac{\Delta x}{|\widehat{f}_-(x_{i-1})|} = \tilde{\tau}_{i-1,-}^{k-1} \\ \geq &t_k. \end{aligned}$$

We then deduce that $\tilde{\tau}_{i-1,-}^{k-1} = t_k$ and so $i_b^{m+1} - 1 = i - 1 \in \mathrm{NA}^k$ This implies that $t_b^{m+2,\Delta} = t_k = t^*$. This is absurd.

**Case 3:** $t_a^{n+1,\Delta} < t_b^{m+1,\Delta} = t^*$
This case can be treated in the same way as Case 2.

□

The following lemma claims that when the discontinuities $X_{a,+}^{S,\Delta}(t)$, $X_{a,+}^{S}(t)$, $X_{b,-}^{S,\Delta}(t)$ and $X_{b,-}^{S}$ move in the same direction, then $X_{a,+}^{S,\Delta}(t)$ and $X_{b,-}^{S,\Delta}(t)$ meet before $X_{a,+}^{S}(t)$ and $X_{b,-}^{S}$. This comes from the fact that the numerical discontinuities cannot be in the same mesh.

**Lemma 4.7. (Numerical discontinuities meet before approached discontinuities)**
*Let $a, b \in (\mathbb{Z} + \frac{1}{2})\Delta x$ be such that $a < b$. Let $\theta^0 = 1_{[a,b[}$. Assume that $f_+(a)f_-(b) \geq 0$. Then*

$$T_a^{\Delta} = T_b^{\Delta} \leq T_a = T_b.$$

**Proof of Lemma 4.7**
By contradiction, assume that $T_a^{\Delta} = T_b^{\Delta} > T_a$. We assume that $f_+(a) \geq 0$ and $f_-(b) \geq 0$ (the other case is similar). This implies that the two discontinuities will move to the right. Let us define $n, m$ such that

$$\begin{cases} t_a^n < T_a \leq t_a^{n+1} \\ t_b^m < T_a \leq t_b^{m+1} \end{cases}$$

In particular, since the discontinuities move to the right, we have for all $t \in [t_a^{n,\Delta}, t_a^{n+1,\Delta}[$

$$X_{a,+}^{S,\Delta}(t_a^n) = X_{a,+}^S(t_a^n) \leq X_{a,+}^S(t) < X_{a,+}^S(t_a^n) + \Delta x$$

and

$$X_{b,-}^{S,\Delta}(t_b^m) = X_{b,-}^S(t_b^m) \leq X_{b,-}^S(t) < X_{b,-}^S(t_b^m) + \Delta x$$

Let us define $i = i_a^n$ and $j = i_b^n$. In particular, we have

$$X_{a,+}^S(t) \in I_i \quad \text{and} \quad X_{b,-}^S(t) \in I_j$$

for $t \in [t_a^n, t_a^{n+1}[$.

Since $X_{a,+}^{S,\Delta}(t^*) \neq X_{b,-}^{S,\Delta}(t^*)$ for $t^* = \sup(t_a^n, t_b^n)$, we deduce that $i < j$. This implies in particular that $t_a^{n+1} = T_a = T_b$. Moreover, since $X_{a,+}^S(t) \in I_i$, $X_{b,-}^S(t) \in I_j$ for $t \in [t_a^n, t_a^{n+1}[$, $i < j$ and $X_{a,+}^S(t_a^{n+1}) = X_{b,-}^S(t_a^{n+1})$, we deduce that $i = j - 1$ and

$$X_{a,+}^S(t_a^n) + \Delta x = X_{a,+}^S(T_a) = X_{b,-}^S(t_b^n) = X_{b,-}^S(t^*) = x_{i+\frac{1}{2}}.$$

This implies that the discontinuity $X_{b,-}^S(t_b^n)$ do not move and then $\widehat{f}^-(x_{i+1}) = 0$. We then deduce that $X_{b,-}^{S,\Delta}(t) = x_{i+\frac{1}{2}}$ for all $t \geq t_b^n$.

Moreover, by definition of $t_a^{n+1}$, we have $T_a = t_a^{n+1} = t_a^n + \frac{\Delta x}{\widehat{f}_+(x_i)} = t_a^{n+1,\Delta}$. So at time $t_a^{n+1,\Delta}$, we have

$$X_{a,+}^{S,\Delta}(t_a^{n+1,\Delta}) = X_{b,-}^{S,\Delta}(t_a^{n+1,\Delta}) = x_{i+\frac{1}{2}}$$

and so $T_a^\Delta = T_b^\Delta = t_a^{n+1,\Delta} = T_a$ which is absurd. $\square$

**Proposition 4.8. (Error estimate after the meeting of discontinuities)** *Let $a, b \in (\mathbb{Z} + \frac{1}{2})\Delta x$ be such that $a < b$. Let $\theta^0 = 1_{[a,b[}$. For all $t \in [\inf(T_a, T_a^\Delta), \sup(T_a, T_a^\Delta)[$, we have*

$$\int_{\mathbb{R}} \left| 1_{[X_{a_i,+}^S(t), X_{b_i,-}^S(t)]}(x) - 1_{[X_{a_i,+}^{S,\Delta}(t), X_{b_i,-}^{S,\Delta}(t)]}(x) \right| dx \leq 3\Delta x e^{Lt}. \tag{4.31}$$

**Proof of Proposition 4.8**

If $f_+(a) f_-(b) \geq 0$, then (4.31) is a consequence of Lemma 4.7 and Lemma 3.1 $(i)$.

If $f_+(a) \leq 0$ and $f_-(b) \geq 0$, then the discontinuities $X_{a,+}^S$ and $X_{a,+}^{S,\Delta}$ will move to the left while the discontinuities $X_{b,-}^S$ and $X_{b,-}^{S,\Delta}$ will move to the right. We then deduce that $T_a = T_a^\Delta = \infty$ and so the result is trivial.

Let us then assume that $f_+(a) \geq 0$ and $f_-(b) \leq 0$. Then the discontinuities $X_{a,+}^S$ and $X_{a,+}^{S,\Delta}$ will move to the right while the discontinuities $X_{b,-}^S$ and $X_{b,-}^{S,\Delta}$ will move to the left. Let us assume that $T_a < T_a^\Delta$ (the other case being similar) and let us define $n$, $m$ such that

$$\begin{cases} t_a^n < T_a \leq t_a^{n+1} \\ t_b^m < T_a \leq t_b^{m+1}. \end{cases}$$

We recall that, by Proposition 4.5, we have

$$X_{a,+}^S(t_a^n) = X_{a,+}^{S,\Delta}(t_a^n) = x_{i_a^n} = x_i \quad \text{and} \quad X_{b,-}^S(t_b^m) = X_{b,-}^{S,\Delta}(t_b^m) = x_{i_b^m} = x_j.$$

Moreover, we have $\forall t \in [\sup(t_a^n, t_b^m), T_a)$

$$X_{a,+}^S(t) \in I_i \quad \text{and} \quad X_{b,-}^S(t) \in I_j \quad \text{and} \quad j = i + 1.$$

We then deduce that

$$X^{S,\Delta}_{b,-}(t^*) - X^{S,\Delta}_{a,+}(t^*) \leq \Delta x$$

for $t^* = \sup(t^n_a, t^m_b)$. Since $X^{S,\Delta}_{b,-}$ moves to the left and $X^{S,\Delta}_{a,+}$ moves to the right, we deduce (4.31). $\square$

We are now able to give the proof of Proposition 4.2:

**Proof of Proposition 4.2 (continued).** Recall that we assumed $w_0$ of the form (4.27). By definition of $w^S$ and $\vartheta^\rho$, we have

$$w^S(x,t) = \sum_{i=1}^{d} 1_{[X^S_{a_i,+}(t), X^S_{b_i,-}(t)]}(x) \quad \text{and} \quad \vartheta^\rho(x,t) = \sum_{i=1}^{d} 1_{[X^{S,\Delta}_{a_i,+}(t), X^{S,\Delta}_{b_i,-}(t)]}(x)$$

We then deduce that

$$\|w^S(\cdot,t) - \vartheta^\rho(\cdot,t)\|_{L^1(\mathbb{R})} \leq \sum_{i=1}^{d} \int_{\mathbb{R}} \left| 1_{[X^S_{a_i,+}(t), X^S_{b_i,-}(t)]}(x) - 1_{[X^{S,\Delta}_{a_i,+}(t), X^{S,\Delta}_{b_i,-}(t)]}(x) \right| dx$$

For each $i \in \{1, ..d\}$, we then have to estimate

$$\mathcal{I}_i = \int_{\mathbb{R}} \left| 1_{[X^S_{a_i,+}(t), X^S_{b_i,-}(t)]}(x) - 1_{[X^{S,\Delta}_{a_i,+}(t), X^{S,\Delta}_{b_i,-}(t)]}(x) \right| dx$$

We distinguish three cases:

**Case 1:** $t \geq \sup(T_{a_i}, T^\Delta_{a_i})$
In this case, we have

$$[X^S_{a_i,+}(t), X^S_{b_i,-}(t)] = \emptyset \quad \text{and} \quad [X^{S,\Delta}_{a_i,+}(t), X^{S,\Delta}_{b_i,-}(t)] = \emptyset$$

and so $\mathcal{I}_1 = 0$.

**Case 2:** $\inf(T_{a_i}, T^\Delta_{a_i}) \leq t < \sup(T_{a_i}, T^\Delta_{a_i})$
In this case, by Proposition 4.8, we have $\mathcal{I}_1 \leq 3\Delta x e^{Lt}$.

**Case 3:** $t < \inf(T_{a_i}, T^\Delta_{a_i})$
In this case, we have

$$\begin{aligned}
\mathcal{I}_1 =& \int_{\mathbb{R}} 1_{[X^S_{a_i,+}(t), X^S_{b_i,-}(t)] \Delta [X^{S,\Delta}_{a_i,+}(t), X^{S,\Delta}_{b_i,-}(t)]}(x) dx \\
\leq& |X^S_{a_i,+}(t) - X^{S,\Delta}_{a_i,+}(t)| + |X^S_{b_i,-}(t) - X^{S,\Delta}_{b_i,-}(t)| \\
\leq& 2\Delta x
\end{aligned}$$

where $A\Delta B$ is the symmetric difference of the sets $A$ and $B$ and where we have used Proposition 4.5 for the last line.

We then deduce that we always have

$$\mathcal{I}_i \leq 3e^{Lt}\Delta x$$

and so

$$\|w^S(t,\cdot) - \vartheta^\rho(t,\cdot)\|_{L^1(\mathbb{R})} \leq 3de^{Lt}\Delta x = \frac{3}{2}e^{Lt} TV(v_0)\Delta x$$

since $TV(v_0) = 2d$.

$\square$

# 5 Numerical Simulations

In all the following examples, we consider an equation on $(x_{\min}, x_{\max})$ in the form of:

$$\begin{cases} \vartheta_t + \max(f_-(x)\vartheta_x, f_+\vartheta_x) = 0 \qquad t \geq 0,\ x \in (x_{\min}, x_{\max}), \\ \vartheta(0,x) = v_0(x); \quad x \in (x_{\min}, x_{\max}), \end{cases} \tag{5.32}$$

with periodic boundary conditions. We will denote by $N_x$ the number of mesh points considered in $(x_{\min}, x_{\max})$, and by $p$ the number of levels used in the level-set decomposition of $v_0$ (see (2.3)-(2.4a)).

***Example 1:*** We first consider an advection equation with constant velocity, $f_- = f_+ \equiv 1$, on $(x_{\min}, x_{\max}) = (-2, 2)$ with periodic boundary conditions, with $v_0$ defined as follows:

$$v_0(x) := \begin{cases} 0.64 & \text{if } x \in [0.2, 0.6], \\ \max(1 - x^2, 0) & \text{otherwise} \end{cases}$$

We show in Figure 4, the numerical solution compared to the exact solution, with parameters $N_x = p = 50$ (and CFL number 0.75 for the Ultra-bee Scheme). The exact solution is periodic of period $T = 4$ and we show the solution at time $t = 12$ (3 periods, thus the exact solution recovers its initial position). For this example, we remark a very good behaviour of the both schemes even for long time. The $L^1$ error produced by FMM comes only from the level-set decomposition of $v_0$. Then the advection of each level-set function is exact (constant velocity). The Ultra-Bee scheme provides also a very good solution, and the $L^1$ error corresponding to this algorithm comes from the decomposition of $v_0$ and also from the truncation made around the maxima.

In Table 1 we show the $L^1$-error for the two schemes and see that they are both of first order.

| $Nx = p$ | FMM | UB |
|---|---|---|
| 25 | 3.74e-2 | 4.38e-2 |
| 50 | 1.52e-2 | 1.67e-2 |
| 100 | 0.74e-2 | 0.78e-2 |
| 200 | 0.41e-2 | 0.42e-2 |

Table 1: Advection with constant velocity

***Example 2:*** We consider now the case of $f_- = 0.9$ and $f_+ = 1$. The domain is $(x_{\min}, x_{\max}) = (-2, 2)$, and the initial data is given by

$$v_0(x) := \max(\max(0, 1 - |x|), \max(0, 0.7 - |x - 0.2|)).$$

In Figure 5 we compare the FMM method with the exact solution, and same comparison involving UB scheme is done in Figure 6. In these two tests the discretisation parameters are $N_x = 50$ and $p = 30$. We also summarize the $L^1$-error estimates in Table 2.

***Example 3:*** In this example, we consider an advection equation on the domain $(x_{\min}, x_{\max}) = (0, 1)$ with periodic boundary conditions, and with variable velocity

$$f_-(x) = f_+(x) = 1 + 0.5\sin(2\pi x),$$

The initial data is given by

$$v_0(x) := \begin{cases} \max(1 - 16(x - 0.5)^2, 0) & \text{if } x \in [0, 0.5] \cup [0.6, 1], \\ 0.84 & \text{otherwise,} \end{cases}$$

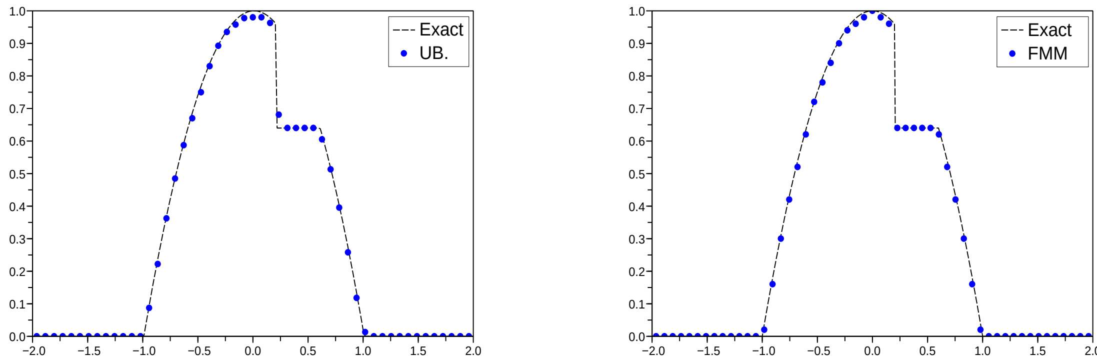



Figure 4: Example 1 at $t = 12$, with $Nx = p = 50$.

| $\Delta x$ | $h$ | FMM | UB |
|---|---|---|---|
| 0.1 | 1/20 | 5.00e-2 | 5.66e-2 |
| 0.05 | 1/40 | 2.37e-2 | 2.70e-2 |
| 0.025 | 1/80 | 1.47e-2 | 1.42e-2 |
| 0.0125 | 1/160 | 0.71e-2 | 0.71e-2 |

Table 2: Example 2, CFL= 0.5

The solution can be shown to be periodic of period $T = 2/\sqrt{3}$. As we can see in Figure 7, we recover very well the initial data after one period, except a little loss of precision at the maximum.

***Example 4:*** In this example, we consider the case of variable velocity functions:

$$f_-(x) = -sin(\pi x) - 0.5, \quad f_+(x) = -\sin(\pi x),$$

on the domain $(x_{\min}, x_{\max}) = (-1, 1)$ with periodic boundary conditions, and with the initial data:

$$v_0(x) := |\sin(\pi(x - 0.5)/2)|.$$

The classical Ultra-Bee scheme (see [4]) tends to project the solution on a class of step functions. The Ultrabee scheme combined with a level-set decomposition does not show this particular behavior (see Figure 8). It furthermore gives a good approximation that is not amplified for longer times.

# A Proof of Proposition 2.3

We first give a stability result by deriving an $L^1$-error estimate between the solution $\vartheta$ of (1.1), corresponding to the initial data $v_0$, and the solution $u$ of (1.1) associated to another initial data $u_0$.

**Proposition A.1.** *Assume (H1) and (H2). Let $u_0$ and $v_0$ be two real valued l.s.c. functions such that $v_0 - u_0 \in L^1(\mathbb{R})$, $v_0$ satisfies (H3), and let $u$ (resp. $\vartheta$) be the l.s.c. solutions of* (1.1) *with*

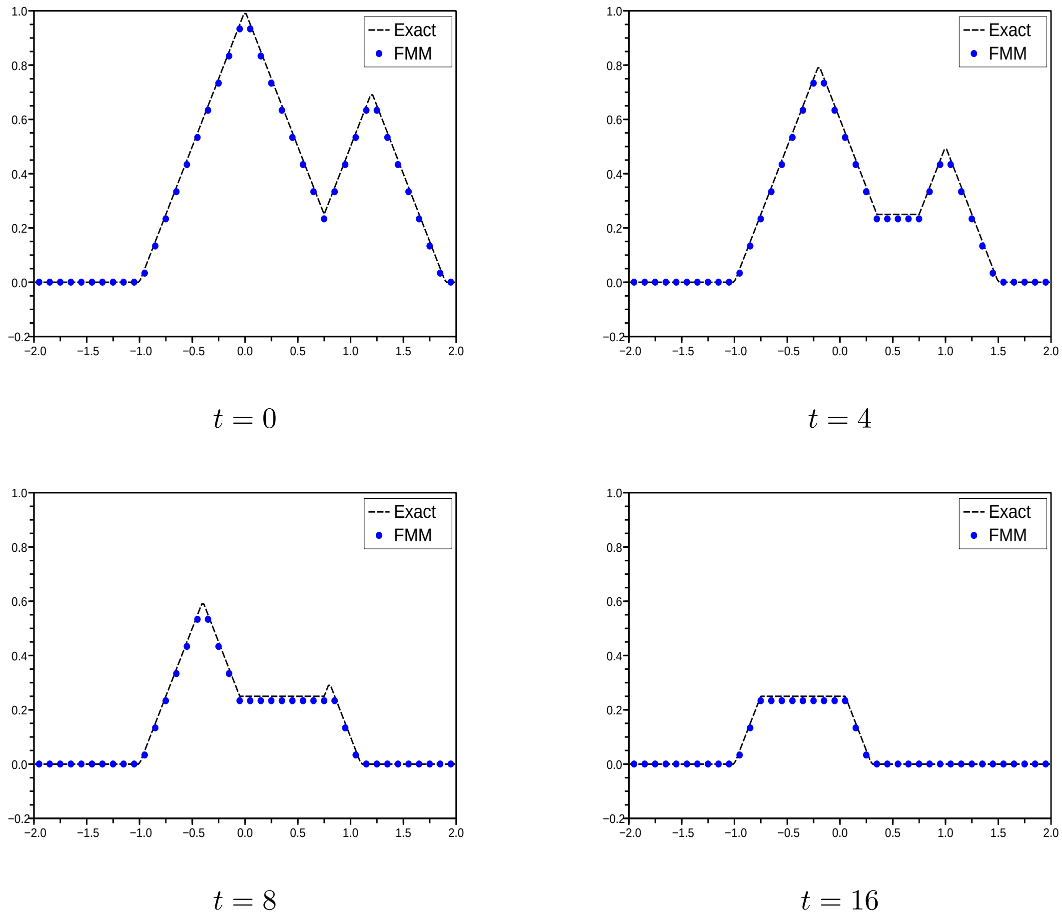



Figure 5: Example 2, FMM scheme with $N_x = 50$ and $p = 30$.

*initial data $u_0$ (resp. $v_0$). We suppose that*

$$\textit{for all interval } I \subset \mathbb{R}, \quad \left\{ \begin{array}{l} v_0 \nearrow \ \textit{ on } I \Rightarrow u_0 \nearrow \ \textit{ on } I, \\ v_0 \searrow \ \textit{ on } I \Rightarrow u_0 \searrow \ \textit{ on } I; \end{array} \right. \tag{1.33}$$

*Then, for all $t \geq 0$, we have*

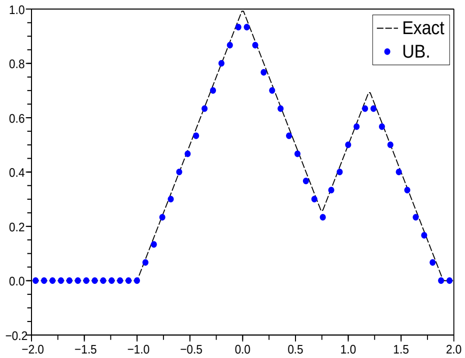
$$\|\vartheta(t,.) - u(t,.)\|_{L^1(\mathbb{R})} \leq e^{Lt} \, \|v_0 - u_0\|_{L^1(\mathbb{R})} + M_0 \, t \, e^{Lt} \max_{i=2,\ldots,q} |v_0(A_i) - u_0(A_i)|. \tag{1.34}$$

*where $M_0 = M_0(v_0, f) := \sum_{i=2,\ldots,q} |f_+(A_i)| + |f_-(A_i)|$.*

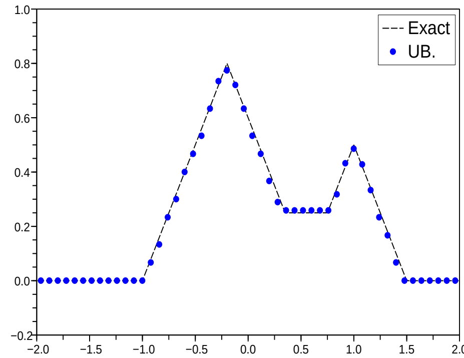
**Remark A.1.** *Indeed in Proposition A.1, instead of (H2) it suffices that $f_-(x) \leq f_+(x)$.*

**Proof of Proposition A.1** We recall that the case when $\forall i = 2, \ldots, q$, $v_0(A_i) = u_0(A_i)$ has already been proved in [4] (in view of assumption (1.33), this case means that the values of $v_0$ and $u_0$ are the same at local minima of $v_0$). In order to treat the general case, we start with the following lemma.

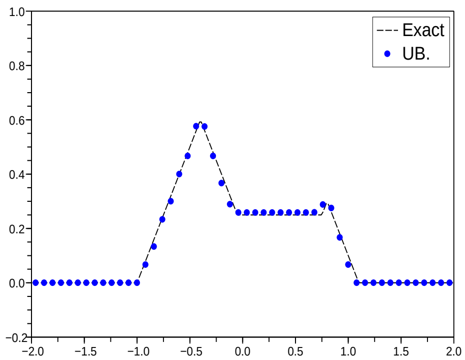
**Lemma A.2.** *Assume (H1)-(H4). Let $\bar{v}_0(x) := v_0(x)$ if $x \notin (A_i)_{i=2,\ldots,q}$ and $\bar{v}_0(A_i) := v_0(A_i) - \delta_i$, with $\delta_i \geq 0$, for $i = 2, \ldots, q$. Let $\bar{\vartheta}$ be the viscosity solution associated to (1.1) with initial data*



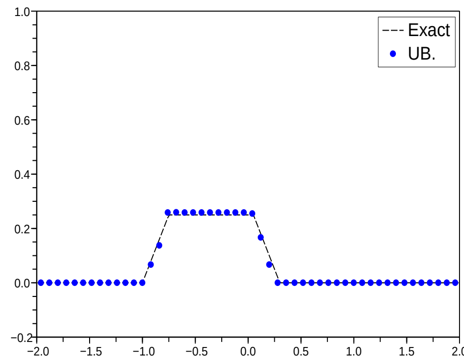

$t = 0$




$t = 4$

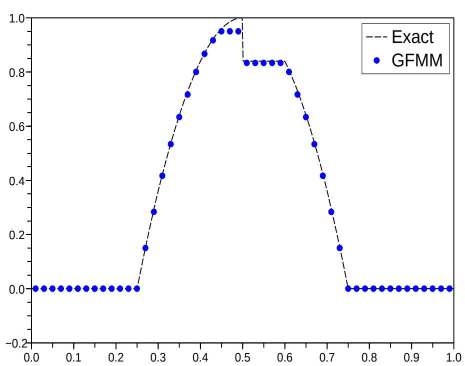



$t = 8$




$t = 16$

Figure 6: Example 2, UB scheme with $N_x = 50$ and $p = 30$ and CFL=0.625



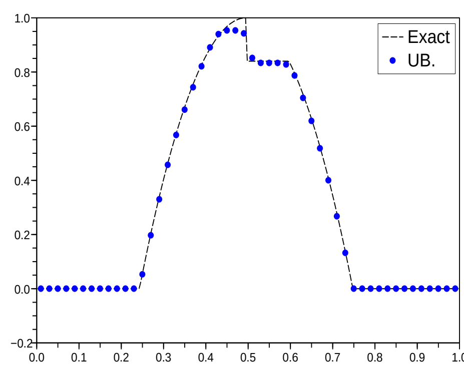
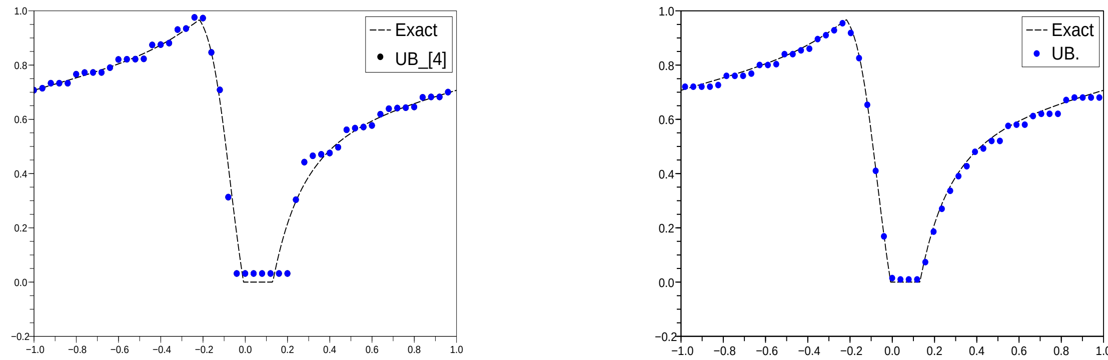




Figure 7: Example 3, UB and FMM schemes at $t = T := 2/\sqrt{3}$, with $N_x = 50$ and $p = 50$ (CFL=0.75 for UB)




Classical Ultra-Bee scheme UB algorithm (combined to level-set decomposition)

Figure 8: Example 4, with $Nx = 50,\ p = 50$ and CFL=0.75

$\bar{v}_0$. *Then*

$$||\bar{\vartheta}(t,.) - \vartheta(t,.)||_{L^1(\mathbb{R})} \le M_0\, t\, e^{Lt} \left( \max_i \delta_i \right).$$

**Proof.** Let $I_x(t) := [X_{x,+}(-t), X_{x,-}(-t)]$. We first have

$$\bar{\vartheta}(t,x) := \min_{y \in I_x(t)} \bar{v}_0(y) = \min \left( \min_{y \in I_x(t)} v_0(y), \quad \min_{i, A_i \in I_x(t)} (v_0(A_i) - \delta_i) \right)$$

and

$$\vartheta(t,x) := \min_{y \in I_x(t)} v_0(y) = \min \left( \min_{y \in I_x(t)} v_0(y), \quad \min_{i, A_i \in I_x(t)} v_0(A_i) \right).$$

Hence by difference,

$$|\bar{\vartheta}(t,x) - \vartheta(t,x)| \le \max_{i,\ A_i \in I_x(t)} |(v_0(A_i) - \delta_i) - v_0(A_i)| \le \max_{i,\ A_i \in I_x(t)} \delta_i.$$

In the case $\forall i,\ A_i \notin I_x(t)$ then $\bar{\vartheta}(t,x) = \vartheta(t,x)$. Hence

$$||\bar{\vartheta}(t,.) - \vartheta(t,.)||_{L^1(\mathbb{R})} \le (\max_i \delta_i) \sum_i \mathcal{L}\Big( \{x,\ A_i \in I_x(t)\} \Big) \le (\max_i \delta_i) \sum_i |X_{A_i,+}(t) - X_{A_i,-}(t)|. \tag{1.35}$$

where $\mathcal{L}$ is Lebesgue's measure. On the other hand, if $X_{a,+}(0) = X_{a,-}(0) = A_i$, then

$$\begin{aligned}
|X_{a,+}(t) - A_i| &= |\int_0^t f_+(X_{a,+}(s)\ ds| \\
&\le \int_0^t (|f_+(X_{a,+}(s)) - f_+(A_i)| + |f_+(A_i)|)\, ds \\
&\le t|f_+(A_i)| + \int_0^t L|X_{a,+}(s) - A_i|\ ds.
\end{aligned}$$

Using a Gronwall Lemma, we obtain

$$|X_{a,+}(t) - A_i| \le |f_+(A_i)|\ \frac{e^{Lt} - 1}{L} \le |f_+(A_i)|\ te^{Lt}.$$

(where we have used that $(e^x-1)/x \le e^x$ with $x = Lt$). In the same way,

$$|X_{a,-}(t) - A_i| \le |f_-(A_i)|\ te^{Lt}.$$

Hence the desired result. $\square$

We come back to the proof of Proposition A.1. We consider now $\delta_i^u := (u_0(A_i) - v_0(A_i))_+$, and $\delta_i^v := (v_0(A_i) - u_0(A_i))_+$. As in the previous Lemma, we define $\bar{u}_0$ by $\bar{u}_0(x) = u_0(x)$ if $x \notin (A_i)$, and $\bar{u}_0(A_i) := u_0(A_i) - \delta_i^u$. In the same way we consider $\bar{v}_0$ such that $\bar{v}_0(x) = v_0(x)$ if $x \notin (A_i)$ and $\bar{v}_0(A_i) := v_0(A_i) - \delta_i^v$. We finally denote by $\bar{u}$ and $\bar{\vartheta}$ the solution of (1.1) with initial condition $\bar{u}_0$ and $\bar{v}_0$. Then we have

$$\|\vartheta - u\|_{L^1(\mathbb{R})} \le \|\bar{\vartheta} - \vartheta\|_{L^1(\mathbb{R})} + \|\bar{\vartheta} - \bar{u}\|_{L^1(\mathbb{R})} + \|\bar{u} - u\|_{L^1(\mathbb{R})}.$$

Then $\|\bar{\vartheta}(t,.) - \bar{u}(t,.)\|_{L^1(\mathbb{R})} \le e^{Lt}\|\bar{v}_0 - \bar{u}_0\|_{L^1(\mathbb{R})} \le e^{Lt}\|v_0 - u_0\|_{L^1(\mathbb{R})}$ since $\bar{v}_0(A_i) = \bar{u}_0(A_i)$ $\forall i = 2,\dots,q$ and using the result from [4, Proposition 3]. Also both terms $\|\bar{\vartheta} - \vartheta\|_{L^1(\mathbb{R})}$ and $\|\bar{u} - u\|_{L^1(\mathbb{R})}$ are controlled by Lemma A.2:

$$\|\bar{u}(t,.) - u(t,.)\|_{L^1(\mathbb{R})} + \|\bar{\vartheta}(t,.) - \vartheta(t,.)\|_{L^1(\mathbb{R})} \le M_0\, t\, e^{Lt} \max_i(\delta_i^u, \delta_i^v)$$

Since $\max(\delta_i^u, \delta_i^v) \le |v_0(A_i) - u_0(A_i)|$ we conclude to the desired result. $\square$

**Proof of Proposition 2.3:**

$(i)$ We proceed by recursion on the number $p$ in the level set decomposition of $w_0$.

First we notice that using Lemma 3.2 the viscosity solution of (1.1) with initial data $\lambda w_{0,1}$ (for a given $\lambda > 0$) is given by $\lambda w_1$. This proves the result when $p = 1$.

Now we assume that $p \ge 2$ and that the result is true for up to $p-1$ levels. Let $w_0^{(1)}(x) := \sum_{k=1}^{p-1} h_k w_{0,k}(x)$ and $w_0^{(2)}(x) := h_p w_{0,p}(x)$. We denote by $w^{(1)}$ (resp $w^{(2)}$) the viscosity solution of (1.1) with initial data $w_0^{(1)}$ (resp $w_0^{(2)}$), and also by $w$ the viscosity solution of (1.1) with initial data $w_0^{(1)} + w_0^{(2)}$. We want to prove that $w \equiv w^{(1)} + w^{(2)}$.

Using the representation of Lemma 3.2 we have for a given $t \ge 0$ and a given $x$, $w(t,x) = \inf_{y\in I} w_0^{(1)}(y) + w_0^{(2)}(y)$, where $I = [X_{x,+}(-t), X_{x,-}(-t)]$. We can assume that $w_0^{(1)}$ is not constant on $I$ otherwise the result is obvious.

Let $a \in I$ be such that $w(t,x) = w_0^{(1)}(a) + w_0^{(2)}(a)$. We shall prove that $w_0^{(1)}(a) = \inf_{y\in I} w_0^{(1)}(y)$ and $w_0^{(2)}(a) = \inf_{y\in I} w_0^{(2)}(y)$ (in the case $w_0^{(1)}$ is not constant on $I$).

Suppose that $w_0^{(1)}(a)$ is not minimal on $I$, i.e., $w_0^{(1)}(a) > w_0^{(1)}(\bar{a})$ for some $\bar{a}$ in $I$. If $v_0(\bar{a}) > \sum_{i=1}^{p} h_i$, then by definition of $w_{0,k}$ we have $w_{0,k}(\bar{a}) = 1\ \forall k = 1,\dots,p$, and thus $w_0^{(1)}(\bar{a}) = \sum_{k=1}^{p-1} h_k$. This contradicts the fact that $w_0^{(1)}(a) > w_0^{(1)}(\bar{a})$ (since $w_0^{(1)} \le \sum_{k=1}^{p-1} h_k$ in all cases). Hence $v_0(\bar{a}) < \sum_{i=1}^{p} h_i$, and thus $w_0^{(2)}(\bar{a}) = 0$, from which we obtain that $w(t,x) = w_0^{(1)}(a) + w_0^{(2)}(a) > w_0^{(1)}(\bar{a}) + w_0^{(2)}(\bar{a})$. Since $a \in I$, this contradicts the fact that $a$ is a minimum of $w_0^{(1)} + w_0^{(2)}$ on $I$. This proves that $w_0^{(1)}(a) = \inf_{y\in I} w_0^{(1)}(y)$.

Now we have also $w_0^{(2)}(a) = 0$ because $w_0^{(1)}$ is non constant on $I$ (and we have $v_0(a) \le \sum_{k=1}^{p} h_k$). Hence $w_0^{(2)}(a)$ is minimal on $I$.

$(ii)$ In view of Proposition A.1, we have to estimate $|v_0(A_i) - w_0(A_i)|$. Let $j$ be such that $A_i \in I_j$. We then have

$$|v_0(A_i) - w_0(A_i)| \le |v_0(A_i) - v_0(x_j)| + |v_0(x_j) - w_0(x_j)| \le ||v_0'||_{L^\infty(I_j)}\Delta x + h \tag{1.36}$$

The result is then a direct consequence of Proposition A.1 and estimates (2.5) and (1.36). $\square$

# B Proof of the discrete comparison principle

**Proposition B.1. (Comparison principle on the times)**
*Let $1 < k_1 < k_2 \le p$. For all $n \in \mathbb{N}$ and all $i \in \mathbb{Z}$, we have either*

$$\theta_i^{n,k_1} > \theta_i^{n,k_2}$$

*or*

$$\theta_i^{n,k_1} = \theta_i^{n,k_2} =: \sigma_i = \pm 1$$

*and if $i \in \mathcal{U}_\alpha^{n,k_1} \cap \mathcal{U}_\alpha^{n,k_2}$, then*

$$\begin{cases} \tau_{i,\alpha}^{n,k_1} \le \tau_{i,\alpha}^{n,k_2} & \text{if} \quad \sigma_i = +1 \\ \tau_{i,\alpha}^{n,k_1} \ge \tau_{i,\alpha}^{n,k_2} & \text{if} \quad \sigma_i = -1 \end{cases}$$

**Proof of Proposition B.1**
By contradiction, let $n$ be the first index such that the condition does not hold and we denote by $i$ a node where the condition is not fullfilled. In particular, we have

$$\begin{cases} & \theta_i^{n,k_1} < \theta_i^{n,k_2} \\ \text{or} & \\ & \theta_i^{n,k_1} = \theta_i^{n,k_2} = \sigma_i, \ i \in \mathcal{U}_\alpha^{n,k_1} \cap \mathcal{U}_\alpha^{n,k_2} \text{ and } \sigma_i \tau_{i,\alpha}^{n,k_1} > \sigma_i \tau_{i,\alpha}^{n,k_2} \end{cases} \tag{2.37}$$

The proof is decomposed in several cases:
**Case 1:** $i \in \mathrm{NA}^{n,k_1} \setminus \mathrm{NA}^{n,k_2}$
Since the condition is true at step $n-1$, we have in particular $\theta_i^{n-1,k_1} \ge \theta_i^{n-1,k_2}$.
**Sub-case 1.1:** $\theta_i^{n-1,k_1} > \theta_i^{n-1,k_2}$
Then $\theta_i^{n-1,k_1} = 1$ and $\theta_i^{n-1,k_2} = -1$. Since $i \in \mathrm{NA}^{n,k_1} \setminus \mathrm{NA}^{n,k_2}$, we deduce that $\theta_i^{n,k_1} = \theta_i^{n,k_2} = -1 = \sigma_i$. We then have

$$\tau_{i,\alpha}^{n,k_2} \le t_n = \tau_{i,\alpha}^{n,k_1}$$

where we have used the fact that $i \in \mathcal{U}_\alpha^{n,k_2}$ and Proposition 4.1($i$) for the first inequality and the fact that $i \in \mathrm{NA}^{n,k_1} \cap \mathcal{U}_\alpha^{n,k_1}$ joint to Proposition 4.1($ii$) for the last equality. This contradicts (2.37).

**Sub-case 1.2:** $\theta_i^{n-1,k_1} = \theta_i^{n-1,k_2}$
**Sub-case 1.2.1:** $\theta_i^{n-1,k_1} = \theta_i^{n-1,k_2} = -1$
Since $i \in \mathrm{NA}^{n,k_1} \setminus \mathrm{NA}^{n,k_2}$, we deduce that $\theta_i^{n,k_1} = 1$ and $\theta_i^{n,k_2} = -1$. This contradicts (2.37).

**Sub-case 1.2.1:** $\theta_i^{n-1,k_1} = \theta_i^{n-1,k_2} = 1$
Since $i \in \mathrm{NA}^{n,k_1} \setminus \mathrm{NA}^{n,k_2}$, we have $\theta_i^{n,k_1} = -1$ and $\theta_i^{n-1,k_1} = \theta_i^{n-1,k_2} = \theta_i^{n,k_2} = 1$. Let $\bar{\alpha}$ be such that $\tilde{\tau}_{i,\bar{\alpha}}^{n-1,k_1} = \tilde{\tau}_i^{n-1,k_1} = t_n$. Let $\bar{\imath} \in \mathcal{U}_{\bar{\alpha}}^{n-1,k_1}(i)$. This implies that $\theta_{\bar{\imath}}^{n-1,k_1} = -1$. Since $\theta^{n-1,k_1} \ge \theta^{n-1,k_2}$, we get that $\theta_{\bar{\imath}}^{n-1,k_2} = -1$ and so $\bar{\imath} \in \mathcal{U}_{\bar{\alpha}}^{n-1,k_2}(i)$. We then deduce that

$$\tau_{\bar{\imath},\bar{\alpha}}^{n-1,k_1} \ge \tau_{\bar{\imath},\bar{\alpha}}^{n-1,k_2}.$$

Using Step 1 of the algorithm, we deduce that

$$t_n = \tilde{\tau}_i^{n-1,k_1} = \tilde{\tau}_{i,\bar{\alpha}}^{n-1,k_1} \ge \tilde{\tau}_{i,\bar{\alpha}}^{n-1,k_2} \ge \tilde{\tau}_i^{n-1,k_2}.$$

Contradiction since $i \notin \mathrm{NA}^{n,k_2}$.

**Case 2:** $i \in \mathrm{NA}^{n,k_2} \setminus \mathrm{NA}^{n,k_1}$
This case can be treated in the same way of Case 1.

**Case 3:** $i \in \mathrm{NA}^{n,k_1} \cap \mathrm{NA}^{n,k_2}$
**Sub-case 3.1:** $\theta_i^{n-1,k_1} = \theta_i^{n-1,k_2}$
Since $i \in \mathrm{NA}^{n,k_1} \cap \mathrm{NA}^{n,k_2}$, we deduce that $\theta_i^{n,k_1} = \theta_i^{n,k_2}$, and so by (2.37), $i \in \mathcal{U}_\alpha^{n,k_1} \cap \mathcal{U}_\alpha^{n,k_2}$. By Proposition 4.1($ii$), we deduce that

$$\tau_{i,\alpha}^{n,k_1} = \tau_{i,\alpha}^{n,k_2} = t_n.$$

Contradiction.

**Sub-case 3.2:** $\theta_i^{n-1,k_1} > \theta_i^{n-1,k_2}$
In this case, we have $\theta_i^{n-1,k_1} = 1$ and $\theta_i^{n-1,k_2} = -1$. Since $i \in \mathrm{NA}^{n,k_1} \cap \mathrm{NA}^{n,k_2}$, we have

$$i \in \mathrm{NB}_+^{n-1,k_1} \cap \mathrm{NB}_-^{n-1,k_2} \quad \text{or} \quad i \in \mathrm{NB}_-^{n-1,k_1} \cap \mathrm{NB}_+^{n-1,k_2}$$

(we cannot have $i \in NB_+^{n-1,k_1} \cap NB_+^{n-1,k_2}$ or $i \in NB_-^{n-1,k_1} \cap NB_-^{n-1,k_2}$ because $\theta_i^{n-1,k_1} \neq \theta_i^{n-1,k_2}$ and the velocity is the same). Let us treat the first case, the other being similar. The fact that $i \in \mathrm{NB}_-^{n-1,k_2}$ and $\theta_i^{n-1,k_2} = -1$ implies that $\theta_{i-1}^{n-1,k_2} = 1$. Similarly, $i \in \mathrm{NB}_+^{n-1,k_1}$ and $\theta_i^{n-1,k_1} = 1$ implies that $\theta_{i-1}^{n-1,k_1} = -1$. This is contradicts the fact that $\theta_i^{n-1,k_2} \leq \theta_i^{n-1,k_1}$.

**Case 4:** $i \notin \mathrm{NA}^{n,k_1} \cup \mathrm{NA}^{n,k_2}$
**Sub-case 4.1:** $\theta_i^{n,k_1} < \theta_i^{n,k_2}$
Since $i \notin \mathrm{NA}^{n,k_1} \cup \mathrm{NA}^{n,k_2}$, we have

$$\theta_i^{n-1,k_1} = \theta_i^{n,k_1} < \theta_i^{n,k_2} = \theta_i^{n-1,k_2}.$$

This is absurd.

**Sub-case 4.2:** $\theta_i^{n,k_1} = \theta_i^{n,k_2}$
Since $i \notin \mathrm{NA}^{n,k_1} \cup \mathrm{NA}^{n,k_2}$, we have

$$\theta_i^{n-1,k_1} = \theta_i^{n,k_1} = \theta_i^{n,k_2} = \theta_i^{n-1,k_2} = \sigma_i = \pm 1.$$

From (2.37),

$$i \in \mathcal{U}_\alpha^{n,k_1} \cap \mathcal{U}_\alpha^{n,k_2} \quad \text{and} \quad \sigma_i \tau_{i,\alpha}^{n,k_1} > \tau_{i,\alpha}^{n,k_2}. \tag{2.38}$$

**Sub-case 4.2.1:** $i \in \mathcal{U}_\alpha^{n-1,k_1} \cap \mathcal{U}_\alpha^{n-1,k_2}$
In this case, we have, since the condition holds at step $n-1$

$$\sigma_i \tau_{i,\alpha}^{n-1,k_1} \leq \sigma_i \tau_{i,\alpha}^{n-1,k_2}.$$

Using the fact that $i \in \mathcal{U}_\alpha^{n-1,k_1} \cap \mathcal{U}_\alpha^{n,k_1}$, $i \in \mathcal{U}_\alpha^{n-1,k_2} \cap \mathcal{U}_\alpha^{n,k_2}$ joint to Proposition 4.1($iii$), we get that $\tau_{i,\alpha}^{n-1,k_1} = \tau_{i,\alpha}^{n,k_1}$ and $\tau_{i,\alpha}^{n-1,k_2} = \tau_{i,\alpha}^{n,k_2}$. This implies that

$$\sigma_i \tau_{i,\alpha}^{n,k_1} \leq \sigma_i \tau_{i,\alpha}^{n,k_2}.$$

This contradicts (2.38).

**Sub-case 4.2.2:** $i \in \mathcal{U}_\alpha^{n-1,k_1} \backslash \mathcal{U}_\alpha^{n-1,k_2}$
**Sub-case 4.2.2.1:** $\sigma_i = 1$
In this case, we have

$$\tau_{i,\alpha}^{n,k_2} = t_n \geq \tau_{i,\alpha}^{n,k_1}$$

where we have used the fact that $i \in \mathcal{U}_\alpha^{n,k_2} \setminus \mathcal{U}_\alpha^{n-1,k_2}$ joint to Proposition 4.1($iv$) for the first equality and the fact that $i \in \mathcal{U}_\alpha^{n-1,k_1} \cap \mathcal{U}_\alpha^{n,k_1}$ joint to Proposition 4.1($iii$) for the last inequality. This contradicts (2.38).

**Sub-case 4.2.2.2:** $\sigma_i = -1$
Let us treat the case $\alpha = +$, the case $\alpha = -$ being similar. We have that $i \in \mathcal{U}_+^{n,k_1}(i+1) \cap \mathcal{U}_+^{n,k_2}(i+1)$
This implies that

$$\widehat{f}_+(x_{i+1}) > 0 \quad \text{and} \quad \theta_{i+1}^{n,k_1} = \theta_{i+1}^{n,k_2} = 1.$$

Moreover, since $\theta_i^{n-1,k_1} = -1$ and $i \in \mathcal{U}_+^{n-1,k_1}$, we get that $i \in \mathcal{U}_+^{n-1,k_1}(i+1)$. This implies that $\theta_{i+1}^{n-1,k_1} = 1$. Since $i \notin \mathcal{U}_+^{n-1,k_2}(i+1)$, $\theta_i^{n-1,k_2} = -1$ and $\widehat{f}_+(x_{i+1}) > 0$, we deduce that $\theta_{i+1}^{n-1,k_2} = -1$.

Since $\widehat{f}_+(x_{i+1}) > 0$ and $\theta_{i+1}^{n-1,k_2} = -1$, we deduce that $\mathcal{U}_+^{n-1,k_2}(i+1) = \emptyset$. Since $\theta_i^{n-1,k_2} = \theta_{i+1}^{n-1,k_2} = -1$, we also deduce that $\mathcal{U}_-^{n-1,k_2}(i+1) = \emptyset$. This implies that $j \notin \mathrm{NB}^{n-1,k_2}$. This contradicts the fact that $\theta_j^{n-1,k_2} = -1 = -\theta_j^{n,k_2}$.

**Sub-case 4.2.3:** $i \in \mathcal{U}_\alpha^{n-1,k_2} \backslash \mathcal{U}_\alpha^{n-1,k_1}$
This case can be treated in the same way of Sub-case 4.2.2.
**Sub-case 4.2.4:** $i \notin \mathcal{U}_\alpha^{n-1,k_1} \cup \mathcal{U}_\alpha^{n-1,k_2}$
In this case, we have

$$\tau_i^{n,k_1} = t_n = \tau_i^{n,k_2}$$

where we have used the fact that $i \in \mathcal{U}_\alpha^{n,k_1} \backslash \mathcal{U}_\alpha^{n-1,k_1}$, $i \in \mathcal{U}_\alpha^{n,k_2} \backslash \mathcal{U}_\alpha^{n-1,k_2}$ joint to Proposition 4.1($iv$)
This contradicts (2.38). □

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