Image Segmentation using a Generalized Fast Marching Method

Nicolas Forcadel (ENPC-CERMICS, Paris), Carole Le Guyader (INSA Rennes), Christian Gout (INRIA, France)

Multivariate Approximation: Theory and Applications

April 26 - May 01, 2007

Plan

Plan

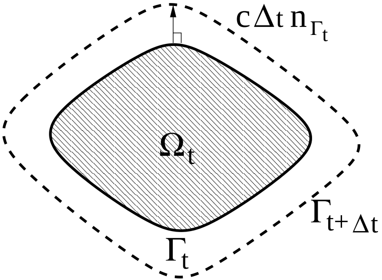
1. Fast Marching Method
2. Generalized Fast Marching Method
3. Image segmentation
4. Numerical results

Plan

Plan

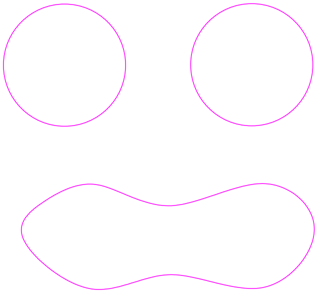
Plan

Evolution of a front



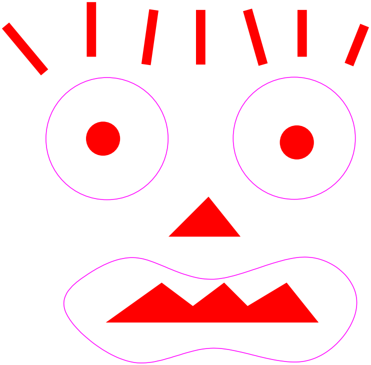
$$\frac{d\Gamma_t}{dt} = c\, n_{\Gamma_t}$$

Mathematical difficulty



How to define the evolution with change of topology?

Mathematical difficulty



Change of topology = frightening

Level set Method

- Level set formulation:

$$\Gamma_t = \{x,\ u(x,t) = 0\}.$$

- Formally :

$$u(\Gamma_t, t) = 0 \Rightarrow u_t + \nabla u \cdot \frac{\partial \Gamma_t}{\partial t} = 0.$$

with $\dfrac{\partial \Gamma_t}{\partial t} = c(x)\vec{n},\ \vec{n} = -\dfrac{\nabla u}{|\nabla u|}.$

- Level set equation for moving fronts:

$$u_t = c(x)|\nabla u|.$$

Level set Method

- Level set formulation:

$$\Gamma_t = \{x,\ u(x,t) = 0\}.$$

- Formally :

$$u(\Gamma_t, t) = 0 \Rightarrow u_t + \nabla u \cdot \frac{\partial \Gamma_t}{\partial t} = 0.$$

with $\frac{\partial \Gamma_t}{\partial t} = c(x)\vec{n},\ \vec{n} = -\frac{\nabla u}{|\nabla u|}.$

- Level set equation for moving fronts:

$$u_t = c(x)|\nabla u|.$$

Level set Method

- Level set formulation:

$$\Gamma_t = \{x,\ u(x,t) = 0\}.$$

- Formally :

$$u(\Gamma_t, t) = 0 \Rightarrow u_t + \nabla u \cdot \frac{\partial \Gamma_t}{\partial t} = 0.$$

with $\dfrac{\partial \Gamma_t}{\partial t} = c(x)\vec{n},\ \vec{n} = -\dfrac{\nabla u}{|\nabla u|}.$

- Level set equation for moving fronts:

$$u_t = c(x)|\nabla u|.$$

Level set Method

- Level set formulation:

$$\Gamma_t = \{x,\ u(x,t) = 0\}.$$

- Formally :

$$u(\Gamma_t, t) = 0 \Rightarrow u_t + \nabla u \cdot \frac{\partial \Gamma_t}{\partial t} = 0.$$

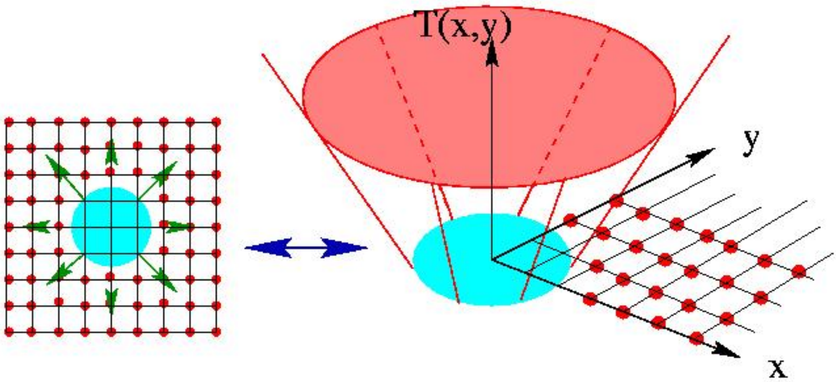
with $\dfrac{\partial \Gamma_t}{\partial t} = c(x)\vec{n},\ \vec{n} = -\dfrac{\nabla u}{|\nabla u|}.$

- Level set equation for moving fronts:

$$u_t = c(x)|\nabla u|.$$

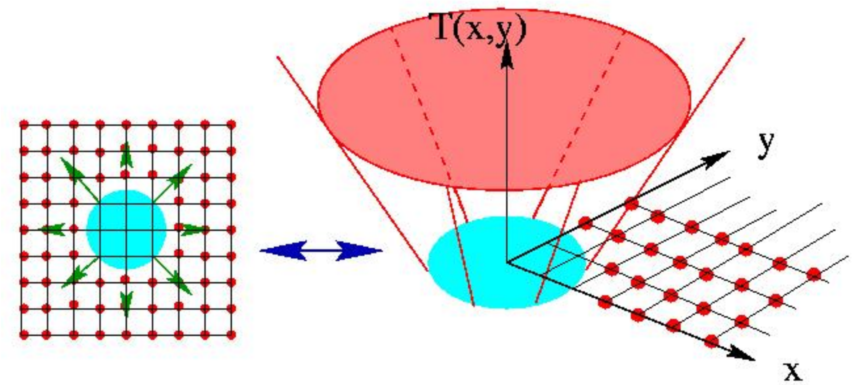
Stationary approach

- Setting $u(x,t) = t - T(x) \Rightarrow |\nabla T(x)| = \dfrac{1}{c(x)}$.
- $T(x)$ = time when the front cross the point x.

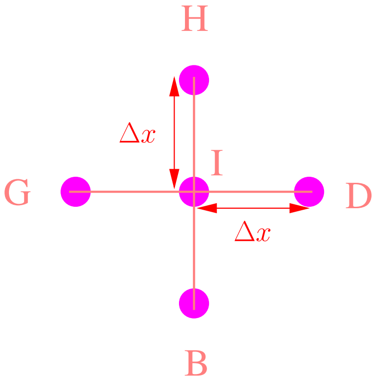


Stationary approach

- Setting $u(x,t) = t - T(x) \Rightarrow |\nabla T(x)| = \dfrac{1}{c(x)}$.
- $T(x) =$ time when the front cross the point x.



Discretization



$$\max(T_I - T_G, T_I - T_D, 0)^2 + \max(T_I - T_H, T_I - T_B, 0)^2 = \left(\frac{\Delta x}{c_I}\right)^2 \quad (1)$$

Introduction of the Narrow Band

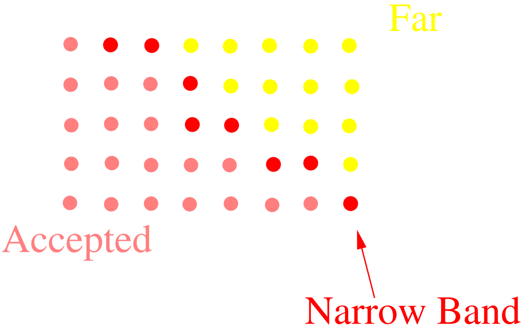
- Processing the nodes in a special ordering, one can compute the solution in just one iteration.
  This special ordering corresponds to the increasing values of T
- Idea : Compute the time only near the front.

Introduction of the Narrow Band

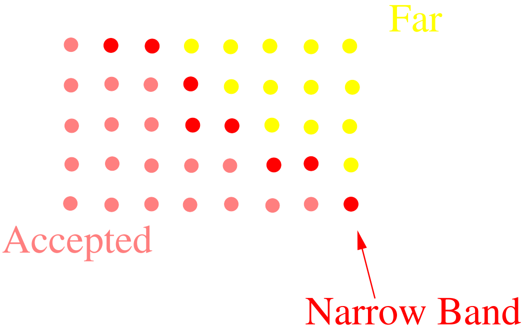
- Processing the nodes in a special ordering, one can compute the solution in just one iteration.
  This special ordering corresponds to the increasing values of T
- Idea : Compute the time only near the front.

Introduction of the Narrow Band

- Processing the nodes in a special ordering, one can compute the solution in just one iteration.
  This special ordering corresponds to the increasing values of T
- Idea : Compute the time only near the front.

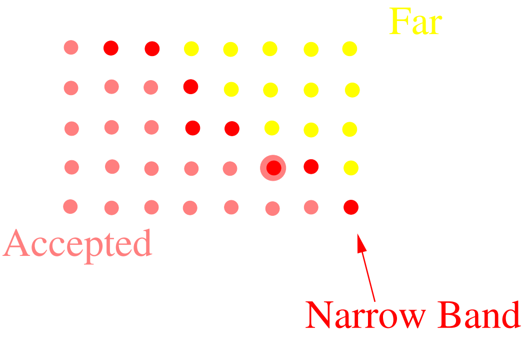


Introduction of the Narrow Band



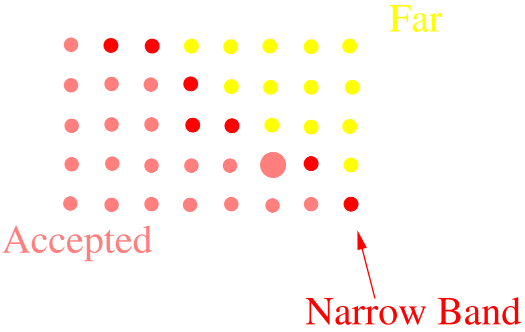
- Computation of T_I on NB
- Call t_n the minimal time T_I on the NB and accept at the time t_n the minimizing points I
- The new NB is defined as the boundary of the new accepted region

Introduction of the Narrow Band



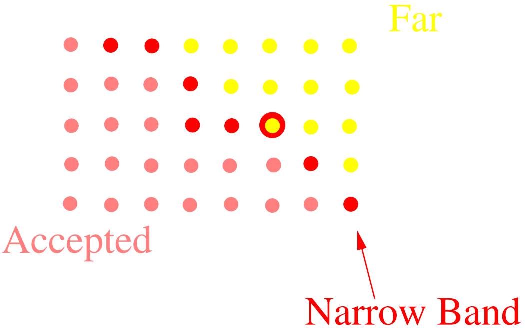
- Computation of T_I on NB
- Call t_n the minimal time T_I on the NB
  and accept at the time t_n the minimizing points I
- The new NB is defined as the boundary of the new accepted region

Introduction of the Narrow Band



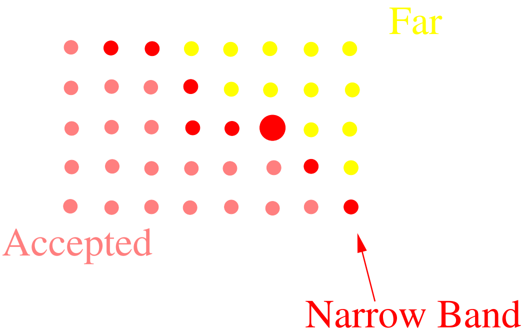
- Computation of T_I on NB
- Call t_n the minimal time T_I on the NB
  and accept at the time t_n the minimizing points I
- The new NB is defined as the boundary of the new accepted region

Introduction of the Narrow Band



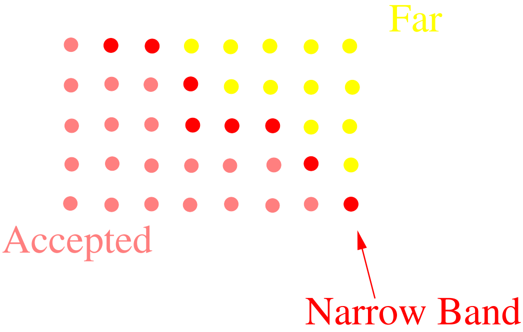
- Computation of T_I on NB
- Call t_n the minimal time T_I on the NB
  and accept at the time t_n the minimizing points I
- The new NB is defined as the boundary of the new accepted region

Introduction of the Narrow Band



- Computation of T_I on NB
- Call t_n the minimal time T_I on the NB
  and accept at the time t_n the minimizing points I
- The new NB is defined as the boundary of the new accepted region

Introduction of the Narrow Band



- Computation of T_I on NB
- Call t_n the minimal time T_I on the NB
  and accept at the time t_n the minimizing points I
- The new NB is defined as the boundary of the new accepted region

References

- [Sethian]: `http://math.berkeley.edu/~sethian/`
- [Vladimirsky] : case $c = c(x,t) > 0$
- [Cristiani, Falcone] : Proof of convergence

Plan

Dependence in time of the velocity

- The time is given implicitly by the algorithm!

$$t_n = \min_{I \in NB} T_I$$

- Two difficulties:
  - t_n cans be smaller than t_{n-1}:

$$t_n := t_{n-1}$$

  - t_n can be very large:

$$\Rightarrow \text{ Introduction of a time step } \Delta t \text{ :}$$

$$\text{if } t_n \geq t_{n-1} + \Delta t \Rightarrow t_n := t_{n-1} + \Delta t$$

Dependence in time of the velocity

- The time is given implicitly by the algorithm!

$$t_n = \min_{I \in NB} T_I$$

- Two difficulties:
  - t_n cans be smaller than t_{n-1}:

$$t_n := t_{n-1}$$

  - t_n can be very large:

$$\Rightarrow \text{ Introduction of a time step } \Delta t :$$

$$\text{if } t_n \geq t_{n-1} + \Delta t \Rightarrow t_n := t_{n-1} + \Delta t$$

Dependence in time of the velocity

- The time is given implicitly by the algorithm!

$$t_n = \min_{I \in NB} T_I$$

- Two difficulties:
  - t_n cans be smaller than t_{n-1}:

$$t_n := t_{n-1}$$

  - t_n can be very large:

$$\Rightarrow \text{ Introduction of a time step } \Delta t :$$

$$\text{if } t_n \geq t_{n-1} + \Delta t \Rightarrow t_n := t_{n-1} + \Delta t$$

Dependence in time of the velocity

- The time is given implicitly by the algorithm!

$$t_n = \min_{I \in NB} T_I$$

- Two difficulties:
  - t_n cans be smaller than t_{n-1}:

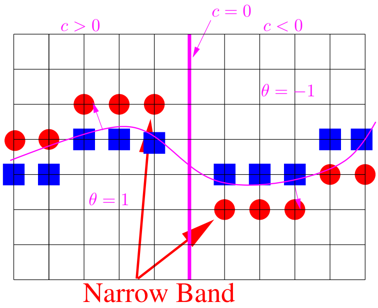
$$t_n := t_{n-1}$$

  - t_n can be very large:

$$\Rightarrow \text{ Introduction of a time step } \Delta t :$$

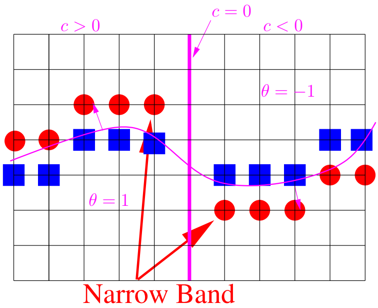
$$\text{if } t_n \geq t_{n-1} + \Delta t \Rightarrow t_n := t_{n-1} + \Delta t$$

Change of the sign of the velocity



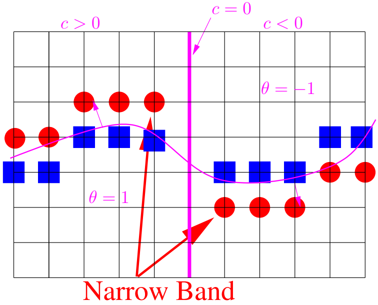
- Introduction of a field $\theta_I^n \in \{-1, 1\}$
- Regularisation of the velocity in space
- Double Narrow band
- The time is defined in each side of the front

Change of the sign of the velocity



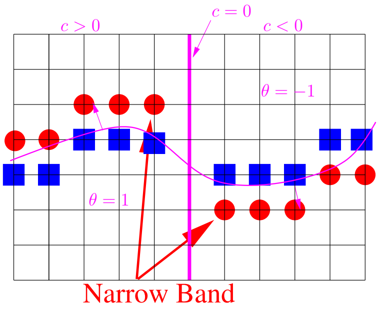
- Introduction of a field $\theta_I^n \in \{-1, 1\}$
- Regularisation of the velocity in space
- Double Narrow band
- The time is defined in each side of the front

Change of the sign of the velocity



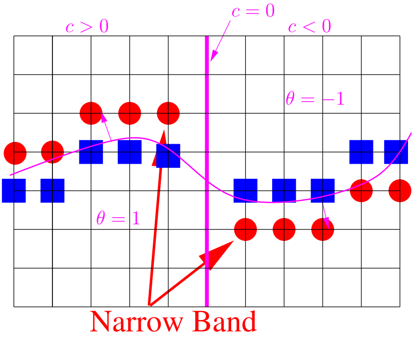
- Introduction of a field $\theta_I^n \in \{-1, 1\}$
- Regularisation of the velocity in space
- Double Narrow band
- The time is defined in each side of the front

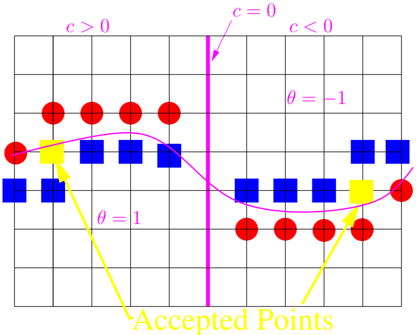
Change of the sign of the velocity



- Introduction of a field $\theta_I^n \in \{-1, 1\}$
- Regularisation of the velocity in space
- Double Narrow band
- The time is defined in each side of the front

Exemple


$c > 0$
$c = 0$
$c < 0$
$\theta = -1$
$\theta = 1$
Narrow Band


$c > 0$
$c = 0$
$c < 0$
$\theta = -1$
$\theta = 1$
Accepted Points

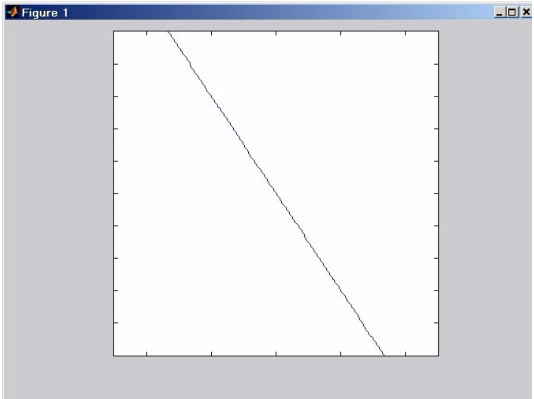
Reference

1 E. Carlini, M. Falcone, N. Forcadel, R. Monneau, *Convergence of a Generalized Fast Marching Method for a non-convex eikonal equation* (submitted, http://cermics.enpc.fr/ forcadel/publications.html).

2 E. Carlini, E. Cristiani, N. Forcadel, *A non-monotone Fast Marching scheme for a Hamilton-Jacobi equation modeling dislocation dynamics*, ENUMATH 2005.

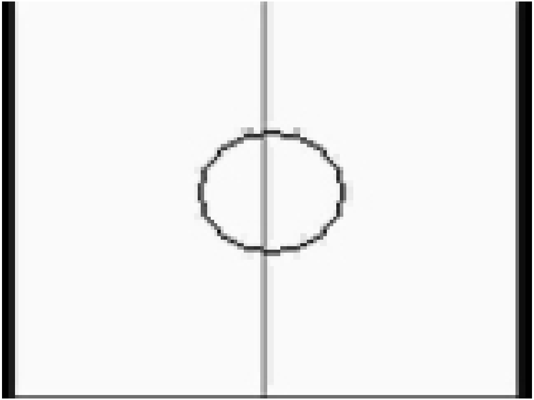
A straight line

$$c(x,t) = x_1$$



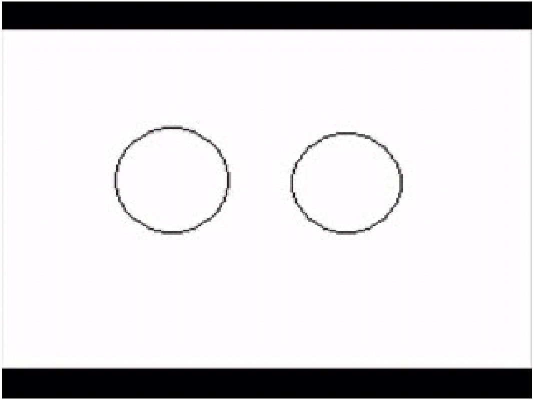
A circle

$$c(x,t) = -x_1$$



Two circles

$$c(x,t) = 1 - t$$



Plan

Image Segmentation with classical FMM

- Given an image I, define the velocity as

$$c(x) = g(|\nabla I(x)|)$$

  where g is an edge-detector function satisfying $g : [0, +\infty[\to [0, +\infty[,\ g(0) = 1,\ g$ strictly decreasing and $lim_{r \to \infty} g(r) = 0$.
- Very efficient method
- But problem of the sign of the velocity implies constaints on initial data.

Image Segmentation with classical FMM

- Given an image I, define the velocity as

$$c(x) = g(|\nabla I(x)|)$$

  where g is an edge-detector function satisfying $g : [0, +\infty[\rightarrow [0, +\infty[,\ g(0) = 1,\ g$ strictly decreasing and $lim_{r\rightarrow\infty} g(r) = 0$.
- Very efficient method
- But problem of the sign of the velocity implies constaints on initial data.

Image Segmentation with classical FMM

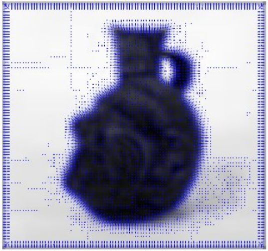
- Given an image I, define the velocity as

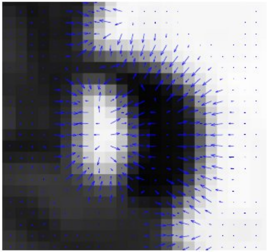
$$c(x) = g(|\nabla I(x)|)$$

  where g is an edge-detector function satisfying $g : [0, +\infty[\to [0, +\infty[,\ g(0) = 1,\ g$ strictly decreasing and $lim_{r\to\infty} g(r) = 0.$
- Very efficient method
- But problem of the sign of the velocity implies constaints on initial data.

Image Segmentation with generalized FMM

- The gradient field of $-\nabla g(|\nabla(I)|)$ give the direction of the edges:

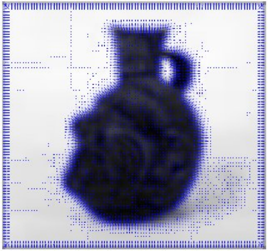


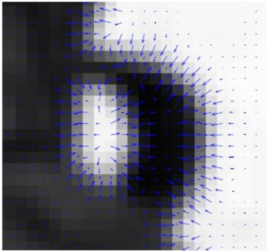


- The vectors point towards the middle of the boundaries.

Image Segmentation with generalized FMM

- The gradient field of $-\nabla g(|\nabla(I)|)$ give the direction of the edges:



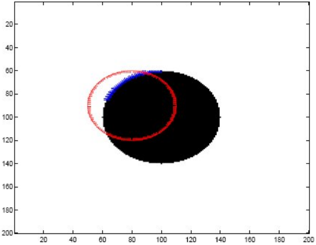


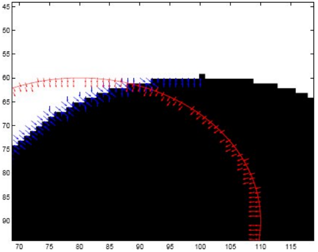
- The vectors point towards the middle of the boundaries.

Image Segmentation with generalized FMM

- Define the velocity as

$$c(x,t) = g(|\nabla I(x)|) \frac{\int_{\mathcal{Q}(x)} \langle -\nabla g(|\nabla I(r)|), \vec{n}(x,t) \rangle \, exp\left(-\frac{||r-x||^2}{2\sigma^2} \right) dr}{\left| \int_{\mathcal{Q}(x)} \langle -\nabla g(|\nabla I(r)|), \vec{n}(x,t) \rangle \, exp\left(-\frac{||r-x||^2}{2\sigma^2} \right) dr \right|}$$





Plan

Geophysics data

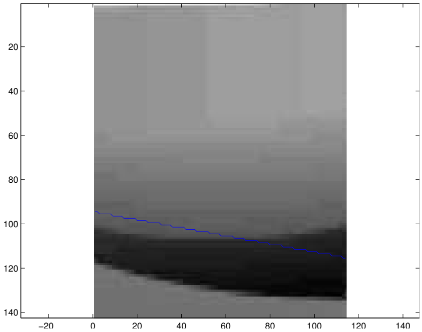


Figure: Initial data

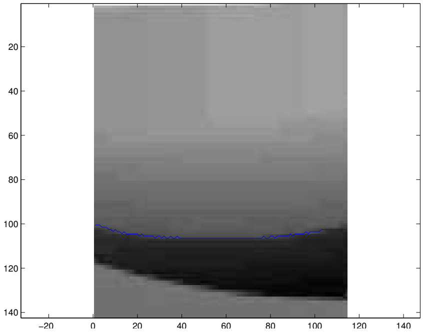


Figure: Final result

Medical data

Medical data

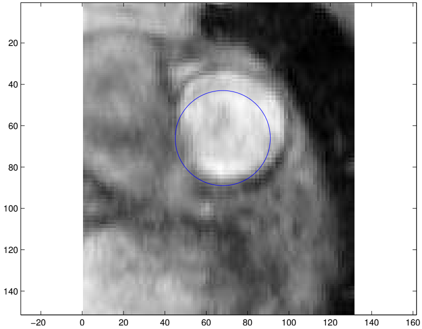


Figure: Initial data

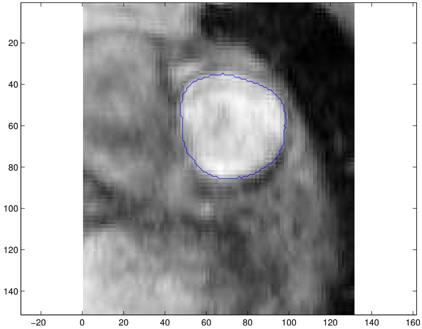


Figure: Final result

From the initial condition to the final result