

A GENERALIZED FAST MARCHING METHOD FOR DISLOCATION DYNAMICS*

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Abstract. In this paper, we consider a generalized fast marching method (GFMM) as a numerical method to compute dislocation dynamics. The dynamics of a dislocation hypersurface in \mathbb{R}^N (with $N = 2$ for physical applications) is given by its normal velocity which is a nonlocal function of the whole shape of the hypersurface itself. For this dynamics, we show a convergence result of the GFMM as the mesh size goes to zero. We also provide some numerical simulations in dimension $N = 2$.

Key words. Hamilton–Jacobi equations, fast marching scheme, convergence, viscosity solutions, dislocation dynamics, nonlocal equations

AMS subject classifications. 65M06, 65M12, 49L25

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1. Introduction. In this paper, we present a numerical approach, the generalized fast marching method, to compute the dynamics of a dislocation. Dislocations are line defects in crystal, and their motion is at the origin of plastic properties of metals (we refer to Alvarez et al. [5] for more details on dislocations and for a description of the mathematical model). A smooth dislocation can be represented as the boundary of an open set Ω_t (where t represents the time dependence). The evolution equation can then be written on the characteristic function of Ω_t , namely, $\theta(x, t)$, which is defined by

$$\theta(x, t) = \begin{cases} 1 & \text{if } x \in \Omega_t, \\ -1 & \text{if } x \notin \Omega_t. \end{cases}$$

Moreover, θ is assumed to satisfy the following *nonlocal equation* (see Alvarez et al. [5] for an equivalent formulation):

$$(1.1) \quad \begin{cases} \theta_t = ((c_0 \star \theta(\cdot, t))(x) + c_1(x, t)) |D\theta| & \text{in } \mathbb{R}^N \times (0, +\infty), \\ \theta(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c} & \text{on } \mathbb{R}^N. \end{cases}$$

Here and throughout the paper, θ_t denotes the time derivative of θ , $D\theta$ is its gradient in space, c_1 is a given function, $c_0 = c_0(x)$ is a given kernel, and \star denotes the convolution in space.

In [3, 4], the authors studied numerically a level set formulation of (1.1). They proposed a convergent scheme and proved a Crandall–Lions-type error estimate [9]. Nevertheless, the level set formulation used in those paper is not completely satisfactory for numerical simulations for dislocation dynamics. This comes from the fact

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that, on the dislocation line, the gradient of the level set function can get very small very quickly, which creates some difficulties in localizing precisely the numerical level set. In particular, this creates numerical oscillations of the front, as we point out in section 8 (Test 2, Figure 5). Let us note that this difficulty might be overcome by using techniques such as reinitialization or velocity extensions [11, 8, 1].

In the present paper, to overcome this problem and to get a better computational cost, we want to replace the level set method with another method which avoids the difficulty of vanishing gradients. A first attempt in that direction was introduced in [12] and was one of our main motivations for developing a new theory, the conclusion of which is the present paper. The method we consider here is the generalized fast marching method (GFMM). This algorithm can be seen as an extension of the well-known fast marching method (FMM), introduced independently by Sethian [13, 14] and Tsitsiklis [15]. The classical FMM is a single pass scheme to compute the solution of a stationary eikonal equation with a positive and independent-on-time velocity $c(x)$. The solution computed gives the arrival time of the front propagating with normal velocity $c(x)$. Recently in [16], this method has been extended to the case of positive velocities $c(x, t)$ depending also on time. Later in [7], a generalization to the case of sign changing velocities $c(x, t)$ was proposed. Precisely, a new scheme has been designed to compute numerically the discontinuous solution of the following evolutive local equation:

$$(1.2) \quad \begin{cases} \theta_t = c(x, t)|D\theta| & \text{in } \mathbb{R}^N \times (0, +\infty), \\ \theta(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c} & \text{on } \mathbb{R}^N. \end{cases}$$

In this problem the velocity $c(x, t)$ is assumed to be a given function (contrary to (1.1)). Here the evolution of the front is given by the evolution of the interface between the two phases of θ , and then the problem of the vanishing gradient is automatically solved.

Even if an evolution equation is approximated by the GFMM, in the case where the velocity $c(x, t)$ is Lipschitz in time (with a reasonable Lipschitz constant), the computational complexity is $O(M \log(M))$, where M is the total number of grid points, as in the classical FMM (see [7]).

On the one hand, the main result of [7] is a convergence result of the method as the mesh size goes to zero. On the other hand, one drawback of the scheme proposed in [7] is the fact that the method is not completely monotone, which makes the analysis difficult for the application of this scheme to nonlocal equations such as for dislocation dynamics. Major progress was made very recently in [10], where an improvement of the GFMM was proposed with a similar computational complexity $O(M \log(M))$. This new version of the GFMM is indeed *truly monotone*, and it is this version of GFMM that we will use in this paper to perform our analysis. Let us mention that one original aspect of this scheme is that several different notions of times can be defined at the same point of the front for the same step of the algorithm.

The following is the main result of the present paper (presented here in very vague terms; see Theorem 3.4 for a precise version).

THEOREM (convergence of the numerical method). *Under suitable assumptions, the numerical solution given by a suitable GFMM is convergent as the mesh size goes to zero to the solution θ of (1.1) of dislocation dynamics.*

This shows in particular that a GFMM can be considered as a good alternative to the level set method. This conclusion is moreover supported by the numerical simulations presented at the end of the paper, which are compared to analogous simulations obtained using the level set method (see Test 2).

Organization of the paper. In section 2 we first recall the monotone GFMM given in [10] for the local equation and then give its extension to compute the numerical solution of the nonlocal equation of dislocation dynamics. In section 3 we present our main results about the convergence of the numerical method to the solution of the nonlocal equation. To prove the results we will use some known properties (monotonicity, convergence) about the GFMM presented in [10] for local equations. These properties are recalled in section 4.

In section 5 we prove the general convergence result, and in the two following sections we apply the result in two particular cases: smooth evolutions (Lipschitz curves in \mathbb{R}^2) with sign changing (nonlocal) velocities in section 6, and nonsmooth evolutions (with the condition of interior ball in \mathbb{R}^N) with positive (nonlocal) velocities in section 7.

In section 8 we provide some numerical simulations for the nonlocal equation describing dislocation dynamics. Finally, in Appendix A we give the precise version of the GFMM, which has been used to perform certain numerical simulations of section 8. This version is not monotone in the general case of speed changing sign in time, but it is simpler to implement in practice than the one presented in section 2.

2. GFMM algorithms for local and nonlocal equations.

2.1. A monotone GFMM for the local equation. We present here the monotone GFMM algorithm studied in [10] (which appears to be a modified version of the GFMM algorithm introduced in [7]).

2.1.1. Preliminaries. Let us consider a lattice $Q = \{x_I = (x_{i_1}, \dots, x_{i_N}) = (i_1\Delta x, \dots, i_N\Delta x), I = (i_1, \dots, i_N) \in \mathbb{Z}^N\}$ with space step Δx . We will also use a time step $\Delta t > 0$ (which does not satisfy any CFL condition).

The following definitions will be useful in the following.

DEFINITION 2.1. *The neighborhood of the node $I \in \mathbb{Z}^N$ is the set*

$$V(I) \equiv \{J \in \mathbb{Z}^N : |J - I| \leq 1\}.$$

DEFINITION 2.2. *The numerical boundary ∂E of a set $E \subset \mathbb{Z}^N$ is*

$$\partial E \equiv V(E) \setminus E$$

with

$$V(E) = \bigcup_{I \in E} V(I).$$

DEFINITION 2.3. *Given a field θ_I^n with values $+1$ and -1 , we define the two phases*

$$\Theta_{\pm}^n \equiv \{I : \theta_I^n = \pm 1\}$$

and the fronts

$$F_{\pm}^n \equiv \partial\Theta_{\pm}^n, \quad F^n \equiv F_+^n \cup F_-^n.$$

As in [7, 10], we have to “regularize” the numerical velocity in space in order to avoid the duplication of the front.

DEFINITION 2.4. *Given the speed $c_I^n \equiv c(x_I, t_n)$ we define the function*

$$\widehat{c}_I^n \equiv \begin{cases} 0 & \text{if there exists } J \in V(I) \text{ such that } (c_I^n c_J^n < 0 \text{ and } |c_I^n| \leq |c_J^n|), \\ c_I^n & \text{otherwise.} \end{cases}$$

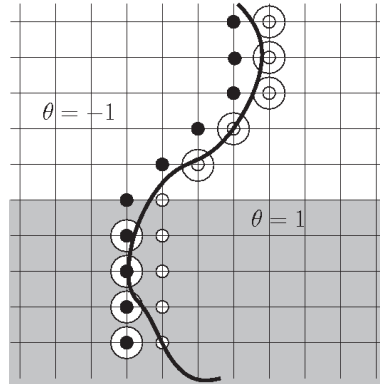


FIG. 1. The front F^+ (white spot), the front F^- (black spot), the narrow band NB^+ (circled white spot), and the narrow band NB^- (circled black spot). In the white domain the speed is negative, and in the gray domain the speed is positive.

We now describe the monotone GFMM algorithm for velocity changing sign. As in the classical FMM, we define the narrow band (NB), which consists of the points $I \in \mathbb{Z}^N$ that can be immediately reached by the front:

$$NB^n = \{I \in \mathbb{Z}^N, \exists J \in V(I), \theta_I^n = -\theta_J^n \text{ and } \theta_I^n \hat{c}_I^n < 0\}, \quad NB_{\pm}^n = NB^n \cap \{I, \theta_I^n = \pm 1\}.$$

We observe that the narrow bands NB_{\pm}^n are related to the previous definition of front sets as follows:

$$NB_+^n = F_+^n \cap \{I, \hat{c}_I^n < 0\}, \quad NB_-^n = F_-^n \cap \{I, \hat{c}_I^n > 0\}.$$

Figure 1 shows the front and the narrow band; the white part of the picture represents the domain where the speed is negative, and the gray part represents the domain with positive speed.

As in the FMM, for all $I \in NB^n$, we have to compute a tentative value (which we denote by \tilde{u}_I^n in what follows) of the arrival time of the front at point I . To compute this tentative value, we define the points that are useful for I , i.e., that we will use in the computation of \tilde{u}_I^n :

$$\text{for } I \in NB^n, \quad \mathcal{U}^n(I) = \{J \in V(I), \theta_I^n = -\theta_J^n\}, \quad \mathcal{U}^n = \bigcup_{I \in NB^n} \mathcal{U}^n(I).$$

For all the points J that are useful for a point $I \in NB^n$ (i.e., $J \in \mathcal{U}^n(I)$) we will introduce a time $u_{J \rightarrow I}^n$. This time $u_{J \rightarrow I}^n$ can be interpreted as the time when the front F^n starts to go from point J to point I and will be used to compute the tentative value at point I . One point of originality in the GFMM that we consider here is that we may have $u_{J \rightarrow I}^n \neq u_{J \rightarrow K}^n$ for $I \neq K$ with $I, K \in NB^n$. This means that we may need, a priori, several times at the same point $J \in \mathcal{U}^n$. This possibility of multiple values of the time at the same point J is one of the key properties needed to prove that the GFMM algorithm is monotone; see [10] and section 4. Then the monotonicity property of the GFMM allows us to obtain convergence.

Once we have computed the tentative value for all points of the narrow band, we denote by \tilde{t}_n the minimum of all these values. Unfortunately, this sequence of

“candidate” time is not necessarily nondecreasing, so we have to truncate \tilde{t}_n to define t_n such that (see step 6 of the algorithm)

$$0 \leq t_n - t_{n-1} \leq \Delta t$$

for a fixed Δt (independent of Δx). The algorithm is now very similar to the classical FMM. We accept (in most of the cases; see step 7 of the algorithm) all the points that realize the minimum (i.e., we change the value of the θ), we update the narrow band NB^n and the useful front \mathcal{U}^n , then we redefine the value $u_{J \rightarrow I}^n$ and we iterate.

2.1.2. The monotone GFMM step by step. We now give the precise formulation of the algorithm.

Initialization

1. Set $n = 1$.
2. Initialize the field θ^0 as

$$\theta_I^0 = \begin{cases} 1 & \text{for } x_I \in \Omega_0, \\ -1 & \text{elsewhere.} \end{cases}$$
3. Initialize the time for points I :

$$u_{I \rightarrow K}^0 = \begin{cases} t_0 = 0 & \text{if } K \in NB^0 \text{ and } I \in \mathcal{U}^0(K), \\ +\infty & \text{otherwise.} \end{cases}$$

Loop

4. Compute \tilde{u}^{n-1} on NB^{n-1} .
Let $I \in NB^{n-1}$; then we compute \tilde{u}_I^{n-1} as the solution of the following quadratic equation:

$$(2.1) \quad \sum_{k=1}^N \left(\max_{\pm} (0, \tilde{u}_I^{n-1} - u_{I^{k,\pm} \rightarrow I}^{n-1}) \right)^2 = \frac{(\Delta x)^2}{|\widehat{C}_I^{n-1}|^2},$$

where

$$I^{k,\pm} = (i_1, \dots, i_{k-1}, i_k \pm 1, i_{k+1}, \dots, i_N).$$

5. $\tilde{t}_n = \min \{ \tilde{u}_I^{n-1}, I \in NB^{n-1} \}$.
6. Truncate \tilde{t}_n :
 $t_n = \max(t_{n-1}, \min\{\tilde{t}_n, t_{n-1} + \Delta t\})$
7. If $t_n < \tilde{t}_n$ go to 10 with $\theta^n = \theta^{n-1}$.
8. Initialize the new accepted points:
 $NA_{\pm}^n = \{I \in NB_{\pm}^{n-1}, \tilde{u}_I^{n-1} = \tilde{t}_n\}$, $NA^n = NA_+^n \cup NA_-^n$.
9. Reinitialize θ^n :

$$\theta_I^n = \begin{cases} -\theta_I^{n-1} & \text{for } I \in NA^n, \\ \theta_I^{n-1} & \text{elsewhere.} \end{cases}$$

10. Reinitialize $u_{I \rightarrow K}^n$:

$$u_{I \rightarrow K}^n = \begin{cases} \min(u_{I \rightarrow K}^{n-1}, t_n) & \text{if } K \in NB^n \text{ and } I \in \mathcal{U}^n(K), \\ +\infty & \text{otherwise.} \end{cases}$$

11. Set $n : +1$ and go to 4.

Remark 2.5. We refer the reader to [7, subsection 2.3] for a detailed discussion on the complexity and implementation of an algorithm very similar to the present one. The only difference is that in the present algorithm we may have local times $u_{I \rightarrow J}$ for each neighbor $J \in V(I) \setminus \{I\}$, which in the worse case may introduce a factor $2N$ (the number of nearest neighbors of a point in \mathbb{Z}^N) in the computation of the total complexity of the algorithm.

2.1.3. Reconstruction of the numerical solution. We now explain how to build the numerical solution, using the output of the monotone GFMM algorithm.

We first remark that the physical sequence of time $\{t_n, n \in \mathbb{N}\}$, defined at step 6 of the algorithm, is nondecreasing, and we can extract a subsequence $\{t_{n_k}, k \in \mathbb{N}\}$ strictly increasing such that

$$(2.2) \quad t_{n_k} = t_{n_{k+1}} = \dots = t_{n_{k+1}-1} < t_{n_{k+1}}.$$

Then we denote by S_I^k the square cell $S_I^k = [x_I, x_I + \Delta x[\times [t_{n_k}, t_{n_{k+1}}[$ with

$$[x_I, x_I + \Delta x[= \prod_{\alpha=1}^N [x_{i_\alpha}, x_{i_\alpha} + \Delta x[$$

and by ε the couple

$$\varepsilon = (\Delta x, \Delta t).$$

We now define the following function:

$$(2.3) \quad \theta^\varepsilon(x, t) = \theta_I^{n_{k+1}-1} \text{ if } (x, t) \in S_I^k,$$

which can be interpreted as the numerical approximation of the solution θ of the local equation (1.2), reconstructed using the monotone GFMM.

2.2. A nonlocal algorithm modeled on the monotone GFMM. Our aim is to adapt the previous algorithm to compute a numerical approximation of a solution of the nonlocal equation (1.1).

Given a data $\theta = (\theta_I)_{I \in \mathbb{Z}^N}$, we define its extension on \mathbb{R}^N by

$$(2.4) \quad \theta^\Delta(x) = \sum_{I \in \mathbb{Z}^N} \theta_I 1_{[x_I, x_I + \Delta x[}(x)$$

and the corresponding nonlocal velocity by

$$c[\theta^\Delta](x, t) = (c_0 \star \theta^\Delta)(x) + c_1(x, t).$$

The nonlocal algorithm modeled on the monotone GFMM (in short, the nonlocal GFMM algorithm) is then defined from the local GFMM, adding an initializing step before step 3 and adding a new step before step 10:

2b. *Initialize the speed:* $c_I^0 \equiv c[(\theta^0)^\Delta](x_I, t_0)$, with $t_0 = 0$.

9b. *Compute the speed c_I^n :*

Given $t_{n-1} \in [p\Delta T, (p+1)\Delta T)$ for some $p \in \mathbb{N}$, we set

$$c_I^n \equiv \begin{cases} c_I^{n-1} & \text{if } t_n < (p+1)\Delta T, \\ c[(\theta^{n-1})^\Delta](x_I, t_n) & \text{if } t_n \geq (p+1)\Delta T. \end{cases}$$

Here $\Delta T > 0$ is a time step that has been introduced to avoid recomputing the nonlocal velocity at each step of the algorithm. This time step ΔT is not connected to the time step Δt (nor to the space step Δx). Indeed, we can choose $\Delta t \ll \Delta T \ll 1$ or, for instance, $\Delta t > \Delta T$.

When we deal with the nonlocal equation, we redefine the parameter of discretization ε as

$$\varepsilon = (\Delta x, \Delta t, \Delta T)$$

that will be required to converge to zero, to ensure the convergence of the numerical method in that case.

Remark 2.6. Indeed, we could choose “ $\Delta T = 0$,” which means recomputing the velocity at each iteration of the algorithm such that $t_n > t_{n-1}$.

Remark 2.7. The fact that we use θ^{n-1} to recompute the velocity at time t_n (even if we know θ^n) could seem curious. The reason is that, once we have computed the solution $(\theta_I^n)_{n,I}$ to the nonlocal GFMM and then constructed θ^ε , it is possible to interpret the nonlocal GFMM as the local GFMM with a velocity which is now recomputed using the solution θ^ε only.

More technically, this is due to the fact that we can write (see (5.3)) the velocity as a convolution with the function θ^ε . Indeed, let us consider a sequence $\{n_k, k \in \mathbb{N}\}$ with the following time iterations:

$$t_{n_k} = t_{n_{k+1}} = \dots = t_{n_{k+1}-1} < t_{n_{k+1}}.$$

Since $t_n \geq (p + 1)\Delta T$ and $t_{n-1} < (p + 1)\Delta T$, then $n - 1 = n_{k+1} - 1$ for some index k . Using now the definition (2.3), we get

$$\theta^\varepsilon(x, t) = \theta_I^{n_{k+1}-1} = \theta_I^{n-1} \text{ if } (x, t) \in S_I^k,$$

and this fact allows us to write the speed as in (5.3):

$$c_I^n = c[\theta^\varepsilon(\cdot, t_{n_k})](x_I, t_{n_{k+1}}).$$

3. Main results. We consider an open set $\Omega_0 \subset \mathbb{R}^N$ and the nonlocal eikonal equation

$$(3.1) \quad \begin{cases} \theta_t(x, t) = c[\theta(\cdot, t)](x, t)|D\theta(x, t)| & \text{on } \mathbb{R}^N \times (0, +\infty), \\ \theta(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c}, \end{cases}$$

where

$$c[\theta(\cdot, t)](x, t) = (c_0 \star \theta(\cdot, t))(x) + c_1(x, t).$$

We make the following assumption:

- (A1) *Regularity of the velocity:* $c_1 \in W^{1,\infty}(\mathbb{R}^N \times [0, +\infty))$, $c_0 \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

We denote by L_{c_1} the time Lipschitz constant of c_1 such that

$$(3.2) \quad |c_1(x, t') - c_1(x, t)| \leq L_{c_1}|t' - t| \quad \text{for any } x \in \mathbb{R}^N, \quad t', t \geq 0.$$

3.1. First main result. Our first main result concerns the case of a dislocation curve in dimension $N = 2$, for which a short time existence and uniqueness result is known (see [5]).

THEOREM 3.1 (dislocation loop for short time with nonsigned velocity). *Assume (A1) and that $\Omega_0 \subset \mathbb{R}^2$ is a C^3 bounded connected and simply connected open set. Then there exists $T^* > 0$ such that for $\varepsilon = (\Delta x, \Delta t, \Delta T)$ converging to zero, we have*

$$\theta^\varepsilon \rightarrow \theta \quad \text{in } L^\infty([0, T^*]; L^1(\mathbb{R}^N)),$$

where θ^ε is given in (2.3) using the nonlocal GFMM algorithm given in subsection 2.2 and θ is the unique solution of (3.1) on the time interval $[0, T^*)$.

In higher dimensions, in the case where the nonlocal velocity is positive, a global existence result is known under suitable assumptions (see [2]). In that case, we get the following result.

THEOREM 3.2 (large time dislocation dynamics with nonnegative velocity). *Assume that $c_0 \in W^{2,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $c_1 \in W^{2,\infty}(\mathbb{R}^N \times [0, +\infty))$, and $c_1(x, t) \geq |c_0|_{L^1(\mathbb{R}^N)} + \delta$ with $\delta > 0$. Assume also that $\Omega_0 \subset \mathbb{R}^N$ is a C^2 bounded open set. Then for $\varepsilon = (\Delta x, \Delta t, \Delta T)$ converging to zero, we have*

$$\theta^\varepsilon \rightarrow \theta \quad \text{in } L^\infty([0, +\infty); L^1(\mathbb{R}^N)),$$

where θ^ε is given in (2.3) using the nonlocal GFMM algorithm given in subsection 2.2 and θ is the unique solution of (3.1).

3.2. A general convergence result. In this subsection, we will present a general convergence result whose Theorems 3.1 and 3.2 will be seen as consequences. To this end, we recall the following definitions of an upper semicontinuous envelope,

$$\theta^*(x, t) = \limsup_{(y,s) \rightarrow (x,t)} \theta(y, s),$$

and of a lower semicontinuous envelope,

$$\theta_*(x, t) = \liminf_{(y,s) \rightarrow (x,t)} \theta(y, s).$$

Then we introduce the following assumptions:

(A2) *Existence, uniqueness, and Lipschitz-in-time nonlocal velocity:* There exists a finite time $T^* > 0$ and a unique solution $\theta \in C([0, T^*), L^1(\mathbb{R}^N))$ which is a discontinuous viscosity solution of equation (3.1). Moreover, the velocity is Lipschitz in time, i.e.,

$$(3.3) \quad |c[\theta(\cdot, t')](x, t') - c[\theta(\cdot, t)](x, t)| \leq L_c |t' - t|$$

for any $x \in \mathbb{R}^N$ and $t', t \in [0, T^*)$

Remark 3.3. Here the uniqueness of the discontinuous viscosity solution θ has to be understood as the fact that all the solutions have the same upper (resp., lower) semicontinuous envelopes.

(A3') *L^1 stability estimate for a perturbed local problem:* For a given local velocity $c \in W^{1,\infty}(\mathbb{R}^N \times [0, T^*))$, there exists $e_0 > 0$, such that for any function $e \in C([0, T^*))$ satisfying $|e|_{L^\infty([0, T^*))} \leq e_0$, the solution θ^e of the perturbed problem

$$(3.4) \quad \begin{cases} \theta_t^e(x, t) = c^e(x, t) |D\theta^e(x, t)| & \text{on } \mathbb{R}^N \times (0, T^*) \text{ with } c^e(x, t) = c(x, t) + e(t), \\ \theta^e(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c} & \text{on } \mathbb{R}^N \end{cases}$$

exists, is unique, and satisfies

$$|(\theta^e)^*(\cdot, t) - (\theta^e)_*(\cdot, t)|_{L^1(\mathbb{R}^N)} = 0 \quad \text{for any } 0 \leq t < T^*.$$

We also assume that there exist some constants $C_0 > 0, \bar{T} > 0$ such that for every $0 \leq T_0 < T^*$, and if moreover $|(\theta^e - \theta)(\cdot, T_0)|_{L^1(\mathbb{R}^N)} \leq e_0$, then

$$(3.5) \quad |\theta^e - \theta|_{L^\infty([T_0, T_0+T]; L^1(\mathbb{R}^N))} \leq C_0 \left(|e|_{L^\infty([T_0, T_0+T])} T + |(\theta^e - \theta)(\cdot, T_0)|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}} \right)$$

for any $0 < T \leq \min(\bar{T}, T^* - T_0)$.

(A3) *L¹ stability estimate for a perturbed nonlocal problem:* For θ the solution given in assumption (A2), we assume that (A3') holds with the velocity $c(x, t) := c[\theta(\cdot, t)](x, t)$.

THEOREM 3.4 (general convergence result for nonlocal dynamics). *Let us consider problem (3.1) with a C^2 bounded open set $\Omega_0 \subset \mathbb{R}^N$, under assumptions (A1), (A2), (A3). Then for $\varepsilon = (\Delta x, \Delta t, \Delta T)$ converging to zero, we have*

$$\theta^\varepsilon \rightarrow \theta \quad \text{in } L^\infty([0, T^*]; L^1(\mathbb{R}^N)),$$

where θ^ε is given in (2.3) using the nonlocal GFMM algorithm given in subsection 2.2 and θ is the unique solution of (3.1) on the time interval $[0, T^*)$.

4. Preliminary results for the monotone GFMM for the local equation.

In this section, we recall some key properties of the monotone GFMM for the local equation (for which we refer the reader to [10]) and also present some consequences.

The following result is stated in Theorem 3.4 of [10].

THEOREM 4.1 (comparison principle for the θ^ε). *Let $T > 0$. Given two velocities c_u and c_v , and given $\theta_{u,J}^0$ (resp., $\theta_{v,J}^0$) for all $J \in \mathbb{Z}^N$, we denote by (t_{n_k}) and (s_{m_p}) the increasing sequences of times defined as in (2.2) for the GFMM associated to c_u and c_v , respectively. We assume that for all $t_{n_k} < T$, $s_{m_p} < T$, such that $[t_{n_k}, t_{n_{k+1}}) \cap [s_{m_p}, s_{m_{p+1}}) \neq \emptyset$, the two velocities satisfy*

$$(4.1) \quad c_v(x, s_{m_p}) \geq c_u(x, t_{n_k})$$

for all $x \in \mathbb{R}^N$. If $\theta_{v,J}^0 \geq \theta_{u,J}^0$ for all $J \in \mathbb{Z}^N$, then

$$\theta_v^\varepsilon(x, t) \geq \theta_u^\varepsilon(x, t)$$

for all $(x, t) \in \mathbb{R}^N \times [0, T)$.

Remark 4.2. The assumption on the comparison of the velocities is slightly different from the one of [10], but this is in fact what is used in the proof. Indeed, the algorithm used only the velocity at the time of the form $t_{n_k} < T$ and $s_{m_p} < T$. Hence, it suffices to compare the velocities at those times.

We now give the convergence result for the monotone GFMM (presented in subsection 2.1). To do this, we have to define the half-relaxed limits of θ^ε :

$$(4.2) \quad \bar{\theta}^0(x, t) = \limsup_{\varepsilon \rightarrow 0, y \rightarrow x, s \rightarrow t} \theta^\varepsilon(y, s), \quad \underline{\theta}^0(x, t) = \liminf_{\varepsilon \rightarrow 0, y \rightarrow x, s \rightarrow t} \theta^\varepsilon(y, s).$$

Then the following result is stated in Theorem 3.6 of [10].

THEOREM 4.3 (convergence). *Let $c \in W^{1,\infty}(\mathbb{R}^N \times [0, T^*))$ and let Ω_0 be a C^2 bounded open set. Then $\bar{\theta}^0$ (resp., $\underline{\theta}^0$) is a viscosity subsolution (resp., supersolution) of (1.2). In particular, if (1.2) satisfies a comparison principle, then $\bar{\theta}^0 = (\underline{\theta}^0)^*$ and $(\bar{\theta}^0)_* = \underline{\theta}^0$ is the unique discontinuous viscosity solution of (1.2).*

For simplicity of notation, given a constant $e_0 > 0$, we now define

$$(4.3) \quad \mathcal{E} = \{e \in W^{1,\infty}([0, T^*)), |e|_{W^{1,\infty}([0, T^*))} \leq e_0\}.$$

We then have the following result, which is a straightening of the convergence result Theorem 4.3 and whose proof is exactly the same and therefore is skipped.

LEMMA 4.4 (uniform convergence). *Let Ω_0 be a C^2 bounded open set and let $c \in W^{1,\infty}(\mathbb{R}^N \times [0, T^*))$ be such that assumption (A3') holds for c . Let us consider*

two sequences $(e_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ and a function $e \in \mathcal{E}$ such that $\varepsilon_n \rightarrow 0$ and $e_n \rightarrow e$ in $L^\infty([0, T^*])$ as $n \rightarrow \infty$. We call θ^e the solution of problem (3.4) and $\theta^{\varepsilon_n, e_n}$ the numerical solution given in (2.3) (with $\varepsilon = \varepsilon_n$) using the GFMM algorithm (see subsection 2.1) with speed $c^{\varepsilon_n, e_n}(x, t) = c(x, t) + e_n(t)$.

Then we have for all $(x, t) \in \mathbb{R}^N \times [0, T^*]$

$$\limsup_{y \rightarrow x, s \rightarrow t, n \rightarrow \infty} \theta^{\varepsilon_n, e_n}(y, s) \leq (\theta^e)^*(x, t), \quad \liminf_{y \rightarrow x, s \rightarrow t, n \rightarrow \infty} \theta^{\varepsilon_n, e_n}(y, s) \geq (\theta^e)_*(x, t).$$

We then have the following result.

PROPOSITION 4.5 (uniform error estimate for the perturbed local dynamics). *Let Ω_0 be a C^2 bounded open set and let $c \in W^{1, \infty}(\mathbb{R}^N \times [0, T^*])$ be such that assumption (A3') holds for c . Let $e \in \mathcal{E}$, where the set \mathcal{E} is built with the constant e_0 given in assumption (A3'). Let θ^e be the solution of (3.4). Let us consider $\theta^{\varepsilon, e}$ to be the numerical approximation of θ^e given in (2.3) using the GFMM algorithm (see subsection 2.1) with speed $c^\varepsilon(x, t) = c(x, t) + e(t)$.*

Then there exists a modulus of continuity $\omega_1(\varepsilon)$ such that

$$(4.4) \quad |\theta^e - \theta^{\varepsilon, e}|_{L^\infty([0, T^*]; L^1(\mathbb{R}^N))} \leq \omega_1(\varepsilon) \quad \text{uniformly in } e \in \mathcal{E}.$$

Proof of Proposition 4.5. Let us consider two sequences $(e_n)_{n \in \mathbb{N}} \subset \mathcal{E}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ and a function $e \in \mathcal{E}$ such that $\varepsilon_n \rightarrow 0$ and $e_n \rightarrow e$ for $n \rightarrow \infty$. We call θ^e the solution of problem (3.4) and $\theta^{\varepsilon_n, e_n}$ the numerical solution given in (2.3) (with $\varepsilon = \varepsilon_n$) using the GFMM algorithm (see subsection 2.1) with speed $c^{\varepsilon_n, e_n}(x, t) = c(x, t) + e_n(t)$. We will prove that for $n \rightarrow \infty$

$$|\theta^{\varepsilon_n, e_n} - \theta^e|_{L^\infty([0, T^*]; L^1(\mathbb{R}^N))} \rightarrow 0.$$

We define

$$\bar{\theta}^e(x, t) = \limsup_{y \rightarrow x, s \rightarrow t, n \rightarrow \infty} \theta^{\varepsilon_n, e_n}(y, s) \quad \text{and} \quad \underline{\theta}^e(x, t) = \liminf_{y \rightarrow x, s \rightarrow t, n \rightarrow \infty} \theta^{\varepsilon_n, e_n}(y, s).$$

From Lemma 4.4, we have for all $(x, t) \in \mathbb{R}^N \times [0, T^*]$

$$(\theta^e)_*(x, t) \leq \underline{\theta}^e(x, t) \leq \bar{\theta}^e(x, t) \leq (\theta^e)^*(x, t).$$

Using Assumption (A3'), we deduce that for any fixed $t \in [0, T^*]$ and $t_n \rightarrow t$, we have for $n \rightarrow \infty$

$$(4.5) \quad |\theta^{\varepsilon_n, e_n}(x, t_n) - \theta^e(x, t_n)| \rightarrow 0 \quad \text{a.e. } x \in \mathbb{R}^N.$$

From the boundedness of Ω_0 and the finite velocity property of the eikonal equation, we deduce that there exists $R > 0$ (depending on T^*) such that

$$\text{supp}(\theta^e(\cdot, t_n) + 1) \subset B_R(0).$$

Moreover, using the barrier functions as in [7, Proposition 5.2], we also get (up to increasing R independently on ε and e bounded)

$$\text{supp}(\theta^{\varepsilon_n, e_n}(\cdot, t_n) + 1) \subset B_R(0).$$

Therefore

$$\text{supp}(\theta^{\varepsilon_n, e_n}(\cdot, t_n) - \theta^e(\cdot, t_n)) \subset B_R(0),$$

and since $|\theta^{\varepsilon_n, e_n} - \theta^e| \leq 2$, we conclude, using (4.5) and Lebesgue’s theorem, that for any fixed $t \in [0, T^*)$ and $t_n \rightarrow t$

$$|\theta^{\varepsilon_n, e_n}(\cdot, t_n) - \theta^e(\cdot, t_n)|_{L^1(\mathbb{R}^N)} \rightarrow 0.$$

Proceeding by contradiction, we see that this proof gives the existence of a modulus of continuity $\omega_1(\varepsilon)$ such that inequality (4.4) holds. This ends the proof of the lemma. \square

5. Proof of Theorem 3.4. We first recall that a function $\omega : [0, +\infty] \rightarrow [0, +\infty]$ is said to be a modulus of continuity if ω is nondecreasing and if $\omega(r) \rightarrow 0^+$ as $r \rightarrow 0^+$. More generally, a function $\omega : \mathbb{R}^d \rightarrow [0, +\infty]$ is said to be a modulus of continuity if $r \mapsto \sup_{|\varepsilon| \leq r} \omega(\varepsilon)$ is a modulus of continuity.

Before we give the proof of Theorem 3.4, we need some preliminaries. First, we define

$$\omega_0(r) = \begin{cases} C_0 r^{\frac{1}{N}} + r & \text{if } r \leq e_0, \\ +\infty & \text{if } r > e_0 \end{cases}$$

such that estimate (3.5) in assumption (A3) could be replaced by (for any $0 < T \leq \min(\bar{T}, T^* - T_0)$)

$$(5.1) \quad |\theta^e - \theta|_{L^\infty([0, T_0+T]; L^1(\mathbb{R}^N))} \leq C_0 |e|_{L^\infty([T_0, T_0+T])} T + \omega_0(|\theta^e - \theta|_{L^\infty([0, T_0]; L^1(\mathbb{R}^N))}).$$

We also have the following global-in-time estimate between θ^e and θ , whose proof is postponed.

PROPOSITION 5.1 (global estimate between θ^e and θ). *Let $0 < T \leq T^*$ and assume (A1), (A2), (A3). Then there exists a modulus of continuity $\tilde{\omega}$ (depending only on T^*) such that*

$$|\theta^e - \theta|_{L^\infty([0, T]; L^1(\mathbb{R}^N))} \leq \tilde{\omega}(|e|_{L^\infty([0, T])}).$$

To simplify the notation, we will set

$$|c_0|_\infty = |c_0|_{L^\infty(\mathbb{R}^N)}.$$

To prove Theorem 3.4, we need the following proposition.

PROPOSITION 5.2 (iterative estimate). *Under the assumptions of Theorem 3.4, up to decreasing \bar{T} , we assume that $\bar{T} \leq \min(\frac{1}{4C_0|c_0|_\infty}, 1)$ and that there exists an integer $K^* \geq 2$ such that $\frac{K^*}{2}\bar{T} = T^*$. We also assume that $\Delta t \leq \frac{\bar{T}}{6}$ and that there exist an integer $m \geq 0$ and a modulus of continuity $\bar{\omega}_m$ such that*

$$(5.2) \quad \begin{cases} |\theta^\varepsilon(\cdot, 0) - \theta(\cdot, 0)|_{L^1(\mathbb{R}^N)} \leq \bar{\omega}_0(\varepsilon) & \text{if } m = 0, \\ |\theta^\varepsilon - \theta|_{L^\infty([0, \frac{m}{2}\bar{T}]; L^1(\mathbb{R}^N))} \leq \bar{\omega}_m(\varepsilon) & \text{if } 1 \leq m \leq K^* - 1. \end{cases}$$

Then there exists $\varepsilon_{m+1} > 0$ such that we can define the following modulus of continuity:

$$\widehat{\omega}_{m+1}(\varepsilon) = \begin{cases} 4 \left(\omega_1(\varepsilon) + C_0 \omega_2(\varepsilon) + \omega_0(\tilde{\omega}(|c_0|_\infty \bar{\omega}_m(\varepsilon) + \omega_2(\varepsilon))) \right) & \text{if } |\varepsilon| \leq \varepsilon_{m+1}, \\ +\infty & \text{if } |\varepsilon| > \varepsilon_{m+1} \end{cases}$$

and

$$\bar{\omega}_{m+1}(\varepsilon) = \max(\bar{\omega}_m(\varepsilon), \hat{\omega}_{m+1}(\varepsilon)),$$

where ω_1 , ω_0 , $\tilde{\omega}$, and C_0 are defined, respectively, in Proposition 4.5, in (5.1), in Proposition 5.1, and in assumption (A3), and $\omega_2(\varepsilon) = L_c(\Delta T + 2\Delta t) + L_{c_1}\Delta t$. Then we have

$$|\theta^\varepsilon - \theta|_{L^\infty([0, \frac{m+1}{2}\bar{T}]; L^1(\mathbb{R}^N))} \leq \bar{\omega}_{m+1}(\varepsilon).$$

Proof of Theorem 3.4. First we remark that by our assumptions on the C^2 regularity of the bounded set Ω^0 , there exists a modulus of continuity $\bar{\omega}_0(\varepsilon)$ such that (5.2) holds for $m = 0$. Then we deduce from Proposition 5.2 that

$$|\theta^\varepsilon - \theta|_{L^\infty([0, T^*]; L^1(\mathbb{R}^N))} \leq \bar{\omega}_{K^*}(\varepsilon).$$

This ends the proof of the theorem. \square

Before getting to the proof of Proposition 5.2, let us first explain the main idea, which is to use a barrier argument. To this aim, we have to introduce, for e which will be chosen later, the function $\theta^{\varepsilon, e}$ (resp., $\theta^{\varepsilon, -e}$) given in (2.3) using the solution of the GFMM algorithm in subsection 2.1 with a given local velocity $c[\theta(\cdot, t)](x, t) + e(t)$ (resp., $c[\theta(\cdot, t)](x, t) - e(t)$), where θ is the solution of (3.1) given in assumption (A2). We also define θ^e (resp., θ^{-e}), the exact solution of (3.4) with speed $c[\theta(\cdot, t)](x, t) + e(t)$ (resp., $c[\theta(\cdot, t)](x, t) - e(t)$).

The idea is to choose e such that

$$\theta^{\varepsilon, -e} \leq \theta^\varepsilon \leq \theta^{\varepsilon, e}.$$

To show this result (see Lemma 5.3 below for a precise statement), we will use the comparison principle of the monotone GFMM (Theorem 4.1). To estimate the difference between θ and θ^ε , we will then use the fact that $\theta^{\varepsilon, \pm e}$ is closed to $\theta^{\pm e}$ (by the convergence result of the monotone GFMM; see Lemma 4.5) and the fact that $\theta^{\pm e}$ is closed to θ (by Assumption (A3)).

For simplicity of notation we define $c(x, t) = c[\theta(\cdot, t)](x, t)$. We also need to define the velocity c^ε that is used in the nonlocal GFMM (see step 10b of the nonlocal GFMM). To do this, we define iteratively $c^\varepsilon(x, t_{n_k})$, where $(t_{n_k})_k$ is the increasing sequence defined in (2.2) for the nonlocal GFMM given in subsection 2.2, by setting the initial velocity

$$c^\varepsilon(x, 0) \equiv c[\theta^\varepsilon(\cdot, 0)](x, 0),$$

and if $t_{n_{k-1}} \in [p\Delta T, (p+1)\Delta T)$ for some $p \in \mathbb{N}$, then

$$(5.3) \quad c^\varepsilon(x, t_{n_k}) = \begin{cases} c^\varepsilon(x, t_{n_{k-1}}) & \text{if } t_{n_k} < (p+1)\Delta T, \\ c[\theta^\varepsilon(\cdot, t_{n_{k-1}})](x, t_{n_k}) & \text{if } t_{n_k} \geq (p+1)\Delta T. \end{cases}$$

We extend the velocity in time by considering

$$c^\varepsilon(x, t) = c^\varepsilon(x, t_{n_k}) \quad \text{for } t \in [t_{n_k}, t_{n_{k+1}}).$$

Using the definition (2.3) of θ^ε , we see that the function θ^ε is also given by the (local) GFMM algorithm with the new velocity c^ε .

Before giving the proof of Proposition 5.2, we need the following lemma, whose proof is postponed.

LEMMA 5.3 (barriers on θ^ε). *Let $\tilde{e} : [0, +\infty) \mapsto [0, +\infty)$ be a nondecreasing function. We assume that*

$$\tilde{e}(0) \geq |(\theta^\varepsilon - \theta)(\cdot, 0)|_{L^1(\mathbb{R}^N)}.$$

Moreover if $t_{n_k} > 0$, we assume that

$$(5.4) \quad \text{for all } t_{n_l} \in (0, t_{n_k}), \quad \tilde{e}(\max(t_{n_l} - \Delta t, 0)) \geq |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_l}]; L^1(\mathbb{R}^N))}.$$

We define

$$e(t) = |c_0|_\infty \tilde{e}(t) + \omega_2(\varepsilon) \quad \text{with } \omega_2(\varepsilon) = L_c(\Delta T + 2\Delta t) + L_{c_1} \Delta t.$$

Then for all $(x, t) \in \mathbb{R}^N \times [0, t_{n_{k+1}})$, we have

$$(5.5) \quad \theta^{\varepsilon, -e}(x, t) \leq \theta^\varepsilon(x, t) \leq \theta^{\varepsilon, e}(x, t).$$

Proof of Proposition 5.2. Let us denote by $(t_{n_k})_k$ the sequence of increasing time corresponding to the nonlocal GFMM for θ^ε defined in (2.2). For any integer m satisfying $0 \leq m \leq K^* - 1$, we define p and k such that

$$0 \leq t_{n_k} \leq \frac{m}{2} \bar{T} < t_{n_{k+1}} < \dots < t_{n_{k+p}} \leq \frac{m+1}{2} \bar{T} < t_{n_{k+p+1}}.$$

In particular, we have $t_{n_{k+p+1}} \in (\frac{m+1}{2} \bar{T}, \frac{m+1}{2} \bar{T} + \Delta t]$.

We proceed by recurrence. We assume that there exists $l \in \{0, \dots, p\}$ such that

$$(5.6) \quad |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l}}]; L^1(\mathbb{R}^N))} \leq \bar{\omega}_{m+1}(\varepsilon) \quad \text{if } t_{n_{k+l}} > 0.$$

We exclude the case $m = l = 0$, because we recall that $t_{n_0} = 0 = t_0$. Note also that inequality (5.6) is always satisfied for $l = 0$ when $m \geq 1$, because of our assumption (5.2), the fact that $t_{n_k} \leq \frac{m}{2} \bar{T}$, and the fact that by definition of $\bar{\omega}_{m+1}(\varepsilon)$, we have $\bar{\omega}_m(\varepsilon) \leq \bar{\omega}_{m+1}(\varepsilon)$.

Then we will show that

$$(5.7) \quad |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l+1}}]; L^1(\mathbb{R}^N))} \leq \bar{\omega}_{m+1}(\varepsilon),$$

where in the special case in which $m = K^* - 1$ we redefine $t_{n_{k+p+1}}$ to be equal to $\frac{K^*}{2} \bar{T} = T^*$ in order to ensure that for any integer $0 \leq m \leq K^* - 1$ we have $t_{n_{k+p+1}} \leq T^*$.

We define ε_{m+1} such that for all $|\varepsilon| \leq \varepsilon_{m+1}$ we have

$$(5.8) \quad \begin{cases} \bar{\omega}_{m+1}(\varepsilon) \leq \frac{e_0 \bar{T}}{6|c_0|_\infty}, \\ |c_0|_\infty \bar{\omega}_{m+1}(\varepsilon) + \omega_2(\varepsilon) \leq e_0, \\ \tilde{\omega}(|c_0|_\infty \bar{\omega}_m(\varepsilon) + \omega_2(\varepsilon)) \leq e_0. \end{cases}$$

The proof is broken down into two cases.

Case 1: $m \geq 1$. In this case, note that $t_{n_k} > 0$ (otherwise $t_{n_{k+1}} \leq \Delta t \leq \frac{\bar{T}}{6} < \frac{m}{2} \bar{T}$). We claim that for $|\varepsilon| \leq \varepsilon_{m+1}$, $|\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l+1}}]; L^1(\mathbb{R}^N))}$ satisfies (5.7). To show

this, let us define a nondecreasing function $\tilde{e} \in W^{1,\infty}([0, \infty))$ such that in the case $l = 0$

$$\tilde{e}(t) = |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_k}); L^1(\mathbb{R}^N))} \quad \text{if } t \geq 0,$$

and we set in the case $l \geq 1$

$$\tilde{e}(t) = \begin{cases} |\theta^\varepsilon - \theta|_{L^\infty([0, \frac{m}{2}\bar{T}); L^1(\mathbb{R}^N))} & \text{if } t \leq (\frac{m}{2} - \frac{1}{3})\bar{T}, \\ |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l}}]); L^1(\mathbb{R}^N))} & \text{if } t \geq (\frac{m}{2} - \frac{1}{6})\bar{T} \end{cases}$$

with \tilde{e} linear on $[(\frac{m}{2} - \frac{1}{3})\bar{T}, (\frac{m}{2} - \frac{1}{6})\bar{T}]$. We also define $e(t) = |c_0|_\infty \tilde{e}(t) + \omega_2(\varepsilon)$. Using estimate (5.8), we easily check that $e \in \mathcal{E}$ (where \mathcal{E} is defined in (4.3) with e_0 given in assumption (A3)).

Using Lemma 5.3 and the fact that $t_{n_{k+l+1}} \leq \min((\frac{m}{2} + \frac{2}{3})\bar{T}, T^*)$, we then get

$$\begin{aligned} |(\theta^\varepsilon - \theta)^+|_{L^\infty([0, t_{n_{k+l+1}}]); L^1(\mathbb{R}^N))} &\leq |(\theta^{\varepsilon, e} - \theta)^+|_{L^\infty([0, t_{n_{k+l+1}}]); L^1(\mathbb{R}^N))} \\ &\leq |(\theta^{\varepsilon, e} - \theta^e)^+|_{L^\infty([0, T^*]); L^1(\mathbb{R}^N))} \\ &\quad + |(\theta^e - \theta)^+|_{L^\infty([0, \min((\frac{m}{2} + \frac{2}{3})\bar{T}, T^*)); L^1(\mathbb{R}^N))} \\ &\leq \omega_1(\varepsilon) + C_0 |e|_{L^\infty([0, \infty))} \bar{T} \\ &\quad + \omega_0(|\theta^e - \theta|_{L^\infty([0, (\frac{m}{2} - \frac{1}{3})\bar{T}]; L^1(\mathbb{R}^N))}), \end{aligned} \tag{5.9}$$

where, in the last line, we have used Proposition 4.5 to bound the first term and have used estimate (5.1) (implied by Assumption (A3)) to bound the second term.

In the same way we obtain

$$\begin{aligned} |(\theta^\varepsilon - \theta)^-|_{L^\infty([0, t_{n_{k+l+1}}]); L^1(\mathbb{R}^N))} \\ \leq \omega_1(\varepsilon) + C_0 |e|_{L^\infty([0, \infty))} \bar{T} + \omega_0(|\theta^e - \theta|_{L^\infty([0, (\frac{m}{2} - \frac{1}{3})\bar{T}]; L^1(\mathbb{R}^N))}). \end{aligned} \tag{5.10}$$

Summing (5.9) and (5.10), we get

$$\begin{aligned} |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l+1}}]); L^1(\mathbb{R}^N))} &\leq 2C_0(|c_0|_\infty |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l}}]); L^1(\mathbb{R}^N))} + \omega_2(\varepsilon)) \bar{T} \\ &\quad + 2\omega_1(\varepsilon) + 2\omega_0(|\theta^e - \theta|_{L^\infty([0, (\frac{m}{2} - \frac{1}{3})\bar{T}]; L^1(\mathbb{R}^N))}) \\ &\leq \frac{1}{2} |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l}}]); L^1(\mathbb{R}^N))} \\ &\quad + 2 \left(C_0 \omega_2(\varepsilon) + \omega_1(\varepsilon) + \omega_0 \right. \\ &\quad \left. \cdot \left(\tilde{\omega} \left(|c_0|_\infty |\theta^\varepsilon - \theta|_{L^\infty([0, \frac{m}{2}\bar{T}]; L^1(\mathbb{R}^N))} + \omega_2(\varepsilon) \right) \right) \right) \\ &\leq \frac{1}{2} |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l}}]); L^1(\mathbb{R}^N))} + \frac{1}{2} \bar{\omega}_{m+1}(\varepsilon) \\ &\leq \bar{\omega}_{m+1}(\varepsilon), \end{aligned}$$

where we have used the fact that $\bar{T} \leq \min(\frac{1}{4C_0|c_0|_\infty}, 1)$ and Proposition 5.1 for the second inequality, the fact that $|\theta^\varepsilon - \theta|_{L^\infty([0, \frac{m}{2}\bar{T}]; L^1(\mathbb{R}^N))} \leq \bar{\omega}_m(\varepsilon)$ with the definition of $\bar{\omega}_{m+1}$ for the third one, and (5.6) for the last one. This proves (5.7).

Case 2: $m = 0$. The proof is completely the same to the previous one. The only difference is that we take the constant function $\tilde{e} = |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l}}]); L^1(\mathbb{R}^N)}$ (or the

constant function $\tilde{e} = |\theta^\varepsilon(\cdot, 0) - \theta(\cdot, 0)|_{L^1(\mathbb{R}^N)}$ if $t_{n_{k+l}} = 0$) and that (5.9) is replaced by

$$\begin{aligned} & |(\theta^\varepsilon - \theta)^+|_{L^\infty([0, t_{n_{k+l+1}}]; L^1(\mathbb{R}^N))} \\ & \leq \omega_1(\varepsilon) + C_0 e^{\bar{T}} + \omega_0(|\theta^\varepsilon(\cdot, 0) - \theta(\cdot, 0)|_{L^1(\mathbb{R}^N)}) = \omega_1(\varepsilon) + C_0 e^{\bar{T}}. \end{aligned}$$

Hence, we will get

$$\begin{aligned} |\theta^\varepsilon - \theta|_{L^\infty([0, t_{n_{k+l+1}}]; L^1(\mathbb{R}^N))} & \leq 2C_0(|c_0|_\infty \bar{\omega}_1(\varepsilon) + \omega_2(\varepsilon))\bar{T} + 2\omega_1(\varepsilon) \\ & \leq \frac{1}{2}\bar{\omega}_1(\varepsilon) + 2\left(C_0\omega_2(\varepsilon) + \omega_1(\varepsilon)\right) \\ & \leq \bar{\omega}_1(\varepsilon). \end{aligned}$$

This proves (5.7) and ends the proof of the theorem. \square

Proof of Lemma 5.3. Let $x \in \mathbb{R}^N$ and $0 \leq t_{n_l} \leq t_{n_k}$. We define p such that $t_{n_l} \in [p\Delta T, (p+1)\Delta T)$.

We claim that

$$(5.11) \quad c^\varepsilon(x, t_{n_l}) \leq \inf_{s \in [\max(t_{n_l} - \Delta t, 0), \min(t_{n_l} + \Delta t, t_{n_{k+1}})]} c^\varepsilon(x, s).$$

In the case $p = 0$, we extend θ^ε for negative times as

$$\theta^\varepsilon(x, t) = \theta_I^0 \quad \text{if } (x, t) \in [x_I, x_I + \Delta x) \times (-\infty, 0)$$

Then, by the definition of c^ε , there exist two times $t^* \in [p\Delta T - \Delta t, p\Delta T)$ and $t^{**} \in [t^*, t^* + \Delta t]$ with $t^{**} \geq p\Delta T$ such that

$$c^\varepsilon(x, t_{n_l}) = c_0 \star \theta^\varepsilon(\cdot, t^*)(x) + c_1(x, t^{**}).$$

We define \bar{s} such that $\inf_{s \in [\max(t_{n_l} - \Delta t, 0), \min(t_{n_l} + \Delta t, t_{n_{k+1}})]} (c(x, s) + e(s)) = c(x, \bar{s}) + e(\bar{s})$; then we have

$$\begin{aligned} c^\varepsilon(x, t_{n_l}) & = c_0 \star (\theta^\varepsilon(\cdot, t^*) - \theta(\cdot, t^*))(x) + c[\theta(\cdot, t^*)](x, t^*) \\ & \quad + c_1(x, t^{**}) - c_1(x, t^*) - c[\theta(\cdot, \bar{s})](x, \bar{s}) + c[\theta(\cdot, \bar{s})](x, \bar{s}) \\ & \leq |c_0|_\infty |\theta^\varepsilon(\cdot, t^*) - \theta(\cdot, t^*)|_{L^1(\mathbb{R}^N)} + L_c |t^* - \bar{s}| \\ & \quad + L_{c_1} |t^* - t^{**}| + c[\theta(\cdot, \bar{s})](x, \bar{s}) \\ & \leq |c_0|_\infty \tilde{e}(\max(t^* - \Delta t, 0)) + L_c(\Delta T + 2\Delta t) + L_{c_1} \Delta t + c(x, \bar{s}) \\ & \leq |c_0|_\infty \tilde{e}(\bar{s}) + L_c(\Delta T + 2\Delta t) + L_{c_1} \Delta t + c(x, \bar{s}) \\ & = e(\bar{s}) + c(x, \bar{s}) = \inf_{s \in [\max(t_{n_l} - \Delta t, 0), \min(t_{n_l} + \Delta t, t_{n_{k+1}})]} (c(x, s) + e(s)), \end{aligned}$$

where we have used for the fourth inequality the fact $\max(t^* - \Delta t, 0) \leq \max(t_{n_l} - \Delta t, 0) \leq \bar{s}$ and the monotonicity of the function \tilde{e} , and for the third inequality the fact that $0 \leq \bar{s} - t^* \leq |t^* - p\Delta T| + |p\Delta T - t_{n_l}| + |t_{n_l} - \bar{s}| \leq \Delta T + 2\Delta t$, that $|t^* - t^{**}| \leq \Delta t$, and that $t^* < t_{n_l} \leq t_{n_k}$ to apply (5.4). This proves the desired result (5.11).

We can then apply the comparison principle, Theorem 4.1, to the couple $(\theta^{\varepsilon, e}, \theta^\varepsilon)$ and get for all $(x, t) \in \mathbb{R}^N \times [0, t_{n_{k+1}})$

$$\theta^\varepsilon(x, t) \leq \theta^{\varepsilon, e}(x, t).$$

Similarly, we prove that

$$c^\varepsilon(x, t_{n_l}) \geq \sup_{s \in [\max(t - \Delta t, 0), \min(t + \Delta t, t_{n_k})]} c^{-\varepsilon}(x, s)$$

and then that $\theta^{\varepsilon, -\varepsilon} \leq \theta^\varepsilon$.

This ends the proof of the lemma. \square

We finish this section with the proof of Proposition 5.1.

Proof of Proposition 5.1. Up to decreasing \bar{T} , we assume that there exists an integer $K \geq 3$ such that $\frac{K}{2}\bar{T} = T$ with $\bar{T} \leq 1$ (note that K can be bounded by a constant depending only on T^* and \bar{T}).

We argue by recurrence. First using (A3), we get that

$$\begin{aligned} |\theta^e - \theta|_{L^\infty([0, \bar{T}]; L^1(\mathbb{R}^N))} &\leq C_0 |e|_{L^\infty([0, \bar{T}])} \bar{T} \\ &\leq C_0 |e|_{L^\infty([0, T])} =: \tilde{\omega}_2(|e|_{L^\infty([0, T])}), \end{aligned}$$

where we have used the fact that $\bar{T} \leq 1$.

Now, let us assume that there exists $k \in \{2, \dots, K - 1\}$ and a modulus of continuity $\tilde{\omega}_k$ such that

$$|\theta^e - \theta|_{L^\infty([0, \frac{k}{2}\bar{T}]; L^1(\mathbb{R}^N))} \leq \tilde{\omega}_k(|e|_{L^\infty([0, T])}),$$

and we will prove that

$$(5.12) \quad |\theta^e - \theta|_{L^\infty([0, \frac{k+1}{2}\bar{T}]; L^1(\mathbb{R}^N))} \leq \tilde{\omega}_{k+1}(|e|_{L^\infty([0, T])}),$$

where

$$\tilde{\omega}_{k+1}(r) = \begin{cases} C_0 r + \omega_0(\tilde{\omega}_k(r)) & \text{if } r \leq e_{k+1}, \\ +\infty & \text{if } r > e_{k+1} \end{cases}$$

for some constant $e_{k+1} \leq e_0$ chosen such that

$$r \leq e_{k+1} \implies \tilde{\omega}_k(r) \leq e_0.$$

The results will then follow by taking

$$\tilde{\omega}(r) := \omega_K(r).$$

To show (5.12), we use assumption (A3) and (5.1), and we get for $2 \leq k \leq K - 1$ that

$$\begin{aligned} |\theta^e - \theta|_{L^\infty([0, \frac{k+1}{2}\bar{T}]; L^1(\mathbb{R}^N))} &\leq C_0 |e|_{L^\infty([0, T])} \bar{T} + \omega_0(|\theta^e - \theta|_{L^\infty([0, \frac{k-1}{2}\bar{T}]; L^1(\mathbb{R}^N))}) \\ &\leq C_0 |e|_{L^\infty([0, T])} \bar{T} + \omega_0(|\theta^e - \theta|_{L^\infty([0, \frac{k}{2}\bar{T}]; L^1(\mathbb{R}^N))}) \\ &\leq \tilde{\omega}_{k+1}(|e|_{L^\infty([0, T])}), \end{aligned}$$

where we have used the fact that $\bar{T} \leq 1$ to get the last line.

This proves (5.12) and ends the proof of the proposition. \square

6. Proof of Theorem 3.1: Short time evolution with nonsigned velocity.

We consider the case of a closed dislocation loop which is a smooth curve $\partial\Omega_0$ in \mathbb{R}^2 at the initial time. For short time evolution, it is known that the curve stays Lipschitz (see [5]). In this subsection, we prove that under an additional regularity assumption on $\partial\Omega_0$, (A1) implies (A2)–(A3).

LEMMA 6.1 (checking assumption (A2)). Assume (A1) and that $\Omega_0 \subset \mathbb{R}^2$ is a C^3 bounded connected and simply connected open set. Then there exists $T_1^* > 0$ (depending only on Ω_0 and on bounds on $|c_1|_{L^\infty(\mathbb{R}^2 \times [0, +\infty))} + |\nabla c_1|_{L^\infty(\mathbb{R}^2 \times [0, +\infty))}$ and on $|c_0|_{L^\infty(\mathbb{R}^2)} + |c_0|_{W^{1,1}(\mathbb{R}^2)}$) for which the following holds:

(i) (Existence and uniqueness) There exists a unique function $\theta \in C([0, T_1^*]; L^1(\mathbb{R}^2))$ which is a discontinuous viscosity solution of (3.1).

The function θ can be written as

$$\theta(\cdot, t) = 1_{\Omega_{f(\cdot, t)}} - 1_{\Omega_{f(\cdot, t)}^c},$$

where $\Omega_{f(\cdot, t)}$ is an open set of boundary Γ_f , where $\Gamma_{f(\cdot, t)}(s) = \Gamma(s) + f(s, t)n(s)$ with $\Gamma(s)$ a parametrization of $\partial\Omega_0$ with unit normal $n(s)$. Here the function f satisfies the equation

$$(6.1) \quad \begin{cases} f_t(s, t) = \left(c_1 - \int_{\mathbb{R}^2} c_0 + 2c_0 \star 1_{\Omega_{f(\cdot, t)}} \right) (\Gamma_{f(\cdot, t)}(s), t) \sqrt{1 + \left(\left(\frac{1}{1 - fK(s)} \right) f_s \right)^2} \\ f(s + L, t) = f(s, t) \quad \text{on } \mathbb{R} \times (0, T_1^*), \\ f(\cdot, 0) = 0 \quad \text{on } \mathbb{R}, \end{cases}$$

where $f_s = \frac{\partial f}{\partial s}$, $K(s)$ is the curvature of $\partial\Omega_0$ at the point $\Gamma(s)$ and L is the length of $\partial\Omega_0$. Moreover

$$|f|_{L^\infty(\mathbb{R} \times [0, T_1^*])} \leq \frac{1}{2|K|_{L^\infty(\mathbb{R})}}$$

and there exists a constant $L_1 > 0$ (depending only on Ω_0 and on the bounds on c_1 and c_0) such that

$$(6.2) \quad |f|_{W^{1,\infty}(\mathbb{R} \times [0, T_1^*])} \leq L_1.$$

(ii) (Velocity Lipschitz in time) The speed $c[\theta(\cdot, t)]$ satisfies (3.3).

Before we prove Lemma 6.1, let us write the following result, whose the proof is postponed (and which will be used in the proofs in what follows).

LEMMA 6.2 (L^∞ versus L^1 estimate). We use the notation of Lemma 6.1.

(i) L^∞ - L^1 estimate. There exists a constant $C_0 > 0$ such that for any functions f^1, f^2 satisfying

$$|f^i|_{L^\infty(\mathbb{R})} \leq \frac{1}{2|K|_{L^\infty(\mathbb{R})}} \quad \text{for } i = 1, 2$$

we have

$$(6.3) \quad |1_{\Omega_{f^2}} - 1_{\Omega_{f^1}}|_{L^1(\mathbb{R}^2)} \leq C_0 |f^2 - f^1|_{L^\infty(\mathbb{R})}.$$

(ii) L^1 - L^∞ estimate. Given a constant $L_1 > 0$, there exists a constant $C_1 > 0$ such that if we have

$$|f_s^i|_{L^\infty(\mathbb{R})} \leq L_1 \quad \text{and} \quad |f^i|_{L^\infty(\mathbb{R})} \leq \frac{1}{2|K|_{L^\infty(\mathbb{R})}} \quad \text{for } i = 1, 2,$$

then

$$|f^2 - f^1|_{L^\infty(\mathbb{R})} \leq C_1 |1_{\Omega_{f^2}} - 1_{\Omega_{f^1}}|_{L^1(\mathbb{R})}^{\frac{1}{2}}.$$

Proof of Lemma 6.1. (i) Let us define $\bar{c}_0(x) = \sup_{y \in B_1(y)} |c_0(y)|$. Under the additional assumption

$$(6.4) \quad \bar{c}_0 \in L^1(\mathbb{R}^2)$$

and

$$(6.5) \quad c_1 \in W^{1,\infty}(\mathbb{R}^2 \times [0, +\infty)),$$

point (i) follows from the statement of Theorem 1.3 in [5] (and estimate (6.2) follows from the proofs of Theorems 3.7 and 4.3 of [5]). Assumption (6.4) was introduced in [5] to be able to deal with an unbounded open set Ω_0 . But looking at the proof of Theorem 1.3 in [5] (see in particular the proof of Theorem 4.3), we see that the result is still true without assumption (6.4) (for a bounded open set Ω_0).

Looking more carefully at the proof of [5], we see that we only use the weaker condition

$$c_1 \in C(\mathbb{R}^2 \times [0, +\infty)) \quad \text{and} \quad \nabla c_1 \in L^\infty(\mathbb{R}^2 \times [0, +\infty)).$$

This shows point (i) for some $T_1^* > 0$.

(ii) Using moreover Corollary 4.4 of [5], we see that $f \in W^{1,\infty}(\mathbb{R} \times (0, T_1^*))$, and then we have with $\theta(x, t) = 2 \mathbf{1}_{\Omega_{f(\cdot, t)}}(x) - 1$ and $s > 0$

$$\begin{aligned} & |c[\theta(\cdot, t + s)](x, t + s) - c[\theta(\cdot, t)](x, t)| \\ & \leq |c_1(x, t + s) - c_1(x, t)| + 2|c_0|_{L^\infty(\mathbb{R}^2)} |\mathbf{1}_{\Omega_{f(\cdot, t+s)}} - \mathbf{1}_{\Omega_{f(\cdot, t)}}|_{L^1(\mathbb{R}^2)} \\ & \leq L_c s \end{aligned}$$

with

$$L_c = |(c_1)_t|_{L^\infty(\mathbb{R}^2 \times (0, T_1^*))} + 2|c_0|_{L^\infty(\mathbb{R}^2)} C_0 L_1,$$

where L_1 is given in (6.2) and we have used estimate (6.3). This shows that $c[\theta(\cdot, t)]$ satisfies (3.3) and ends the proof of point (ii). \square

Proof of Lemma 6.2. (i) Estimate (6.3) is simple to prove (see estimate (4.44) in the proof of Theorem 4.3 in [5]).

(ii) Let $\delta = |f^2 - f^1|_{L^\infty(\mathbb{R})}$. Up to exchanging the role of f^2 and f^1 , we can assume that $\delta = f^2(s_0) - f^1(s_0)$ for some point $s_0 \in \mathbb{R}$. Then for any $|h| \leq h^* := \delta/(2L_1)$, we have

$$f^2(s_0 + h) \geq f^2(s_0) - L_1|h| \geq f^1(s_0) + L_1|h| \geq f^1(s_0 + h)$$

Then it is natural to define the set

$$E = \{(r, h), \quad h \in (-h^*, h^*), r \in (f^1(s_0) + L_1|h|, f^2(s_0) - L_1|h|)\}$$

and the map $\Psi(r, h) := \Gamma(s_0 + h) + rn(s_0 + h)$. Therefore, we see that by assumption we have

$$(6.6) \quad \Psi(E) \subset \Omega_{f^2} \setminus \Omega_{f^1}.$$

On the other hand the set E is a lozange of diagonals of lengths $2h^*$ and δ , which implies

$$|E| = \delta h^*.$$

Therefore, we have with $C_2 = \inf_E |\text{Jac } \Psi|$

$$|\Psi(E)| \geq C_2|E| = C_2\delta h^* = (\delta/C_1)^2,$$

which defines the constant C_1 . Then the result follows from (6.6). This ends the proof of the lemma. \square

LEMMA 6.3 (checking assumption (A3)). *Given Ω_0 as in Lemma 6.1 and a velocity $c \in W^{1,\infty}(\mathbb{R}^2 \times [0, T_1^*])$, for any function e , we consider the velocity*

$$c^e(x, t) = c(x, t) + e(t).$$

Then there exist $e_0 > 0$ and $T^ \in (0, T_1^*]$ such that for any $e \in C([0, T^*])$ satisfying $|e|_{L^\infty([0, T^*])} \leq e_0$, there exists a unique viscosity solution θ^e of the local equation*

$$(6.7) \quad \begin{cases} \theta_t^e(x, t) = c^e(x, t)|D\theta^e(x, t)| & \text{in } \mathbb{R}^2 \times (0, T^*), \\ \theta^e(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c} & \text{on } \mathbb{R}^2. \end{cases}$$

Moreover

$$(6.8) \quad |(\theta^e)^*(\cdot, t) - (\theta^e)_*(\cdot, t)|_{L^1(\mathbb{R}^2)} = 0 \quad \text{for all } t \in [0, T^*]$$

and there exist constants $C_0 > 0$ and $\bar{T} > 0$ such that for every $0 \leq T_0 < T^*$, if $|(\theta^e - \theta)(\cdot, T_0)|_{L^1(\mathbb{R}^2)} \leq e_0$, then we have

$$(6.9) \quad |\theta^e - \theta|_{L^\infty([T_0, T_0+T]; L^1(\mathbb{R}^2))} \leq C_0 \left(|e|_{L^\infty([T_0, T_0+T])} T + |(\theta^e - \theta)(\cdot, T_0)|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \right)$$

with $0 < T \leq \min(\bar{T}, T^* - T_0)$.

Proof of Lemma 6.3.

Step 1: Existence and uniqueness. We simply apply Lemma 6.1 in the case $c_0 = 0$, i.e., a local equation, and $c_1 = c^e$. This gives the existence and uniqueness of a solution on a time interval $[0, T^*]$ with $T^* \in (0, T_1^*]$, which depends only on Ω_0 and on bounds on $|c^e|_{L^\infty(\mathbb{R}^2 \times [0, T^*])} + |\nabla c^e|_{L^\infty(\mathbb{R}^2 \times [0, T^*])}$, i.e., on bounds on $|c|_{W^{1,\infty}(\mathbb{R}^2 \times [0, T^*])}$ and on $e_0 \geq |e|_{L^\infty([0, T^*])}$. Then (6.8) follows from the fact that

$$\theta^e = 1_{\Omega_{f^e(\cdot, t)}} - 1_{\Omega_{f^e(\cdot, t)}^c}.$$

Here the function $f^e(s, t)$ is a solution of

$$(6.10) \quad \begin{cases} f_t^e = H^e(s, t, f^e, f_s^e) & \text{on } \mathbb{R} \times (0, T^*), \\ f^e(s + L, t) = f^e(s, t) & \text{on } \mathbb{R} \times (0, T^*), \\ f^e(\cdot, 0) = 0 & \text{on } \mathbb{R} \end{cases}$$

with

$$H^e(s, t, f, p) = c^e(\Gamma_f(s), t) \sqrt{1 + \left(\frac{p}{1 - fK(s)} \right)^2}$$

and

$$\Gamma_f(s) = \Gamma(s) + fn(s),$$

where $\Gamma(s), n(s), K(s)$, and f are defined in Lemma 6.1. Moreover, up to redefining the constant L_1 , we have

$$|f^e|_{W^{1,\infty}(\mathbb{R} \times [0, T^*])} \leq L_1$$

for all $|e|_{L^\infty([0, T^*])} \leq e_0$.

Step 2: Bound on $|f^e - f|_{L^\infty(\mathbb{R} \times [T_0, T_0+T])}$ for any $0 \leq T_0 < T^*$ and $0 < T \leq \min(\bar{T}, T^* - T_0)$. Let us define

$$\begin{aligned} \hat{f}^e(s, t) &= f(s, t) + C\bar{e}(t - T_0) + a \\ &\text{with } \bar{e} = |e|_{L^\infty([T_0, T_0+T])} + a \\ &\text{and } a = |f^e(\cdot, T_0) - f(\cdot, T_0)|_{L^\infty(\mathbb{R})} \end{aligned}$$

For $|f_2|, |f_1| \leq \delta_0 := \frac{1}{2|K|_{L^\infty(\mathbb{R})}}$, we have with $H := H^0$

$$(6.11) \quad |H(s, t, f_2, p) - H(s, t, f_1, p)| \leq L_2(1 + |p|)|f_2 - f_1|$$

for some constant L_2 (see, for instance, in [5], both the end of the proof of Theorem 4.3 and inequality (3.27)). Up to reducing, respectively, T^* and e_0 , we can assume that

$$(6.12) \quad |f|_{L^\infty(\mathbb{R} \times [0, T^*])} \leq \delta_0/3 \quad \text{and} \quad e_0 \leq \delta_0/3.$$

Finally, assuming that

$$a \leq e_0 \quad \text{and} \quad 2e_0C\bar{T} \leq \delta_0/3,$$

we compute for $T^* \geq T_0 + T > t \geq T_0$

$$\begin{aligned} \hat{f}_t^e &\geq H(s, t, f, f_s) + C\bar{e} \\ &\geq H(s, t, \hat{f}^e, f_s) + C\bar{e} - L_2(1 + L_1)(C\bar{e}(t - T_0) + a) \\ &\geq H^e(s, t, \hat{f}^e, f_s) - |e|_{L^\infty([T_0, T_0+T])}(1 + 2L_1) \\ &\quad + C\bar{e}(1 - (t - T_0)L_2(1 + L_1)) - aL_2(1 + L_1) \\ &\geq H^e(s, t, \hat{f}^e, \hat{f}_s^e) + C\bar{e}/4, \end{aligned}$$

where we have used (6.11) for the second line, (6.12) and the fact that $|e|_{L^\infty([0, T^*])} \leq e_0$ for the third line, and we have assumed $\bar{T} \leq (2L_2(1 + L_1))^{-1}$ and $C > 0$ large enough for the last line ($C \geq 4 \max(1 + 2L_1, L_2(1 + L_1))$). This is a classical exercise to check that we have in the viscosity sense

$$\hat{f}_t^e \geq H^e(s, t, \hat{f}^e, \hat{f}_s^e) \quad \text{on } \mathbb{R} \times (T_0, T_0 + T).$$

Then from the comparison principle we deduce that

$$f^e \leq \hat{f}^e \quad \text{on } \mathbb{R} \times [T_0, T_0 + T].$$

Similarly, we can prove that

$$f^e \geq \hat{f}^{-e} \quad \text{with} \quad \hat{f}^{-e}(s, t) = f(s, t) - C\bar{e}(t - T_0) - a,$$

which implies for $|e|_{L^\infty([0, T^*])} \leq e_0$

$$(6.13) \quad |f^e - f|_{L^\infty(\mathbb{R} \times [T_0, T_0+T])} \leq C\bar{e}T + a \quad \text{for any } 0 \leq T_0 < T_0+T \leq T^* \quad \text{with } 0 < T \leq \bar{T}.$$

Step 3: Conclusion. Estimate (6.13) together with (6.3) and Lemma 6.2 (ii) implies estimate (6.9) with another constant $C > 0$. \square

Proof of Theorem 3.1. By Lemmas 6.1 and 6.3, assumptions (A2)–(A3) are verified, and then the result follows from Theorem 3.4. \square

7. Proof of Theorem 3.2: Large time evolution with positive velocity.

In this section, we consider the case of dislocation dynamics with positive velocity. In this case, under a strengthened assumption (A1), we prove that (A2) and (A3) hold.

LEMMA 7.1 (checking assumption (A2)). *Assume that $c_0 \in W^{2,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $c_1 \in C(\mathbb{R}^N \times [0, +\infty))$ with $\nabla c_1, D^2 c_1 \in L^\infty(\mathbb{R}^N \times [0, +\infty))$, and $c_1(x, t) \geq |c_0|_{L^1}$. Assume also that $\Omega_0 \subset \mathbb{R}^N$ is a C^2 bounded open set. Then the following holds:*

(Existence and uniqueness) *There exists a unique discontinuous viscosity solution $\theta \in C([0, \infty); L^1(\mathbb{R}^N))$ of (3.1). Moreover, we have*

$$(7.1) \quad |\theta^*(\cdot, t) - \theta_*(\cdot, t)|_{L^1(\mathbb{R}^2)} = 0 \quad \text{for all } t \in [0, +\infty).$$

(Velocity Lipschitz in time) *The speed $c[\theta(\cdot, t)]$ satisfies (3.3).*

Proof of Lemma 7.1. If we assume that the velocity has a higher local regularity, namely, $c_0 \in W^{2,\infty}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $c_1 \in W^{2,\infty}(\mathbb{R}^N \times [0, +\infty))$, then the existence and uniqueness is given by [2, Theorem 4.3]. Equality (7.1) follows from [2, Lemma 3.1]. The Lipschitz-in-time property (3.3) of $c[\theta(\cdot, t)](x, t)$ comes from [2, Proposition 3.3] applied to the case with two velocities $\tilde{c}_1(x, s) = c[\theta(\cdot, s + t)](x, s + t)$ and $\tilde{c}_2 = 0$ with initial condition $\theta(\cdot, t)$ together with the inequality

$$(7.2) \quad |c[\theta^1(\cdot, t)](\cdot, t) - c[\theta^2(\cdot, t)](\cdot, t)|_{L^\infty(\mathbb{R}^N)} \leq |c_0|_\infty |\theta^1(\cdot, t) - \theta^2(\cdot, t)|_{L^1(\mathbb{R}^N)}.$$

From the proofs of the previously mentioned results, it is possible to see that the same result holds when we weaken the assumptions on c_0, c_1 as in the statement of Lemma 7.1 (see in particular Theorem 1.3, case (ii) of [6], where this result is true for the level set formulation of the problem). \square

DEFINITION 7.2 (interior ball property). *We say that a compact set K of \mathbb{R}^N has the interior ball property of radius $r > 0$ if and only if for any $y \in \partial K$ there exists a ball $B_r(x) \subset K$ such that $y \in \partial B_r(x)$.*

DEFINITION 7.3 (Hausdorff distance). *Let K^1 and K^2 be two compact sets of \mathbb{R}^N . Then we define the Hausdorff distance between K^1 and K^2 as*

$$\text{dist}_{\mathcal{H}}(K^1, K^2) = \max \left(\sup_{x \in K^1} \inf_{y \in K^2} |x - y|, \sup_{x \in K^2} \inf_{y \in K^1} |x - y| \right).$$

PROPOSITION 7.4 (Hausdorff distance controlled by the Lebesgue measure). *Let the radii $R > r > 0$ be fixed. Then there exists a constant $C = C(R, r, N) > 0$ such that the following holds. Let K^1 and K^2 be two compact sets contained in the ball $B_R(0)$ of \mathbb{R}^N . Assume that K^1 and K^2 have the interior ball property of radius r . Then we have*

$$(7.3) \quad \text{dist}_{\mathcal{H}}(K^1, K^2) \leq C |K^1 \Delta K^2|^{\frac{1}{N}},$$

where $|A|$ is the Lebesgue measure of a set A and $K^1 \Delta K^2 = (K^1 \setminus K^2) \cup (K^2 \setminus K^1)$.

Proof of Proposition 7.4. Without loss of generality, let us assume that

$$\rho = \text{dist}_{\mathcal{H}}(K^1, K^2) = \sup_{x \in K^2} \text{dist}(x, K^1) = \text{dist}(x_0, K^1),$$

with $x_0 \in K^2$. This implies in particular that

$$B_\rho(x_0) \cap K^1 = \emptyset.$$

The proof is broken down into two cases.

Case 1: there exists $x_1 \in (\partial K^2) \cap B_{\rho/2}(x_0)$. This implies that

$$B_{\rho/2}(x_1) \cap K^1 = \emptyset.$$

Using the fact that K^2 satisfies the interior ball property of radius r , we deduce that there exists $z_1 \in K^2$ such that

$$B_r(z_1) \subset K^2 \quad \text{and} \quad x_1 \in \partial B_r(z_1).$$

Subcase 1.1: $\frac{\rho}{2} \leq r$. In this case, we deduce that

$$B_{\rho/2}(z_0) \subset B_r(z_1),$$

where $z_0 = x_1 + \frac{\rho}{2} \frac{z_1 - x_1}{|z_1 - x_1|}$. Hence

$$B_{\rho/2}(z_0) \cap B_{\rho/2}(x_1) \subset B_r(z_1) \cap B_{\rho/2}(x_1) \subset K^2 \setminus K^1 \subset K^1 \Delta K^2.$$

This implies that there exists a constant C_2 (depending only on N) such that

$$|K^1 \Delta K^2| \geq C_2 \rho^N$$

and so

$$\text{dist}_{\mathcal{H}}(K^1, K^2) = \rho \leq \left(\frac{|K^1 \Delta K^2|}{C_2} \right)^{\frac{1}{N}}.$$

Subcase 1.2: $\frac{\rho}{2} \geq r$. In this case, we deduce that

$$B_r(z_1) \cap B_r(x_1) \subset K^2 \setminus K^1 \subset K^1 \Delta K^2.$$

This implies that there exists a constant C_3 (depending only on N) such that

$$|K^1 \Delta K^2| \geq C_3 r^N$$

and so

$$\text{dist}_{\mathcal{H}}(K^1, K^2) \leq 2R \leq \frac{2R}{r} \left(\frac{|K^1 \Delta K^2|}{C_3} \right)^{\frac{1}{N}}.$$

Case 2: $B_{\rho/2}(x_0) \subset K^2 \setminus K^1$. In this case, there exists a constant C_4 such that

$$|K^1 \Delta K^2| \geq C_4 \rho^N$$

and so

$$\text{dist}_{\mathcal{H}}(K^1, K^2) = \rho \leq \left(\frac{|K^1 \Delta K^2|}{C_4} \right)^{\frac{1}{N}}.$$

Finally, in the general case we see that (7.3) holds with a constant C depending only on the radii R, r and the dimension N . \square

LEMMA 7.5 (checking assumption (A3)). Assume that $\Omega_0 \subset \mathbb{R}^N$ is a C^2 bounded open set and let $c \in C(\mathbb{R}^N \times [0, \infty))$ with $\nabla c, D^2 c \in L^\infty(\mathbb{R}^N \times [0, +\infty))$, satisfying $c \geq \delta > 0$, and for any $e \in C([0, \infty))$ with $|e|_{L^\infty([0, \infty))} < \delta$, let us set

$$c^e(x, t) = c(x, t) + e(t).$$

Then there exists a unique viscosity solution θ^e of the local equation

$$(7.4) \quad \begin{cases} \theta_t^e(x, t) = c^e(x, t)|D\theta^e(x, t)| & \text{in } \mathbb{R}^N \times (0, +\infty), \\ \theta^e(\cdot, 0) = 1_{\Omega_0} - 1_{\Omega_0^c} & \text{on } \mathbb{R}^N \end{cases}$$

and

$$(7.5) \quad |(\theta^e)^*(\cdot, t) - (\theta^e)_*(\cdot, t)|_{L^1(\mathbb{R}^N)} = 0 \quad \text{for all } t \in [0, +\infty).$$

Moreover for every $T^* > 0$, there exist $e_0 \in (0, \delta)$, $C_0 > 0$, and $\bar{T} > 0$ such that for any $0 \leq T_0 < T^*$, and if

$$|e|_{L^\infty([0, T^*])} \leq e_0 \quad \text{and} \quad |\theta^e - \theta|_{L^\infty([0, T^*]; L^1(\mathbb{R}^N))} \leq e_0$$

holds, we have

$$(7.6) \quad |\theta^e - \theta|_{L^\infty([T_0, T_0+T]; L^1(\mathbb{R}^N))} \leq C_0 \left(|e|_{L^\infty([T_0, T_0+T])} T + |(\theta^e - \theta)(\cdot, T_0)|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}} \right)$$

for $0 < T \leq \min(\bar{T}, T^* - T_0)$.

Proof of Lemma 7.5. Because $c \geq \delta$ and $|e|_{L^\infty([0, +\infty))} < \delta$, Lemma 7.1 applies and proves the existence and uniqueness of a solution satisfying (7.5). Moreover, for any $T^* > 0$, applying [2, Propositions 3.3] to (7.4) with speeds c and c^e gives the existence of $e_0, C_1 > 0$ such that

$$(7.7) \quad |\theta^e - \theta|_{L^\infty([T_0, T_0+T]; L^1(\mathbb{R}^N))} \leq C_1 \left(|e|_{L^\infty([T_0, T_0+T])} T + \text{dist}_{\mathcal{H}}(\{\theta^e(\cdot, T_0) = 1\}, \{\theta(\cdot, T_0) = 1\}) \right)$$

for $T_0 < T_0 + T \leq T^*$ with $T \leq \bar{T}$ for some $\bar{T} > 0$. Now using Proposition 7.4 together with the estimate on the radius $r = r(T_0)$ of the interior ball given in [2, Lemma 3.1], we deduce that

$$(7.8) \quad \text{dist}_{\mathcal{H}}(\{\theta^e(\cdot, T_0) = 1\}, \{\theta(\cdot, T_0) = 1\}) \leq C_2 |\theta^e(\cdot, T_0) - \theta(\cdot, T_0)|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}}.$$

Finally (7.7) and (7.8) imply (7.6). \square

Proof of Theorem 3.2. By Lemmas 7.1 and 7.5, assumptions (A2), (A3) are verified, and then the result follows from Theorem 3.4. \square

8. Numerical tests for dislocation dynamics. In this section, we present some numerical tests in the nonlocal case performed with an algorithm modeled either on the monotone GFMM (section 2.2) or on the nonmonotone GFMM (Appendix A). Let us point out that the algorithm based on the nonmonotone GFMM is not proved to be convergent. These tests simulate dislocation dynamics in dimension $N = 2$. Test 1 is performed with the nonlocal, nonmonotone GFMM, while the two others are performed with the two algorithms.

The nonlocal part of the speed is given by $c_0 \star \theta^+$, where $\theta^+ = \frac{\theta+1}{2}$ is the positive part of θ . This problem is equivalent to (3.1), up to rescaling the kernel c_0 and modifying the local speed c_1 . In what follows, for simplicity we will denote θ^+ by θ .

To compute the nonlocal discrete velocity $c^\varepsilon(x, t) = c[\theta^\varepsilon(\cdot, t)](x, t) + c_1(x, t)$, we have to compute a convolution of the form

$$(c_0 \star \theta^\varepsilon(\cdot, t_{n_k}))(x_I).$$

This convolution can be rewritten in the following sum:

$$(8.1) \quad (c_0 \star \theta^\varepsilon(\cdot, t_{n_k}))(x_I) = (c_0 \star (\theta^{n_{k+1}-1})^\Delta)(x_I) = \sum_{J \in \mathbb{Z}^2} \bar{c}_{0,I-J} \theta_J^{n_{k+1}-1},$$

where $(\theta)^\Delta$ has been defined in (2.4) and

$$\bar{c}_{0,I} = \frac{1}{|Q_I|} \int_{Q_I} c_0(x) dx,$$

with Q_I the square cell centered in x_I :

$$Q_I = [x_{i_1} - \Delta x/2, x_{i_1} + \Delta x/2] \times [x_{i_2} - \Delta x/2, x_{i_2} + \Delta x/2].$$

The kernel $c_0 = c_{0,\delta}$ is chosen as the kernel proposed in [5] for the Peierls–Nabarro model, whose Fourier transform is given by

$$\widehat{c}_{0,\delta}(\xi_{x_1}, \xi_{x_2}) = -\frac{1}{2} \left(\frac{\xi_{x_1}^2 + (\frac{1}{1-\nu})\xi_{x_2}^2}{\sqrt{\xi_{x_1}^2 + \xi_{x_2}^2}} \right) e^{-\delta \sqrt{\xi_{x_1}^2 + \xi_{x_2}^2}},$$

where ν is the Poisson ratio that takes values in $(-1, 0.5)$, and δ is a constant proportional to the size of the core of the dislocations. In our simulations we consider cases with $\delta = C\Delta x$, where $0.1 \leq C \leq 11$. Since c_0 tends to 0 for $|x| \rightarrow \infty$, we truncate $c_{0,\delta}$ outside the numerical domain.

We consider a bounded square $D = [-l, l] \times [-l, l]$ as numerical domain. In the case $\theta(x_1, x_2)$ has a compact support contained in the numerical domain (Test 2), and we assume $\theta(x_1, x_2)$ $2l$ -periodic in both directions. Then the discrete convolution is computed on D by the inverse discrete Fourier transform of the product of the Fourier transform of c_0 with the Fourier transform of $\theta(x_1, x_2)$ (for more details about the computation of the convolution, see [4]).

In other cases (Tests 1 and 3) we choose an initial data which is $2l$ -periodic in the x_2 -direction. Then the discrete convolution (8.1) is calculated on a larger numerical domain $D_K = [-(2K+1)l, (2K+1)l] \times [-(2K+1)l, (2K+1)l]$, where K is a positive integer. The kernel c^0 is now truncated outside D_K and the following extension of θ is used in the computation of the speed. First, $\theta(x_1, x_2)$ is extended over the domain $[-(2K+1)l, -l] \times [-l, l]$ by the constant in x_2 value $\theta(-l, \cdot)$ and on the domain $[l, (2K+1)l] \times [-l, l]$ by the constant in x_2 value $\theta(l, \cdot)$. Then it is extended over all of domain D_K periodically in the x_2 -direction.

The nonlocal speed is computed each time step size ΔT , whose value will be specified in each test. The step Δt is chosen equal to the space step Δx , when it is not specified.

Test 1: A propagating line with obstacles. We choose as initial data a line and three different speeds describing three different obstacles. In all cases the continuous speed is given by $c(x, t) = (c_{0,\delta} \star \theta(\cdot, t))(x) + c_1(x, t)$ with $\delta = 0.5$, $\nu = 0$ and local speed

$$c_1(x, t) = \begin{cases} \alpha & \text{if } |x|^2 < 0.3, \\ 2 & \text{otherwise.} \end{cases}$$

The presence of the external force c_1 allows the line to propagate. For $\alpha < 2$, the speed c_1 on the domain $|x|^2 < 0.3$ can be thought of as an obstacle which slows down

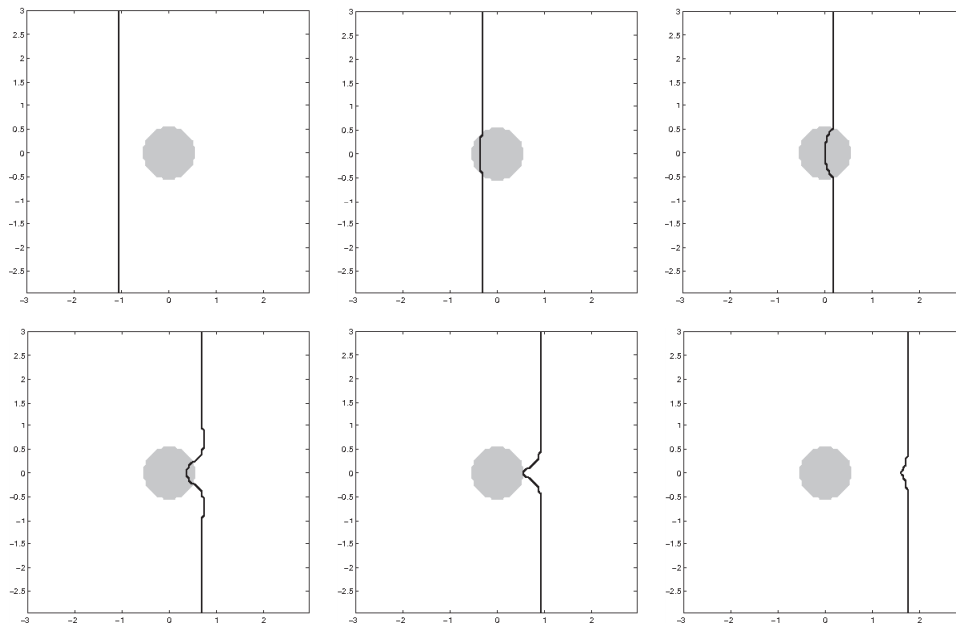


FIG. 2. *A propagating line with an obstacle, case $\alpha = 1.5$.*

the propagation of the line. The smaller α is, the deeper the obstacle is. We compute the discrete solution in the numerical domain $[-3, 3] \times [-3, 3]$, with 129 nodes on each side, and we update the speed each $\Delta T = 0.02$. Here we use the nonmonotone GFMM (presented in the appendix), and the discrete convolution is computed in the extended domain D_K with $K = 1$ and $l = 3$.

Case $\alpha = 1.5$: the line passes over the obstacle. The line slows down in the zone of the obstacle, but it manages to exceed it without disconnecting itself. In Figure 2 we show the line evolution and the obstacle at times $t = 0, 0.42, 0.66, 0.93, 1.03, 1.44$.

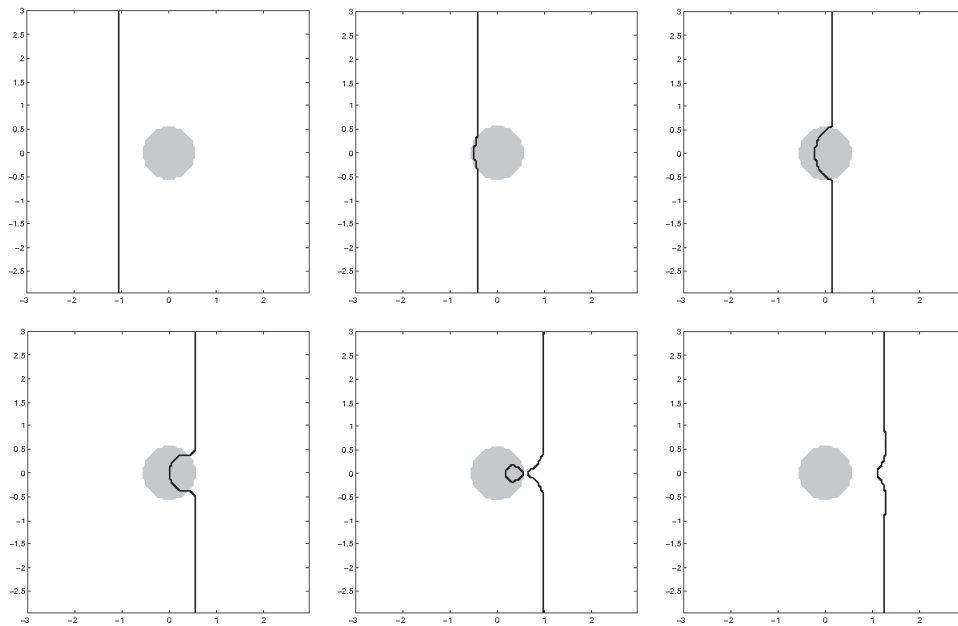
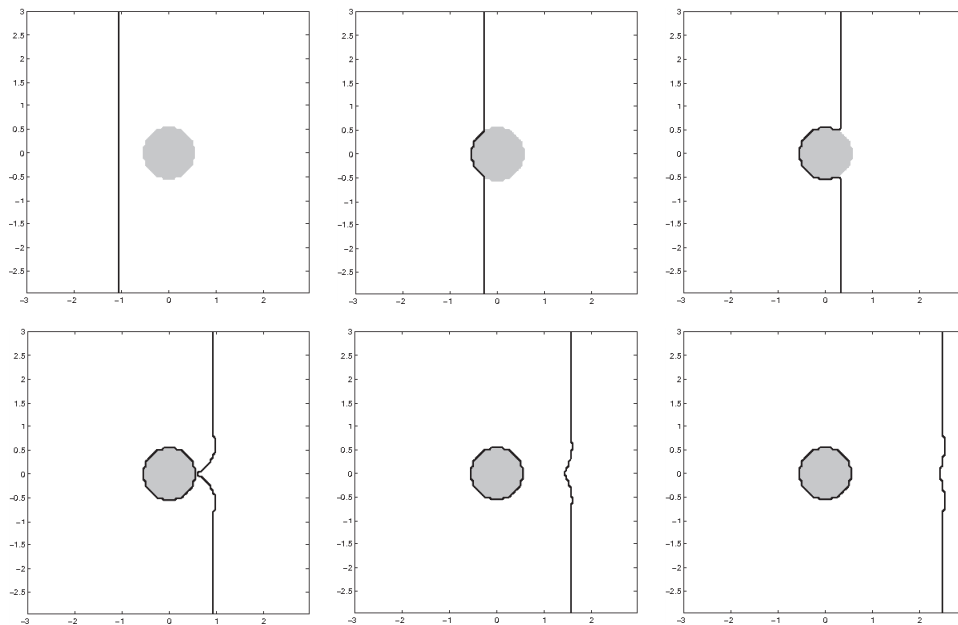
Case $\alpha = 1$: the obstacle breaks the line. Lowering the value of α but holding it still positive, we simulate the presence of a deeper obstacle that slows down the propagation of the line. Due to the presence of this deeper obstacle, the line disconnects. Therefore there is a change of topology. One part of the dislocation still propagates to the left, the other part remains captured by the obstacle and collapses over time. In Figure 3 we show the line evolution and the obstacle at times $t = 0, 0.38, 0.69, 0.85, 1.05, 1.20$.

Case $\alpha = -0.5$: the obstacle captures the line. Choosing a deeper obstacle, the line does not manage to exceed the obstacle. There is a change of topology: a part exceeds the obstacle and continues propagating to the left, while another part remains captured around to the obstacle. In Figure 4, we show the line evolution and the obstacle at times $t = 0, 0.47, 0.74, 1.04, 1.35, 1.81$.

Test 2: Shrinking of a square. Here, we are looking at the collapse of a nonsmooth front: the square.

We choose as continuous speed $c(x, t) = (c_{0,\delta} \star \theta)(\cdot, t)(x)$ with $\delta = 0.02$ and $\nu = 0$. In this case the shrinking of the square approaches a circle; this is expected since for $\delta \rightarrow 0$ the dislocation dynamics approaches the mean curvature motion.

We compute the discrete solution in the numerical domain $D \equiv [-3, 3] \times [-3, 3]$, with 160 nodes on each side, and we update the speed each $\Delta T = 0.001$. We plot the contour line each 0.05 time steps.

FIG. 3. A propagating line with an obstacle, case $\alpha = 1$.FIG. 4. A propagating line with an obstacle, case $\alpha = -0.5$.

This test has been done in a preceding paper with a finite difference scheme for the level set formulation; see [4].

Since for very small core δ , the speed had a large gradient, the finite difference scheme was not satisfactory: the gradient of the solution becomes too small and it is hard to follow the level set evolution.

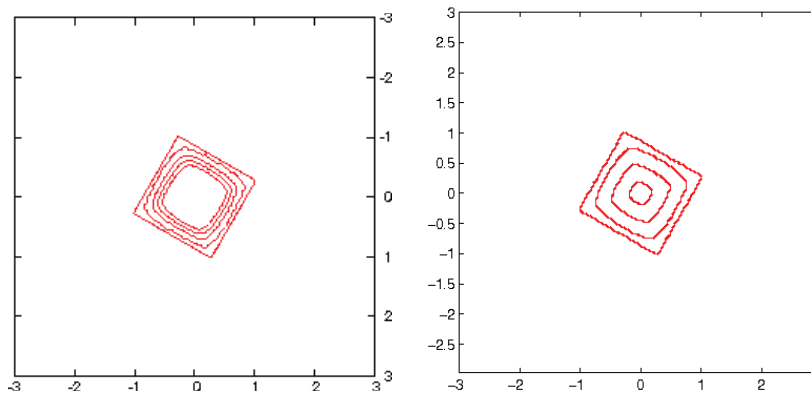


FIG. 5. Shrinking of a square approaching a circle with the level sets method (left) and GFMM (right).

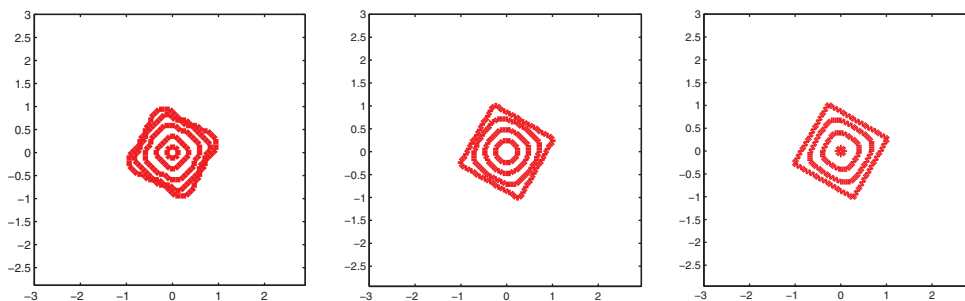


FIG. 6. Shrinking of a square approaching a circle with decreasing space steps Δx .

We compare the numerical solution given by the GFMM algorithm (subsection 2.2) with the one given by the level sets method with a finite difference scheme. Let us point out that techniques such as extension velocity or reinitialization (see [11, 8, 1]) might be used in improving the numerical results (but with more computational complexity).

The finite difference scheme produces oscillations in the level set line representing the dislocation, and it is not possible to follow the complete evolution (see Figure 5) on the left. This fact happens because the continuous function, whose 0 contour line represents the fronts, has become flat.

The GFMM does not suffer from this problem: no oscillations are produced in the evolution, and it is possible to follow the evolution until the square collapse; see Figure 5 on the right.

In Figure 6, we show the increase in accuracy in the approximation of the contour lines with respect to different space steps. From the top, we used, respectively, $\Delta x = 0.12$, $\Delta x = 0.06$, $\Delta x = 0.03$.

Remark 8.1. The stair-step appearance in Figures 5–6 comes from the fact that the computed solution is discontinuous (and takes only value -1 or 1). Hence we cannot interpolate the solution (as for the classical level set method), and in the graphical plot one can see a stair-step pattern of the size of the mesh. As we point out in Figure 6, this phenomena will disappear as Δx goes to zero.

We study numerical convergence by comparing the numerical solution with respect to the exact solution, where we assume exactly the numerical solution computed with

TABLE 1
Area and Hausdorff distances for Test 2 with nonmonotone GFMM.

Δx	$ \mathcal{A}(\Omega_T^+) - \mathcal{A}(\Omega_m^+) $	$\mathcal{H}(\mathcal{C}, \tilde{\mathcal{C}})$
0.06	$2.71 \cdot 10^{-1}$	$1.32 \cdot 10^{-1}$
0.03	$1.27 \cdot 10^{-1}$	$7.63 \cdot 10^{-2}$
0.015	$3.90 \cdot 10^{-1}$	$1.38 \cdot 10^{-1}$

TABLE 2
Area and Hausdorff distances for Test 2 with monotone GFMM.

Δx	$ \mathcal{A}(\Omega_T^+) - \mathcal{A}(\Omega_m^+) $	$\mathcal{H}(\mathcal{C}, \tilde{\mathcal{C}})$
0.06	$5.03 \cdot 10^{-1}$	$1.89 \cdot 10^{-1}$
0.03	$2.14 \cdot 10^{-1}$	$1.14 \cdot 10^{-1}$
0.015	$4.21 \cdot 10^{-3}$	$3.31 \cdot 10^{-2}$

$\Delta x = 0.0075$, $\delta = 0.005$, $\Delta t = \Delta x/10$, and $\Delta T = 10^{-5}$ at $T = 0.02$; let us denote it by $\theta^T(x)$.

We define the approximated set $\Omega_n^+ = \{x_I \in Q \cap D : \theta_I^n > 0\}$, and $\tilde{\mathcal{C}} = \{x \in \mathbb{R}^2 : I[\theta^n](x) = 0\}$ the approximated front, where $I[\cdot]$ is a linear interpolation. Moreover, we indicate by $\Omega_T^+ = \{x_I \in Q \cap D : \theta^T(x_I) > 0\}$ and $\mathcal{C} = \{x \in \mathbb{R}^2 : I[\theta^T](x) = 0\}$ the exact set and the exact front.

We measure the errors in terms of the Hausdorff distance $\mathcal{H}(\cdot, \cdot)$ between the exact front and the approximated front. Although that distance cannot be computed exactly, a good approximation can be obtained by computing it on only a finite number of points belonging to the fronts. Moreover we measured the difference of the discrete area $\mathcal{A}(\cdot)$ of the sets Ω_n^+ and Ω_T^+ .

In Table 1 we show the errors obtained by the nonmonotone version of GFMM, and in Table 2 we show the errors obtained by the monotone version. The parameters are chosen as following: $\delta = 0.005$, $\Delta t = \Delta x/10$, $\Delta T = 5 \cdot 10^{-5}$, and $T = 0.02$. We see that the second algorithm is more accurate than the first one, and it shows numerical convergence.

Test 3 : Collapse of a sinusoidal line dislocation. We choose as initial data a sinusoidal line $x_1 = -0.15 \sin(\pi x_2)$ and as speed $c(x, t) = (c_{0,\delta} \star \theta)(\cdot, t)(x)$ with $\delta = 0.01$ and $\nu = 0$. In this case the speed changes sign with respect to the space variable. It is expected that a sinusoidal line will collapse into a straight line.

We compute the discrete solution in the numerical domain $[-1, 1] \times [-1, 1]$, with 100 nodes each side. The discrete convolution is computed in the extended domain D_K with $K = 1$ and it is updated each $\Delta T = 10^{-4}$.

In Figure 7 we show the numerical solution at times $t = 0, 1.90 \cdot 10^{-3}, 3.30 \cdot 10^{-3}, 4.50 \cdot 10^{-3}, 6.70 \cdot 10^{-3}, 2.67 \cdot 10^{-2}$.

In Figure 8 we compare the numerical solution at time $t = 5 \cdot 10^{-3}$ with respect to different space steps. More precisely, we use, respectively, $N = 50, N = 100$, and $N = 150$ number of nodes on each side of the square numerical domain. One can see that, together with the decreasing of the space step, we get a more accurate representation of the curve.

We perform an error analysis by comparing the areas $\mathcal{A}(\cdot)$ of the sets Ω_T^+ and Ω_m^+ and computing the approximated Hausdorff distance between the exact and approximated fronts, as defined in the previous test. We consider exact solution a numerical solution computed using $\Delta x = 0.005$, $\Delta t = \Delta x/20$, $\Delta T = 10^{-6}$, $\delta = 10^{-3}$ and at the final time $T = 2.50 \cdot 10^{-3}$.

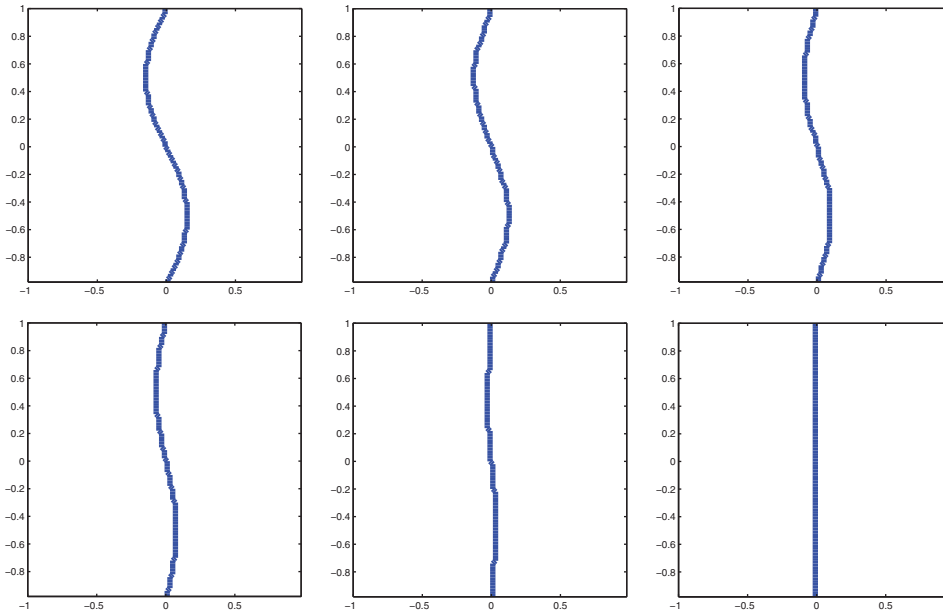


FIG. 7. *Shrinking of a sinusoidal line dislocation.*

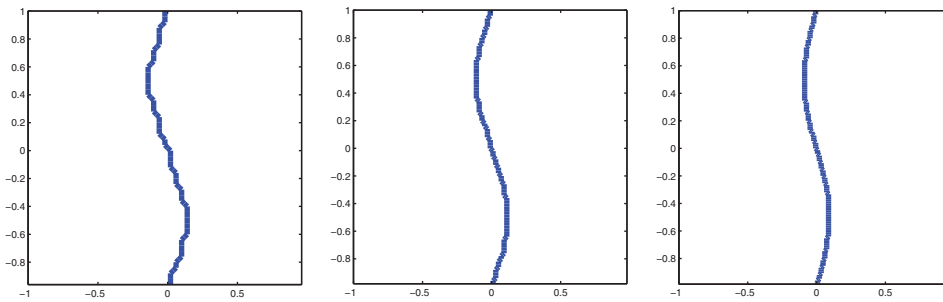


FIG. 8. *A collapsing sinusoidal line computed at $t = 5 \cdot 10^{-3}$ with, respectively, $\Delta x = 0.04$, $\Delta x = 0.02$, and $\Delta x = 0.0133$.*

TABLE 3
Area and Hausdorff distances for Test 3 with nonmonotone GFMM.

Δx	$ \mathcal{A}(\Omega_T^+) - \mathcal{A}(\Omega_m^+) $	$\mathcal{H}(\mathcal{C}, \bar{\mathcal{C}})$
0.04	$4.13 \cdot 10^{-2}$	$9.71 \cdot 10^{-2}$
0.02	$1.30 \cdot 10^{-2}$	$4.12 \cdot 10^{-2}$
0.0133	$4.20 \cdot 10^{-3}$	$1.80 \cdot 10^{-2}$

In Tables 3 and 4 we show the errors obtained by using, respectively, the non-monotone version and the monotone version of the scheme; the parameters used to compute the solution are $\Delta t = \Delta x/20$, $\Delta T = 10^{-5}$, $\delta = 10^{-3}$ and at the final time $T = 5 \cdot 10^{-3}$. Both tables show numerical convergence, and the errors result very similar.

TABLE 4
Area and Hausdorff distances for Test 3 with monotone GFMM.

Δx	$ \mathcal{A}(\Omega_T^+) - \mathcal{A}(\Omega_n^+) $	$\mathcal{H}(\mathcal{C}, \tilde{\mathcal{C}})$
0.04	$2.86 \cdot 10^{-2}$	$1.01 \cdot 10^{-1}$
0.02	$1.30 \cdot 10^{-2}$	$4.53 \cdot 10^{-2}$
0.0133	$4.00 \cdot 10^{-3}$	$2.59 \cdot 10^{-2}$

Appendix A. A nonlocal algorithm modeled on a nonmonotone GFMM.

In certain tests we have used a nonlocal algorithm modeled on a nonmonotone GFMM (in short, nonmonotone nonlocal GFMM).

There are two main differences; the first one concerns the time $u_{J \rightarrow I}$. As in [7], we take the same time $u_{J \rightarrow I}$ in all directions and denote it by $u_{J \rightarrow \pm}$.

Initialization

1. Set $n = 1$.
2. Initialize the field θ^0 :

$$\theta_I^0 = \begin{cases} 1 & \text{for } x_I \in \Omega_0, \\ -1 & \text{elsewhere.} \end{cases}$$
3. Initialize the speed: $c_I^0 \equiv c[(\theta^0)^\Delta](x_I, t_0)$, with $t_0 = 0$.
4. Initialize the time for points I :

$$u_{I \rightarrow \pm}^0 = \begin{cases} t_0 = 0 & \text{if } I \in \mathcal{U}^0 \cap F_{\mp}^0, \\ +\infty & \text{otherwise.} \end{cases}$$

Loop

5. Compute \tilde{u}^{n-1} on NB^{n-1} .
 Let $I \in NB^{n-1}$; then we compute \tilde{u}_I^{n-1} as the solution of the following quadratic equation:

$$\sum_{k=1}^N \left(\max_{\pm} \left(0, \tilde{u}_I^{n-1} - u_{I^{k,\pm \rightarrow -}}^{n-1} \right) \right)^2 = \frac{(\Delta x)^2}{|\tilde{c}_I^{n-1}|^2} \quad \text{if } I \in F_-^{n-1},$$

$$\sum_{k=1}^N \left(\max_{\pm} \left(0, \tilde{u}_I^{n-1} - u_{I^{k,\pm \rightarrow +}}^{n-1} \right) \right)^2 = \frac{(\Delta x)^2}{|\tilde{c}_I^{n-1}|^2} \quad \text{if } I \in F_+^{n-1},$$

where

$$I^{k,\pm} = (i_1, \dots, i_{k-1}, i_k \pm 1, i_{k+1}, \dots, i_N).$$

6. $\tilde{t}_n = \min \{ \tilde{u}_I^{n-1}, I \in NB^{n-1} \}$.
7. Truncate t_n :
 $t_n = \max(t_{n-1}, \min\{\tilde{t}_n, t_{n-1} + \Delta t\})$.
8. If $t_n < \tilde{t}_n$ go to 11 with $\theta^n = \theta^{n-1}$.
9. Initialize the new accepted points:
 $NA_{\pm}^n = \{ I \in NB_{\pm}^{n-1}, \tilde{u}_I^{n-1} = \tilde{t}_n \}$, $NA^n = NA_+^n \cup NA_-^n$.
10. Reinitialize θ^n :

$$\theta_I^n = \begin{cases} -\theta_I^{n-1} & \text{for } I \in NA^n, \\ \theta_I^{n-1} & \text{elsewhere.} \end{cases}$$

11. Compute the speed c_I^n :

Given $t_{n-1} \in [p\Delta T, (p+1)\Delta T)$ for some $p \in \mathbb{N}$, we set

$$c_I^n \equiv \begin{cases} c_I^{n-1} & \text{if } t_n < (p+1)\Delta T, \\ c[(\theta^{n-1})^\Delta](x_I, t_n) & \text{if } t_n \geq (p+1)\Delta T. \end{cases}$$

12. Reinitialize $u_{I \rightarrow K}^n$:

$$u_{I \rightarrow \pm}^n = \begin{cases} \min(u_{I \rightarrow K}^{n-1}, t_n) & \text{if } I \in \mathcal{U}^n \cap F_{\mp}^n, \\ +\infty & \text{otherwise.} \end{cases}$$

13. Set $n : +1$ and go to 5.

Remark A.1. Let us note that, since the algorithm described in subsection 2.2 has constant-in-time speed on each interval ΔT , it is not necessary to recompute the values of \tilde{u}^{n-1} on the whole narrow band NB^n in this time interval; the computation is needed only on the neighborhood of the accepted nodes.

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