# DISLOCATION DYNAMICS WITH A MEAN CURVATURE TERM: SHORT TIME EXISTENCE AND UNIQUENESS

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**Abstract** In this paper, we study a new model for dislocation dynamics with a mean curvature term. The model is a non-local Hamilton-Jacobi equation. We prove a short time existence and uniqueness result for this equation. We also prove a Lipschitz estimate in space and an estimate of the the modulus of continuity in time for the solution.
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## 1 Introduction

Plastic deformation is mainly due to the movement of linear defects called dislocations, whose typical length in metallic alloys is of the order of $10^{-6}m$ and thickness of the order of $10^{-9}m$. Since the beginning of the 90's, the research field of dislocation is enjoying a new boom, in particular thanks to the power of computers which now makes possible to simulate dislocations in a 3D domain. The concept of dislocations in crystals was put forward in the XXth century as the main microscopic explanation of the macroscopic behaviour of metallic crystals (see the physical monograph Hirth and Lothe [28]).

More recently, a new approach has been introduced: *phase field model of dislocations* (see for example Rodney, Le Bouar, Finel [31]). One of the advantage of this method is that the possible topological changes during the dislocation movement are automatically taken into account. In the face centered cubic structure, the dislocation line in the crystal moves in it slip plane with a normal velocity which is proportional to the Peach-Koeller force acting on the line. This force is the self force created by the elastic field generated by the dislocation itself. In [4] and [5], using a level set formulation, Alvarez, Hoch, Le Bouar and Monneau propose a non-local Hamilton Jacobi equation to model this approach. Having such an equation allows in particular to study mathematically and numerically dislocations dynamics. Since the equation is nonlinear, a natural framework to study this kind of equation is the theory of viscosity solutions (see for instance the monographs of Barles [8] and Bardi and Capuzzo-Dolcetta [7] for a presentation of first order equations and the article of Crandall, Ishii and Lions [19] for the second order case). The theory of viscosity solutions has been first introduced by Crandall and

Lions [20]. The main difficulty of this equations is that the comparison principle, which is a crucial argument in this theory, does not hold. Nevertheless, using the geometric property of the equation (we refer to Barles, Soner, Souganidis [11] for a detailed presentation of this theory) Alvarez *et al.* prove short time existence and uniqueness for this equation. Then Alvarez, Cardaliaguet and Monneau [1] and Barles and Ley [10] prove a long time existence and uniqueness result under certain assumptions on the monotony of the velocity. This model was also numerically studied by Alvarez, Carlini, Monneau and Rouy [2], [3].

Here, we consider a new model in which the energy also contains a line tension term which approximates better what happen near the dislocation. That amounts to adding a mean curvature term in the equation. This line tension term appears in a lot of physical models (see for instance Gavazza, Barnett [13] and Brown [14]). We also refer to Garroni, Müller [27] for a mathematical reference. Non-local equation with mean curvature term have also been studied by Chen, Hilhorst and Logak [17].

For this model we show a short time existence and uniqueness result. Since the comparison principle does not hold, the strategy of the proof is the same as the one used by Alvarez *et al.* in [4], *i.e.* is to use a fix point method by freezing the nonlocal term. Here, the main difficulties come from the fact that the equation is a second order one and so the regularity estimates are really more difficult to obtain. Indeed, for the fix point to work, we need fines estimates on the Lipschitz constant in space and on the modulus of continuity in time of the solution. To get round these difficulties, we use a regularisation of the initial condition to obtain an estimate of the modulus of continuity in time of the solution for a class of operator. Using the geometric property of the equation, we also prove that the estimate we get is like $\sqrt{t}$. We also prove a Lipschitz estimate in space for the solution. The estimate we obtain is the same as in the first order case (*i.e.*, without the mean curvature term).

The long time existence and uniqueness result for this non-local equation is really more difficult to obtain. Indeed, due to the non validity of the comparison principle, defining a large time solution is rather difficult and this problem is still open. Moreover, even in the the first order case, the problem is still open for general velocity. We refer to Alvarez, Cardaliaguet and Monneau [1] and Barles and Ley [10] for such results with monotony assumptions on the velocity.

Let us now explain how this paper is organised: in section 2, we present the model. The main result is state in Section 3. In Section 4, we give some preliminary results on a local problem. Finally, we prove existence, uniqueness and regularity results for the non-local problem (see Theorem 3.1) in section 5.

## 2 Presentation of the model

We describe the model in an heuristic way. Let us consider an orthonormal basis $(e_1, e_2, e_3)$ of $\mathbb{R}^3$ and denote by $x = (x_1, x_2, x_3)$ the coordinates. The energy of the dislocation along the line is singular. To solve this problem, Brown [14], [15] then Barnett [12] and Gavazza, Barnett [13] propose to surround the dislocation $\Gamma$ by a tube $T_\epsilon$ of size $\epsilon$ and to consider the energy of the form:

$$\mathcal{E} = \int_{\mathbb{R}^3 \setminus T_\epsilon} \frac{1}{2} \Lambda e^{class}.e^{class} + \int_{T_\epsilon} \gamma_0(\vec{n}), \tag{1}$$

where $\Lambda$ represents the elastic coefficients, $\gamma_0$ is an energy of tension line, $\vec{n}$ is the outward normal to the curve and $e^{class}$ is a deformation and is solution of:

$$\begin{cases} \operatorname{div}(\Lambda e^{class}) = 0, \\ \operatorname{inc}(e^{class}) = -\operatorname{inc}(\rho\delta_0(x_3)e^0) \text{ where } e^0 = \frac{1}{2}(b \otimes e_3 + e_3 \otimes b), \\ e^{class}(x) \to 0 \text{ when } |x| \to \infty. \end{cases} \tag{2}$$

Here, $\Gamma$ belongs to the plane $(e_1, e_2)$, the vector $e_3$ is the vector normal to the plane, $b \in \mathbb{R}^3$ is a constant vector (called Burgers' vector) associated with the dislocation line and the operator of incompatibility "inc" is defined for a field $e = (e_{ij}) \in S^3$, the set of symmetric $3 \times 3$ matrix, by:

$$(\operatorname{inc}(e))_{ij} = \sum_{i_1, j_1 = 1}^{3} \varepsilon_{i i_1 i_2} \varepsilon_{j j_1 j_2} \partial_{i_1} \partial_{j_1} e_{i_2 j_2}, \quad i, j = 1, 2, 3$$

where we note as usual

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if} \quad (ijk) \quad \text{is a positive permutation of} \quad (123), \\ -1 & \text{if} \quad (ijk) \quad \text{is a negative permutation of} \quad (123), \\ 0 & \text{if} \quad \text{two indices are the same.} \end{cases}$$

The solution $e^{class}$ of (2) satisfies $e^{class} \sim \frac{1}{r}$, for $r$ small, where $r$ is the distance to the dislocation (cf Alvarez *et al.* [4] for a description of the physical model). Finally, the cutoff tube is represented by Figure 1. We consider an approximate model of this one where the field $e$ is given by $e = \chi \star e^{class}$,

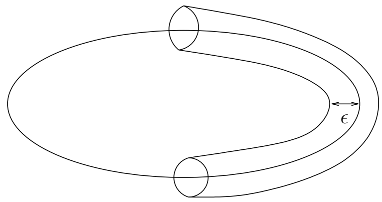


Figure 1: The cutoff tube of radius $\epsilon$.

with $\chi$ a regularising core function (connected to $\epsilon$) to be adjusted, and the energy (1) is replaced by the following one:

$$\mathcal{E} = \int_{\mathbb{R}^3} \frac{1}{2} \Lambda e.e + \eta \int_{\Gamma} \gamma_0(\vec{n}),$$

where $\eta$ is to be adjusted and connected to $\epsilon$. We set $\gamma(\vec{n}) = \eta\gamma_0(\vec{n})$. In order to model the movement of a dislocation $\Gamma$ in its crystallographic plane, we assume that $\Gamma$ is the edge of a smooth bounded set $\Omega \subset \mathbb{R}^2$ and we compute the first variation of the energy (see Alvarez *et al.* [4]). We define $\Gamma_\delta(s) = \Gamma(s) + \delta h(s).\vec{n}_\Gamma(s)$. Then, the following holds

$$-\left.\frac{d\mathcal{E}(\Gamma_\delta)}{d\delta}\right|_{\delta=0} = \int_\Gamma c.h,$$

with $c = c_0 \star \rho + \lambda(\vec{n})\kappa$, where $\lambda(\vec{n}) = (\gamma(\vec{n}) + \gamma''(\vec{n}))$ the kernel $c_0 = c_0(x_1, x_2)$ only depends on $\Lambda$ and $\chi$, $\star$ denotes the convolution in space and $\rho$ is defined as follows:

$$\rho = \begin{cases} 1 & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathbb{R}^2/\Omega. \end{cases} \tag{3}$$

Thus, the evolution is postulated to be $\frac{\partial \rho}{\partial t} = c|D\rho|$. We can then reformulate the problem by a "level set" equation on the set $\{u \le 0\}$ of a smooth function $u$ which then satisfies:

$$u_t = (c_0 \star [u] + \lambda(\vec{n})\kappa)|Du|, \tag{4}$$

with:

$$\begin{cases} \rho = [u] = \begin{cases} 1 & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u \le 0 \end{cases}, \\ \vec{n} = \frac{Du}{|Du|}, \\ \kappa = \operatorname{div}\left(\frac{Du}{|Du|}\right). \end{cases} \tag{5}$$

**Remark 2.1 (Explicit example for $c_0$ and $\gamma$)**
*If $b = |b|e_1$, then, for the isotropic elasticity, one can give the value of $c_0$:*

$$\widehat{c_0}(\xi) = \frac{-\mu b^2}{2} e^{-\zeta\sqrt{\xi_1^2+\xi_2^2}} \left( \frac{\xi_1^2 + \frac{1}{1-\nu}\xi_2^2}{\sqrt{\xi_1^2+\xi_2^2}} \right)$$

*and the form of $\gamma$:*

$$\gamma(\vec{n}) = C\left(n_1^2 + \frac{1}{1-\nu}n_2^2\right),$$

*where $\vec{n} = (n_1, n_2)$ is the normal vector to the curve, $C$ is a prefactor (which depends on the Burgers vector and elasticity coefficients), $\zeta > 0$ is a physical parameter (depending on the material), $\nu$ is the Poisson ratio and $\mu$ is the Lamé coefficient. We refer to Alvarez et al. [4] section 6 for the expression of $c_0$ and to Hirth, Lothe [28] chapter 6 and 7 for the form of $\gamma$.*

**Remark 2.2** *Formally, we have:*

$$\frac{d\mathcal{E}}{dt} = \int_\Gamma -c^2 \le 0.$$

# 3 Main result

The goal of the paper is to prove short time existence and uniqueness for the problem (4). Here we consider more general second order term and we study the n-dimensional case (see 6). Since the Hamiltonian intervening in the equation is not continuous and singular, a natural framework for the study is the theory of viscosity solutions (for a good introduction to this theory, we refer to Barles [8], [9], Crandall, Ishii, Lions [19], Crandall, Lions [21], [22], Ishii [29] and Ishii, Lions [30] and for an introduction to viscosity solution for evolving fronts, we refer to Ambrosio [6], Barles, Soner, Souganidis [11], Chen, Giga, Goto [16], Evans [25], Evans, Spruck [26] and Souganidis [33]). We consider the following problem: find $u(x,t)$ solution of

$$\begin{cases} u_t = (c_0 \star [u])|Du| - F(Du, D^2u) \text{ in } \mathbb{R}^n \times (0,T), \\ u(x, t=0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{6}$$

where $[u]$ is the characteristic function of the set $\{u > 0\}$ (see (5)). Moreover, we assume that

$$c_0 \in L^\infty_{\text{int}}(\mathbb{R}^n) \cap BV(\mathbb{R}^n), \tag{7}$$

where $BV(\mathbb{R}^n)$ is the space of bounded variations functions and

$$L^\infty_{\rm int}(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_{L^\infty_{\rm int}(\mathbb{R}^n)} < \infty\}$$

with

$$\|f\|_{L^\infty_{\rm int}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \|f\|_{L^\infty(Q(x))}$$

and $Q(x)$ is the unit square centered at $x$:

$$Q(x) = \left\{x' \in \mathbb{R}^n : |x_i - x_i'| \le \frac{1}{2}\right\}.$$

The assumptions $(HF)$ on the operator $F$ are the following ones:

(i) The operator $F$ is elliptic, *i.e.*, $\forall X,\ Y \in S^n$, $\forall p \in \mathbb{R}^n$,

$$\text{if } X \le Y \text{ then } F(p,X) \ge F(p,Y), \tag{8}$$

where $S^n$ (the set of symmetric $n \times n$ matrices) is equipped with its natural partial order.

(ii) $F$ is locally bounded on $\mathbb{R}^n \times S^n$, continuous on $\mathbb{R}^n\backslash\{0\} \times S^n$ and

$$F^*(0,0) = F_*(0,0) = 0, \tag{9}$$

where $F^*$ (resp. $F_*$) is the upper-semicontinous (usc) envelope (resp. lower semicontinuous (lsc) envelope) of $F$, *i.e.* the smallest usc function greater than $F$ (resp. the greatest lsc function smaller than $F$).

(iii) $F$ is geometric, *i.e.*

$$F(\nu p, \nu A + \mu p \otimes p) = \nu F(p,A), \qquad \forall \nu > 0, \mu \in \mathbb{R}, A \in S^n. \tag{10}$$

The main result is:

**Theorem 3.1 (Short time existence and uniqueness)**
*Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a Lipshitz continuous function on $\mathbb{R}^n$ such that*

$$|Du_0| < B_0 \quad in\ \mathbb{R}^n \tag{11}$$

*and*

$$\frac{\partial u_0}{\partial x_n} > b_0 > 0 \quad in\ \mathbb{R}^n. \tag{12}$$

*Let $c_0$ satisfying (7). Then, under assumptions $(HF)$, there exists a unique viscosity solution of the problem (6) in $\mathbb{R}^n \times [0,T^*)$ with*

$$T^* = \inf\left\{\frac{1}{|c_0|_{BV(\mathbb{R}^n)}} \ln\left(1 + \frac{b_0}{2B_0}\right),\quad \frac{b_0}{B_0}\,\frac{1}{16\|c_0\|_{L^\infty_{\rm int}(\mathbb{R}^n)}},\quad \frac{1}{|c_0|_{BV(\mathbb{R}^n)}} \ln\left(1 + \frac{b_0}{B_0}\,\frac{|c_0|_{BV(\mathbb{R}^n)}}{8\|c_0\|_{L^\infty_{\rm int}}}\right)\right\}.$$

*Moreover, the solution satisfies*

$$|Du(x,t)| < 2B_0 \quad on \quad \mathbb{R}^n \times [0,T^*), \tag{13}$$

$$\frac{\partial u}{\partial x_n}(x,t) > b_0/2 > 0 \quad on \quad \mathbb{R}^n \times [0,T^*). \tag{14}$$

*and $u$ is uniformly continuous in time and its modulus of continuity behaves like $\sqrt{t}$.*

**Remark 3.2** *This theorem gives, in particular, in the two dimensional case, and for*

$$F(p,X) = -tr\left(\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right)\left(\lambda\left(\frac{p}{|p|}\right)\right)\right), \tag{15}$$

*with* $\lambda > 0$ *and smooth, short time existence and uniqueness for dislocation dynamics with a mean curvature term.*

**Remark 3.3** *Due to the non validity of the comparison principle, defining a large time solution is rather difficult and this problem is still open. Even in the one order case this problem is still open for general velocity.*

# 4 Preliminary results for a local problem

Given $T > 0$, we consider the following problem:

$$\begin{cases} u_t + G(x,t,Du,D^2u) = 0 \text{ in } (0,T)\times\mathbb{R}^n, \\ u(x,t=0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{16}$$

with the following assumptions $(H_0)$:

(i) $G(x,t,p,X) = -c(x,t)|p| + F(p,X)$ and $F$ satisfies the assumptions $(HF)$,

(ii) $c : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ is bounded, Lipschitz continuous in space (we note $L_c$ its Lipschitz constant) and uniformly continuous in time (we note $\omega_c$ its modulus of continuity, defined by: $\forall x \in \mathbb{R}^n,\ \forall s,\ t \in [0,T),\ |c(x,t) - c(x,s)| \le \omega_c(|t-s|)$),

(iii) $u_0$ is Lipschitz continuous (we note $B_0$ its Lipschitz constant).

## 1 Existence and uniqueness for the problem (16)

For the reader's convenience, we recall the classical definition for viscosity solution of (16):

**Definition 4.1 (Viscosity subsolution, supersolution and solution)**
*A locally bounded upper semi-continuous (usc)(resp. lower semi-continuous (lsc)) function u is a viscosity subsolution (resp. supersolution) of (16) if it satisfies:*

*(i)* $u(x,t=0) \le u_0(x)$ *(resp.* $u(x,t=0) \ge u_0(x)$*) in* $\mathbb{R}^n$,

*(ii) for every* $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$ *and for every test function* $\Phi : (\mathbb{R}^n \times (0,T)) \to \mathbb{R}$, $C^1$ *in time and* $C^2$ *in space, that is tangent from above (resp. below) to u at* $(x_0,t_0)$*, the following holds:*

$$\frac{\partial\Phi}{\partial t}(x_0,t_0) + G_*\left(x_0,t_0,D\Phi,D^2\Phi\right) \le 0.$$

$$(\textit{resp. } \frac{\partial\Phi}{\partial t}(x_0,t_0) + G^*\left(x_0,t_0,D\Phi,D^2\Phi\right) \ge 0.$$

*A function* $u \in C^0(\mathbb{R}^n\times[0,T))$ *is a viscosity solution of (16) if, and only if, it is a sub and a supersolution of (16).*

We recall that we have an equivalent definition using the sub and superdifferentials (see Crandall *et al.* [19]).

We also recall the fundamental property of geometric equations:

***Lemma* 4.2 (Fundamental property of geometric equations)**
*Let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous, non decreasing scalar function and $u$ be a subsolution (respectively a supersolution) of (16), then $\theta(u)$ is also a subsolution (resp. a supersolution).*

For the proof of this *Lemma*, we refer to Soner [32] (Theorem 1.11).
We have the following comparison principle:

**Theorem 4.3 (Comparison principle)**
*Let $u$, a locally bounded usc function, be a subsolution and $v$, a locally bounded lsc function, be a supersolution of (16). Assume that $u_0(x) = u(0,x) \leq v(0,x) = v_0(x)$ in $\mathbb{R}^n$, then, under the assumptions $(H_0)$, $u \leq v$ in $\mathbb{R}^n \times [0,T)$.*

**Proof of theorem 4.3**
The proof of this theorem is rather classical when the functions $u$ and $v$ are bounded (see for instance Chen, Giga, Goto [16]). When the functions are not bounded, it suffices to use the fundamental property of geometric equations. We then consider the truncature functions $T_k = \max(\min(x,k), -k)$. For every $k$, we then have $T_k(u) \leq T_k(v)$ and by letting $k$ go to infinity, we obtain the result. □

**Theorem 4.4 (Existence and uniqueness for the local problem)**
*Let $T > 0$. Then, under the assumptions $(H_0)$, there exists a unique viscosity solution of the problem (16) in $\mathbb{R}^n \times [0,T)$. Moreover, the solution satisfies, for every $(x,t) \in \mathbb{R}^n \times (0,T)$:*

$$u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0t \leq u(x,t) \leq u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))}B_0t.$$

*where $\omega_F$ is the modulus of continuity of the solution of (17) and behaves like $\sqrt{t}$.*

**Proof of theorem 4.4**
To prove this theorem, by Perron's method (see Crandall, Ishii, Lions [19]), it suffices to construct a subsolution $U^-$ (resp. a supersolution $U^+$) which satisfy $U^-(x,0) \leq u_0(x) \leq U^+(x,0)$. Since $u_0$ is not bounded, constant cannot be sub or supersolution. We begin with studying the problem

$$\begin{cases} u_t + F(Du, D^2u) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{17}$$

We assume, in a first time, that $u_0 \in C_b^2 = \{u\, C^2, \exists\, C,\ ||Du||_{L^\infty},\ ||D^2u||_{L^\infty} \leq C\}$. We set $u^\pm = u_0 \pm C_1 t$ with $C_1 = \inf_{x\in R^n}\{-F^*(Du_0, D^2u_0), F_*(Du_0, D^2u_0)\}$ ($C_1$ depends only on the bounds of $Du_0$ and $D^2u_0$). It then easy to check that $u^+$ is a supersolution and $u^-$ is a subsolution. Then, there exists a unique solution of (17) and, by the comparison principle, the following holds:

$$\forall t \in [0,T),\ \forall x \in \mathbb{R}^n,\ |u(x,t) - u_0(x)| \leq C_1 t. \tag{18}$$

Moreover, $u(x,t+h)$ is solution of (17) so, by the comparison principle, we obtain:

$$|u(x,t+h)-u(x,t)| \leq \sup(u(x,h)-u_0) \leq C_1 h.$$

We now assume that $u_0$ is only Lipschitz continuous. We set $u^0_\epsilon = u_0 \star \rho_\epsilon$ where $\rho_\epsilon$ is a regularising sequence, *i.e.* $\rho_\epsilon = \frac{1}{\epsilon^n}\rho(\frac{\cdot}{\epsilon})$ where $\rho \in C^\infty_c(\mathbb{R}^n, \mathbb{R})$ and satisfies:

$$\rho \geq 0, \quad \mathrm{supp}(\rho) \subset \bar{B}(0,1), \quad \int_{\mathbb{R}^n} \rho(x)dx = 1.$$

Then, it is easy to check that $u^0_\epsilon \in C^2_b$ and $\|Du^0_\epsilon\|_{L^\infty(\mathbb{R}^n)}$, $\|D^2 u^0_\epsilon\|_{L^\infty(\mathbb{R}^n)} \leq \frac{B_0 C_2}{\epsilon}$. Moreover, $\|u_0 - u^0_\epsilon\|_{L^\infty(\mathbb{R}^n)} \leq B_0 \epsilon$. Indeed, since $\int_{\mathbb{R}^n} \rho_\epsilon(x)dx = 1$

$$\begin{aligned}
|u_0 - u^0_\epsilon(x)| \leq & \int_{\mathbb{R}^n} |u_0(x) - u_0(x-y)|\rho_\epsilon(y)dy \\
\leq & B_0 \int_{\bar{B}(0,\epsilon)} |y|\rho_\epsilon(y)dy \\
\leq & \epsilon B_0 \int_{\bar{B}(0,\epsilon)} \rho_\epsilon(y)dy = \epsilon B_0.
\end{aligned}$$

We note $u_\epsilon$ the solution with initial condition $u^0_\epsilon$. Then, by the comparison principle, $\|u_{\epsilon'}(\cdot,t) - u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u^0_{\epsilon'} - u^0_\epsilon\|_{L^\infty(\mathbb{R}^n)}$, and so $u_\epsilon$ converge uniformly (since $u^0_\epsilon$ converge uniformly) to $u$ which is, by stability (see for instance Theorem 2.3 of Barles [8]), the solution of (17) with initial condition $u_0$. We then have, by the comparison principle, $\|u_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u^0_\epsilon - u_0\|_{L^\infty(\mathbb{R}^n)}$. We then deduce:

$$\begin{aligned}
\|u(\cdot,t+h) - u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq & 2\|u_0 - u^0_\epsilon\|_{L^\infty(\mathbb{R}^n)} + \|u_\epsilon(\cdot,t+h) - u_\epsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\
\leq & 2B_0\epsilon + C_1\left(B_0, \frac{B_0 C_2}{\epsilon}\right) h.
\end{aligned}$$

By taking the minimum on $\epsilon$, we obtain the modulus of continuity of $u$, $\omega_F$, which depends only on $B_0$. Moreover, using the geometric property of the equation, one deduces that $C_1\left(B_0, \frac{B_0 C_2}{\epsilon}\right) \sim \frac{1}{\epsilon}$ and so $\omega_F(h)$ behaves like $\sqrt{h}$.

**Remark 4.5** *In the case of dislocation dynamics, i.e. with the function $F$ given by (15), an alternative proof can be found in Chen, Giga, Goto [16], based on self-similar solutions (Wulff Shape) of the mean curvature motion.*

We also remark that the solution of (17) is Lipschitz continuous in space with Lipschitz constant $\|Du_0\|_{L^\infty(\mathbb{R}^n)} = B_0$ (because the equation is independent in space).

We now construct sub and supersolution for the general case.

We set $U^+(x,t) = u(x,t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t$. Then, $\|DU^+\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq \|Du\|_{L^\infty(\mathbb{R}^n\times(0,T))} \leq B_0$ and $U^+$ is solution of:

$$\begin{cases} v_t - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 + F(Dv, D^2 v) = 0, \\ v(x,0) = u_0, \end{cases}$$

and so $U^+$ is supersolution of (16) and satisfies:

$$\begin{aligned} U^+(x,t) =& u(x,t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t \\ \leq& u_0(x) + \omega_F(t) + \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t. \end{aligned}$$

Similarly, we construct a subsolution $U^-$ such that $U^-(x,t) \geq u_0(x) - \omega_F(t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t$ by setting $U^-(x,t) = u(x,t) - \|c\|_{L^\infty(\mathbb{R}^n\times[0,T))} B_0 t$. To achieve the proof, it suffices to apply the comparison principle to $U^-$ and $U^+$. □

## 2 Regularity results for the local problem

***Lemma* 4.6 (Regularity results for the local problem)**
*Assume that $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \leq B_0$ and $\frac{\partial u_0}{\partial x_n} \geq b_0$, with $B_0 > 0$ and $b_0 > 0$. Then, the solution of (16) given by Theorem 4.4 satisfies*

$$\|Du(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq B(t) \text{ and } \frac{\partial u}{\partial x_n} \geq b(t),$$

*with $B(t) = B_0 e^{L_c t}$ and $b(t) = b_0 - B_0(e^{L_c t} - 1)$. Moreover, $u$ is uniformly continuous in time and its modulus of continuity in time $\omega_u$, defined by:*

$$\forall x \in \mathbb{R}^n,\ \forall s,\ t \in [0,T),\ |u(x,t) - u(x,s)| \leq \omega_u(|t-s|),$$

*satisfies:*

$$\omega_u(\delta) \leq \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^T B(s)ds,$$

*where $\omega_c$ is the modulus of continuity in time of $c$, and $\omega_F$ is the modulus of continuity in time of the solution of (17) and behaves like $\sqrt{t}$.*

**Proof of *Lemma* 4.6** For the proof of the Lipschitz estimate in space, we assume in a first time that $u$ is bounded. We set $\phi^\epsilon(x,y,t) = B(t)\left(|x-y|^2 + \epsilon^2\right)^{1/2}$. We prove that $u(x,t) - u(y,t) \leq \phi^\epsilon$. We set:

$$M = \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} \{u(x,t) - u(y,t) - \phi^\epsilon(x,y,t)\},$$

Assume that $M > 0$. Then we set:

$$\bar{M} = \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} \left\{u(x,t) - u(y,t) - \phi^\epsilon(x,y,t) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{\gamma}{T-t}\right\}.$$

For $\alpha > 0,\ \gamma > 0$ small enough, we have $\bar{M} > 0$. Moreover $u$ is bounded, so the supremum is reached in $(\bar{x}, \bar{y}, \bar{t})$ (with $\bar{x} \neq \bar{y}$) and

$$\frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C$$

and so $\alpha\bar{x} \to 0$ and $\alpha\bar{y} \to 0$. We prove that $\bar{t} > 0$. Indeed, assume the contrary. Then, we have

$$u_0(\bar{x}) - u_0(\bar{y}) - \phi^\epsilon(\bar{x}, \bar{y}, 0) > 0,$$

*i.e.*

$$u_0(\bar{x}) - u_0(\bar{y}) > B_0\left(|\bar{x} - \bar{y}|^2 + \epsilon^2\right)^{\frac{1}{2}} > B_0|\bar{x} - \bar{y}|,$$

what is absurd since $\|Du_0\|_{L^\infty(\mathbb{R}^n)} \le B_0$. We set

$$\bar p = D_x\phi^\epsilon = \left(|\bar x-\bar y|^2+\epsilon^2\right)^{-1/2}(\bar x-\bar y)B(t) = -D_y\phi^\epsilon \neq 0 \text{ (because } \bar x \neq \bar y),$$

$$Z = D_x^2\phi^\epsilon = \left(\left(|\bar x-\bar y|^2+\epsilon^2\right)^{-1/2} I - \left(|\bar x-\bar y|^2+\epsilon^2\right)^{-3/2}(\bar x-\bar y)\otimes(\bar x-\bar y)\right)B(t) = D_y^2\phi^\epsilon,$$

$$A = D^2\phi^\epsilon = \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}.$$

Then, by parabolic version of Ishii's *Lemma* (see Crandall, Ishii and Lions [19]), applied to $\tilde u = u(x,t) - \frac{\alpha}{2}|x|^2$, $\tilde v(y,t) = v(y,t) + \frac{\alpha}{2}|y|^2$ and $\phi(x,y,t) = \phi^\epsilon(x,y,t) + \frac{\gamma}{T-t}$, for every $\beta$ such that $\beta A < I$, there exists $\tau_1,\ \tau_2 \in \mathbb{R}$ and $X,\ Y \in S^n$ such that:

$$\tau_1 - \tau_2 = \frac{\gamma}{(T-\bar t)^2} + L_c B(t)\left(|\bar x-\bar y|^2+\epsilon^2\right)^{\frac12},$$

$$(\tau_1, p+\alpha\bar x, X+\alpha I) \in \bar{\mathcal{P}}^+ u(\bar x,\bar t),$$

$$(\tau_2, p-\alpha\bar y, Y-\alpha I) \in \bar{\mathcal{P}}^- v(\bar y,\bar t),$$

$$\frac{-1}{\beta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le (I-\beta A)^{-1}A.$$

So, the following holds

$$\tau_1 - c(\bar x,\bar t)|\bar p+\alpha\bar x| + F_*(\bar p+\alpha\bar x, X+\alpha I) \le 0,$$

$$\tau_2 - c(\bar y,\bar t)|\bar p-\alpha\bar y| + F^*(\bar p-\alpha\bar y, Y-\alpha I) \ge 0.$$

The matrix inequality implies in particular that $X \le Y$, so by using the ellipticity of $F$, we deduce:

$$\tau_2 - c(\bar y,\bar t)|\bar p-\alpha\bar y| + F^*(\bar p-\alpha\bar y, X-\alpha I) \ge 0.$$

From that, by subtracting:

$$\begin{aligned}&\frac{\gamma}{(T-\bar t)^2} + L_c B(t)\left(|\bar x-\bar y|^2+\epsilon^2\right)^{\frac12} - c(\bar x,\bar t)|\bar p+\alpha\bar x| + c(\bar y,\bar t)|\bar p-\alpha\bar y| \\ &+F_*(\bar p+\alpha\bar x, X+\alpha I) - F^*(\bar p-\alpha\bar y, X-\alpha I) \le 0.\end{aligned}$$

We let $\alpha$ go to 0 ($\bar p$ and $X$ are bounded so we can extract a converging subsequence and we still note $\bar p$ and $X$ their limit):

$$\frac{\gamma}{(T-\bar t)^2} + \lim_{\alpha\to0}\left(L_c B(t)\left(|\bar x-\bar y|^2+\epsilon^2\right)^{\frac12} + \left(-c(\bar x,\bar t)+c(\bar y,\bar t)\right)|\bar p|\right) + F_*(\bar p, X) - F^*(\bar p, X) \le 0.$$

Now, $\bar p \neq 0$, therefore $F_*(\bar p, X) = F^*(\bar p, X)$. Moreover,

$$\begin{aligned}&L_c B(t)\left(|x-y|^2+\epsilon^2\right)^{1/2} - c(x,t)|\bar p| + c(y,t)|\bar p| \\ &= \left(|x-y|^2+\epsilon^2\right)^{1/2}\left(L_c B(t) - \frac{|x-y|B(t)}{|x-y|^2+\epsilon^2}\left(c(x,t)-c(y,t)\right)\right) \\ &\ge \left(|x-y|^2+\epsilon^2\right)^{1/2}\left(L_c B(t) - \frac{|x-y|^2 L_c B(t)}{|x-y|^2+\epsilon^2}\right) \\ &\ge \left(|x-y|^2+\epsilon^2\right)^{1/2}\left(L_c B(t) - L_c B(t)\right) \\ &\ge 0,\end{aligned}$$

so

$$\frac{\gamma}{(T-\bar{t})^2} \leq 0,$$

what is absurd. So $u(x,t)-u(y,t) \leq \phi^\epsilon$. By letting $\epsilon$ go to 0, we obtain:

$$u(x,t)-u(y,t) \leq B(t)|x-y|.$$

Exchanging $x$ and $y$, yields

$$|u(x,t)-u(y,t)| \leq B(t)|x-y|,$$

what gives the first result in the case where $u$ is bounded. If $u$ is not bounded, we consider the truncature functions $T_k = \max(\min(x,k),-k)$. Then $T_k(u)$ is bounded and solution of the problem, and so:

$$|T_k(u(x,t)) - T_k(u(y,t))| \leq B(t)|x-y|.$$

Letting $k$ go to infinity, yields:

$$|u(x,t)-u(y,t)| \leq B(t)|x-y|,$$

and we obtain the first estimate.

For the second estimate, we set, for $x=(x',x_n)$, $u^\lambda(x,t) = u(x',x_n+\lambda,t) - \lambda b(t)$. We have

$$\begin{aligned} u^\lambda(x',x_n,0) =& u(x',x_n+\lambda) - \lambda b_0 \\ \geq& u(x',x_n,0). \end{aligned}$$

Moreover,

$$\begin{aligned} & u_t^\lambda + G^*\left(x',x_n,t,Du^\lambda,D^2u^\lambda\right) \\ =& u_t - \lambda b'(t) - c(x',x_n,t)|Du| + F^*\left(Du,D^2u\right) \\ =& u_t + \lambda B_0 L_c e^{L_c t} - c(x',x_n,t)|Du| + F^*\left(Du,D^2u\right) \\ \geq& u_t + \lambda B_0 L_c e^{L_c t} - \left(c(x',x_n+\lambda,t) + \lambda L_c\right)|Du| + F^*\left(Du,D^2u\right) \\ \geq& \lambda B_0 L_c e^{L_c t} - \lambda B_0 L_c e^{L_c t} + u_t + G^*\left(x',x_n+\lambda,t,Du,D^2u\right) \\ \geq& 0, \end{aligned}$$

where $u_t$, $Du$, $D^2u$ are taken at the point $(x',x_n,t)$. This is written in a formal way and it can be justified by using a test function. So, we obtain that $u^\lambda$ is a supersolution. By the comparison principle, we deduce $u^\lambda \geq u$, and so

$$u(x',x_n+\lambda,t) - u(x',x_n,t) \geq \lambda b(t).$$

what proves the second estimate.

It thus remains to be shown that $u$ is uniformly continuous in time. We set $\delta > 0$. For every $(x,t) \in \mathbb{R}^n \times (0,T)$ such that $t+\delta \leq T$, we set $v(x,t) = u(x,t+\delta)$. Then, $v$ is a subsolution of

$$w_t - \omega_c(\delta)B(t+\delta) - c(x,t)|Dw| + F\left(Dw,D^2w\right) = 0$$

on $\mathbb{R}^n \times (0,T-\delta)$ in the sense of definition 4.1 $(ii)$. Indeed, we have

$$v_t - c\left(x,t+\delta\right)|Dv| + F\left(Dv,D^2v\right) = 0,$$

and

$$-c\,(x,t+\delta)\,|Dv| \geq -\omega_c(\delta)B(t+\delta) - c(x,t)|Dv|,$$

what gives in a formal way:

$$v_t - \omega_c(\delta)B(t+\delta) - c(x,t)|Dv| + F\left(Dv, D^2v\right) \leq 0.$$

Moreover, $u+\omega_c(\delta)\int_0^{t+\delta} B(s)ds$ is solution of the same problem. So $\tilde{u} = u+\sup_{x\in\mathbb{R}^n}\left(u(x,\delta)-u_0(x)\right)^+ + \omega_c(\delta)\int_0^{t+\delta} B(s)ds$ is a supersolution and $v(x,0) \leq \tilde{u}(x,0)$. By Theorem 4.4 and the comparison principle, we then have:

$$\begin{aligned} u(x,t+\delta) - u(x,t) \leq & \sup_{x\in\mathbb{R}^n}\left(u(x,\delta)-u_0(x)\right)^+ + \omega_c(\delta)\int_0^{t+\delta} B(s)ds \\ \leq & \omega_F(\delta) + \|c\|_{L^\infty}B_0\delta + \omega_c(\delta)\int_0^T B(s)ds. \end{aligned}$$

Similarly, $v$ is a supersolution of

$$w_t + \omega_c(\delta)B(t+\delta) - c(x,t)|Dw| + F\left(Dw, D^2w\right) = 0$$

and $\tilde{u} = u - \sup_{x\in\mathbb{R}^n}\left(u(x,\delta)-u_0(x)\right)^- - \omega_c(\delta)\int_0^{t+\delta} B(s)ds$ is subsolution. So, by the comparison principle, we have

$$\begin{aligned} u(x,t) - u(x,t+\delta) \leq & \omega_F(\delta) + \|c\|_{L^\infty}B\delta + \omega_c(\delta)\int_0^{t+\delta} B(s)ds \\ \leq & \omega_F(\delta) + \|c\|_{L^\infty}B_0\delta + \omega_c(\delta)\int_0^T B(s)ds, \end{aligned}$$

*i.e.*

$$|u(x,t) - u(x,t+\delta)| \leq \omega_F(\delta) + \|c\|_{L^\infty}B_0\delta + \omega_c(\delta)\int_0^T B(s)ds,$$

what achieves the proof of the *lemma*. □

## 5 The non local problem: proof of Theorem 3.1

For the proof of Theorem 3.1, we will need the three following *lemmata*:

***Lemma* 5.1 (Estimate on the characteristic functions)**
*Let $u^1 \in C(\mathbb{R}^n)$ satisfying*

$$\frac{\partial u^1}{\partial x_n} \geq b$$

*in the distributions sense for some $b>0$ and $u^2 \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfying the same condition. Then, we have the following estimate:*

$$\left\|[u^2]-[u^1]\right\|_{L^1_{\text{unif}}} \leq \frac{2}{b}\left\|u^2-u^1\right\|_{L^\infty}. \tag{19}$$

For the proof of this *Lemma*, we refer to the proof of Alvarez *et al.* [2] in the case $n=2$, which adapts without difficulty to the case of any dimension.

***Lemma* 5.2 (Convolution inequality)**
*For every $f \in L^1_{\text{unif}}(\mathbb{R}^n)$ and $g \in L^\infty_{\text{int}}(\mathbb{R}^n)$, the convolution product $f \star g$ is bounded and satisfies*

$$\|f \star g\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1_{\text{unif}}(\mathbb{R}^n)} \|g\|_{L^\infty_{\text{int}}(\mathbb{R}^n)}.$$

For the proof, we refer to Alvarez *et al.* [4].

***Lemma* 5.3 (Stability of the solution with respect to the velocity)**
*Let $T > 0$. We consider for $i = 1, 2$ two different equations:*

$$\begin{cases} u^i_t = c^i(x,t)|Du^i| - F\left(Du^i, D^2u^i\right) & \text{in } \mathbb{R}^n \times (0,T), \\ u^i(x,0) = u_0(x). \end{cases} \tag{20}$$

*where $c^i$ satisfy the assumption $(H_0)(ii)$, $u_0$ satisfies $(H_0)(iii)$ and $F$ satisfies the assumptions $(HF)$. Then, for every $t \in [0,T)$, we have*

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))} \int_0^t B(s)ds,$$

*where $u^i$ are the solutions of (20) (see Theorem 4.4), $B(t) = B_0 e^{L_c t}$ with $L_c = \sup_i L_{c^i}$ ($L_{c^i}$ is the Lipcshitz constant of $c^i$).*

**Proof of *Lemma* 5.3**
We set $K = \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))}$. We remark that $u^1$ is subsolution of

$$u_t - c^2(x,t)|Du| + F\left(Du, D^2u\right) - KB(t) = 0.$$

Indeed, we have:

$$\begin{aligned} u^1_t - c^2(x,t)|Du^1| + F\left(Du^1, D^2u^1\right) \leq & c^1(x,t)|Du^1| - F\left(Du^1, D^2u^1\right) - c^2(x,t)|Du^1| + F\left(Du^1, D^2u^1\right) \\ \leq & \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))} B(t) \\ \leq & KB(t). \end{aligned}$$

It is a routine exercise to check that the differential inequality actually holds in the viscosity sense. Moreover, $u^2 + K\int_0^t B(s)ds$ is solution of the same problem. By the comparison principle (Theorem 4.3), we deduce

$$u^1 \leq u^2 + K \int_0^t B(s)ds.$$

From what

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^n \times (0,T))} \int_0^t B(s)ds.$$

□

We now prove Theorem 3.1.

**Proof of Theorem 3.1**
We set $\omega(\delta) = \omega_F(\delta) + \|c_0\|_{L^1} B_0 \delta$, where $\omega_F$ is the modulus of continuity of (17) and behaves like $\sqrt{t}$.

We define the space

$$E = \left\{ u \in L^\infty_{loc}(\mathbb{R}^n \times [0,T^*)), s.t. \quad \middle| \begin{array}{l} |Du(x,t)| \le 2B_0, \\ \dfrac{\partial u}{\partial x_n}(x,t) \ge \dfrac{b_0}{2} \\ u \text{ is uniformly continuous in time and } \omega_u(\delta) \le 2\omega(\delta) \end{array} \right\}$$

where $\omega_u$ is the modulus of continuity in time of $u$.

For $u \in E$, we set $c(x,t) = (c_0 \star [u(\cdot,t)])(x)$. We see that $c$ is bounded, Lipschitz continuous in space (with $L_c = |c_0|_{BV}$ as Lipschitz constant) and uniformly continuous in time. Indeed,

$$\begin{aligned} \|c\|_{L^\infty(\mathbb{R}^n\times[0,T^*))} \le& \sup_{t\in\mathbb{R}} \|c_0\|_{L^1} \|[u(\cdot,t)]\|_{L^\infty(\mathbb{R}^n)} \\ \le& \|c_0\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Moreover, for every $t$

$$\begin{aligned} \|Dc(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} =& \|Dc_0 \star [u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \\ \le& |c_0|_{BV} \|[u(\cdot,t)]\|_{L^\infty(\mathbb{R}^n)} \\ \le& |c_0|_{BV}. \end{aligned}$$

Finally, for $0 < t, s < T^*$ :

$$\begin{aligned} |c(x,t) - c(x,s)| =& |(c_0 \star [u(\cdot,t)])(x) - (c_0 \star [u(\cdot,s)])(x)| \\ =& |c_0 \star ([u(\cdot,t)] - [u(\cdot,s)])(x)| \\ \le& \|c_0\|_{L^\infty_{\text{int}}} \|[u(\cdot,t)] - [u(\cdot,s)]\|_{L^1_{\text{unif}}(\mathbb{R}^n)} \\ \le& \frac{4\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \|u(\cdot,t) - u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)} \\ \le& \frac{4\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \omega_u(|t-s|) \\ \le& \frac{8\|c_0\|_{L^\infty_{\text{int}}}}{b_0} \omega(|t-s|), \end{aligned}$$

so $c$ is uniformly continuous in time and $\omega_c(\delta) \le \frac{8\|c_0\|_{L^\infty_{\text{int}}}}{b_0}\omega(\delta)$.

For $u \in E$, we then define $v = \Phi(u)$ as the unique viscosity solution (see Theorem 4.4) of

$$\begin{cases} v_t = (c_0 \star [u])|Dv| - F(Dv, D^2v) \text{ in } \mathbb{R}^n \times (0,T^*), \\ v(x,t=0) = u_0(x) \text{ in } \mathbb{R}^n. \end{cases} \tag{21}$$

We show that $\Phi : E \to E$ is a contraction. First, we show that $\Phi$ is well defined. We have $\|Dv(\cdot,t)\| \le B(t) \le B_0 e^{L_c T^*} \le 2B_0$, by definition of $T^*$ (see *Lemma* 4.6). Moreover, $\frac{\partial v}{\partial x_n} \ge b(t) = b_0 - B_0(e^{L_c t} - 1)$ (see *Lemma* 4.6), and we want $\frac{\partial v}{\partial x_n} \ge \frac{b_0}{2}$, so it suffices to ensure that

$$B_0(e^{L_c t} - 1) \le \frac{b_0}{2}$$

$$e^{L_c t} \le \frac{b_0}{2B_0} + 1$$

$$t \le \frac{\ln\left(\frac{b_0}{2B_0}+1\right)}{L_c},$$

which is true according to the choice of $T^*$. It thus remains to be shown that $v$ is uniformly continuous with $\omega_v(\delta) \le 2\omega(\delta)$. Now, by the estimate of *Lemma* 4.6 on the modulus of continuity in time of the solution, we have:

$$\omega_v(\delta) \le \omega_F(\delta) + \|c\|_{L^\infty} B_0 \delta + \omega_c(\delta) \int_0^{T^*} B(s)ds.$$

Since $\|c\|_{L^\infty(\mathbb{R}^n\times[0,T^*))} \le \|c_0\|_{L^1}$, it suffices to show that $\omega_c(\delta)\int_0^{T^*} B(s)ds \le \omega(\delta)$, *i.e.*

$$\frac{8\|c_0\|_{L^\infty_{\text{int}}}}{b_0}\omega(\delta)\int_0^{T^*} B(s)ds \le \omega(\delta)$$

$$\int_0^{T^*} B(s)ds \le \frac{b_0}{8\|c_0\|_{L^\infty_{\text{int}}}}$$

$$\frac{1}{L_c}\left(e^{L_cT^*}-1\right) \le \frac{b_0}{8B_0\|c_0\|_{L^\infty_{\text{int}}}}$$

$$T^* \le \frac{\ln\left(\frac{L_c b_0}{8B_0\|c_0\|_{L^\infty_{\text{int}}}}+1\right)}{L_c},$$

which is true according to the choice of $T^*$ and so $v \in E$.

It thus remains to be shown that $\Phi$ is a contraction. For $v^i = \Phi(u^i)$, according to the *Lemmata* 5.3, 5.2 and 5.1, we have

$$\begin{aligned}
\|v^2 - v^1\|_{L^\infty(\mathbb{R}^n\times(0,T^*))} &\le 2B_0T^*\|c_0 \star [u^2] - c_0 \star [u^1]\|_{L^\infty(\mathbb{R}^n\times(0,T^*))}\\
&\le 2B_0T^*\|c_0\|_{L^\infty_{\text{int}}(\mathbb{R}^n)} \sup_{t\in(0,T^*)} \|[u^2(\cdot,t)] - [u^1(\cdot,t)]\|_{L^1_{\text{unif}}(\mathbb{R}^n)}\\
&\le \frac{8B_0T^*}{b_0}\|c_0\|_{L^\infty_{\text{int}}(\mathbb{R}^n)}\|u^2-u^1\|_{L^\infty(\mathbb{R}^n\times(0,T^*))}\\
&\le \frac{1}{2}\|u^2-u^1\|_{L^\infty(\mathbb{R}^n\times(0,T^*))}.
\end{aligned}$$

And so $\Phi$ is a contraction on $E$ which is a closed set for the $L^\infty$ topology. So, there exists a unique viscosity solution of (6) in $E$ on $(0,T^*)$. □

**Remark 5.4** *To be rigourous, we should consider the intersection of $E$ with a ball of center $u_0$ and write the elements of $E$ as $u = \tilde{u} + u_0$ with $\tilde{u}$ bounded. Then we could make the same computations on $\tilde{u}$ and we will obtain the same result.*

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