Dislocations dynamics and mean curvature motion

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ENPC, CERMICS

New trends in viscosity solutions and nonlinear PDEs

26, July 2006

Plan

1. Convergence of dislocations dynamics to mean curvature motion
2. Numerical scheme for dislocations dynamics
3. Numerical Scheme for the mean curvature motion

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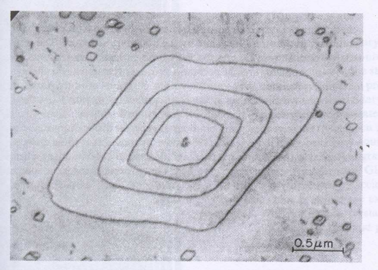
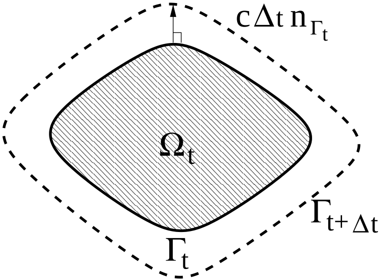
Observation of dislocations



Definition: a dislocation is a line of crystal defects.

Dynamics of a dislocation



$$\frac{d\Gamma_t}{dt} = c\, n_{\Gamma_t} \quad \text{with} \quad \mathrm{c} = \mathrm{c}(\Gamma_\mathrm{t})$$

Slepčev Formulation

- We consider the solutions u^ε of:

$$\left\{ \begin{array}{l} u^\varepsilon_t(x,t) = \left(\left(c^\varepsilon_0 \star 1_{\{u^\varepsilon(\cdot,t)>u^\varepsilon(x,t)\}}\right)(x) - \frac{1}{2}\int_{\mathbb{R}^n} c^\varepsilon_0\right)|Du^\varepsilon| \\ \\ u^\varepsilon(\cdot,0) = u_0(\cdot) \end{array} \right. \quad (1)$$

where

$$c^\varepsilon_0 = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} c_0\left(\frac{x}{\varepsilon}\right)$$

with the particular kernel

$$c_0 \in C^\infty(\mathbb{R}^n), \quad \left\{ \begin{array}{ll} c_0(x) = \frac{1}{|x|^{n+1}} g\left(\frac{x}{|x|}\right) & \text{if} \quad |x| > 1 \\ c_0(-x) = c_0(x) \geq 0 \end{array} \right.$$

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Definition of the solutions

Definition (Slepčev)

- u^ε *is sub-solution if for* Φ *test function:*

$$\Phi_t \leq \left(\left(c_0^\varepsilon \star 1_{\{u^\varepsilon(\cdot,t_0) \geq u^\varepsilon(x_0,t_0)\}} \right)(x_0) - \frac{1}{2}\int_{\mathbb{R}^n} c_0^\varepsilon \right) |D\Phi|$$

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Convergence to anisotropic MCM

Theorem (Da Lio, F., Monneau)

Under certain regularity assumptions, when $\varepsilon \to 0$, u^ε *converges locally uniformly on compact sets to* u^0*, which is the unique solution of the limit problem:*

$$\begin{cases} u_t^0 + F(D^2 u^0, Du^0) = 0 \\ u^0(\cdot, 0) = u_0 \end{cases}$$

with

$$F(M,p) = -\mathrm{trace}\left(MA\left(\frac{p}{|p|}\right)\right)$$

$$A\left(\frac{p}{|p|}\right) = \int_{\theta \in \mathbf{S}^{n-1} \cap \{p^\perp\}} \frac{1}{2} g(\theta)\theta \otimes \theta d\theta$$

Idea of the proof

- Error estimate for regular function s.t. $|D\varphi| > 0$,
- [Barles, Georgelin]: For test function: $|D\varphi| > 0$ or $D\varphi = 0$ and $D^2\varphi = 0$.

Similar Results

- [Garroni, Muller] (Gamma convergence, stationnary problem)
- Algorithm of [Merriman, Bence, Osher]
- [Evans], [Barles, Georgelin], [Ishii], [Ishii, Pires, Souganidis], [Bellitini,Novaga], [Chambolle, Novaga]

Error Estimate

We have the following error estimate:

Theorem (F.)

Under certain regularity assumptions, we have the following error estimate between u^ε and u^0, for $T \leq 1$:

$$|u^\varepsilon - u^0| \leq C \left(\frac{T}{\sqrt{|\ln \varepsilon|}} \right)^{\frac{1}{3}}.$$

Proof: Use the argument of [Barles, Georgelin] in a quantitative way.

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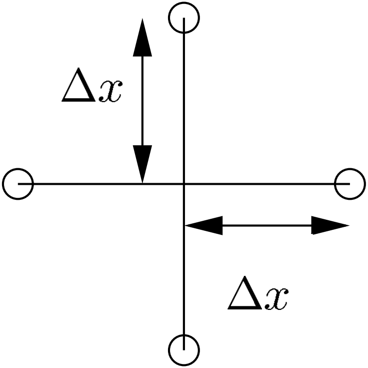
Discretization



Monotone scheme

$$c_0^\varepsilon = \frac{1}{\varepsilon^{n+1}|\ln \varepsilon|} c_0\left(\frac{x}{\varepsilon}\right)$$

- Equation: $u_t(x,t) = \left(c[u](x,t) - \frac{1}{2}\int_{\mathbb{R}^n} c_0\right)|Du|$
- Scheme: $\frac{v_I^{n+1} - v_I^n}{\Delta t} = \left(c_I^{n+1}[v] - \frac{1}{2}\int_{\mathbb{R}^n} c_0\right)$ "$|Dv|_I^{n+1}$"
- Non-local velocity: $c[u](x,t) = \left(c_0 \star 1_{\{u(\cdot,t)>u(x,t)\}}\right)(x)$
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Property of the scheme

Property of the scheme:

- Implicit, but practically computable $\Rightarrow$ no CFL condition
- Discrete Slepcev formulation
- Monotone

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Discrete-continuous error estimate

Theorem (F.)

Under certain regularity assumptions, we have the following error estimate between the continuous solution u of the dislocations dynamics equation (with c_0) and its numerical approximation v:

$$\sup_{\mathbb{R}^N \times (0,T)} |u - v| \leq K\sqrt{T}\,(\Delta x + \Delta t)^{1/2}$$

provided $|\Delta X| + \Delta t \leq \dfrac{1}{K^2}$.

Rescaling

$$u^\varepsilon(x,t) = \varepsilon u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2|\ln\varepsilon|}\right)$$

$$\begin{aligned} |u^\varepsilon - v^\varepsilon| \leq & K\frac{\sqrt{T}}{\sqrt{|\ln\varepsilon|}}\sqrt{\Delta x + \frac{\Delta t}{\varepsilon|\ln\varepsilon|}} \\ \leq & K\frac{\sqrt{T}}{\sqrt{\varepsilon}|\ln\varepsilon|}\sqrt{\Delta x + \Delta t} \end{aligned}$$

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Plan

Numerical scheme for mean curvature motion

- Using the previous result, we get, for $T \leq 1$:

$$
\begin{aligned}
|u^0 - v^\varepsilon| \leq & |u^0 - u^\varepsilon| + |u^\varepsilon - v^\varepsilon| \\
\leq & C\left(\frac{T}{\sqrt{|\ln\varepsilon|}}\right)^{\frac{1}{3}} + \frac{K}{\sqrt{\varepsilon}|\ln \varepsilon|}\sqrt{T}\sqrt{\Delta x + \Delta t} \\
\leq & C\left(\frac{T}{\sqrt{|\ln\varepsilon|}}\right)^{\frac{1}{3}}
\end{aligned}
$$

for $\varepsilon \sim \Delta x$ and $\Delta t \leq \Delta x$.

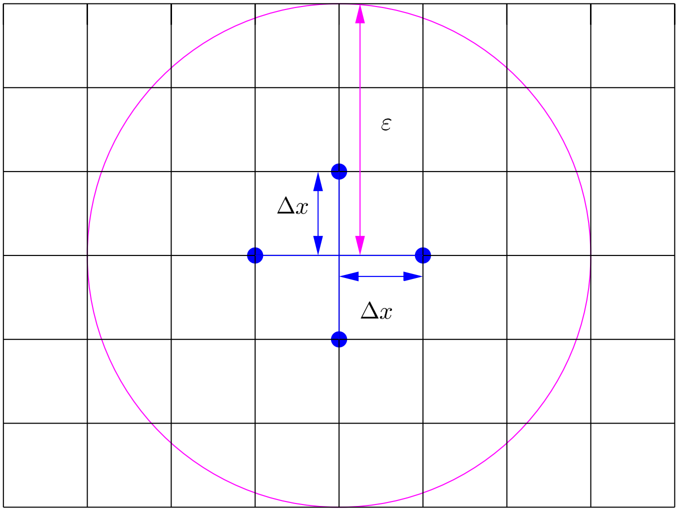
Numerical scheme for mean curvature motion

Theorem (NF)

Let $T \leq 1$. Under certain regularity assumptions, we have the following error estimate between the continuous solution u^0 of the mean curvature motion and its numerical approximation v^ε:

$$\sup_{\mathbb{R}^N \times (0,T)} |u^0 - v^\varepsilon| \leq C \left(\frac{T}{\sqrt{|\ln(\varepsilon)|}} \right)^{\frac{1}{3}} \quad \textit{where } \varepsilon \sim \Delta x, \; \Delta t \leq \Delta x$$

Discretization



References

- [Crandall, Lions], [Oberman]
- Algorithm of [Merriman, Bence, Osher]
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Monotone scheme

- Scheme:

$$v_I^0 = \tilde{u}_0(x_I),$$

$$v_I^{n+1} = v_I^n + \Delta t c^\Delta[v](x_I, t_{n+1}) E^{sign(c^\Delta[v](x_I, t_{n+1}))}(D^+ v_I^{n+1}, D^- v_I^{n+1}) \quad (2)$$

- Non-local velocity:

$$c^\Delta[v](x_I, t_{n+1}) = \sum_{J\in\mathbb{Z}^N} \bar{c}_0(x_{I-J}) 1_{\{v_J^{n+1} \geq v_I^{n+1}\}} \Delta x_1 \ldots \Delta x_N - \frac{1}{2} \sum_{J\in\mathbb{Z}^N} \bar{c}_0(x_J) \Delta x_1 \ldots \Delta x_N \quad (3)$$

- Kernel:

$$\bar{c}_0(x_I) = \frac{1}{|Q_I|} \int_{Q_I} c_0(x)\, dx \quad (4)$$

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Monotone scheme

where Q_I is the square cell

$$Q_I = [x_{i_1} - \Delta x_1/2, x_{i_1} + \Delta x_1/2] \times ... \times [x_{i_N} - \Delta x_N/2, x_{i_N} + \Delta x_N/2]$$

- $E^{\pm}$; approximation of the Euclidean norm ([Osher-Sethian]):

$$E^+ = \left\{ \sum_i \max(D^+_{x_i} v^n_I, 0)^2 + \sum_i \min(D^-_{x_i} v^n_I, 0)^2 \right\}^2$$

$$E^- = \left\{ \sum_i \min(D^+_{x_i} v^n_I, 0)^2 + \sum_i \max(D^-_{x_i} v^n_I, 0)^2 \right\}^2$$

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$$E^{-} = \left\{ \sum_i \min(D^{+}_{x_i} v^n_I, 0)^2 + \sum_i \max(D^{-}_{x_i} v^n_I, 0)^2 \right\}^2$$

Definition of the solutions

Definition (Numerical sub, super and solution of the scheme)

We say that v is a discrete sub-solution (resp super-solution) of the scheme (2) if for all $I \in \mathbb{Z}^N$, $n \in \mathbb{N}$, we have

$$v_I^{n+1} \leq v_I^n + \Delta t\; c^\Delta[v](x_I, t_{n+1}) E^{Sign(c^\Delta[v](x_I,t_{n+1}))}(D^+ v_I^{n+1}, D^- v_I^{n+1})$$

(resp.

$$v_I^{n+1} \geq v_I^n + \Delta t\; \tilde{c}^\Delta[v](x_I, t_{n+1}) E^{Sign(\tilde{c}^\Delta[v](x_I,t_{n+1}))}(D^+ v_I^{n+1}, D^- v_I^{n+1}))$$

$$\text{where} \quad \tilde{c}^\Delta[v](x_I, t_{n+1}) = \sum_{J\in\mathbb{Z}^N} \bar{c}_0(x_{I-J}) 1_{\{v_J^{n+1} > v_I^{n+1}\}} \Delta x_1 \ldots \Delta x_N$$

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