

A short time existence/uniqueness result for a nonlocal topology-preserving segmentation model

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Abstract

Motivated by a prior applied work of Vese and the second author dedicated to segmentation under topological constraints, we derive a slightly modified model phrased as a functional minimization problem, and propose to study it from a theoretical viewpoint. The mathematical model leads to a second order nonlinear PDE with a singularity at $\nabla u = 0$ and containing a nonlocal term. A suitable setting is thus the one of the viscosity solution theory and, in this framework, we establish a short time existence/uniqueness result as well as a Lipschitz regularity result for the solution.

1 Introduction

In [17], Le Guyader and Vese propose a topology-preserving segmentation model based on an implicit level-set formulation and on the geodesic active contours. The goal of this paper is to prove a short time existence and uniqueness result for a slightly modified model. The necessity of designing topology-preserving processes arises in medical imaging, for instance, in the human cortex reconstruction: it is well-known that the human cortex has a spherical topology and this anatomical feature must be preserved through the segmentation process accordingly. The need for topology-preserving models also occurs when the shape to be detected must be homeomorphic to the initial one. To fix ideas, we propose some examples to illustrate what the results should be when running such a kind of algorithm. The implicit framework of the level-set method (see [18] for instance) has several advantages when tracking propagating fronts. In particular, it easily handles topological changes such as merging and breaking.

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Thus, in Fig. 1, when no topological constraints are enforced, the evolving contour splits into two components. On the other hand, in the topology-preserving framework (see Fig. 2), we aim at segmenting the two disks while maintaining the same topology throughout the process, which means that we expect to get one path-connected component.

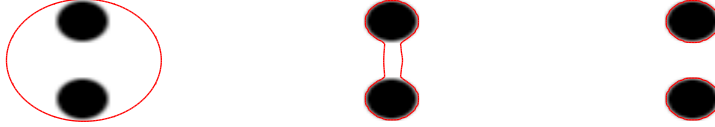


Figure 1: Segmentation of the synthetic image with two disks when no topological constraint is enforced: the contour has split into two components. Iterations 0, 140, 180.

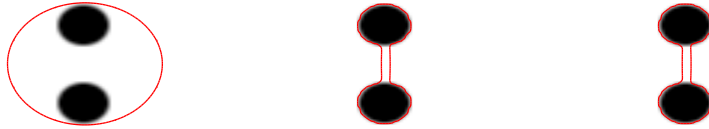


Figure 2: Segmentation of the synthetic image with two disks when topological constraints are applied. Iterations 0, 180, 210.

For the sake of completeness, we refer the reader to other prior related works dedicated to segmentation models under topological constraints: the work of Han *et al.* ([14]) based on the concept of simple point in digital topology, the work of Alexandrov and Santosa ([1]), which is a curve evolution method based on level sets for shape optimization problems in material sciences and which prevents the narrow band from overlapping, the work of Sundaramoorthi and Yezzi ([20]) based on the knot energy and which includes a penalty in the segmentation functional in the form of a double integral as done in [17], the work of Rochery *et al.* ([19]) which uses a similar idea to [17] to avoid that pieces of the evolving curve come in contact to merge or break, and lastly the work of Cecil *et al.* ([8]).

1.1 Description of the model

The model proposed in [17] is as follows. Let Ω be a bounded open subset of \mathbb{R}^2 , $\partial\Omega$ its boundary and let I be a given bounded image function defined by $I: \bar{\Omega} \rightarrow \mathbb{R}$. Let $g: [0, +\infty[\rightarrow [0, +\infty[$ be an edge-detector function satisfying $g(0) = 1$, g strictly decreasing, and $\lim_{r \rightarrow +\infty} g(r) = 0$. The evolving contour \mathcal{C} is embedded in a higher-dimensional Lipschitz continuous function Φ defined by $\Phi: \Omega \times [0, +\infty[\rightarrow \mathbb{R}$ with $(x, t) \mapsto \Phi(x, t)$ such that $\mathcal{C}(t, \cdot) = \{x \in \Omega \mid \Phi(x, t) = 0\}$ and

$$\begin{cases} \Phi < 0 & \text{on } w \text{ the interior of } \mathcal{C}, \\ \Phi > 0 & \text{on } \Omega \setminus \bar{w}. \end{cases}$$

Generally, this function Φ is preferred to be a signed-distance function for the stability of numerical computations.

The segmentation model of [17] combines the classical geodesic active contour functional (see [7]) with a topological constraint phrased in terms of a double integral. More precisely, it consists in minimizing the following functional:

$$F(\Phi) + \mu E(\Phi),$$

where $\mu > 0$ is a tuning parameter. The functional F stems from the geodesic active contour model and is defined by:

$$F(\Phi) = \int_{\Omega} g(|\nabla I(x)|) \delta(\Phi(x)) |\nabla \Phi(x)| dx,$$

with δ the 1-D Dirac measure. The functional E , related to the topological constraint, is defined by:

$$E(\Phi) = - \int_{\Omega} \int_{\Omega} \left[\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right) \langle \nabla \Phi(x), \nabla \Phi(y) \rangle H(\Phi(x)+l) H(l-\Phi(x)) \right. \\ \left. H(\Phi(y)+l) H(l-\Phi(y)) \right] dx dy,$$

with H the 1-D Heaviside function, $\langle \cdot, \cdot \rangle$ denoting the Euclidean scalar product in \mathbb{R}^2 and $\|\cdot\|_2$ the associated norm. A geometrical observation motivates the introduction of E . Indeed, in the case where Φ is a signed-distance function, $|\nabla \Phi| = 1$ and the unit outward normal vector to the zero level line at point x is $\nabla \Phi(x)$. Let us now consider two points $(x, y) \in \Omega \times \Omega$ belonging to the zero level line of Φ , close enough to each other, and let $\nabla \Phi(x)$ and $\nabla \Phi(y)$ be the two unit outward normal vectors to the contour at these points. As shown in Fig. 3, when the contour is about to merge or split, that is, when the topology of the evolving contour is to change, then $\langle \nabla \Phi(x), \nabla \Phi(y) \rangle \simeq -1$. This remark justifies the construction of E . Also, instead of working with only the points of the zero level line, the authors propose to focus on the points contained in a narrow band around the zero level line, more precisely, on the set of points $\{x \in \Omega \mid -l \leq \Phi(x) \leq l\}$, l being a level parameter. Lastly, the function $(x, y) \mapsto \exp\left(-\frac{\|x-y\|_2^2}{d^2}\right)$ measures the nearness of the two points x and y .

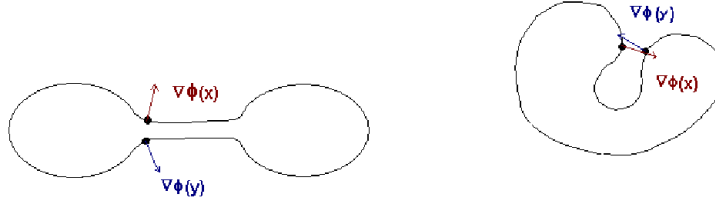


Figure 3: Geometrical characterization of points in a zone where the curve is to split, merge or have a contact point.

Thus if the unit outward normal vectors to the level lines passing through x and y have opposite directions, the functional is not minimal.

The Euler-Lagrange equation is derived and is solved by a gradient descent method. A rescaling is made by replacing $\delta(\Phi)$ by $|\nabla\Phi|$ and the evolution equation is complemented by Neumann homogeneous boundary conditions. It leads to the following evolution problem:

$$\begin{cases} \frac{\partial\Phi}{\partial t} = |\nabla\Phi| \left[\operatorname{div} \left(g(|\nabla I|) \frac{\nabla\Phi}{|\nabla\Phi|} \right) \right] + 4 \frac{\mu}{d^2} H(\Phi(x)+l) H(l-\Phi(x)) \\ \int_{\Omega} \left[\langle x-y, \nabla\Phi(y) \rangle e^{-\|x-y\|_2^2/d^2} H(\Phi(y)+l) H(l-\Phi(y)) \right] dy, \\ \Phi(x,0) = \Phi_0(x), \\ \frac{\partial\Phi}{\partial \bar{\nu}} = 0, \text{ on } \partial\Omega. \end{cases}$$

This problem is hard to handle from a theoretical point of view. A suitable setting would be the one of the viscosity solution theory (due to the nonlinearity induced by the modified mean-curvature term) but the dependency of the nonlocal term to the gradient ($\nabla\Phi(y)$) and the failure to fulfill the monotony property in Φ make it impossible. For this reason, we consider a slightly modified problem. We propose to focus on the following minimization problem for which the topological constraint is only applied to the zero level line (we still assume that $|\nabla\Phi|=1$).

$$\inf_{\Phi} \int_{\Omega} g(|\nabla I(x)|) \delta(\Phi(x)) |\nabla\Phi(x)| dx - \mu \int_{\Omega} \int_{\Omega} \left[\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right) \langle \nabla\Phi(x), \nabla\Phi(y) \rangle \delta(\Phi(x)) \delta(\Phi(y)) \right] dx dy.$$

We compute the Euler-Lagrange equation and apply a gradient descent method.

We get the following evolution equation:

$$\begin{aligned}
\frac{\partial \Phi}{\partial t} &= \delta(\Phi) \operatorname{div} \left(g(|\nabla I|) \frac{\nabla \Phi}{|\nabla \Phi|} \right) \\
&\quad - 2\mu \int_{\Omega} \frac{\partial}{\partial x_1} \left[\exp \left(-\frac{\|x-y\|_2^2}{d^2} \right) \right] \delta(\Phi(x)) \delta(\Phi(y)) \frac{\partial \Phi}{\partial y_1}(y) dy \\
&\quad - 2\mu \int_{\Omega} \frac{\partial}{\partial x_2} \left[\exp \left(-\frac{\|x-y\|_2^2}{d^2} \right) \right] \delta(\Phi(x)) \delta(\Phi(y)) \frac{\partial \Phi}{\partial y_2}(y) dy. \\
&= \delta(\Phi) \left\{ \operatorname{div} \left(g(|\nabla I|) \frac{\nabla \Phi}{|\nabla \Phi|} \right) \right. \\
&\quad + \frac{4\mu}{d^2} \int_{\Omega} (x_1 - y_1) \exp \left(-\frac{\|x-y\|_2^2}{d^2} \right) \frac{\partial}{\partial y_1} [H(\Phi(y))] dy \\
&\quad \left. + \frac{4\mu}{d^2} \int_{\Omega} (x_2 - y_2) \exp \left(-\frac{\|x-y\|_2^2}{d^2} \right) \frac{\partial}{\partial y_2} [H(\Phi(y))] dy \right\}.
\end{aligned}$$

Doing an integration by parts in the second part of the PDE and setting the necessary boundary conditions to zero, it yields:

$$\begin{aligned}
\frac{\partial \Phi}{\partial t} &= \delta(\Phi) \left\{ \operatorname{div} \left(g(|\nabla I|) \frac{\nabla \Phi}{|\nabla \Phi|} \right) \right. \\
&\quad \left. + \frac{4\mu}{d^2} \int_{\Omega} \left(2 - \frac{2}{d^2} \|x-y\|_2^2 \right) \exp \left(-\frac{\|x-y\|_2^2}{d^2} \right) H(\Phi(y)) dy \right\}, \\
&= \delta(\Phi) \left\{ \operatorname{div} \left(g(|\nabla I|) \frac{\nabla \Phi}{|\nabla \Phi|} \right) + c_0 * [\Phi] \right\},
\end{aligned}$$

with $[\Phi]$ the characteristic function of the set $\{\Phi > 0\}$ and

$$c_0 : \begin{cases} \mathbb{R}^2 \rightarrow \mathbb{R} \\ x \mapsto \frac{4\mu}{d^2} \left(2 - \frac{2}{d^2} \|x\|_2^2 \right) \exp \left(-\frac{\|x\|_2^2}{d^2} \right). \end{cases}$$

A rescaling can be made by replacing $\delta(\Phi)$ by $|\nabla \Phi|$ in order to apply the same motion to all level sets. Also, for the sake of simplicity, we assume, in the sequel, that the problem is formulated on \mathbb{R}^2 for the spatial coordinates.

1.2 Main results

Given $T > 0$, we consider the following problem: find $\Phi(x, t)$ solution of:

$$\begin{cases} \frac{\partial \Phi}{\partial t} = |\nabla \Phi| \left\{ \operatorname{div} \left(g(|\nabla I|) \frac{\nabla \Phi}{|\nabla \Phi|} \right) + c_0 * [\Phi] \right\} & \text{in } \mathbb{R}^2 \times (0, T), \\ \Phi(x, 0) = \Phi_0(x) & \text{in } \mathbb{R}^2, \end{cases} \quad (1.1)$$

with $\Phi_0 \in W^{2,\infty}(\mathbb{R}^2)$ (we denote by B_0 its Lipschitz constant). We need the following assumptions on function g :

- (H1) $\exists \delta > 0, \forall x \in \mathbb{R}^2, \delta < g(x) \leq 1$.
- (H2) $g, g^{\frac{1}{2}}$ and ∇g are bounded and Lipschitz continuous on \mathbb{R}^2 with Lipschitz constant $\kappa_g, \kappa_{\nabla g}$ and $\kappa_{g^{\frac{1}{2}}}$ respectively. For simplicity of notation, we set $L_g = \max(\kappa_g, \kappa_{\nabla g}, \kappa_{g^{\frac{1}{2}}})$.

This model is a nonlocal Hamilton-Jacobi equation. We propose, in this paper, to prove a short time existence and uniqueness result for this equation.

Theorem 1 (Short time existence and uniqueness).

Assume (H1)-(H2) and let $u_0: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a $W^{2,\infty}(\mathbb{R}^2)$ function such that:

$$|Du_0| < B_0 \text{ in } \mathbb{R}^2 \quad \text{and} \quad \frac{\partial u_0}{\partial x_2} > b_0 > 0 \text{ in } \mathbb{R}^2.$$

Then there exists $T^* > 0$ (depending only on b_0, B_0, c_0 and g) such that there exists a unique viscosity solution of problem (1.1) in $\mathbb{R}^2 \times [0, T^*)$. Moreover, the solution is Lipschitz continuous in space and time.

Since the equation is nonlinear, as previously mentioned, a natural framework is the one of the viscosity solution theory (see for instance the monographs of Barles [5] and Bardi and Capuzzo-Dolcetta [4] for a presentation of first order equations and the article of Crandall, Ishii and Lions [11] for the second order case). Our work is much motivated by a previous article of the first author ([12]), which is dedicated to the mathematical study of a model for dislocation dynamics with a mean curvature term. The main difference with the model in [12] is that in our case, the PDE explicitly depends on the space variable x , which induces substantial adaptations of the proof. The strategy of the proof is the same as the one applied in [3] or [12], *i.e.*, using a fixed point method by freezing the nonlocal term. To do this, the key point is to get estimates on the Lipschitz constant in space and time of the solution as well as a bound from below on the gradient in space. The main difficulties come from the fact that the mean curvature term is balanced by a function of the space variable x and so, to obtain the estimate from below on the gradient, we have to bound the mean curvature term. This is done using the Lipschitz regularity of the solution.

The outline of the paper is as follows. Section 2 is devoted to the mathematical study of a related preliminary local problem, which is useful to establish the existence/uniqueness of the solution of the nonlocal problem. We give an existence/uniqueness result for the solution of the local problem and provide some results on the regularity of this solution. Section 3 presents the main result of the paper, that is, a short time existence/uniqueness result for the nonlocal problem.

2 Study of a related local problem

In the following, for the sake of simplicity, we denote by g the function $g(|\nabla I|)$. Also, the gradient of u is now denoted by Du . Given $T > 0$, we consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = c(x,t)|Du| + |Du| \operatorname{div} \left(g(x) \frac{Du}{|Du|} \right) & \text{on } \mathbb{R}^2 \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \quad (2.2)$$

with $c: \mathbb{R}^2 \times [0, T] \mapsto c(x, t)$ bounded, Lipschitz continuous in space (we denote by L_c its Lipschitz constant in space), and in time (we denote by L_{c_t} its Lipschitz constant in time).

The evolution equation can be rewritten in the form

$$\frac{\partial u}{\partial t} + G(x, t, Du, D^2u) = 0,$$

with $G: \mathbb{R}^2 \times [0, T] \times \mathbb{R}^2 \times \mathcal{S}^2$ (\mathcal{S}^2 being the set of symmetric 2×2 matrices equipped with its natural partial order) defined by:

$$\begin{aligned} G(x, t, p, X) &= -c(x, t)|p| + F(x, p, X), \\ &= -c(x, t)|p| + g(x)H(p, X) - \langle \nabla g(x), p \rangle, \end{aligned}$$

with the following properties:

1. The operators G , F and $H: (p, X) \mapsto -\operatorname{trace} \left(\left(I - \frac{p \otimes p}{|p|^2} \right) X \right)$ are independent of u and are elliptic, *i.e.* $\forall X, Y \in \mathcal{S}^2, \forall p \in \mathbb{R}^2$,

$$\text{if } X \leq Y \text{ then } F(x, p, X) \geq F(x, p, Y).$$

2. F is locally bounded on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}^2$, continuous on $\mathbb{R}^2 \times \mathbb{R}^2 - \{0_{\mathbb{R}^2}\} \times \mathcal{S}^2$, and $F^*(x, 0, 0) = F_*(x, 0, 0) = 0$, where F^* (resp. F_*) is the upper semi-continuous (usc) envelope (resp. lower semicontinuous (lsc) envelope) of F .

2.1 Existence and uniqueness

Theorem 2 (Comparison principle).

Assume (H1)-(H2) and let $u: \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ be a bounded and upper semi-continuous subsolution and $v: \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ be a bounded and lower semi-continuous supersolution of (2.2). Assume that $u(x, 0) \leq u_0(x) \leq v(x, 0)$ in \mathbb{R}^2 , then $u \leq v$ in $\mathbb{R}^2 \times [0, T]$.

Proof. This proof is rather classical. For the reader's convenience, we refer to [13], in which the authors prove comparison theorems for viscosity solutions of related degenerate parabolic equations of general form in a domain not necessarily bounded. \square

We now turn to the existence of a solution. To do so, we construct barriers.

Proposition 1 (Existence of barriers).

Assume (H1)-(H2) and let $u_0 \in W^{2,\infty}(\mathbb{R}^2)$. Then there exists a constant $C_1 > 0$ depending only on $\|c\|_{L^\infty}$, g and u_0 such that $u^\pm = u_0 \pm C_1 t$ are resp. super- and sub-solution of (2.2).

Proof. Let us check that u^+ is a supersolution (the proof for u^- being similar). We have:

$$\begin{aligned} c(x,t)|Du^+| - F^*(x,Du^+,D^2u^+) &= c(x,t)|Du_0| - g(x)H^*(Du_0,D^2u_0) \\ &\quad + \langle \nabla g(x), Du_0 \rangle \\ &\leq \|c\|_{L^\infty} \|Du_0\|_{L^\infty} + \|\nabla g\|_{L^\infty} \|Du_0\|_{L^\infty} \\ &\quad + \sup_{x \in \mathbb{R}^2} (-g(x)H^*(Du_0,D^2u_0)) \\ &\leq C_1 = (u^+)_t, \end{aligned}$$

if we choose

$$C_1 \geq \|c\|_{L^\infty} \|Du_0\|_{L^\infty} + \sup_{x \in \mathbb{R}^2} (-g(x)H^*(Du_0,D^2u_0)) + \|\nabla g\|_{L^\infty} \|Du_0\|_{L^\infty}.$$

□

Theorem 3 (Existence/Uniqueness).

Assume (H1)-(H2) and that $u_0 \in W^{2,\infty}(\mathbb{R}^2)$. Then there exists a unique bounded continuous solution of (2.2) on $\mathbb{R}^2 \times [0, T)$. Moreover, the solution satisfies for $(x, t) \in \mathbb{R}^2 \times [0, T)$,

$$u_0(x) - C_1 t \leq u(x, t) \leq u_0(x) + C_1 t,$$

where C_1 is defined in Proposition 1.

Proof. This is a direct application of the Perron's method (see [5]) joint to the comparison principle (Theorem 2).

□

2.2 Regularity results

We now prove that the solution of problem (2.2) is Lipschitz continuous in space and time, and derive a lower bound on the partial derivative $\frac{\partial u}{\partial x_2}$.

Theorem 4 (Lipschitz regularity in space).

Assume (H1)-(H2) and that $\|Du_0\|_{L^\infty(\mathbb{R}^2)} \leq B_0$ with $B_0 > 0$. Then the solution of (2.2) is Lipschitz continuous in space and satisfies:

$$\|Du(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq B(t),$$

with $B(t) = e^{C_2 t} B_0$ and $C_2 = L_c + L_g + 5L_g^2$.

Proof.

The function u is bounded and continuous on $\mathbb{R}^2 \times [0, T)$. We set

$$\Phi^\epsilon(x, y, t) = B(t)(|x - y|^2 + \epsilon^2)^{\frac{1}{2}}$$

and aim at proving that $u(x, t) - u(y, t) \leq \Phi^\epsilon(x, y, t)$.

Let us set

$$M = \sup_{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, t \in [0, T)} (u(x, t) - u(y, t) - \Phi^\epsilon(x, y, t)).$$

Let us assume by contradiction that $M > 0$. Then we denote by \bar{M} :

$$\bar{M} = \sup_{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, t \in [0, T)} \left\{ u(x, t) - u(y, t) - \Phi^\epsilon(x, y, t) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{\gamma}{T-t} \right\}.$$

For α, γ small enough, we have $\bar{M} \geq \frac{M}{2} > 0$. The function u being bounded, the supremum is reached. Let $(\bar{x}, \bar{y}, \bar{t})$ be a point of maximum with obviously $\bar{x} \neq \bar{y}$. Note that $\frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2) \leq C$ (with a constant C depending only on $\|u\|_{L^\infty}$) and so $\lim_{\alpha \rightarrow 0} \alpha \bar{x} = \lim_{\alpha \rightarrow 0} \alpha \bar{y} = 0$.

Let us prove that $\bar{t} > 0$. Assuming the contrary, we have:

$$\begin{aligned} 0 < \frac{M}{2} &\leq u_0(\bar{x}) - u_0(\bar{y}) - \Phi^\epsilon(\bar{x}, \bar{y}, 0), \\ &\leq u_0(\bar{x}) - u_0(\bar{y}) - B_0(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}}, \\ &\leq u_0(\bar{x}) - u_0(\bar{y}) - B_0|\bar{x} - \bar{y}|, \end{aligned}$$

which contradicts the assumptions on u_0 .

A mere calculus gives:

$$\bar{p} = D_x \Phi^\epsilon(\bar{x}, \bar{y}, \bar{t}) = B(\bar{t}) \frac{(\bar{x} - \bar{y})}{(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}}} = -D_y \Phi^\epsilon(\bar{x}, \bar{y}, \bar{t}) \neq 0,$$

$$\partial_t \Phi^\epsilon(\bar{x}, \bar{y}, \bar{t}) = B'(\bar{t})(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} = C_2 e^{C_2 \bar{t}} B_0 (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}},$$

and

$$A = D^2 \Phi^\epsilon(\bar{x}, \bar{y}, \bar{t}) = B(\bar{t}) \begin{pmatrix} U & -U \\ -U & U \end{pmatrix},$$

with $U = (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{1}{2}} I - (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{3}{2}} (\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y})$.

Using the parabolic version of Ishii's lemma with $\Phi(x, y, t) = \Phi^\epsilon(x, y, t) + \frac{\gamma}{T-t}$,

for all $\eta > 0$, there exist $\tau_1, \tau_2 \in \mathbb{R}$ and $X, Y \in \mathcal{S}(2)$ such that:

$$\begin{aligned}\tau_1 - \tau_2 &= B'(\bar{t}) (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} + \frac{\gamma}{(T - \bar{t})^2}, \\ (\tau_1, \bar{p} + \alpha \bar{x}, X + \alpha I) &\in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}), \quad (\tau_2, \bar{p} - \alpha \bar{y}, Y - \alpha I) \in \bar{\mathcal{P}}^- u(\bar{y}, \bar{t}), \\ -\left(\frac{1}{\eta} + \|A\|\right) I &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \eta A^2.\end{aligned}$$

So the following holds:

$$\tau_1 \leq c(\bar{x}, \bar{t}) |\bar{p} + \alpha \bar{x}| - F_*(\bar{x}, \bar{p} + \alpha \bar{x}, X + \alpha I),$$

and

$$\tau_2 \geq c(\bar{y}, \bar{t}) |\bar{p} - \alpha \bar{y}| - F^*(\bar{y}, \bar{p} - \alpha \bar{y}, Y - \alpha I).$$

By subtracting the two previous inequalities, it yields:

$$\begin{aligned}B'(\bar{t}) (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} + \frac{\gamma}{T^2} &\leq c(\bar{x}, \bar{t}) |\bar{p} + \alpha \bar{x}| - c(\bar{y}, \bar{t}) |\bar{p} - \alpha \bar{y}| + \langle \nabla g(\bar{x}) - \nabla g(\bar{y}), \bar{p} \rangle \\ &\quad + \langle \nabla g(\bar{x}), \alpha \bar{x} \rangle + \langle \nabla g(\bar{y}), \alpha \bar{y} \rangle \\ &\quad - g(\bar{x}) H_*(\bar{p} + \alpha \bar{x}, X + \alpha I) + g(\bar{y}) H^*(\bar{p} - \alpha \bar{y}, Y - \alpha I), \\ &\leq \alpha (\|c\|_{L^\infty} + L_g) (\bar{x} + \bar{y}) + (c(\bar{x}, \bar{t}) - c(\bar{y}, \bar{t})) |\bar{p}| \\ &\quad + L_g \frac{B(\bar{t}) |\bar{x} - \bar{y}|^2}{(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}}} \\ &\quad - g(\bar{x}) [H_*(\bar{p} + \alpha \bar{x}, X + \alpha I) - H_*(\bar{p}, X)] \\ &\quad + g(\bar{y}) [H^*(\bar{p} - \alpha \bar{y}, Y - \alpha I) - H^*(\bar{p}, Y)] \\ &\quad - g(\bar{x}) H(\bar{p}, X) + g(\bar{y}) H(\bar{p}, Y).\end{aligned}\tag{2.3}$$

We choose $\eta = \frac{1}{B(\bar{t})} (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{1/2} > 0$, so $A + \eta A^2$ can be written:

$$A + \eta A^2 = B(\bar{t}) \begin{pmatrix} C & -C \\ -C & C \end{pmatrix}$$

with

$$\begin{aligned}C &= 3(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{1}{2}} I - 5(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{3}{2}} (\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y}) \\ &\quad + 2|\bar{x} - \bar{y}|^2 (|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{5}{2}} (\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y}).\end{aligned}$$

Moreover, denoting by $E(p)$ the matrix defined by $E(p) = I - \frac{p \otimes p}{|p|^2}$, $p \in \mathbb{R}^2 - \{0_{\mathbb{R}^2}\}$, one has:

$$\begin{aligned} & g(\bar{x})\text{trace}(E(\bar{p})X) - g(\bar{y})\text{trace}(E(\bar{p})Y) \\ &= \text{trace} \left(\underbrace{\begin{pmatrix} g(\bar{x})E(\bar{p}) & \sqrt{g(\bar{x})g(\bar{y})}E(\bar{p}) \\ \sqrt{g(\bar{x})g(\bar{y})}E(\bar{p}) & g(\bar{y})E(\bar{p}) \end{pmatrix}}_G \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \right). \end{aligned}$$

The matrix G is symmetric semi-positive definite so it can be decomposed into $G = \chi\chi^T$. This implies that

$$\text{trace} \left(G \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \right) = \text{trace} \left(\chi^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \chi \right),$$

so

$$\begin{aligned} \text{trace} \left(G \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \right) &\leq B(\bar{t})\text{trace} \left(G \begin{pmatrix} C & -C \\ -C & C \end{pmatrix} \right), \\ &\leq B(\bar{t})(\sqrt{g(\bar{x})} - \sqrt{g(\bar{y})})^2 \text{trace}(E(\bar{p})C), \\ &\leq B(\bar{t})L_g^2|\bar{x} - \bar{y}|^2 \text{trace}(E(\bar{p})C). \end{aligned}$$

A mere calculus gives that

$$\text{trace}(E(\bar{p})C) \leq 3(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{1}{2}} + 2|\bar{x} - \bar{y}|^4(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{5}{2}},$$

so finally

$$\begin{aligned} g(\bar{x})\text{trace}(E(\bar{p})X) - g(\bar{y})\text{trace}(E(\bar{p})Y) &\leq 3L_g^2B(\bar{t})|\bar{x} - \bar{y}|^2(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{1}{2}} \\ &\quad + 2L_g^2B(\bar{t})|\bar{x} - \bar{y}|^6(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{5}{2}}, \\ &\leq 5L_g^2B(\bar{t})|\bar{x} - \bar{y}|^2(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{-\frac{1}{2}}. \end{aligned}$$

Thus, from relation (2.3),

$$\begin{aligned} \frac{\gamma}{T^2} + B'(\bar{t})(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}} &\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) \\ &\quad + (L_c + L_g + 5L_g^2)B(\bar{t})\frac{|\bar{x} - \bar{y}|^2}{(|\bar{x} - \bar{y}|^2 + \epsilon^2)^{\frac{1}{2}}} \\ &\quad - g(\bar{x})[H_*(\bar{p} + \alpha\bar{x}, X + \alpha I) - H_*(\bar{p}, X)] \\ &\quad + g(\bar{y})[H^*(\bar{p} - \alpha\bar{y}, Y - \alpha I) - H^*(\bar{p}, Y)]. \end{aligned} \quad (2.4)$$

Using the fact that $B'(t) = (L_c + L_g + 5L_g^2)B(t)$, we deduce that:

$$\begin{aligned} \frac{\gamma}{T^2} &\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) \\ &\quad - g(\bar{x})[H_*(\bar{p} + \alpha\bar{x}, X + \alpha I) - H_*(\bar{p}, X)] \\ &\quad + g(\bar{y})[H^*(\bar{p} - \alpha\bar{y}, Y - \alpha I) - H^*(\bar{p}, Y)]. \end{aligned}$$

It is easy to see that $|\bar{p}| \leq B(T)$. Also, it can be checked that:

$$-\frac{3}{\epsilon}B(T)I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{10}{\epsilon}B(T)I.$$

We let α go to 0 in (2.4): \bar{p} , X and Y are bounded independently of α so we can extract a convergent subsequence $\bar{p} \rightarrow p_\infty$, $X \rightarrow X_\infty$ and $Y \rightarrow Y_\infty$. Using the continuity of u and the fact that $u(\bar{x}, \bar{t}) - u(\bar{y}, \bar{t}) \geq \frac{\gamma}{T - \bar{t}} \geq \frac{\gamma}{T}$, we deduce that $p_\infty \neq 0$. Sending α to 0 in the previous inequality and using the fact that g is bounded, it yields:

$$\frac{\gamma}{T^2} \leq 0,$$

which is absurd. So

$$u(x, t) - u(y, t) \leq \Phi^\epsilon(x, y, t)$$

and letting ϵ tend to 0, we obtain:

$$u(x, t) - u(y, t) \leq B(t)|x - y|.$$

Exchanging x and y yields:

$$|u(x, t) - u(y, t)| \leq B(t)|x - y|.$$

□

Proposition 2 (Lipschitz regularity in time).

Let $u_0 \in W^{2, \infty}(\mathbb{R}^2)$. Then the solution u of (2.2) is Lipschitz continuous in time and satisfies:

$$\|u_t(x, \cdot)\|_{L^\infty(0, T)} \leq C_1 + L_{c_t} \int_0^T B(s) ds,$$

where C_1 is defined in Proposition 1.

Proof. We recall, from Theorem 3, that

$$|u(x, t) - u_0(x)| \leq C_1 t.$$

Let $h > 0$ be such that $t + h \leq T$. We denote by

$$M = \sup_{x \in \mathbb{R}^2} |u(x, h) - u_0(x)| \leq C_1 h$$

and

$$u_h(x, t) = u(x, t + h) - L_{c_t} h \int_0^{t+h} B(s) ds - M.$$

Then u_h is still a sub-solution of (2.2). Indeed, formally, we have

$$\begin{aligned} (u_h)_t(x, t) &= u_t(x, t + h) - L_{c_t} h B(t + h), \\ &= c(x, t + h) |Du(x, t + h)| - F(x, Du(x, t + h), D^2 u(x, t + h)) - L_{c_t} h B(t + h), \\ &\leq c(x, t) |Du_h(x, t)| - F(x, Du_h(x, t), D^2 u_h(x, t)). \end{aligned}$$

Hence, using the comparison principle, one has $u_h(x, t) \leq u(x, t)$, that is

$$u(x, t+h) - u(x, t) \leq M + L_{c_t} h \int_0^{t+h} B(s) ds \leq C_1 h + L_{c_t} h \int_0^T B(s) ds.$$

Similarly, one obtains that

$$|u(x, t+h) - u(x, t)| \leq C_1 h + L_{c_t} h \int_0^T B(s) ds.$$

In conclusion, u is Lipschitz continuous in time with Lipschitz constant equal to $C_1 + L_{c_t} \int_0^T B(s) ds$. \square

We now turn to the prescribing of a lower bound on the gradient. We need the following lemma:

Lemma 1 (Estimate on the curvature).

Let $(p, Y) \in \mathbb{R}^2 \times \mathcal{S}^2$ such that $\exists \tau \in \mathbb{R}$ such that $(\tau, p, Y) \in \bar{\mathcal{P}}^- u(y, t)$ (respectively $(\tau, p, Y) \in \bar{\mathcal{P}}^+ u(y, t)$). Then

$$-H(p, Y) \leq \frac{C_1 + L_{c_t} T B(T) + \|c\|_{L^\infty(\mathbb{R}^2 \times [0, T])} B(T) + \|\nabla g\|_{L^\infty(\mathbb{R}^2)} B(T)}{\delta} =: C_3,$$

$$\left(\text{resp. } H(p, Y) \leq C_3 \right),$$

where C_1 denotes the Lipschitz constant in time of u and $B(\cdot)$ is defined in Theorem 4.

Proof. We only do the proof for $(\tau, p, Y) \in \bar{\mathcal{P}}^- u(y, t)$, the other one being similar. By definition,

$$\tau - c(y, t)|p| + g(y)H(p, Y) - \langle \nabla g(y), p \rangle \geq 0.$$

That is,

$$g(y)H(p, Y) \geq -\tau + c(y, t)|p| + \langle \nabla g(y), p \rangle.$$

But $-\tau \geq -C_1 - L_{c_t} T B(T)$ and $|p| \leq B(t) \leq B(T)$. Consequently,

$$g(y)H(p, Y) \geq -C_1 - L_{c_t} T B(T) - \|c\|_{L^\infty(\mathbb{R}^2 \times [0, T])} B(T) - \|\nabla g\|_{L^\infty(\mathbb{R}^2)} B(T),$$

and

$$-H(p, Y) \leq \frac{C_1 + L_{c_t} T B(T) + \|c\|_{L^\infty(\mathbb{R}^2 \times [0, T])} B(T) + \|\nabla g\|_{L^\infty(\mathbb{R}^2)} B(T)}{\delta}.$$

\square

Theorem 5 (Lower bound on the gradient).

Let $u_0 \in W^{2,\infty}(\mathbb{R}^2)$ be such that $\frac{\partial u_0}{\partial x_2} \geq b_0$ with $b_0 > 0$. Then the solution of (2.2) satisfies:

$$\frac{\partial u}{\partial x_2} \geq b(t),$$

with $b(t) = b_0 - 2 \left(L_c + L_g \right) \frac{B_0}{C_2} \left(e^{C_2 t} - 1 \right) - L_g C_3 t$, where C_2 and C_3 are defined respectively in Theorem 4 and Lemma 1.

Proof. We aim to prove that for $x_n < y_n$, $u(x', y_n, t) - u(x', x_n, t) \geq b(t)(y_n - x_n)$. In this purpose, let us introduce

$$M = \sup_{(x', x_n, y_n, t) | x_n < y_n} \{ u(x', x_n, t) - u(x', y_n, t) - b(t)(x_n - y_n) \},$$

and let us prove that $M \leq 0$. We argue by contradiction. Let us assume that $M > 0$. Then we set:

$$\bar{M} = \sup_{(x', x_n, y', y_n, t) | x_n < y_n} \left\{ u(x', x_n, t) - u(y', y_n, t) - b(t)(x_n - y_n) - \frac{|x' - y'|^2}{2\epsilon} - \frac{\gamma}{T-t} - \frac{\alpha}{2} (|x|^2 + |y|^2) \right\}.$$

For α and γ small enough, $\bar{M} \geq \frac{M}{2} > 0$. Moreover, the solution u being bounded, the supremum is reached in $(x', \bar{x}_n, \bar{y}', \bar{y}_n, \bar{t})$.

Let us prove that $\bar{t} \neq 0$. By contradiction, let us assume that $\bar{t} = 0$. We then have:

$$\begin{aligned} 0 < \frac{M}{2} \leq \bar{M} &\leq u_0(\bar{x}', \bar{x}_n) - u_0(\bar{y}', \bar{y}_n) - b_0(\bar{x}_n - \bar{y}_n) - \frac{|\bar{x}' - \bar{y}'|^2}{2\epsilon}, \\ &\leq B_0 |\bar{x}' - \bar{y}'| - \frac{|\bar{x}' - \bar{y}'|^2}{2\epsilon} + u_0(\bar{y}', \bar{x}_n) - u_0(\bar{y}', \bar{y}_n) - b_0(\bar{x}_n - \bar{y}_n). \end{aligned}$$

A study of the function $h: r \mapsto B_0 r - \frac{r^2}{2\epsilon}$ gives us that it is bounded by $\frac{B_0^2 \epsilon}{2}$. Thus,

$$0 < \frac{M}{2} \leq \bar{M} \leq \frac{B_0^2 \epsilon}{2} + u_0(\bar{y}', \bar{x}_n) - u_0(\bar{y}', \bar{y}_n) - b_0(\bar{x}_n - \bar{y}_n),$$

where we have that $u_0(\bar{y}', \bar{x}_n) - u_0(\bar{y}', \bar{y}_n) - b_0(\bar{x}_n - \bar{y}_n) \leq 0$ according to the assumptions on u_0 . We clearly raise a contradiction for ϵ small enough, so $\bar{t} \neq 0$.

Let us now prove that $\bar{x}_n \neq \bar{y}_n$. We have:

$$\begin{aligned}
0 < \frac{M}{2} &\leq \bar{M} = u(\bar{x}', \bar{x}_n, \bar{t}) - u(\bar{y}', \bar{x}_n, \bar{t}) + u(\bar{y}', \bar{x}_n, \bar{t}) - u(\bar{y}', \bar{y}_n, \bar{t}) - b(\bar{t})(\bar{x}_n - \bar{y}_n) \\
&\quad - \frac{|\bar{x}' - \bar{y}'|^2}{2\epsilon} - \frac{\gamma}{T - \bar{t}} - \frac{\alpha}{2} (|\bar{x}|^2 + |\bar{y}|^2), \\
&\leq B(\bar{t})|\bar{x}' - \bar{y}'| - \frac{|\bar{x}' - \bar{y}'|^2}{2\epsilon} + u(\bar{y}', \bar{x}_n, \bar{t}) - u(\bar{y}', \bar{y}_n, \bar{t}) - b(\bar{t})(\bar{x}_n - \bar{y}_n), \\
&\leq \frac{B(\bar{t})^2\epsilon}{2} + u(\bar{y}', \bar{x}_n, \bar{t}) - u(\bar{y}', \bar{y}_n, \bar{t}) - b(\bar{t})(\bar{x}_n - \bar{y}_n).
\end{aligned}$$

Thus, for ϵ small enough,

$$u(\bar{y}', \bar{x}_n, \bar{t}) - u(\bar{y}', \bar{y}_n, \bar{t}) - b(\bar{t})(\bar{x}_n - \bar{y}_n) \geq \frac{M}{3}.$$

Consequently, $\bar{x}_n \neq \bar{y}_n$.

We consider $\Phi(x, y, t) = b(t)(x_n - y_n) + \frac{|x' - y'|^2}{2\epsilon} + \frac{\gamma}{T - t}$ and we set $\bar{p} = \bar{x}' - \bar{y}'$.

We use the parabolic version of Ishii's lemma and we set:

$$\left\{ \begin{array}{l} p_1 = D_x \Phi(\bar{x}, \bar{y}, \bar{t}) = p_2 = -D_y \Phi(\bar{x}, \bar{y}, \bar{t}) = \begin{pmatrix} \epsilon^{-1} \bar{p} \\ b(\bar{t}) \end{pmatrix} \neq 0 \text{ for } T \text{ small enough,} \\ A = D^2 \Phi(\bar{x}, \bar{y}, \bar{t}) = \begin{pmatrix} \frac{1}{\epsilon} & 0 & -\frac{1}{\epsilon} & 0 \\ \frac{\epsilon}{0} & 0 & \frac{\epsilon}{0} & 0 \\ -\frac{1}{\epsilon} & 0 & \frac{1}{\epsilon} & 0 \\ 0 & \frac{\epsilon}{0} & \frac{\epsilon}{0} & 0 \end{pmatrix}. \end{array} \right.$$

Then for all $\eta > 0$, there exist X et Y such that:

$$\left\{ \begin{array}{l} \tau_1 - \tau_2 = b'(\bar{t})(\bar{x}_n - \bar{y}_n) + \frac{\gamma}{(T - \bar{t})^2}, \\ (\tau_1, p_1 + \alpha \bar{x}, X + \alpha I) \in \bar{\mathcal{P}}^+ u(\bar{x}, \bar{t}), \\ (\tau_2, p_1 - \alpha \bar{y}, Y - \alpha I) \in \bar{\mathcal{P}}^- u(\bar{y}, \bar{t}), \\ -\left(\frac{1}{\eta} + \|A\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \eta A^2 = \begin{pmatrix} \frac{1}{\epsilon} + \frac{2\eta}{\epsilon^2} & 0 & -\frac{1}{\epsilon} - \frac{2\eta}{\epsilon^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\epsilon} - \frac{2\eta}{\epsilon^2} & 0 & \frac{1}{\epsilon} + \frac{2\eta}{\epsilon^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{array} \right.$$

Because u is a subsolution and v is a supersolution,

$$\begin{aligned}
\tau_1 - c(\bar{x}, \bar{t})|p_1 + \alpha \bar{x}| + F_*(\bar{x}, p_1 + \alpha \bar{x}, X + \alpha I) &\leq 0, \\
\tau_2 - c(\bar{y}, \bar{t})|p_1 - \alpha \bar{y}| + F^*(\bar{y}, p_1 - \alpha \bar{y}, Y - \alpha I) &\geq 0.
\end{aligned}$$

Then, subtracting the two previous inequalities yields:

$$\begin{aligned}
b'(\bar{t})(\bar{x}_n - \bar{y}_n) + \frac{\gamma}{T^2} &\leq c(\bar{x}, \bar{t})|p_1 + \alpha \bar{x}| - c(\bar{y}, \bar{t})|p_1 - \alpha \bar{y}| \\
&\quad + g(\bar{y})H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - g(\bar{x})H_*(p_1 + \alpha \bar{x}, X + \alpha I) \\
&\quad + \langle \nabla g(\bar{x}) - \nabla g(\bar{y}), p_1 \rangle + \langle \nabla g(\bar{y}), \alpha \bar{y} \rangle + \langle \nabla g(\bar{x}), \alpha \bar{x} \rangle, \\
&\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) + \left(c(\bar{x}, \bar{t}) - c(\bar{y}, \bar{t}) \right) |p_1| \\
&\quad + g(\bar{y})[H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H^*(p_1, Y)] \\
&\quad + g(\bar{x})[H_*(p_1, X) - H_*(p_1 + \alpha \bar{x}, X + \alpha I)] \\
&\quad + g(\bar{y})H^*(p_1, Y) - g(\bar{x})H_*(p_1, X) + L_g|\bar{x} - \bar{y}||p_1|.
\end{aligned}$$

Let us assume that $g(\bar{y}) \leq g(\bar{x})$. In this case,

$$\begin{aligned}
&g(\bar{y})H(p_1, Y) - g(\bar{x})H(p_1, X) \\
&= \underbrace{(g(\bar{x}) - g(\bar{y}))}_{\geq 0} \underbrace{(-H(p_1, Y))}_{\leq C_3} + g(\bar{x}) \underbrace{(H(p_1, Y) - H(p_1, X))}_{\leq 0 \text{ since } X \leq Y}, \\
&\leq C_3 L_g |\bar{x} - \bar{y}|.
\end{aligned}$$

In the case where $g(\bar{y}) \geq g(\bar{x})$, we obtain the same result using the inequality $H(p_1, X) \leq C_3$. We then have:

$$\begin{aligned}
&b'(\bar{t})(\bar{x}_n - \bar{y}_n) + \frac{\gamma}{T^2} \\
&\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) + C_3 L_g (|\bar{x}' - \bar{y}'| + \bar{y}_n - \bar{x}_n) \\
&\quad + (L_c + L_g) \left(\frac{|\bar{x}' - \bar{y}'|^2}{\epsilon} + b(\bar{t})|\bar{x}' - \bar{y}'| + \frac{|\bar{x}' - \bar{y}'|}{\epsilon}(\bar{y}_n - \bar{x}_n) + b(\bar{t})(\bar{y}_n - \bar{x}_n) \right) \\
&\quad + g(\bar{y})[H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H^*(p_1, Y)] \\
&\quad + g(\bar{x})[H_*(p_1, X) - H_*(p_1 + \alpha \bar{x}, X + \alpha I)].
\end{aligned}$$

Moreover, since u is $B(t)$ -Lipschitz continuous in space, we have $|p_1| \leq B(\bar{t})$, hence $|\bar{x}' - \bar{y}'| \leq B(\bar{t})\epsilon$. Thus,

$$\begin{aligned}
&b'(\bar{t})(\bar{x}_n - \bar{y}_n) + \frac{\gamma}{T^2} \\
&\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) + C_3 L_g (B(T)\epsilon + \bar{y}_n - \bar{x}_n) \\
&\quad + (L_c + L_g) (B(T)^2\epsilon + b_0 B(T)\epsilon + B(\bar{t})(\bar{y}_n - \bar{x}_n) + b(\bar{t})(\bar{y}_n - \bar{x}_n)) \\
&\quad + g(\bar{y})[H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H^*(p_1, Y)] \\
&\quad + g(\bar{x})[H_*(p_1, X) - H_*(p_1 + \alpha \bar{x}, X + \alpha I)].
\end{aligned}$$

Taking ϵ sufficiently small, it yields:

$$\begin{aligned}
b'(\bar{t})(\bar{x}_n - \bar{y}_n) + \frac{\gamma}{2T^2} &\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) + C_3 L_g(\bar{y}_n - \bar{x}_n) \\
&\quad + (L_c + L_g)(B(\bar{t})(\bar{y}_n - \bar{x}_n) + b(\bar{t})(\bar{y}_n - \bar{x}_n)) \\
&\quad + g(\bar{y})[H^*(p_1 - \alpha\bar{y}, Y - \alpha I) - H^*(p_1, Y)] \\
&\quad + g(\bar{x})[H_*(p_1, X) - H_*(p_1 + \alpha\bar{x}, X + \alpha I)], \\
&\leq \alpha(\|c\|_{L^\infty} + L_g)(|\bar{x}| + |\bar{y}|) \\
&\quad + (2B(\bar{t})(L_c + L_g) + C_3 L_g)(\bar{y}_n - \bar{x}_n) \\
&\quad + g(\bar{y})[H^*(p_1 - \alpha\bar{y}, Y - \alpha I) - H^*(p_1, Y)] \\
&\quad + g(\bar{x})[H_*(p_1, X) - H_*(p_1 + \alpha\bar{x}, X + \alpha I)].
\end{aligned}$$

But $b'(\bar{t}) = -2B(\bar{t})(\|\nabla c\|_{L^\infty} + L_g) - C_3 L_g$ so,

$$\begin{aligned}
\frac{\gamma}{2T^2} &\leq \alpha(\|c\|_{L^\infty} + \|\nabla g\|_{L^\infty})(|\bar{x}| + |\bar{y}|) \\
&\quad + g(\bar{y})[H^*(p_1 - \alpha\bar{y}, Y - \alpha I) - H^*(p_1, Y)] \\
&\quad + g(\bar{x})[H_*(p_1, X) - H_*(p_1 + \alpha\bar{x}, X + \alpha I)]. \tag{2.5}
\end{aligned}$$

Remark that X and Y are bounded independently of α from the matrix inequality. This is also the case for p_1 . So there exists $\alpha_n \rightarrow 0$ such that $\bar{t} \rightarrow t_\infty$, $p_1 \rightarrow p_\infty$ and $(X, Y) \rightarrow (X_\infty, Y_\infty)$. Also, if we set

$$\begin{aligned}
M_\alpha &= \sup_{(x', x_n, y', y_n, t) | x_n < y_n} \left\{ u(x', x_n, t) - u(y', y_n, t) - b(t)(x_n - y_n) - \frac{|x' - y'|^2}{2\epsilon} \right. \\
&\quad \left. - \frac{\gamma}{T-t} - \frac{\alpha}{2}(|x|^2 + |y|^2) \right\}, \\
&= u(\bar{x}', \bar{x}_n, \bar{t}) - u(\bar{y}', \bar{y}_n, \bar{t}) - b(\bar{t})(\bar{x}_n - \bar{y}_n) - \frac{|\bar{x}' - \bar{y}'|^2}{2\epsilon} - \frac{\gamma}{T-\bar{t}} - \frac{\alpha}{2}(|\bar{x}|^2 + |\bar{y}|^2),
\end{aligned}$$

obviously $M_\alpha \leq M_0$ and $(M_\alpha)_\alpha$ increases when $\alpha \rightarrow 0$, so $(M_\alpha)_\alpha$ is convergent and we denote by L its limit. To finish, it can be easily seen that $M_{\frac{\alpha}{2}} - M_\alpha \geq \frac{\alpha}{4}(|\bar{x}|^2 + |\bar{y}|^2) \geq 0$. So by letting α tend to 0, it follows that $\lim_{\alpha \rightarrow 0} \alpha(|\bar{x}|^2 + |\bar{y}|^2) = 0$. Thus $\lim_{\alpha \rightarrow 0} \alpha\bar{x} = \lim_{\alpha \rightarrow 0} \alpha\bar{y} = 0$.

Sending α_n to 0 in (2.5) and using the fact that $\lim_{\alpha \rightarrow 0} \alpha\bar{x} = \lim_{\alpha \rightarrow 0} \alpha\bar{y} = 0$, $p_1 \neq 0$ and $p_\infty \neq 0$, it yields:

$$\frac{\gamma}{2T^2} \leq 0,$$

which is absurd. □

3 The non local problem

The space $BV(\mathbb{R}^2)$ is the space of bounded variation functions. Let us define by $L^1_{unif}(\mathbb{R}^2)$ the space:

$$L^1_{unif}(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \rightarrow \mathbb{R}, \|f\|_{L^1_{unif}(\mathbb{R}^2)} < \infty \right\},$$

with

$$\|f\|_{L^1_{unif}(\mathbb{R}^2)} = \sup_{x \in \mathbb{R}^2} \int_{Q(x)} |f|,$$

and with $Q(x)$ the unit square centered at x : $Q(x) = \left\{ x' \in \mathbb{R}^2, |x_i - x'_i| \leq \frac{1}{2} \right\}$,

and by $L^\infty_{int}(\mathbb{R}^2)$ the space:

$$L^\infty_{int}(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \rightarrow \mathbb{R}, \|f\|_{L^\infty_{int}(\mathbb{R}^2)} < \infty \right\},$$

with $\|f\|_{L^\infty_{int}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \|f\|_{L^\infty(Q(x))}$

Theorem 6 (Short time existence and uniqueness).

Assume (H1)-(H2) and let $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a $W^{2,\infty}(\mathbb{R}^2)$ function such that:

$$|Du_0| < B_0 \text{ in } \mathbb{R}^2 \quad \text{and} \quad \frac{\partial u_0}{\partial x_n} > b_0 > 0 \text{ in } \mathbb{R}^2.$$

Let c_0 satisfies $c_0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$. Then there exists a unique viscosity solution of problem (1.1) in $\mathbb{R}^2 \times [0, T^*)$ with:

$$T^* = \inf \left(\frac{\ln \left(\frac{b_0 C_2}{8 B_0 (|c_0|_{BV} + L_g)} + 1 \right)}{C_2}, \frac{b_0}{4 C_4}, \frac{b_0}{16 B_0 \|c_0\|_{L^\infty_{int}(\mathbb{R}^2)}}, \frac{\ln 2}{C_2} \right),$$

where

$$C_4 = \frac{L_g}{\delta} (2C_1 + 2\|c_0\|_{L^1} B_0 + 2\|\nabla g\|_{L^\infty} B_0) \quad \text{and} \quad C_2 = |c_0|_{BV} + 5L_g^2 + L_g.$$

Moreover, the solution satisfies:

$$\begin{aligned} |Du(x, t)| &\leq 2B_0 \text{ on } \mathbb{R}^2 \times [0, T^*), & \frac{\partial u}{\partial x_2}(x, t) &> \frac{b_0}{2} > 0 \text{ on } \mathbb{R}^2 \times [0, T^*), \\ |u_t(x, t)| &\leq 2C_1 \text{ on } \mathbb{R}^2 \times [0, T^*). \end{aligned}$$

We need the three following lemmata.

Lemma 2 (Estimate on the characteristic functions).
Let $u^1 \in \mathcal{C}(\mathbb{R}^2)$ satisfying

$$\frac{\partial u^1}{\partial x_2} \geq b$$

in the distribution sense for some $b > 0$ and $u^2 \in L_{loc}^\infty(\mathbb{R}^2)$ satisfying the same condition. Then, we have the following estimate:

$$\|[u^2] - [u^1]\|_{L_{unif}^1} \leq \frac{2}{b} \|u^2 - u^1\|_{L^\infty}.$$

Lemma 3 (Convolution inequality).

For every $f \in L_{unif}^1(\mathbb{R}^2)$ and $g \in L_{int}^\infty(\mathbb{R}^2)$, the convolution product $f * g$ is bounded and satisfies:

$$\|f * g\|_{L^\infty(\mathbb{R}^2)} \leq \|f\|_{L_{unif}^1(\mathbb{R}^2)} \|g\|_{L_{int}^\infty(\mathbb{R}^2)}.$$

A proof of these two lemmata can be found respectively in [2] and [3].

Lemma 4 (Stability of the solution with respect to the velocity).

Let $T > 0$. We consider for $i = 1, 2$ two different equations:

$$\begin{cases} u_t^i = c^i(x, t) |Du^i| - F(x, Du^i, D^2u^i) & \text{in } \mathbb{R}^2 \times (0, T), \\ u^i(x, 0) = u_0(x), \end{cases} \quad (3.6)$$

c^i , u_0 and F satisfying the previous assumptions. Then, for every $t \in [0, T)$, we have:

$$\|u^1(\cdot, t) - u^2(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^2 \times (0, T))} \int_0^t B(s) ds,$$

where u^i are the solutions of (3.6), $B(t) = B_0 e^{(L_c + 5L_g^2 + L_g)t}$ with $L_c = \sup_i L_{c^i}$.

Proof. We set $K = \|c^1 - c^2\|_{L^\infty(\mathbb{R}^2 \times (0, T))}$. We remark that u^1 is a subsolution of

$$u_t - c^2(x, t) |Du| + F(x, Du, D^2u) - KB(t) = 0.$$

Indeed, we have:

$$\begin{aligned} u_t^1 - c^2(x, t) |Du^1| + F(x, Du^1, D^2u^1) &= c^1(x, t) |Du^1| - F(x, Du^1, D^2u^1) \\ &\quad - c^2(x, t) |Du^1| + F(x, Du^1, D^2u^1), \\ &\leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^2 \times (0, T))} B(t), \\ &\leq KB(t). \end{aligned}$$

This differential inequality holds in the viscosity sense. Moreover, the function $u^2 + K \int_0^t B(s) ds$ is solution of the same problem. By the comparison principle, we deduce that:

$$u^1 \leq u^2 + K \int_0^t B(s) ds.$$

Switching the role of u^1 and u^2 , it yields:

$$\|u^1(\cdot, t) - u^2(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq \|c^1 - c^2\|_{L^\infty(\mathbb{R}^2 \times (0, T))} \int_0^t B(s) ds.$$

□

It now brings us to the proof of Theorem 6.

Proof of Theorem 6. We define the space

$$E = \left\{ u \in L_{loc}^\infty(\mathbb{R}^2 \times [0, T^*)) \text{ s.t. } \begin{cases} |Du(x, t)| \leq 2B_0 \\ \frac{\partial u}{\partial x_n}(x, t) \geq \frac{b_0}{2} \\ |u_t(x, t)| \leq 2C_1 \end{cases} \right\},$$

where C_1 is defined in Proposition 1.

For $u \in E$, we set $c(x, t) = (c_0 * [u(\cdot, t)])(x)$. This function is bounded, Lipschitz continuous in space (with $L_c = |c_0|_{BV}$) and time (with $L_{c_t} = \frac{8C_1 \|c_0\|_{L_{int}^\infty}}{b_0}$). Indeed,

$$\begin{aligned} \|c\|_{L^\infty(\mathbb{R}^2 \times [0, T^*))} &\leq \sup_{t \in \mathbb{R}} \|c_0\|_{L^1} \| [u(\cdot, t)] \|_{L^\infty(\mathbb{R}^2)}, \\ &\leq \|c_0\|_{L^1}. \end{aligned}$$

Moreover, for every t ,

$$\begin{aligned} \|Dc(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} &= \|Dc_0 * [u(\cdot, t)]\|_{L^\infty(\mathbb{R}^2)}, \\ &\leq |c_0|_{BV} \| [u(\cdot, t)] \|_{L^\infty(\mathbb{R}^2)}, \\ &\leq |c_0|_{BV}. \end{aligned}$$

Finally, for $0 < t, s < T^*$:

$$\begin{aligned} |c(x, t) - c(x, s)| &= |c_0 * [u(\cdot, t)](x) - c_0 * [u(\cdot, s)](x)|, \\ &= |c_0 * ([u(x, t)] - [u(x, s)])(x)|, \\ &\leq \|c_0\|_{L_{int}^\infty} \| [u(\cdot, t)] - [u(\cdot, s)] \|_{L_{unif}^1(\mathbb{R}^2)}, \\ &\leq \frac{4 \|c_0\|_{L_{int}^\infty}}{b_0} \|u(\cdot, t) - u(\cdot, s)\|_{L^\infty(\mathbb{R}^2)}, \\ &\leq \frac{8C_1 \|c_0\|_{L_{int}^\infty}}{b_0} |t - s|. \end{aligned}$$

For $u \in E$, we then define $v = \Phi(u)$ as the unique viscosity solution of:

$$\begin{cases} v_t = (c_0 * [u]) |Dv| - F(x, Dv, D^2v) & \text{in } \mathbb{R}^2 \times (0, T^*), \\ v(x, t=0) = u_0(x) & \text{in } \mathbb{R}^2. \end{cases}$$

We show that $\Phi : E \rightarrow E$ is a contraction. First, we show that Φ is well-defined. We have:

$$\|Dv(\cdot, t)\| \leq B(t) \leq B_0 e^{(L_c + 5L_g^2 + L_g)T^*} \leq 2B_0,$$

by definition of T^* .

Moreover, by Proposition 2, v is Lipschitz continuous in time and satisfies

$$\begin{aligned} \|v_t\|_{L^\infty} &\leq C_1 + L_{c_t} T^* B(T^*) \leq C_1 + 2L_{c_t} B_0 T^* \\ &\leq C_1 \left(1 + \frac{16B_0 \|c_0\|_{L_{int}^\infty} T^*}{b_0} \right) \leq 2C_1 \end{aligned}$$

by definition of T^* . Finally, by Theorem 5, we have

$$\frac{\partial v}{\partial x_2} \geq b(t) \geq b_0 - 2(L_c + L_g) \frac{B_0}{C_2} (e^{C_2 t} - 1) - C_4 t,$$

where $C_2 = L_c + 5L_g^2 + L_g$ and $C_4 = \frac{L_g}{\delta} (2C_1 + 2\|c_0\|_{L^1} B_0 + 2\|\nabla g\|_{L^\infty} B_0)$. To ensure that $\frac{\partial v}{\partial x_n} \geq \frac{b_0}{2}$, it suffices to have that:

$$C_4 T^* \leq \frac{b_0}{4}$$

and

$$2(L_c + L_g) \frac{B_0}{C_2} (e^{C_2 T^*} - 1) \leq \frac{b_0}{4}.$$

This two inequalities are true owing to the choice of T^* .

It thus remains to be shown that Φ is a contraction. For $v^i = \Phi(u^i)$, according to Lemmata 3 and 4, we have:

$$\begin{aligned} \|v^2 - v^1\|_{L^\infty(\mathbb{R}^2 \times (0, T^*))} &\leq 2B_0 T^* \|c_0 * [u^2] - c_0 * [u^1]\|_{L^\infty(\mathbb{R}^2 \times (0, T^*))}, \\ &\leq 2B_0 T^* \|c_0\|_{L_{int}^\infty} \sup_{t \in (0, T^*)} \|[u^2(\cdot, t)] - [u^1(\cdot, t)]\|_{L_{unif}^1}, \\ &\leq \frac{8B_0 T^*}{b_0} \|c_0\|_{L_{int}^\infty} \|u^2 - u^1\|_{L^\infty(\mathbb{R}^2 \times (0, T^*))}, \\ &\leq \frac{1}{2} \|u^2 - u^1\|_{L^\infty(\mathbb{R}^2 \times (0, T^*))}. \end{aligned}$$

In conclusion, Φ is a contraction on E which is a closed set for the L^∞ -topology. So there exists a unique viscosity solution to the problem in E on $(0, T^*)$. \square

We are now able to prove the short time existence and uniqueness result of Problem (1.1).

Proof of Theorem 1. We recall that

$$c_0 : \begin{cases} \mathbb{R}^2 \rightarrow \mathbb{R} \\ x \mapsto \frac{4\mu}{d^2} \left(2 - \frac{2}{d^2} \|x\|_2^2 \right) \exp \left(-\frac{\|x\|_2^2}{d^2} \right). \end{cases}$$

It is easy to check that $c_0 \in L^1(\mathbb{R}^2)$. It is also $\mathcal{C}^1(\mathbb{R}^2)$ so its total variation $J(c_0)$ is defined by:

$$J(c_0) = \int_{\mathbb{R}^2} |\nabla c_0| dx.$$

It is obvious that $J(c_0) < +\infty$. Consequently, $c_0 \in BV(\mathbb{R}^2)$. To finish, using inequalities, it can be proved that $c_0 \in L_{int}^\infty(\mathbb{R}^2)$. Hence we can apply Theorem 6 to get the desired result. □

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