Stefano Olla

Central Limit Theorems for Tagged Particles and for Diffusions in Random Environment

Notes of the course given at
Etats de la recherche: Milieux Aléatoires
CIRM, Luminy
23-25 November 2000

Correspondence to: Stefano Olla
Université de Cergy Pontoise
Département de Mathématiques
2 Av.Adolphe Chauvin
B.P.222, Pontoise
95302 Cergy-Pontoise-Cedex, France
e-mail: olla@math.u-cergy.fr
http://www.cmap.polytechnique.fr/~olla

Abstract

We will review here some aspects of the approach based on *the point of view of the particle* in the study of random motions in random environments.

Résumé: Nous exposons ici quelque aspects de la technique basée sur *le point de vue de la particule* dans l'étude des mouvements aléatoires dans des environnements aléatoires.

Table of Contents

1. Introduction 7

2. Central Limit Theorems for Markov Processes 9
- 2.1 General Setup 9
- 2.2 Reversible Processes 15
- 2.3 Strong Sector Condition 15
- 2.4 Weak (Graded) Sector Condition 16

3. Applications 21
- 3.1 Exclusion Processes 21
- 3.2 The Tagged Particle in the Exclusion Processes 22
- 3.3 The Symmetric Case 24
- 3.4 Asymmetric Exclusion with zero drift 26
- 3.5 Asymmetric Exclusion in $d \geq 3$ 26
- 3.6 Other Interacting Particles Systems 29
- 3.7 Diffusion in Random Environment 32
- 3.8 Diffusion in Gaussian Random Fields 35

4. Some Models Without Sector Condition 37
- 4.1 Diffusion in a time-dependent Divergence-Free Flow ... 37
- 4.2 Ornstein-Uhlenbeck Process in a Random Potential.... 38

5. Approximation, Regularity and some Open Problems 41
- 5.1 Regularity of Self-Diffusion in Simple Exclusion 41
- 5.2 Finite Dimensional Approximation of Self-Diffusion in Simple Exclusion 42
- 5.3 Open Problem: Breaking the Translation Invariance Symmetry 42

1. Introduction

In the study of the random motion in random environments *the point of view of the particle* has been a powerful tool, in particular in the investigation of the ergodic properties and the diffusive behavior of the motion (central limit theorems).

We will expose here some aspects of this approach: the basic classical results and some of its defaults. The first ingredient is the translation invariance of the model. This permits to define *the point of view of the particle* as a Markov process on the all possible configuration of the environment. Then one needs to know explicitly the invariant measures for this environment process. We will see that the method works very well if this invariant measure is reversible, and that difficulties arise as this process is less and less reversible.

The main ideas of the general approach are taken from [8] and [24].

2. Central Limit Theorems for Markov Processes

2.1 General Setup

In the examples we will treat, the problem can be reduced to the following general setup.

Let η_t be a Markov process, defined on a probability space $(\Omega, \mathcal{F}, P)$, with values in a Polish metric space $\mathcal{X}$. Let $\mathcal{B}(\mathcal{X})$ the σ-algebra of the Borel subsets. In our applications $\mathcal{X}$ will be the space of the configurations of the environment and η_t will be the *environment viewed from the particle at time t*. We will work with continuous time parameter, but all ideas (with a little care) can be worked out for Markov chains with discrete time (cf. [8] and [2] for the reversible case).

We will assume that there exists a probability measure μ on $\mathcal{X}$ that is invariant and ergodic for η_t, i.e. for any $f \in L^2(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$

$$\mathbb{E}_\mu\left(f(\eta_t)\right) = \int f \, d\mu \qquad \forall t > 0,$$

and if

$$\mathbb{E}_\mu\left(f(\eta_t) g(\eta_0)\right) = \int fg \, d\mu \qquad \forall t > 0, \ \forall g \in L^2(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$$

then f is constant $\mu - a.e.$. $\mathbb{E}_\mu$ denotes here the expectation with respect to the process with initial distribution μ.

By P^t we denote the corresponding semigroup of Markov operators on $L^2(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$, i.e.

$$P^t f(\eta) = \mathbb{E}_\eta\left(f(\eta_t)\right)$$

where $\mathbb{E}_\eta$ denotes the expectation with respect to the process starting at η. We assume that this semigroup is strongly continuous. Let $L : \mathcal{D}(L) \to L^2(\mu)$ be the generator of this semigroup, and let $\mathcal{D}(L)$ be its domain dense in $L^2(\mu)$.

Denote by L^* the adjoint operator of L in $L^2(\mu)$, i.e. the generator of the time reversed process with respect to μ. The symmetrized operator $S = \frac{1}{2}[L+L^*]$ is defined on $\mathcal{D}(L)\cap\mathcal{D}(L^*)$. We will assume that $\mathcal{D}(L)\cap \mathcal{D}(L^*)$ is large enough to contain a core for the corresponding Dirichlet form

$$< f, g >_1 = \int f(-S)g \, d\mu \ . \tag{2.1.1}$$

In all the applications we will consider, there is always a core $\mathcal{C}$ of *nice* functions. Typically in the interacting particles systems this core is given by the local "smooth" functions.

We denote by $\|f\|_1 = \sqrt{< f, f >_1}$ the associated Dirichlet norm, and we define $\mathbb{H}_1$ as the Hilbert space generated by the closure of $\mathcal{C}$ with respect to the norm $\| \cdot \|_1$.

The corresponding dual norm is defined by

$$\|g\|_{-1}^2 = \sup_{f\in\mathcal{C}} \left\{ 2\int fg \, d\mu - \|f\|_1^2 \right\} , \tag{2.1.2}$$

or equivalently

$$\|g\|_{-1} = \inf \left\{ C : \left| \int fg \, d\mu \right| \le C\|f\|_1, \ \forall f \in \mathcal{C} \right\} .$$

Observe that $\|g\|_{-1} < \infty$ implies $\int g \, d\mu = 0$. We define $\mathbb{H}_{-1}$ as the Hilbert spaces generated by the closure of $\{g \in \mathcal{C} : \|g\|_{-1} < \infty\}$ with respect to the norm $\| \cdot \|_{-1}$.

We will look for a central limit theorem of the following type: let $g \in L^2 \cap \mathbb{H}_{-1}$, then we would like to prove that

$$\frac{1}{\sqrt{t}} \int_0^t g(\eta_s) \, ds \tag{2.1.3}$$

converges in law to a centered Gaussian random variable. The limit variance will be defined as

$$\begin{aligned} \sigma^2(g) &= \limsup_{t\to\infty} \mathbb{E}_\mu \left(\left[\frac{1}{\sqrt{t}} \int_0^t g(\eta_s) \, ds \right]^2 \right) \\ &= \limsup_{t\to\infty} \frac{2}{t} \int_0^t ds \int_0^s d\tau \; \mathbb{E}_\mu \left(g(\eta_{s-\tau}) g(\eta_0) \right) \\ &= \limsup_{t\to\infty} 2 \int_0^\infty \left(1 - \frac{s}{t} \right)_+ \mathbb{E}_\mu \left(g(\eta_s) g(\eta_0) \right) \, ds \end{aligned} \tag{2.1.4}$$

where $a_+ = \max\{a, 0\}$. In the general case, it is not clear whether this lim sup is in fact a limit without some more restriction on the function g and on the process.

If μ is reversible, then $\{P^t\}_{t\geq 0}$ is a semigroup of symmetric operators and we have

$$\mathbb{E}_\mu(g(\eta_t)g(\eta_0)) =< (P^{t/2}g)^2 > .$$

By the spectral theorem it is easy to see that $< (P^{t/2}g)^2 >$ is monotone decreasing in t, so the limit defining $\sigma^2(g)$ exists and is equal to

$$2\int_0^\infty \mathbb{E}_\mu(g(\eta_t)g(\eta_0))\; dt\ . \tag{2.1.5}$$

If it is finite, then the integral of the time correlation exists and is finite. Notice we used here that time is a continuous parameter. For reversible *discrete time* Markov chains one can find trivial counterexamples, as soon as one allows negative eigenvalues in the spectrum of the transition probability matrix. In these cases the convergence of the integral in (2.1.5) should be understood in the Cesaro sense.

Back to the general *non-reversible* case, the convergence of the integral (2.1.5) to the variance in the sense given by (2.1.4) is equivalent to the convergence of

$$\sigma^2(g) = \lim_{\lambda\to 0} 2\int_0^\infty e^{-\lambda t}\mathbb{E}_\mu(g(\eta_t)g(\eta_0))\; dt\ . \tag{2.1.6}$$

In this situation we have

$$\sigma^2(g) \leq 2\|g\|_{-1}^2\ . \tag{2.1.7}$$

This inequality says that the condition $g \in \mathbb{H}_{-1}$ guarantees the finiteness of the limit variance $\sigma^2(g)$, provided that the limit defining $\sigma^2(g)$ exists. In the reversible case we have equality in (2.1.7), and $g \in \mathbb{H}_{-1}$ becomes a necessary condition.

To prove (2.1.7), let u_λ be the solution of the resolvent equation

$$\lambda u_\lambda - Lu_\lambda = g \tag{2.1.8}$$

then, assuming that the limit given by (2.1.6) exists, the variance $\sigma^2(g)$ can be rewritten as

$$\begin{aligned}\sigma^2(g) &= \lim_{\lambda\to 0} 2\int g(\lambda - L)^{-1} g\, d\mu = \lim_{\lambda\to 0} 2\int g u_\lambda\, d\mu \\ &= \lim_{\lambda\to 0} 2\int u_\lambda(\lambda - L)u_\lambda\, d\mu \\ &= \lim_{\lambda\to 0} 2\lambda\int u_\lambda^2\, d\mu + 2\int u_\lambda(-S)u_\lambda\, d\mu \\ &= 2\lim_{\lambda\to 0} \lambda\|u_\lambda\|_0^2 + \|u_\lambda\|_1^2\end{aligned}$$

Multiplying (2.1.8) by u_λ and integrating in $d\mu$, it is easy to obtain the bound

$$\lambda\|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 \le \|g\|_{-1}^2 \tag{2.1.9}$$

and this proves (2.1.7).

But in general we are not able to prove that the finiteness of the $\mathbb{H}_{-1}$-norm of g implies that (2.1.6) exists nor that the integral (2.1.5) converges. Still one can prove that

$$\sigma^2(g) \le 8\|g\|_{-1}^2\ . \tag{2.1.10}$$

This is a direct consequence of the following proposition, which shows that the finiteness of the $\|\cdot\|_{-1}$–norm gives a uniform control over the *finite time* variances. It will also be a main tool for proving tightness for the corresponding invariance principle.

Proposition 2.1.1.

$$\mathbb{E}_\mu\left(\sup_{0\le t\le T}\left|\int_0^t g(\eta_\tau)\, d\tau\right|^2\right) \le 8T\|g\|_{-1}^2 \tag{2.1.11}$$

Proof. By the assumptions made on the domains of S, L and L^*, there exists a sequence $u_n \in \mathcal{D}(L)\cap\mathcal{D}(L^*)$, such that

$$\|Su_n - g\|_{L^2(\mu)} \to 0$$

and $\|u_n\|_1^2 \le \|g\|_{-1}^2$. Then observe that, for any $s \ge 0$

$$u_n(\eta_t) - u_n(\eta_s) - \int_s^t Lu_n(\eta_\tau)\, d\tau = M_n(t) - M_n(s), \qquad t \ge s$$

are martingales respect to the forward filtration $\{\mathcal{F}_t^\eta, t \ge 0\}$ (where $\mathcal{F}_t^\eta$ is the σ-algebra on Ω generated by $\{\eta_s, s \le t\}$, cf. [3], proposition 4.1.7, pag. 162).

Similarly, for any $t \geq 0$

$$u_n(\eta_s) - u_n(\eta_t) - \int_s^t L^* u_n(\eta_\tau)\, d\tau = M_n^*(s) - M_n^*(t), \qquad s \leq t$$

are martingales respect to the backward filtration $\{\mathcal{B}_s^\eta, 0 \leq s \leq t\}$ (where $\mathcal{B}_s^\eta$ is the σ-algebra on Ω generated by $\{\eta_\tau, s \leq \tau \leq t\}$).

After summing up these two expressions, the boundary terms cancel and we obtain

$$-\int_s^t S u_n(\eta_s)\, ds = \frac{1}{2}\left[M_n(t) - M_n(s) + M_n^*(s) - M_n^*(t)\right] \ .$$

Since

$$\begin{aligned}\mathbb{E}_\mu\left([M_n(t) - M_n(s)]^2\right) &= \mathbb{E}_\mu\left([M_n^*(s) - M_n^*(t)]^2\right) \\ &= 2(t-s)\|u_n\|_1^2 \leq 2(t-s)\|g\|_{-1}^2,\end{aligned}$$

by applying separately Doob's inequality (cf [3], proposition 2.2.16, pag. 63) to the two martingales, after using Schwarz inequality, one obtains an estimate that is uniform in n. The convergence of $Su_n \to g$ in $L^2(\mu)$ concludes the argument. □

Remark 2.1.1. In the non-reversible case, there are interesting examples where the function g involved is not in $\mathbb{H}_{-1}$ but still we expect it has a finite variance. One of these examples is the tagged particle problem (see section 3.2 for the definition) in the asymmetric simple exclusion in dimensions 1 and 2.

Let us look now for some conditions on the generator of the process that will imply the central limit theorem (CLT) and the invariance principle for a function $g \in \mathbb{H}_{-1} \cap L^2(\mu)$.

Assume that the solution of (2.1.8) satisfies

$$\boxed{\lambda\|u_\lambda\|_0^2 \underset{\lambda \to 0}{\longrightarrow} 0} \ . \tag{2.1.12}$$

By a simple functional analysis argument, it follows that u_λ converges strongly in $\mathbb{H}_1$. Consequently the limit of the $\mathbb{H}_1$-norm of u_λ exists and this implies the existence of the limit given by (2.1.6) and

$$\sigma^2(g) = 2 \lim_{\lambda \to 0} \|u_\lambda\|_1^2 \ .$$

One can prove that (2.1.12) is a sufficient condition for proving CLT for $g \in L^2 \cap \mathbb{H}_{-1}$. This is the classical starting point in the Kipnis-Varadhan approach and in the homogenization literature.

What we will use here is the stronger condition

$$\boxed{\sup_{\lambda>0} \|\lambda u_\lambda\|_{-1} \le C} \,. \tag{2.1.13}$$

Since $\|g\|_{-1} < \infty$, (2.1.13) is equivalent to

$$\boxed{\sup_{\lambda>0} \|L u_\lambda\|_{-1} \le C} \,. \tag{2.1.14}$$

Proposition 2.1.2. *The bound (2.1.13) (or (2.1.14)) implies (2.1.12). Furthermore there exists a sequence of functions $v_n \in \mathcal{D}(L)$ such that*

$$\|Lv_n - g\|_{-1} \longrightarrow 0 \tag{2.1.15}$$

Proof. By (2.1.9) there is a subsequence $\lambda_n \to 0$ such that $u_n = u_{\lambda_n}$ has a weak limit $u_0 \in \mathbb{H}_1$, and $\lambda u_\lambda \to 0$ in $L^2(\mu)$. Then by (2.1.13) $\lambda u_\lambda \to 0$ weakly in $\mathbb{H}_{-1}$. This, in turn, implies that $-Lu_\lambda \to g$ weakly in $\mathbb{H}_{-1}$. There exists therefore some convex combination v_n of $u_1, \dots, u_n$ such that v_n converges strongly to u_0 in $\mathbb{H}_1$ and $-Lv_n$ converges strongly to g in $\mathbb{H}_{-1}$.

Additionally this establishes the equation $< g, u_0 >= \|u_0\|_1^2$. In particular

$$\begin{aligned}\|u_0\|_1^2 &= \limsup_n \|u_{\lambda_n}\|_1^2 \\ &\le \limsup_n \left(\lambda_n \|u_{\lambda_n}\|_{L^2}^2 + \|u_{\lambda_n}\|_1^2\right) =< g, u_0 >= \|u_0\|_1^2\end{aligned}$$

which implies $\lim_n \lambda_n \|u_{\lambda_n}\|_{L^2}^2 = 0$. It is easy to prove then the uniqueness of u_0. □

We will show now that (2.1.15) is all we need for the proof of the CLT (in fact also for the complete invariance principle).

Since we can write $g = Lv_n + r_n$

$$\frac{1}{\sqrt{t}} \int_0^t g(\eta_s)\, ds = \frac{1}{\sqrt{t}} \int_0^t Lv_n(\eta_s)\, ds - \frac{1}{\sqrt{t}} \int_0^t r_n(\eta_s)\, ds \tag{2.1.16}$$

By (2.1.11) and (2.1.15)

$$\lim_n \sup_t \mathbb{E}_\mu \left(\left| \frac{1}{\sqrt{t}} \int_0^t r_n(\eta_s)\, ds \right|^2 \right) = 0 \tag{2.1.17}$$

while we can rewrite

$$\frac{1}{\sqrt{t}} \int_0^t Lv_n(\eta_s)\, ds = \frac{1}{\sqrt{t}} \left(v_n(\eta_t) - v_n(\eta_0) \right) + \frac{1}{\sqrt{t}} M_{v_n}(t) \tag{2.1.18}$$

where $M_{u_n}(t)$ is a martingale with quadratic variation given by an additive functional $A_n(t)$ such that $\mathbb{E}[A_n(t)] = 2t\|v_n\|_1^2$.

As a consequence we obtain that

$$\lim_n \lim_{t\to\infty} \mathbb{E}_\mu \left(\left| \frac{1}{\sqrt{t}} \int_0^t g(\eta_s)\, ds - \frac{1}{\sqrt{t}} M_{v_n}(t) \right|^2 \right) = 0 \tag{2.1.19}$$

The quadratic variation of $\frac{1}{\sqrt{t}} M_{v_n}(t)$ is now given by $t^{-1}A_n(t)$ and by the ergodic theorem it converges in probability, as $t \to \infty$, to $2\|v_n\|_1^2$. In the proof above of (2.1.15) it is shown that v_n converges to u_0 strongly in $\mathbb{H}_1$. So $2\|v_n\|_1^2$ converges to $2\|u_0\|_1^2$ which identifies to $\sigma^2(g)$. At this point we conclude by applying the CLT for martingales.

Let us see now some explicit conditions on the generator of the Markov process which imply (2.1.13) or (2.1.14).

2.2 Reversible Processes

If the Markov process η_t is reversible with respect to μ, i.e. L is self-adjoint in $L^2(\mu)$, then condition (2.1.14) is immediately satisfied. In fact in this case $L = S$ and by (2.1.9)

$$\|Lu_\lambda\|_{-1} = \|u_\lambda\|_1 \le \|g\|_{-1}$$

and $\sigma^2(g) = 2\|g\|_{-1}^2$.

2.3 Strong Sector Condition

Assume that L is not self-adjoint but satisfies

$$\left| \int uLv\, d\mu \right| \le K\|u\|_1 \|v\|_1 \tag{2.3.1}$$

for any $u, v \in \mathcal{C}$. This condition basically means that the process is not too far from being reversible, the antisymmetric part of the operator is bounded by the symmetric part. It is an easy exercise to show that (2.3.1) implies that the complex spectrum of L is contained in the cone

$$\{z \in \mathbb{C} : |\mathcal{I}m\ z| \leq K|\mathcal{R}e\ z|,\ \mathcal{R}e\ z \leq 0\}$$

Again condition (2.1.14) is immediately satisfied since

$$\|Lu_\lambda\|_{-1} \leq K\|u_\lambda\|_1 \leq K\|g\|_{-1}$$

2.4 Weak (Graded) Sector Condition

Assume that there exists an orthogonal decomposition

$$L^2(\mu) = \bigoplus_{n=0}^{\infty} H_n, \qquad H_0 = \text{span}\{1\}$$

that satisfies the following properties:

A. $L_n = L_{|\mathcal{D}_n} : \mathcal{D}_n \longrightarrow H_{n-1} \oplus H_n \oplus H_{n+1}$, where $\mathcal{D}_n = \mathcal{D}(L) \cap H_n$. We will write
$$L_n = B_{n,n-1} + B_{n,n} + B_{n,n+1}$$
where $B_{n,n+j} : D_n \to H_{n+j},\ j = -1, 0, 1$.

B. the symmetric part of the generator is diagonal with respect to this decomposition, i.e. $S = \sum_n B_{n,n}$.

C. there exist constants $K > 0$ and $\beta < 1$ such that for any $n \geq 1$, any $v_n \in \mathcal{D}_n$ and $u \in \mathcal{D}(L)$
$$\left| \int u B_{n,n+j} v_n \ d\mu \right| \leq K n^\beta \|u\|_1 \|v_n\|_1 \qquad j = -1, 0, 1$$

The condition B implies that the spaces H_n are orthogonal also with respect to the $\mathbb{H}_1$-norm. In some applications, like the tagged particles problem in the exclusion processes, condition B should be relaxed, allowing some parts of the symmetric operator to be off-diagonal, and some parts of the asymmetric part to be diagonal. So the conditions A,B,C we choose here are just a simple setup, not the most general.

Proposition 2.4.1. *Assume g of finite order, i.e. $g \in \bigoplus_{n=0}^{n_0} H_n$ for some finite n_0. Let $u_\lambda = \sum_n u_{\lambda,n}$ be the orthogonal decomposition of the solution of (2.1.8). Then for any $k \geq 0$:*

$$\sup_\lambda \sum_n n^{2k} \|u_{\lambda,n}\|_1^2 \leq C(k,g) \tag{2.4.1}$$

Proof. For any function $u \in L^2(\mu)$ consider the orthogonal decomposition $u = \sum_n u_n$ with $u_n \in H_n$.

Fix $n_1 < n_2$ and put $t(n) = n_1^k \vee (n^k \wedge n_2^k)$. Consider the operator on $L^2(\mu)$ defined by

$$Tu = \sum_n t(n) u_n$$

Applying T to both sides of the resolvent equation 2.1.8, we have

$$\lambda T u_\lambda - L T u_\lambda = Tg + [T,L] u_\lambda \tag{2.4.2}$$

Observe that since g is of finite order, $Tg = g$ for n_1 big enough. We will show now that $[T,L]$ is a bounded operator from $\mathbb{H}_1$ to $\mathbb{H}_{-1}$. Explicitly this commutator is given by

$$[T,L]u = \sum_n \left[(t(n+1) - t(n)) B_{n,n+1} u_n + (t(n-1) - t(n)) B_{n,n-1} u_n\right]$$

Then using condition C we have

$$\begin{aligned}
< [T,L]u, Tu > &= \sum_n \Big[t(n+1)(t(n+1) - t(n)) < B_{n,n+1} u_n, u_{n+1} > \\
&\qquad + t(n-1)(t(n-1) - t(n)) < B_{n,n-1} u_n, u_{n-1} > \Big] \\
&= \sum_n \Big[\frac{(t(n+1) - t(n))}{t(n)} < B_{n,n+1} (Tu)_n, (Tu)_{n+1} > \\
&\qquad + \frac{(t(n-1) - t(n))}{t(n)} < B_{n,n-1} (Tu)_n, (Tu)_{n-1} > \Big] \\
&\leq K \sum_n \Big[\frac{t(n+1) - t(n)}{t(n)} n^\beta \|(Tu)_n\|_1 \|(Tu)_{n+1}\|_1 \\
&\qquad + \frac{t(n) - t(n-1)}{t(n)} n^\beta \|(Tu)_n\|_1 \|(Tu)_{n-1}\|_1 \Big] \\
&\leq 4K \sum_n \frac{t(n+1) - t(n)}{t(n)} n^\beta \|(Tu)_n\|_1^2
\end{aligned}$$

Since, for $n_1 \leq n \leq n_2 - 1$, by choosing $n_1 > k$

$$\frac{t(n+1)-t(n)}{t(n)} = \frac{(n+1)^k - n^k}{n^k} = \left(1+\frac{1}{n}\right)^k - 1 = \sum_{j=1}^{k} \binom{k}{j} n^{-j}$$
$$\leq \sum_{j=1}^{k} \frac{1}{j!}\left(\frac{k}{n}\right)^j \leq \frac{k}{n}\sum_{j=1}^{k}\frac{1}{j!} \leq \frac{ke}{n}$$

We obtain the bound

$$| < [T,L]u, Tu > | \leq \frac{4Kke}{n_1^{1-\beta}} \sum_{n \geq n_1+1}^{n_2-1} \|Tu_n\|_1^2$$
$$\leq \frac{4Kke}{n_1^{1-\beta}} \|Tu\|_1^2$$

Multiplying the equation 2.4.2 by Tu_λ and integrating one has, using the above bound

$$\lambda < (Tu_\lambda)^2 > + \|Tu_\lambda\|_1^2 = < Tu_\lambda, Tg > + < Tu_\lambda, [T,L]u_\lambda >$$
$$\leq \|Tg\|_{-1}\|Tu_\lambda\|_1 + \frac{4Kke}{n_1^{1-\beta}}\|Tu_\lambda\|_1^2$$

Now we can choose n_1 such that $\frac{4Kke}{n_1^{1-\beta}} < 1$. So we have, choosing n_1 larger than the order on the function g:

$$\|Tu_\lambda\|_1 \leq \frac{1}{1-\frac{4Kke}{n_1^{1-\beta}}}\|Tg\|_{-1} = \frac{n_1}{1-\frac{4Kke}{n_1^{1-\beta}}}\|g\|_{-1} = C(k)\|g\|_{-1}$$

obtaining in this way

$$\sum_n t(n)^2 \|u_{\lambda,n}\|_1^2 \leq C(k)^2 \|g\|_{-1}^2$$

Since the bound obtained does not depend on n_2, we can send $n_2 \to \infty$ and obtain:

$$\sum_{n \geq n_1+1} n^{2k} \|u_{\lambda,n}\|_1^2 \leq 4C(k)^2 \|g\|_{-1}^2$$

while the sum over the first n_1 terms is readily bounded by $\|g\|_{-1}^2$, so one obtains

$$\sum_{n>0} n^{2k} \|u_{\lambda,n}\|_1^2 \leq (C(k)^2+1)\|g\|_{-1}^2$$

□

It is immediate now to show that (2.1.14) follows: since

$$Lu_\lambda = \sum_n (B_{n,n}u_{\lambda,n} + B_{n-1,n}u_{\lambda,n-1} + B_{n+1,n}u_{\lambda,n+1})$$

$$\|Lu_\lambda\|_{-1}^2 \le 3\sum_n \|B_{n,n}u_{\lambda,n}\|_{-1}^2 + \|B_{n-1,n}u_{\lambda,n-1}\|_{-1}^2 + \|B_{n+1,n}u_{\lambda,n+1}\|_{-1}^2$$

$$\le 3K\sum_n n^{2\beta}\|u_{\lambda,n}\|_1^2 \le C(1,g) \tag{2.4.3}$$

Remark 2.4.1. If the constant K in condition C is small, then (2.4.3) is valid even if $\beta = 1$. In fact if $\beta = 1$ and $K < 1/4e$, then we have from the proof of proposition 2.4.1 that $C(1,g) = \frac{\deg(g)}{1-4Ke}$. In this situation the bound (2.4.1) is valid for all $k < 1/4eK$.

3. Applications

The method presented in the previous section finds its most powerful application in establishing central limit theorems in infinite systems of interacting particles. When conservation laws are present typically the time correlations decay slowly, and, in general, we do not have information about this decay.

We will consider first the simple exclusion process, and will indicate to which other models the method extends. Of particular interest is the problem of the self-diffusion of a tagged particle.

3.1 Exclusion Processes

This model is constituted by infinitely many particles performing random walks on $\mathbb{Z}^d$. The only interaction considered between the particles is the exclusion rule: when a particle attempt to jump on a site already occupied by another particle, the jump is suppressed. Consequently at any time there is only one particle per site, if such is the initial configuration.

Let us fix a finite-range probability distribution $p(\cdot)$ on $\mathbb{Z}^d$. The simple exclusion process associated to p is the Markov process on $\mathcal{X} = \{0,1\}^{\mathbb{Z}^d}$ whose generator L acts on cylinder functions f as

$$(Lf)(\eta) = \sum_{x,y\in\mathbb{Z}^d} p(y)\eta(x)[1-\eta(x+y)][f(\sigma^{x,x+y}\eta) - f(\eta)] \, . \quad (3.1.1)$$

Here and below the configurations of $\mathcal{X}$ are denoted by Greek letters. In particular, for x in $\mathbb{Z}^d$, $\eta(x)$ is equal to 1 or 0 whether the site x is occupied or not for the configuration η. Moreover, for a configuration η and x, y in $\mathbb{Z}^d$, $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) \;=\; \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise}\ . \end{cases} \qquad (3.1.2)$$

Fix $0 \le \alpha \le 1$ and denote by ν_α the Bernoulli product measure on $\mathcal{X}$. This is the probability measure on $\mathcal{X}$ obtained by placing a particle with probability α at each site x, independently from the other sites. It is easy to check that, for any $\alpha \in [0,1]$, ν_α is stationary. It can also be proved that it is ergodic (see thm. III.1.17 in [17]).

In the symmetric case, $p(y) = p(-y)$, the stationary measures $\{\nu_\alpha,\, 0 \le \alpha \le 1\}$ are also reversible and the generator can be rewritten as

$$(Lf)(\eta) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d} p(y)[f(\sigma^{x,x+y}\eta) - f(\eta)]\ . \qquad (3.1.3)$$

3.2 The Tagged Particle in the Exclusion Processes

We examine now the evolution of a single tagged particle in the symmetric simple exclusion process. Let η be an initial configuration with a particle at the origin, i.e. with $\eta(0) = 1$. Tag this particle and denote by η_t (resp. X_t) the state of the process (resp. the position of the tagged particle) at time t. Let ξ_t be the state of the environment as seen from the tagged particle:

$$\xi_t = \theta_{X_t}\eta_t\ .$$

Here, for x in $\mathbb{Z}^d$ and a configuration η, θ_x stands for the translation of η by x, i.e. $(\theta_x\eta)(y) = \eta(x+y)$. Notice that the origin is always occupied (by the tagged particle) for the environment as seen from the tagged particle. For this reason, we shall consider the process ξ_t as taking values in $\{0,1\}^{\mathbb{Z}^d_*}$, where $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$.

Whereas X_t is not a Markov process due to the presence of the environment, (X_t, ξ_t) and ξ_t are. A simple computation shows that the generator $\mathcal{L}$ of the Markov process ξ_t is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\tau$, where

$$(\mathcal{L}_0 f)(\xi) \;=\; \sum_{x,y \in \mathbb{Z}^d_*} p(y-x)\xi(x)[1-\xi(y)][f(\sigma^{x,y}\xi) - f(\xi)]\ ,$$

$$(\mathcal{L}_\tau f)(\xi) \;=\; \sum_{z \in \mathbb{Z}^d_*} p(z)[1-\xi(z)][f(\tau_z\xi) - f(\xi)]\ . \qquad (3.2.1)$$

The first part of the generator takes into account the jumps in the environment, while the second one corresponds to jumps of the tagged particle. In the above formula, $\tau_z\xi$ stands for the configuration where the tagged particle, sitting at the origin, is first transferred to the (empty) site z and then the entire environment is translated by $-z$: for all y in $\mathbb{Z}^d_*$

$$(\tau_z\xi)(y) = \begin{cases} \xi(z) & \text{if } y = -z, \\ \xi(y+z) & \text{for } y \neq -z. \end{cases}$$

For $0 \leq \alpha \leq 1$, denote by μ_α the Bernoulli product measure on $\mathcal{X}_* = \{0,1\}^{\mathbb{Z}^d_*}$. A simple computation shows that μ_α is an ergodic stationary measure for the Markov process ξ_t (cf. proposition III.4.8 in [17]).

The position of the tagged particle can be written as

$$\begin{aligned} X_t \cdot v =& M_t + \int_0^t \sum_z p(z)(z\cdot v)[1-\eta_s(X_s+z)]\, ds \\ =& M_t + \int_0^t \sum_z p(z)(z\cdot v)[1-\xi_s(z)]\, ds \end{aligned}$$

where M_t is a martingale with quadratic variation given by

$$\int_0^t \sum_z p(z)(z\cdot v)^2[1-\xi_s(z)]\, ds\ .$$

If we start the process ξ_t with the stationary measure μ_α for a fixed given α, we have that

$$\mathbb{E}_{\mu_\alpha}(X_t\cdot v) = t(1-\alpha)\sum_z p(z)(z\cdot v) = t(1-\alpha)\sum_z a(z)(z\cdot v)$$

where a is the antisymmetric part of p, i.e. $p(z) = s(z) + a(z)$, $s(z) = (p(z)+p(-z))/2$ and $a(z) = (p(z)-p(-z))/2$. It follows that in the symmetric case ($p = s$) the average velocity of the particle is null.

By the ergodicity of μ_α we have

$$\frac{X_t\cdot v}{t} \underset{t\to\infty}{\longrightarrow} (1-\alpha)\sum_z p(z)(z\cdot v) \qquad \mu_\alpha - \text{a.e.}$$

Let us define

$$
\begin{aligned}
Y_v(t) &= \frac{1}{\sqrt{t}}\left[X_t \cdot v - t(1-\alpha)\sum_z p(z)(z\cdot v)\right] \\
&= \frac{1}{\sqrt{t}} M_t + \frac{1}{\sqrt{t}}\int_0^t g(\xi_s(z))\, ds
\end{aligned}
\tag{3.2.2}
$$

where

$$
g(\xi) = \sum_z p(z)(z\cdot v)[\alpha - \xi(z)] \tag{3.2.3}
$$

Of course in the symmetric case g does not depend on α and it is equal to $\sum_z p(z)(z\cdot v)[1-\xi(z)]$.

The problem of self diffusion is then to prove that $Y_v(t)$ converges in law to a Gaussian random variable with variance

$$
D_v = v \cdot Dv \ .
$$

The variance matrix D can be obtained by polarization. We can see immediately that D depends on the density α, i.e. $D = D(\alpha)$.

The first term in (3.2.2) is already a martingale with variance:

$$
\mathbb{E}_{\mu_\alpha}(M_t^2) = t(1-\alpha)\sum_z p(z)(z\cdot v)^2 = t(1-\alpha)\sum_z s(z)(z\cdot v)^2 \equiv t(1-\alpha)\sigma_0^2
$$

Notice that σ_0^2 is the variance of the random walk with transition probability p. So the problem is to examine the second term of (3.2.2). We first treat the symmetric case.

3.3 The Symmetric Case

If p is symmetric the generator of the environment as seen from the particle, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\tau$, is selfadjoint in $L^2(\mu_\alpha)$. Then all we have to check here is that the function g defined above is in $\mathbb{H}_{-1}$. Since now p is symmetric, g can be rewritten as

$$
g(\xi) = \sum_z p(z)(z\cdot v)[1-\xi(z)] = \frac{1}{2}\sum_z p(z)(z\cdot v)[\xi(-z)-\xi(z)] \ . \tag{3.3.1}
$$

Then it is easy to see that for any local function f

$$
\begin{aligned}
\int fg \, d\mu_\alpha &= \frac{1}{2}\sum_z p(z)(z\cdot v)\int (f(\tau_z\xi) - f(\xi))\, \xi(z)\, d\mu_\alpha(\xi) \\
&\le C\|f\|_1
\end{aligned}
$$

so $\|g\|_{-1} \leq C$. This proves the central limit theorem for the tagged particle in the symmetric simple exclusion.

The reversibility property gives the possibility to compute a bit more explicitly the diffusion matrix $D(\alpha)$. Observe that

$$\int_0^t g(\xi_s)\, ds$$

is an additive functional, symmetric with respect to the time reversal

$$\xi_s \longrightarrow \xi_{t-s}, \qquad 0 \leq s \leq t$$

while the position X_t of the tagged particle itself is an antisymmetric additive functional. Since the measure μ_α is reversible, i.e. the corresponding measure on the paths space is invariant under this path transformation, we find that these two functionals are orthogonal:

$$\mathbb{E}_{\mu_\alpha}\left((X_t \cdot v) \int_0^t g(\xi_s)\, ds \right) = 0 \ .$$

It follows that

$$\mathbb{E}_{\mu_\alpha}\left((X_t \cdot v)^2 \right) = \mathbb{E}_{\mu_\alpha}\left(M_t^2 \right) - \mathbb{E}_{\mu_\alpha}\left(\left[\int_0^t g(\xi_s)\, ds \right]^2 \right) \ .$$

Dividing by t and after the limit for $t \to \infty$ we obtain

$$\begin{aligned} v \cdot D(\alpha) v =& (1-\alpha) \sum_z p(z)(z \cdot v)^2 - 2 \int_0^\infty \mathbb{E}_{\mu_\alpha}(g(\xi_t) g(\xi_0))\, dt \\ =& \ (1-\alpha)\sigma_0^2 - 2\|g\|_{-1}^2 \ . \end{aligned} \tag{3.3.2}$$

This formula shows that the interaction with the environment gives, apart the obvious factor $(1-\alpha)$, a further reduction of the variance of the particle due to the autocorrelation term $2\|g\|_{-1}^2$.

It is not immediate to see from (3.3.2) that $D(\alpha)$ is positive, but using the variational definition of the $\|\cdot\|_{-1}$-norm, one can rewrite it as

$$\begin{aligned} v \cdot D(\alpha) v = \inf_f \Big\{ & \sum_{z \in \mathbb{Z}_*^d} p(z) \left\langle [1-\xi(z)]\{v \cdot z - T_z f\}^2 \right\rangle_\alpha \\ & + \sum_{x,y \in \mathbb{Z}_*^d} p(y-x) < \{T_{x,y} f\}^2 >_\alpha \Big\} \end{aligned} \tag{3.3.3}$$

where $T_z f(\xi) = f(\tau_z \xi) - f(\xi)$, and $T_{x,y} f(\xi) = f(\sigma^{x,y}\xi) - f(\xi)$. Variational formula like (3.3.3) can be very useful in approximation problems [14].

3.4 Asymmetric Exclusion with zero drift

If p is not symmetric, but such that

$$\sum_{z \in \mathbb{Z}^d_*} zp(z) = 0$$

then $\mathcal{L}$ is not anymore self adjoint, but it satisfies the strong sector condition (cf. [25] for the proof)

$$\left| \int u\mathcal{L}v \; d\mu \right| \leq K\|u\|_1\|v\|_1$$

and since g can be written again as in the symmetric case like (3.3.1), $g \in \mathbb{H}_{-1}$, so the CLT follows.

Observe that now (3.3.2) and (3.3.3) are not anymore valid.

3.5 Asymmetric Exclusion in $d \geq 3$

If p is asymmetric and $m = \sum_{z \in \mathbb{Z}^d_*} zp(z) \neq 0$, the strong sector condition is not verified, furthermore we will see that

$$g = \sum_z p(z)(z \cdot v)[\alpha - \xi(z)] \tag{3.5.1}$$

is in $\mathbb{H}_{-1}$ only in dimension $d \geq 3$. In [24] it is proven that there exists an orthogonal decomposition $L^2(\mu_\alpha) = \bigoplus_n H_n$ such that $\mathcal{L}$ satisfies a graded sector condition in the sense of the previous chapter (actually here diagonal term $B_{n,n}$ is not bounded or symmetric in this case, and it require an extra coupling argument specific of this model, cf. [24] section 6 for details).

This orthogonal decomposition is given by a duality technique. We will expose here how it is defined and works for the asymmetric simple exclusion process (ASEP) in $d \geq 3$ if we want to study CLT for functionals $\int_0^t g(\eta_s)ds$. The shifts, due to the movements of the tagged particle, will introduce non diagonal symmetric terms in the generator. This is a further complication, so for simplicity we will ignore $\mathcal{L}_\tau$.

The *dual* orthonormal base on $L^2(\mu_\alpha)$ is defined by:

$$\Psi_A \;=\; \Psi_A(\alpha, \xi) \;=\; \prod_{x \in A} \frac{\xi(x) - \alpha}{\sqrt{\alpha(1-\alpha)}} \,,$$

for any finite $A \subset \mathbb{Z}^d_*$. By convention, $\Psi_\emptyset = 1$.

We will denote by $\mathfrak{u}(A)$ the Ψ_A component of u

$$u = \sum_A \mathfrak{u}(A)\Psi_A \ . \tag{3.5.2}$$

Let

$$H_n = \operatorname{span}\{\Psi_A, |A| = n\} \tag{3.5.3}$$

then $L^2(\mu_\alpha) = \bigoplus_{n \geq 0} H_n$ and we denote $u = \sum_n u_n$ the corresponding orthogonal decomposition.

Observe that the subspaces H_n are invariant for the symmetric part of the generator, i.e. for the symmetric simple exclusion, in fact

$$\begin{aligned}
L^s u_n(\eta) &= \frac{1}{2} \sum_{x \neq y} \sum_{|A|=n} s(x-y)[\Psi_A(\eta^{x,y}) - \Psi_A(\eta)]\mathfrak{u}(A) \\
&= \frac{1}{2} \sum_{x \neq y} \sum_{|A|=n} s(x-y)[\Psi_{A^{x,y}} - \Psi_A]\mathfrak{u}(A) \\
&= \frac{1}{2} \sum_{x \neq y} \sum_{|A|=n} s(x-y)[\mathfrak{u}(A^{x,y}) - \mathfrak{u}(A)]\Psi_A \\
&= \sum_{|A|=n} \left\{ \sum_{x \in A, y \notin A} s(x-y)[\mathfrak{u}(A^{x,y}) - \mathfrak{u}(A)] \right\} \Psi_A
\end{aligned}$$

where $A^{x,y} = \begin{cases} A \backslash x \cup y & \text{if } x \in A, y \notin A \\ A \backslash y \cup x & \text{if } y \in A, x \notin A \\ A & \text{if either } x, y \in A \text{ or } x, y \notin A \end{cases}$

The function $\mathfrak{u}_n$ can be seen as a symmetric function of the configuration of these random walks, so we do not need to label the particles. Then the above equation can be written as

$$L^s u_n(\eta) = \sum_{|A|=n} \mathfrak{L}^s_n \mathfrak{u}(A) \ \Psi_A$$

where $\mathfrak{L}^s_n$ is the generator of n symmetric random walks on $\mathbb{Z}^d$ with exclusion rule. This is the Spitzer self-duality property of the **symmetric** simple exclusion.

This system of random walks is transient for $nd \geq 3$ and recurrent for $nd \leq 2$ (cf. the definition in section 4.1 of the introduction). It follows that

$$H_n \subset \mathbb{H}_{-1} \qquad \forall n \text{ if } d \geq 3$$

and all $n \geq 2$ if $d = 2$, $n \geq 3$ if $d = 1$. In fact let us call $p_t^n(A, A')$ the transition probability for these random walks, i.e. the probability that they jump from the configuration A to A' at time t, and let be $g_\lambda^n(A, A') = \int_0^\infty e^{-\lambda t} p_t^n(A, A') dt$. Then one can compute explicitly

$$(\lambda - L^s)^{-1} u_n = \sum_{|A|=n} \mathfrak{u}(A) \sum_{|A'|=n} g_\lambda^n(A, A') \Psi_{A'}$$

and

$$< u_n, (\lambda - L^s)^{-1} u_n >= \sum_{|A|=n, |A'|=n} \mathfrak{u}(A)\mathfrak{u}(A') g_\lambda^n(A, A')$$

that remains finite as $\lambda \to 0$ if $dn \geq 3$.

The function g given by (3.5.1) is in H_1, so if $d \geq 3$ it is automatically in $\mathbb{H}_{-1}$. Observe that now the space $\mathbb{H}_{-1}$ is related to the inner product $< \cdot, (-\mathcal{L}_0^s)^{-1} \cdot >$, that is bigger than $< \cdot, (-\mathcal{L}_0^s - \mathcal{L}_\tau^s)^{-1} \cdot >$ considered for the tagged particle process. So g is also in the $\mathbb{H}_{-1}$ space associated to the tagged particle process.

This duality property depends strongly on the symmetry of p, and is not true for the ASEP. In fact the antisymmetric part of the generator L^a does not preserve the order of the functions, but we will show that it does not mess it up too much. It can be written as

$$\begin{aligned} L^a u(\eta) &= \frac{1}{2} \sum_{x \neq y} a(x-y)(\eta(x) - \eta(y))(u(\eta^{x,y}) - u(\eta)) \\ &= \frac{1}{2} \sum_{x \neq y} \sum_A a(x-y)(\eta(x) - \eta(y))(\Psi_{A^{x,y}} - \Psi_A)\mathfrak{u}(A) \\ &= \frac{1}{2} \sum_{x \neq y} \sum_A a(x-y)\sqrt{\alpha(1-\alpha)}(\xi_x - \xi_y)(\Psi_{A^{x,y}} - \Psi_A)\mathfrak{u}(A) \ . \end{aligned}$$

Since

$$\xi_x^2 = 1 + \frac{1 - 2\alpha}{\sqrt{\alpha(1-\alpha)}} \xi_x$$

after a patient computation we have

$$\begin{aligned} &\sqrt{\alpha(1-\alpha)}(\xi_x - \xi_y)(\xi_{A^{x,y}} - \Psi_A) \\ &\quad = 1_{[x \in A, y \notin A]} \left[2\sqrt{\alpha(1-\alpha)}(\xi_{A \cup y} - \xi_{A \setminus x}) - (1-2\alpha)(\xi_{A \setminus x \cup y} - \Psi_A) \right] \\ &\quad + 1_{[x \notin A, y \in A]} \left[2\sqrt{\alpha(1-\alpha)}(-\xi_{A \cup x} + \xi_{A \setminus x}) + (1-2\alpha)(\xi_{A \setminus y \cup x} - \Psi_A) \right] \ . \end{aligned}$$

Putting all together, we can write the complete generator acting on H_n as the sum of three terms

$$Lu_n = B_{n,n-1}u_n + B_{n,n}u_n + B_{n,n+1}u_n$$

with

$$B_{n,n-1}u_n = \sqrt{\alpha(1-\alpha)} \sum_{x \neq y} a(x-y) \sum_{\substack{A \not\ni x, A \not\ni y \\ |A|=n-1}} (\mathfrak{u}(A \cup y) - \mathfrak{u}(A \cup x))\Psi_A$$

$$B_{n,n+1}u_n = \sqrt{\alpha(1-\alpha)} \sum_{x \neq y} a(x-y) \sum_{\substack{A \ni x, A \ni y \\ |A|=n+1}} (\mathfrak{u}(A \backslash y) - \mathfrak{u}(A \backslash x))\Psi_A$$

$$B_{n,n}u_n = L^s u_n + (2\alpha - 1) \sum_{x \neq y} a(x-y) \sum_{\substack{A \ni x, A \not\ni y \\ |A|=n}} (\mathfrak{u}(A \backslash x \cup y) - \mathfrak{u}(A))\Psi_A.$$

Observe that $B_{n,n\pm1} : H_n \to H_{n\pm1}$ and $B_{n,n} : H_n \to H_n$. By using transience estimates for random walks one can prove the following theorem (cf. [24]):

Theorem 3.5.1. *If* $d \geq 3$

$$\|B_{n,n\pm1}u_n\|_{-1} \leq C\sqrt{n}\sqrt{\alpha(1-\alpha)}\|u_n\|_1 \ . \tag{3.5.4}$$

This shows that, at least if the density $\alpha = 1/2$, the ASEP satisfies the graded sector condition, and consequently CLT holds for any local function with zero average (since we have shown that they are all in $\mathbb{H}_{-1}$).

In the case $\alpha \neq 1/2$, an unbounded term appears in the diagonal term $B_{n,n}$, and some coupling technique are needed to deal with this term (cf. [24], section 6).

3.6 Other Interacting Particles Systems

The method is very robust in all reversible cases, all one needs to prove is ergodicity on the invariant/reversible measure and that the rate function g we are interested is in $\mathbb{H}_{-1}$.

In the lattice case this is the case for all reversible speed change exclusion models, i.e. exclusion models with a local interaction

$$(Lf)(\eta) = \sum_{x,y \in \mathbb{Z}^d} c(y, \tau_x \eta)[f(\sigma^{x,x+y}\eta) - f(\eta)] \ .$$

with the rates satisfying a detailed balance condition with respect to some Hamiltonian $\mathcal{H}$

$$c(y, \sigma^{0,y}\eta) = c(y, \eta) e^{\mathcal{H}(\sigma^{0,y}\eta) - \mathcal{H}(\eta)} \tag{3.6.1}$$

which guarantees that the corresponding Gibbs measure is reversible. The rate function is now given by

$$g(\xi) = \sum_{z} c(z, \xi)(1 - \xi(z)) z \cdot v$$

and using (3.6.1) one can show that $g \in \mathbb{H}_{-1}$.

In general as soon a *drift* is added to these speed change models, the Gibbs measure associated to $\mathcal{H}$ is not anymore invariant, and in fact we do not know the invariant measures explicitly (we have been very lucky in the ASEP!). So for these models, also called *driven lattice gases*, even to prove the law of large numbers or any ergodic behavior is a challenging problem (cf. [4]).

An exception are the so-called *gradient models*: in the speed change class these are models such that the instantaneous *current* between two bonds

$$j_{x,x+y} = c(y, \tau_x \eta)(\eta(x+y) - \eta(x))$$

can be written as $h(\tau_{x+y}\eta) - h(\tau_x \eta)$ for some local function h.

About interacting particles in continuous space, an interesting gradient model in given by the interacting Brownian motions. This is given by the solution of the infinite system of stochastic differential equations on $\mathbb{R}^d$:

$$dx_j(t) = -\frac{1}{\gamma} \sum_{i \neq j} \nabla V(x_j(t) - x_i(t))\, dt \; + \sqrt{\frac{2}{\gamma\beta}}\, dw_j(t) \tag{3.6.2}$$

Here V is a superstable 2-body interaction (cf. [23]), and $w_j(t)$ are independent standard Wiener processes. The grand-canonical Gibbs measures $\mu_{z,\beta}$, associated to the interaction V, temperature β^{-1} and activity z, are reversible and ergodic. This is a model for a system of particles in a fluid in equilibrium at temperature β^{-1}. The parameter γ depends on the strength of the viscous interaction between the fluid and the particles.

Let $x_0(t)$ the position at time t of the tagged particle. The environment as seen from the tag is given by

$$y_j(t) = x_j(t) - x_0(t)$$

This way the environment *drives* the tagged particle, that will satisfy the equation

$$dx_0(t) = -\frac{1}{\gamma}\sum_{j\neq 0} \nabla V(-y_j(t))\, dt + \sqrt{\frac{2}{\gamma\beta}}\, dw_0(t) \tag{3.6.3}$$

The environment itself evolves autonomously following the stochastic differential equations

$$\begin{aligned} dy_j(t) = \\ -\frac{1}{\gamma}\left[\sum_{i\neq j} \nabla V(y_j(t) - y_i(t)) - \sum_{j\neq 0} \nabla V(-y_j(t)) + \nabla V(y_j(t))\right] dt \\ + \sqrt{\frac{2}{\gamma\beta}}\, (dw_j(t) - dw_0(t)) \end{aligned} \tag{3.6.4}$$

Consider now the Hamiltonian

$$\mathcal{H}_0(\eta) = \sum_{j\neq i} V(y_j - y_i) + \sum_j V(y_j) \tag{3.6.5}$$

One can construct a grand canonical Gibbs measure $\mu^0_{z,\beta}$ corresponding to this Hamiltonian, that is called the Palm measure associated to $\mu_{z,\beta}$. Observe that $\mu^0_{z,\beta}$ is not translation invariant. This measure is reversible and ergodic for the process $\eta(t) = \{y_j(t)\}_j$. In fact the generator of the environment process can be written as

$$L^0_{IB} = \frac{1}{\gamma\beta}\left[e^{\beta\mathcal{H}_0}\mathbf{D}\left(e^{-\beta\mathcal{H}_0}\mathbf{D}\right) + \sum_j e^{\beta\mathcal{H}_0}\nabla_{y_j}\left(e^{-\beta\mathcal{H}_0}\nabla_{y_j}\right)\right] \tag{3.6.6}$$

where $\mathbf{D} = \sum_j \nabla_{y_j}$, i.e. the generators of the translations. The first term in (3.6.6) is the contribution given by the tagged particle to the movement of the environment.

We consider now the tagged particle *in equilibrium*, i.e. we start by convention at $x_0(0) = 0$ and we distribute the environment $\eta = \{y_j(0)\}_j$ according to the Palm measure $\mu^0_{z,\beta}$. The diffusely rescaled position of the tagged particle is

$$x_0^\epsilon(t) = \epsilon x_0(\epsilon^{-2}t) = \epsilon \int_0^{\epsilon^{-2}t} \sum_j \frac{1}{\gamma} \nabla V(y_j(s))\, ds + \sqrt{\frac{2}{\gamma\beta}} \epsilon w_0(\epsilon^{-2}t) \tag{3.6.7}$$

We are in the same situation as in the symmetric simple exclusion process, but here the function g is given by

$$g(\eta) = \sum_j \frac{1}{\gamma} \nabla V(y_j(s)) = \mathbf{D}\mathcal{H}_0(\eta)$$

So it follows by an easy integration by parts that

$$\left| \int f(\eta) g(\eta)\, d\mu^0_{z,\beta} \right| = \left| \int \mathbf{D} f(\eta)\, d\mu^0_{z,\beta} \right| \le \sqrt{\int (\mathbf{D} f(\eta))^2\, d\mu^0_{z,\beta}} \le \|f\|_1$$

so that $g \in \mathbb{H}_{-1}$. Since $\mu^0_{z,\beta}$ is reversible, the central limit theorem for $x_0^\epsilon(t)$ follows.

3.7 Diffusion in Random Environment

Consider the previous example of the tagged particle in the interacting Brownian motions. If we *freeze* the environment in a random configuration η distributed by the Palm measure $\mu^0_{z,\beta}$, we obtain a diffusion in a (static) random environment. The CLT for these kind of diffusions in static random environment has been a classic problem in stochastic homogenization before the Kipnis-Varadhan paper (cf. Kozlov [11], Papanicolaou-Varadhan [20]). Here is the general setup:

Let $(\Omega, \mathcal{G}, \nu)$ be a probability space and $G = \{\tau_x\, ; (x) \in \mathbb{R}^d\}$ be a group of measure preserving transformations acting ergodically on Ω. Denote by $L^2(\nu)$ the space of square integrable functions and define on $L^2(\nu)$ the operators $\{T_x\, , (x) \in \mathbb{R}^d\}$ given by

$$T_x f(\omega) = f(\tau_x \omega)\ .$$

Assume that $T_x f(\omega)$ is jointly measurable in $\mathbb{R}^d \times \Omega$ for each measurable function f.

It follows from these assumptions that $\{T_x\, , x \in \mathbb{R}^d\}$ is a group of strongly continuous unitary operators on $L^2\,(\Omega, \mathcal{G}, \nu)$. For every $\tilde{f}$ in $L^2(\nu)$, let $f(x,\omega) = \tilde{f}(\tau_x \omega)$. Each function $\tilde{f}$ in $L^2(\nu)$ defines in this way a stationary ergodic random field on $\mathbb{R}^d$. Reciprocally, given a stationary ergodic random field one can always find a probability space where such a representation is possible.

Denote by D_i, $1 \le i \le d$ the infinitesimal generators of $\{T_x,\, x \in \mathbb{R}^d\}$:

$$D_i \;=\; \frac{\partial}{\partial x_i}\, T_x\Big|_{x=0} \;.$$

These infinitesimal generators are closed and densely defined on $L^2(\nu)$.

For a given random stationary diffusion matrix $\sigma_{i,j}(x,\omega) = \tilde{\sigma}_{i,j}(\tau_x\omega)$ and a random stationary drift $b_i(x,\omega) = \tilde{b}_i(\tau_x\omega)$ we want to consider the SDE:

$$dy(t) = \sqrt{2}\sigma(y(t),\omega)\; dw_t + b(y(t),\omega)\; dt \tag{3.7.1}$$

under the standard conditions for the existence of a global solutions for (3.7.1) verified by σ and b for almost every ω with respect to ν.

Here the process as seen from the particle is given by

$$\eta_t = \tau_{y(t)}\omega$$

which is be a Markov process on Ω with generator

$$L = \sum_{i,j} \tilde{a}_{i,j}(\eta) D_{i,j} + \tilde{b}(\eta)\cdot D$$

where $\tilde{a} = \tilde{\sigma}^*\tilde{\sigma}$. In this generality one does not know the invariant measure for η_t. In order to know explicitly the invariant measure, we assume that there exist a *smooth* function $\tilde{V}(\omega)$ such that $\int e^{-\tilde{V}} d\nu < \infty$ and a *smooth* matrix valued function $\tilde{a}(\omega)$ such that

$$\tilde{b}_j = \sum_i \left(D_i \tilde{a}_{i,j} - \tilde{a}_{j,i} D_i \tilde{V} \right)$$

so the generator L can be rewritten as

$$L = e^{\tilde{V}} D \cdot e^{-\tilde{V}} \tilde{a} D \;.$$

The matrix $\tilde{a}$ can always be written as $\tilde{a} = S + H$, where S is a symmetric matrix that we assume strictly positive, and H is antisymmetric.

The probability measure

$$d\mu = \frac{e^{-\tilde{V}}}{\int e^{-\tilde{V}} d\nu} d\nu$$

is then invariant for η_t. It is immediate to see that

$$L^s = e^{\tilde{V}} D \cdot e^{-\tilde{V}} SD$$

is the symmetric part of L with respect to $d\mu$.

Furthermore one can show that under the condition that $s(\omega) \geq C > 0$, the measure $d\mu$ is also ergodic.

In order to apply the method exposed in the first section, we need first to check that $\tilde{b}_j \in \mathbb{H}_{-1}$. By integrating by parts

$$\int \tilde{f}\tilde{b}_j \, d\mu = \sum_i \int \tilde{f} \left(D_i \tilde{a}_{i,j} - \tilde{a}_{j,i} D_i \tilde{V} \right) \tilde{f} \frac{e^{-\tilde{V}}}{\int e^{-\tilde{V}} d\nu} \, d\nu$$

$$= \sum_i \int \tilde{a}_{i,j} D_i \tilde{f} \, d\mu \leq \|\tilde{a}\|_{L^2(\mu)} \sqrt{\int |D\tilde{f}|^2 \, d\mu} \leq C^{-1} \|\tilde{a}\|_{L^2(\mu)} \|\tilde{f}\|_1 \ .$$

So if we to assume $\tilde{a} \in L^2(\mu)$ we have $\tilde{b}_j \in \mathbb{H}_{-1}$.

If $\tilde{a}$ is symmetric the measure μ is reversible and to prove the CLT for $y(t)$ all we need is $\tilde{a} \in L^2(\mu)$ (beside the regularity condition for the existence of the process). This condition can be weakened further (cf. [2, 19]).

If $\tilde{a}$ is not symmetric, the **strong sector condition** will be verified if there exists a constant $C > 0$ such that

$$|H|(\omega) \leq CS(\omega) \qquad \mu - \text{a.e.} \tag{3.7.2}$$

where $|H|$ indicates the positive matrix $\sqrt{-H^2}$, and the inequality is intended in the sense of the corresponding positive symmetric forms. In fact for any $u, v \in \mathbb{R}^d$

$$|v \cdot Hu| \leq (v \cdot |H|v)^{1/2} (u \cdot |H|u)^{1/2} \leq C(v \cdot Sv)^{1/2} (u \cdot Su)^{1/2} \ .$$

It follows that

$$-\int \tilde{f} L\tilde{g} \, d\mu = \int D\tilde{f} \cdot (S + H) D\tilde{g} \, d\mu$$

$$\leq (1 + C) \sqrt{\int D\tilde{f} \cdot SD\tilde{f} \, d\mu} \sqrt{\int D\tilde{g} \cdot SD\tilde{g} \, d\mu} = (1 + C) \|\tilde{f}\|_1 \|\tilde{g}\|_1 \ . \tag{3.7.3}$$

This is as far the *soft* general methods go. If we look at the situation when $S = Id$, then (3.7.2) implies that $H \in L^\infty(\mu)$. But assuming only $H \in L^2$ on has that $b_j = \sum_i D_i H_{i,j} \in \mathbb{H}_{-1}$, and in fact we can prove the CLT (cf. [18]) by doing some cutoffs.

3.8 Diffusion in Gaussian Random Fields

Assuming in the above model that $S = id$ and that H is a *matrix valued stationary Gaussian field* then we have another example of a process satisfying the **graded sector conditions** (cf. [10]). The Wiener chaos gives the corresponding orthogonal decomposition of $L^2(\mu)$.

By assuming that the random drift is Gaussian we mean that the space $\mathcal{H}$ - the L^2 closure of the random vectors $b(\varphi)(\omega) := \int \varphi(x)b(x;\omega)dx$, with $\varphi \in \mathcal{S}(R^d)$ - is a Gaussian Hilbert space i.e. all finite sets of random vectors from $\mathcal{H}$ are normally distributed, see e.g. [6] Definition 1.2 p. 4.

By $\mathcal{P}_n(\mathcal{H})$ we denote the space of n-th degree polynomials formed over the elements of $\mathcal{H}$. We let H_0 be the space of constants and $H_n := \mathcal{P}_n(\mathcal{H}) \ominus \mathcal{P}_{n-1}(\mathcal{H})$. The elements of H_n are sometimes called Hermite polynomials of degree n. It is well known, see e.g. Theorem 2.6 of [6] that $L^2 = \bigoplus_{n=0}^{\infty} H_n$. Going back to (3.7.3), we need to estimate

$$\int Df_n \cdot HDg \, d\mu$$

for $f \in H_n \cup \mathcal{D}(D)$ and $g \in \mathcal{D}(D)$. By Hölder inequality the absolute value is less than or equal to

$$\|HDf\|_{L^2}\|Dg\|_{L^2} \leq \|H\|_{L^{2n}}\|Df\|_{L^{2n/(n-1)}}\|Dg\|_{L^2} \leq \left(\frac{n+1}{n-1}\right)^{n/2} \|H\|_{L^{2n}}\|Df\|_{L^2}\|Dg\|_{L^2},$$

by virtue of the hyper-contractivity estimate of L^p norms on Gaussian spaces, see [6] Theorem 5.10. Notice that by Stirling's formula $\| H\|_{L^{2n}} \sim \sqrt{n}$, thus

$$\left|\int Df_n \cdot HDg \, d\mu\right| \leq C\sqrt{n}\|f_n\|_1\|g\|_1$$

that implies the graded sector condition with $\beta = 1/2$.

4. Some Models Without Sector Condition

Here are two processes for which we know the invariant measure, but the are kind of degenerate and they do not satisfy any sector condition. In this two cases a CLT is proven, but one needs to use the special features of the processes.

4.1 Diffusion in a time-dependent Divergence-Free Flow

Consider a stationary space-time vector valued random field $b(t, x, \omega)$ realized on a probability space $(\Omega, \mathcal{F}, \mu)$, such that $\nabla_x \cdot b(t, x, \omega) = 0$ μ-a.e. This implies that there exists a *stochastically continuous* group of *measure preserving* transformations $\{\tau_{t,x}; (t, x) \in \mathbb{R} \times \mathbb{R}^d\}$ acting *ergodically* on Ω, and such that $b(t, x, \omega) = \tilde{b}(\tau_{t,x}\omega)$ for some measurable function $\tilde{b}$ on Ω.

Then we consider the SDE

$$dx(t) = b(t, x(t); \omega)\ dt + \sqrt{2}dw(t) \tag{4.1.1}$$

Now the environment as seen from the particle has to be defined as

$$\eta(t) = \tau_{t,x(t)}\omega \tag{4.1.2}$$

We assume here that there exists an antisymmetric matrix valued function $\tilde{H}(\omega)$ on Ω such that $\tilde{H}_{i,j} \in L^2(\Omega)$ and $\tilde{b}_j = \sum_i D_i \tilde{H}_{i,j}$. One has also to assume that $b(t, x; \omega)$ is locally Lipschitz in x for a.e. ω.

We are pretty much in the same framework as in the static field case, but here one has to take into account also the translation in the time direction. Here η_t is still a Markov process with generator

$$L = D^2 + D_t + \tilde{b}(\omega) \cdot D = D^2 + D_t + D \cdot \tilde{H}D$$

where D_t is the generator of the translation in the time direction. The measure μ is stationary and one can prove here that it is ergodic. But one can see immediately that L is degenerate in the time direction. So there is no hope that it can satisfy any sector condition, not even in the graded sense. Furthermore $\tilde{H}$ is only in L^2 and not bounded, so even the static case is not included in the previous theory.

Still, under the above conditions, it can be proven a CLT for $x(t)$ (cf. [9]). Here the strategy is to prove directly that, for the solution of the resolvent equation

$$\lambda u_\lambda - L u_\lambda = \tilde{b}_j$$

one has $\lambda \int u_\lambda^2 \, d\mu \to 0$ as $\lambda \to 0$. (See details in [9]).

4.2 Ornstein-Uhlenbeck Process in a Random Potential

Let $V(x, \omega) = V(\tau_x \omega)$ a stationary random potential on $\mathbb{R}^d$ as in the previous section. Then consider

$$\begin{aligned} dx(t) &= v(t) \, dt \\ dv(t) &= -v(t) \, dt - \nabla_x V(x(t), \omega) dt + \sqrt{2} dw_t \end{aligned}$$

Then one would like to prove the CLT for the rescaled position

$$x^\epsilon(t) = \epsilon x(\epsilon^{-2} t) = \epsilon \int_0^{\epsilon^{-2} t} v(s) ds \tag{4.2.1}$$

The environment as seen from the particle should here keep track of the velocity on the particle, so this is given by $\{\eta_t = \tau_{x(t)} \omega, v(t)\}$, the Markov process on $\Omega \times \mathbb{R}^d$ with generator

$$\begin{aligned} L &= L^s + L^a \\ L^s &= \partial_v^2 - v \cdot \partial_v \\ L^a &= v \cdot D - DV(\omega) \cdot \partial_v \end{aligned} \tag{4.2.2}$$

The invariant measure is given by

$$d\mu(\omega, v) = \frac{e^{-v^2/2}}{(2\pi)^{d/2}} \frac{e^{-V(\omega)}}{Z} \, dv \, d\nu(\omega)$$

where Z is the obvious normalization factor.

As before here one would like to consider the resolvent equation

$$\lambda u_\lambda - L u_\lambda = v$$

and prove that $\lambda \int u_\lambda^2 \, d\mu \to 0$ as $\lambda \to 0$. This is actually proven in [21] but under the strong condition that $DV(\omega)$ is bounded. This condition makes inapplicable the approach of [21] to the corresponding problem of the tagged particle in a system of interacting Ornstein-Uhlenbeck particles.

5. Approximation, Regularity and some Open Problems

In the previous sections I exposed some general methods for obtaining central limit theorems for Markov processes and some applications. These methods gives little information for more concrete questions that arise in the applications. Two natural questions are the following:

REGULARITY. In general the effective diffusion coefficients obtained are complicate and non-explicit functions of the various parameters appearing in the *microscopic* dynamics (like the probability transition rates, the density of the other particles in the tagged particle problems, etc). One would like to know when these effective diffusion coefficients are smooth functions of these parameters. For example the self-diffusion coefficient of the tagged particle appears in the macroscopic non-linear diffusion equations in certain hydrodynamic limits (cf.[22]). So to give strong sense to these non-linear equations one would like to prove that the self-diffusion coefficient, $D(\alpha)$ is, at least, a differentiable function of the density α.

APPROXIMATION. Another related problem is the approximation of these effective diffusion coefficient. If one considers a *finite dimensional* or *periodic* approximation of the microscopic dynamics, do the corresponding effective diffusion coefficients converge to those related to the *infinite* system? This question is relevant for numerical approximations, but also in other contexts (existence of conductivity in percolation, cf.[7, 2], smoothness of the surface tension in massless field models, cf.[5]).

5.1 Regularity of Self-Diffusion in Simple Exclusion

The duality approach exposed in the previous chapter is a good tool for studying these regularity and finite dimensional problems. In fact it gives some more detailed information on the solution of the resolvent equation than just the convergence in an abstract Dirichlet space.

In [15] is proven that the self-diffusion coefficient $D(\alpha)$ for the tagged particle in the symmetric simple exclusion (cf. (3.3.2) and (3.3.3)) is a $\mathcal{C}^\infty$ function of the density α in the interval $[0,1]$. The method extends to the asymmetric case (cf. [16]).

5.2 Finite Dimensional Approximation of Self-Diffusion in Simple Exclusion

Consider a finite dimensional version of the symmetric simple exclusion process on the torus $\{-N,\dots,0,\dots,N\}^d$ (i.e. with periodic boundary conditions, preserving in this manner the translation symmetry). Since we want to work with an ergodic process, we also fix the total number K of particles. Consider now a tagged particle in this finite system. If N is much larger than the size of a single jump, the motion of the tagged particle has a unique canonical lifting to $\mathbb{Z}^d$. We get in this manner a process $X_N(t)$ with values in $\mathbb{Z}^d$. Let us denote by $D_{[N,K]}$ the variance of the Brownian motion which is the limit of the scaled process $\varepsilon X_N(\varepsilon^{-2}t)$ as $\varepsilon \to 0$. We expect that

$$\lim_{\substack{N\to\infty \\ K/(2N)^d\to\alpha}} D_{[N,K]} = D(\alpha) \ . \tag{5.2.1}$$

This is proven in this context in [14].

For random walk in random environment, similar results are recently proven in [1], under the condition of independence on the environments rates of jump. In this case it is possible also to establish an exponential convergence rate.

5.3 Open Problem: Breaking the Translation Invariance Symmetry

Consider the symmetric simple exclusion on the positive integers $\mathbb{Z}_+$ with reflecting boundary at 0. The product measures are still invariant and reversible, and one expects that the tagged particle would converge to the Brownian motion with reflection in 0. But the lack of translation invariance of the system does not allow to apply the Kipnis-Varadhan approach.

References

1. Caputo P., Ioffe D.; Finite volume approximation of the effective diffusion matrix: the case of independent bond disorder. Preprint (2001), to appear in Annals I. H. Poincaré, Probabilites et Statistiques.
2. De Masi A., Ferrari P.A., Goldstein S., Wick W.D.; An Invariance principle for reversible Markov processes. Applications to random motions in random environments. J. Stat. Phys. **55**, nos.3/4, 787–855 (1989).
3. Ethier S. N., Kurtz T.G., *Markov Processes, Characterization and Convergence*, J. Wiley & sons, New York 1986.
4. Eyink G.L., Lebowitz J.L., Spohn H.; Hydrodynamics and fluctuation outside of local equilibrium: driven diffusive systems. J. Stat. Phys. **61**, 533-572 (1996).
5. Giacomin G., Olla S., Spohn H., Equilibrium fluctuations for $\nabla\phi$ interface models. Annals of Probability, Vol. 29, No. 3, 1138-1172 (2001).
6. Janson S., *Gaussian Hilbert Spaces*, Cambridge University Press, 1997.
7. Kesten H.; *Percolation Theory for Mathematicians*, Birkhausen (1982).
8. Kipnis C., Varadhan S. R. S.; Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusion. Commun. Math. Phys. **106** 1–19 (1986).
9. Komorowski T., Olla S.; On Homogenization of Time Dependent Random Flows, Probability Theory and Related Fields, **121**, 98-116, (2001).
10. Komorowski T., Olla S.: On the Sector Condition and Homogenization of Diffusions with a Gaussian Drift, preprint (2001), to appear in J. of Functional Analisys.
11. Kozlov S.M., Averaging of Random Operator, Math. USSR Sbornik, **37**, No.2, 167-180 (1980).
12. Landim C., Olla S., Yau H. T.; Some properties of the diffusion coefficient for asymmetric simple exclusion processes. Annals of Probability **24**, 1779–1807, (1996).
13. Landim C., Yau H. T.; Fluctuation–dissipation equation of asymmetric simple exclusion processes, Probab. Th. Rel. Fields **108**, 321–356, (1997).
14. Landim C., Olla S., Varadhan S. R. S.; Finite-dimensional approximation of the self-diffusion coefficient for the exclusion process. Annals of Prob. vol. 30, No. 2, 1-26 (2002).
15. Landim C., Olla S., Varadhan S. R. S.; Symmetric simple exclusion: regularity of the self-diffusion coefficient, Comm. Math. Phys., **224**, 302-321 (2001).
16. Landim C., Olla S., Varadhan S. R. S.; On the viscosity and fluctuation dissipation in the exclusion processes. preprint (2001).
17. Liggett T.; *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Grund.Math.Wiss. vol. 324, Springer 1999.
18. Oelschlager K., Homogenization of a diffusion process in a divergence–free random field, Ann.Prob. **16**, No.3, 1084-1126, (1988).

19. Osada H., Saitoh, T., An invariance principle for non-symmetric Markov processes and reflecting diffusions in random domains. Prob. Theory and Relat. Fields, **101**, 45-63. (1995).
20. Papanicolau G., Varadhan S.R.S., Boundary Value Problems with Rapidly Oscillating Random Coefficients, Colloquia Mathematica Societatis János Bolay, 27. Random Fields, Esztergom (Hungary),835-873 (1979).
21. Papanicolaou G.C., Varadhan, S.R.S.: Ornstein-Uhlenbeck process in random potential, Comm.Pure Appl.Math. **38**, 819-934, (1985).
22. Quastel J.; Diffusion of color in the simple exclusion process , Comm. Pure Appl. Math, XLV, 623-679 (1992).
23. Ruelle, D.: *Statistical mechanics: Rigorous results*, W.A. Benjamin, Inc., New-York-Amsterdam (1969).
24. Sethuraman S., Varadhan S. R. S., Yau H. T.; Diffusive limit of a tagged particle in asymmetric exclusion process. Comm.Pure Appl.Math. **53**, 972-1006 (2000).
25. Varadhan S.R.S.; Self diffusion of a tagged particle in equilibrium for asymmetric mean zero random walks with simple exclusion, Ann. Inst. H. Poincaré (Probabilités), **31**, 273–285. (1996)
26. Varadhan S.R.S.; Regularity of the self-diffusion coefficient. The Dynkin Festschrift, 387–397, Progr. Probab. **34**, Birkhäuser Boston, Boston, MA, (1994).