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Equilibrium fluctuations of asymmetric simple exclusion processes in dimension $d \geq 3$

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Abstract. We consider an asymmetric exclusion process in dimension $d \geq 3$ under diffusive rescaling starting from the Bernoulli product measure with density $0 < \alpha < 1$. We prove that the density fluctuation field Y_t^N converges to a generalized Ornstein–Uhlenbeck process, which is formally the solution of the stochastic differential equation $dY_t = \mathscr{A} Y_t dt + dB_t^{\nabla}$, where $\mathscr{A}$ is a second order differential operator and B_t^{∇} is a mean zero Gaussian field with known covariances.

Introduction

Consider the asymmetric simple exclusion process. It has been proved by Rezakhanlou [R] that under Euler time scaling the macroscopic behavior of the system is described by the entropy solution of a quasi-linear first order hyperbolic equation. A natural question is then to investigate the equilibrium fluctuations around the hydrodynamic limit. It is not hard to show that the equilibrium fluctuations in this time scale are trivial in the sense they are produced by the initial data and are rigidly transported. Such phenomena were already pointed out by Gärtner and Presutti [GP] and Ferrari and Fontes [FF] for one-dimensional nearest-neighbor asymmetric exclusion processes.

Therefore, to observe non trivial fluctuations of the density field one needs to examine the process in a longer time scale. Recently, Esposito, Marra and Yau [EMY] considered the incompressible limit of the system. They proved that in dimension $d \geq 3$, starting from a small perturbation of an invariant state, under diffusive time scaling, the perturbation evolves according to a parabolic equation. This result leads to the investigation of the fluctuations in the diffusive scaling.

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The main result of this article shows that in equilibrium the density fluctuation field diffusively rescaled evolves according to a stationary generalized Ornstein-Uhlenbeck process with given covariances. It relates the covariance of the equilibrium density fluctuation to the diffusion coefficient of the hydrodynamic equation, a parameter determined by the non equilibrium evolution. In the mathematical physics literature this result is called a fluctuation–dissipation theorem since it connects the non equilibrium dissipative feature of the system to its equilibrium fluctuations.

The first result concerning the equilibrium fluctuations around the hydrodynamic limit was obtained by Brox and Rost [BR] for symmetric zero range processes. The main step in the proof was what the authors called the Boltzmann–Gibbs principle. It starts from the observation that all local fields associated to non-conserved quantities evolve on a faster scale than the density field and it states that the local fields are projected on the density field. Chang and Yau [CY] and Chang [C1] proposed an alternative method for proving the Boltzmann–Gibbs principle for gradient systems. This approach was adapted to nongradient systems by Lu [L], Chang [C2] and Sellami [S]. We extend further the method to asymmetric systems.

The article is conceived as follows. In section 1 we introduce the notation and we state the main result of the article. In section 2 we recall some facts about the diffusive behavior of the asymmetric exclusion process. In section 3 we prove the main result, postponing to section 4 the proof of the Boltzmann–Gibbs principle and to section 5 the problem of tightness.

1. Notation and results

Let $\mathbb{T}^d$ be the d-dimensional torus $[0, 1)^d$. For each positive integer N, denote by $\mathbb{T}_N^d$ the discrete d-dimensional torus with N^d points: $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$. Points of $\mathbb{T}^d$ are denoted by the letter u while sites of $\mathbb{T}_N^d$ are denoted by the letters x, y, z. We consider on $\mathbb{T}_N^d$ a stochastic dynamics that can be informally described as follows. We start placing particles on $\mathbb{T}_N^d$ in such a way that each site is occupied by at most one particle. Denote by η the configuration of particles obtained in this way so that, for each site x in $\mathbb{T}_N^d$, $\eta(x) = 1$ if the site x is occupied and $\eta(x) = 0$ otherwise. The configuration space is thus $\mathcal{E}_N = \{0, 1\}^{\mathbb{T}_N^d}$. Configurations of $\mathcal{E}_N$ are denoted by the letters η, ξ. Fix a transition probability p on $\mathbb{Z}^d$. A particle sitting at site x waits a mean one exponential time at the end of which it tries to jump to site $x + y$ with probability $p(y)$. If the site $x + y$ is occupied, to respect the exclusion rule that forbids more than one particle per site, the jump is suppressed, otherwise the particle jumps. The evolution just described is a Markov process whose generator L_N acts on functions $f : \mathcal{E}_N \to \mathbb{R}$ as

$$(L_N f)(\eta) = \sum_{x,y \in \mathbb{T}_N^d} p(y)\eta(x)[1 - \eta(x + y)][f(\eta^{x,x+y}) - f(\eta)],$$

where $\eta^{x,x+y}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x + y)$:

$$\eta^{x,x+y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, x+y, \\ \eta(x+y) & \text{if } z = x, \\ \eta(x) & \text{if } z = x+y\ . \end{cases}$$

This is the so-called simple exclusion process. We say it is symmetric if the transition probability $p(\cdot)$ is symmetric and asymmetric if not. We denote by η_t the state of the process at time t.

We shall assume that jumps occur only to nearest neighbor sites ($p(y) = 0$ if $|y| > 1$), that $p(e_i) + p(-e_i) = 1$ for all i and that the process is asymmetric. In these formulas $|\cdot|$ stands for the norm of the maximum so that $|(x_1, \dots, x_d)| = \max_i |x_i|$ and $e_1, \dots, e_d$ is the canonical basis of $\mathbb{R}^d$. To keep notation simple, let $p_i = p(e_i)$, $q_i = p(-e_i)$.

It is well known [Li] that the asymmetric exclusion model has a one parameter family of invariant measures. For each $0 \le \alpha \le 1$, denote by $\nu_\alpha = \nu_\alpha^N$ the product measure on $\{0, 1\}^{\mathbb{T}_N^d}$ whose marginals are given by

$$\nu_\alpha\{\eta,\ \eta(x) = 1\} = \alpha \quad \text{for all } x \text{ in } \mathbb{T}_N^d .$$

It is easy to check that ν_α is left invariant under the time evolution.

For a configuration η, denote by $\pi^N = \pi^N(\eta)$ the empirical measure associated to η. This is the measure on $\mathbb{T}^d$ obtained assigning mass N^{-d} to each particle of η:

$$\pi^N = N^{-d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N},$$

where δ_u stands for the Dirac measure concentrated on u. For a measure π on $\mathbb{T}^d$ and a continuous function $G\colon \mathbb{T}^d \to \mathbb{R}$, denote by $< \pi, G >$ the integral of G with respect to π. In particular, $< \pi^N, G >$ is equal to $N^{-d} \sum_{x \in \mathbb{T}_N^d} G(x/N)\eta(x)$.

Consider a sequence of probability measures μ^N on $\mathcal{E}_N$ and assume that, under μ^N, π^N converges in probability to an absolutely continuous measure $\rho_0(u)du$. This means that for every continuous function $G\colon \mathbb{T}^d \to \mathbb{R}$ and every $\delta > 0$,

$$\lim_{N \to \infty} \mu^N \left\{ \eta,\ \left| \langle \pi^N, G \rangle - \int G(u) \rho_0(u)\, du \right| > \delta \right\} = 0. \tag{1.1}$$

Consider the hyperbolic equation

$$\begin{cases} \partial_t \rho + \gamma \cdot \nabla \Phi(\rho) = 0 \\ \rho(0, \cdot) = \rho_0(\cdot) \end{cases} \tag{1.2}$$

In this formula γ stands for the drift: $\gamma = \sum_y y p(y)$, Φ stands for the function $\Phi(\alpha) = \alpha(1-\alpha)$ and ∇ for the gradient so that $\nabla\Phi(\rho) = (\partial_{u_1}\Phi(\rho), \dots, \partial_{u_d}\Phi(\rho))$. For $t > 0$, denote by π_t^N the empirical measure associated to the state of the process at time t: $\pi_t^N = \pi^N(\eta_t)$. Rezakhanlou [R] proved that starting from a state μ^N satisfying (1.1) and some additional hypotheses, for every $t \ge 0$, π_{tN}^N converges in probability to the measure $\rho(t, u)du$, where the density ρ is the entropy solution of the hyperbolic equation (1.2). More precisely, for a measure μ on $\mathcal{E}_N$, denote by $\mathbb{P}_\mu$ the measure on the path space $D(\mathbb{R}_+, \mathcal{E}_N)$ induced by the Markov process η_t

and the measure μ. Then, for every $t \geq 0$, every continuous function G and every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left\{ \eta,\ \left| \langle \pi_{tN}^N, G\rangle - \int G(u)\rho(t,u)\, du \right| > \delta \right\} = 0,$$

where the density ρ is the entropy solution of the hyperbolic equation (1.2). Notice the Euler rescaling of time in the previous formula.

To investigate the evolution of the asymmetric simple exclusion process under diffusive scaling, following [EMY], one is led to consider the *incompressible limit* of the empirical measure. To describe this approach, fix a density α in (0, 1) and a profile $\lambda \colon \mathbb{T}^d \to \mathbb{R}$. Denote by $\mu^N_{\lambda(\cdot)}$ the product measure on $\mathscr{E}_N$ whose marginals are given by

$$\mu^N_{\lambda(\cdot)}\{\eta,\ \eta(x) = 1\} = \alpha + N^{-1}\lambda(x/N) \quad \text{for all } x \text{ in } \mathbb{T}^d_N.$$

This is therefore an order N^{-1} perturbation of the invariant state ν_α. If λ is positive, we can think of distributing first class particles with density α and adding a second class particle at site x with probability $N^{-1}\lambda(x/N)$. The dynamics of the first and the second class particles is as follows. Both types of particles jump according to the transition probability $p(\cdot)$. The first class particles have priority over the second class particles so that if a first class particle jumps over a site occupied by a second class particle, they exchange position. In this way both the first class particles and the addition of the first and second class particles evolve as an asymmetric simple exclusion process.

To investigate the evolution of the second class particles, let Π^N be the empirical measure on $\mathbb{T}^d$ defined by

$$\Pi^N = N^{1-d} \sum_{x\in\mathbb{T}^d_N} \{\eta(x) - \alpha\}\delta_{x/N}.$$

Notice that we are multiplying the difference $\eta(x) - \alpha$ by N to compensate the density of order N^{-1} of second class particles.

To examine the evolution of the second class particles, observe that in the presence of a density α of first class particles, a second class particle jumps from x to $x + e_i$ at rate $p_i(1-\alpha) + q_i\alpha$ and it jumps from x to $x - e_i$ at rate $q_i(1-\alpha) + p_i\alpha$. There is thus a drift of magnitude $v_i = (p_i - q_i)(1 - 2\alpha)$ in the i-th direction. In particular, to obtain a smooth evolution of the empirical measure associated to the second class particles under diffusive rescaling, we need to consider the process Π^N_t defined by

$$\Pi^N_t = N^{1-d} \sum_{x\in\mathbb{T}^d_N} \{\eta_{tN^2}(x) - \alpha\}\delta_{x/N - vNt}.$$

Consider the partial differential equation

$$\begin{cases} \partial_t\lambda = \nabla \cdot D(\alpha)\nabla\lambda + \gamma \cdot \nabla\lambda^2 \\ \lambda(0,\cdot) = \lambda_0(\cdot). \end{cases} \tag{1.3}$$

In this formula $D(\alpha) = \{D_{i,j}(\alpha),\ 1 \le i, j \le d\}$ is the matrix

$$D_{i,j}(\alpha) = \frac{1}{\alpha(1-\alpha)} \lim_{t\to\infty} \frac{1}{t} \sum_{x\in\mathbb{Z}^d} x_i x_j \mathbb{E}_{\nu_\alpha}\left((\eta_t(x - vt) - \alpha)(\eta_0(0) - \alpha)\right), \quad (1.4)$$

where η_t is the asymmetric exclusion in the infinite volume $\mathbb{Z}^d$ in the stationary state ν_α, λ_0 is the density of the second class particles at time 0. Esposito, Marra and Yau [EMY] proved that in dimension $d \ge 3$, starting from the product measure $\mu^N_{\lambda(\cdot)}$, the empirical measure Π^N_t converges in probability to the measure $\lambda(t, u)du$ whose density λ is the solution of the equation (1.3): for every $t \ge 0$, every continuous function G and every $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N_{\lambda(\cdot)}} \left\{\eta,\ \left|\langle \Pi^N_t, G\rangle - \int G(u)\lambda(t, u)\, du\right| > \delta\right\} = 0,$$

where λ is the solution of (1.3). Notice the diffusive rescaling of time in the definition of Π^N_t.

Moreover Landim, Olla and Yau [LOY1] showed that in the Euler scaling, the first order correction to the convergence to (1.2) is given by the viscous Burgers equation with diffusion coefficient given by (1.4).

The purpose of this paper is to investigate the fluctuations of the empirical measure π^N_t around its limit $\rho(t, u)du$, when the initial measure is the equilibrium measure ν_α, i.e., in the case where $\rho(t, u) = \alpha$. To state the main result of the paper we need to introduce some notation. Let $Y^N_\cdot$ be the density fluctuation field that acts on smooth functions H as

$$Y^N_t(H) = N^{-d/2} \sum_{x\in\mathbb{T}^d_N} H(x/N - vtN)(\eta_{tN^2}(x) - \alpha). \quad (1.5)$$

Notice the diffusive rescaling of time on the right hand side of this identity. The aim of this article is to prove that $Y^N_\cdot$ converges to a stationary Gaussian process with a given space–time correlations.

Define $\psi_z : \mathbb{T}^d \to \mathbb{C}$ by $\psi_z(u) = \exp\{2\pi i(u\cdot z)\}$, where $\cdot$ denotes the inner product of $\mathbb{R}^d$. It is well known that the set $\{\psi_z, z \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{T}^d)$: each function f in $L^2(\mathbb{T}^d)$ can be written as

$$f = \sum_{z\in\mathbb{Z}^d} \langle f, \psi_z\rangle \psi_z.$$

In this formula and below $\langle\cdot,\cdot\rangle$ stands for the inner product of $L^2(\mathbb{T}^d)$. Notice that we are using the same symbol $\langle\cdot,\cdot\rangle$ for two different objects (the integral of a smooth function with respect to a measure and the inner product of $L^2(\mathbb{T}^d)$). We shall be explicit in case of ambiguity.

Consider on $L^2(\mathbb{T}^d)$ the positive, symmetric linear operator $\mathscr{L} = (1 - \Delta)$, where Δ is the Laplacian. A simple computation shows that the functions ψ_z are eigenvectors:

$$\mathscr{L}\psi_z = \gamma_z \psi_z,$$

where $\gamma_z = 1 + 4\pi^2 \|z\|^2$. Here $\|\cdot\|$ stands for the Euclidean norm of $\mathbb{R}^d$. For $k \geq 0$, denote by $\mathscr{H}_k$ the Hilbert space obtained as the completion of $C^\infty(\mathbb{T}^d)$ endowed with the inner product $\langle \cdot, \cdot \rangle_k$ defined by

$$\langle f, g \rangle_k = \langle f, \mathscr{L}^k g \rangle.$$

It is easy to check that $\mathscr{H}_k$ is the subspace of $L^2(\mathbb{T}^d)$ consisting of all functions f such that

$$\sum_{z \in \mathbb{Z}^d} \|\langle f, \psi_z \rangle\|^2 \gamma_z^k < \infty. \tag{1.6}$$

Moreover, on $\mathscr{H}_k$ the inner product $\langle \cdot, \cdot \rangle_k$ can be expressed as

$$\langle f, g \rangle_k = \sum_{z \in \mathbb{Z}^d} \langle f, \psi_z \rangle \overline{\langle g, \psi_z \rangle}\, \gamma_z^k .$$

Here, for a complex number a, $\bar{a}$ stands for its conjugate.

For each positive integer k, denote by $\mathscr{H}_{-k}$ the dual of $\mathscr{H}_k$ relative to the inner product $\langle \cdot, \cdot \rangle$. $\mathscr{H}_{-k}$ can be obtained as the completion of $L^2(\mathbb{T}^d)$ with respect to the inner product obtained from the quadratic form $\langle f, f \rangle_{-k}$ defined by

$$\langle f, f \rangle_{-k} = \sup_{g \in \mathscr{H}_k} \left\{ 2\langle f, g \rangle - \langle g, g \rangle_k \right\}.$$

It is again easy to check that $\mathscr{H}_{-k}$ consists of all sequences $\{\langle f, \psi_z \rangle,\ z \in \mathbb{Z}^d\}$ such that

$$\sum_{z \in \mathbb{Z}^d} \|\langle f, \psi_z \rangle\|^2 \gamma_z^{-k} < \infty$$

and that the inner product $\langle f, g \rangle_{-k}$ of two functions f, g in $\mathscr{H}_{-k}$ can be written as

$$\langle f, g \rangle_{-k} = \sum_{z \in \mathbb{Z}^d} \langle f, \psi_z \rangle \overline{\langle g, \psi_z \rangle}\, \gamma_z^{-k}.$$

It follows also from the explicit characterization of $\mathscr{H}_{-k}$ and from (1.6) that

$$\cdots \subset \mathscr{H}_2 \subset \mathscr{H}_1 \subset \mathscr{H}_0 \subset \mathscr{H}_{-1} \subset \mathscr{H}_{-2} \subset \cdots$$

Hereafter, for k in $\mathbb{Z}$, we denote by $\|\cdot\|_k$ the norm of the Hilbert space $\mathscr{H}_k$ so that $\|\cdot\|_0$ stands for the norm of $L^2(\mathbb{T}^d)$.

We shall consider the density fluctuation field Y_t^N as taking values in the Sobolev space $\mathscr{H}_{-k_0}$ for some large enough k_0. Fix a time $T > 0$, a positive integer k_0 and denote by $D([0, T], \mathscr{H}_{-k_0})$ (resp. $C([0, T], \mathscr{H}_{-k_0})$) the space of $\mathscr{H}_{-k_0}$ valued functions, that are right continuous with left limits (resp. continuous), endowed with the uniform weak topology: a sequence $\{Y^j,\ j \geq 1\}$ converges to a path Y if Y_t^j converges weakly to Y_t uniformly in time, i.e., if for all f in $\mathscr{H}_{k_0}$,

$$\lim_{j\to\infty} \sup_{0\le t\le T} \left\| Y_t^j(f) - Y_t(f) \right\| = 0.$$

Denote by Q^N the probability measure on $D([0,T], \mathscr{H}_{-k_0})$ induced by the density fluctuation field Y^N introduced in (1.5) and the product measure ν_α, by $\mathbb{P}_{\nu_\alpha}$ the probability measure on $D([0,T], \mathscr{E}_N)$ induced by the probability measure ν_α and the Markov process η_t speeded up by N^2 and denote by $\mathbb{E}_{\nu_\alpha}$ expectation with respect to $\mathbb{P}_{\nu_\alpha}$.

Recall that we denote by $D_{i,j}(\alpha)$ the matrix associated to the incompressible limit of the asymmetric simple exclusion process, which is bounded below by a multiple of the identity in the matrix sense. Let $\sigma(\alpha)$ be the square root of D: $\sigma^*\sigma = D$. Denote by $\mathscr{A}, \mathscr{B}$ the differential operators defined by $\mathscr{A} = \sum_{1\le i,j\le d} D_{i,j}(\alpha)\partial^2_{u_i,u_j}$, $\mathscr{B} = \sqrt{2\alpha(1-\alpha)}\,\sigma\,\nabla$. Denote by $\{T_t,\ t\ge 0\}$ the semigroup in $L^2(\mathbb{T}^d)$ associated to the operator $\mathscr{A}$.

Theorem 1.1. *Assume that $d \ge 3$. Fix a positive integer $k_0 > 2 + (d/2)$. Let Q be the probability measure concentrated on $C([0,T], \mathscr{H}_{-k_0})$ corresponding to the stationary generalized Ornstein–Uhlenbeck process with mean 0 and covariance*

$$E_Q[Y_t(H)Y_s(G)] = \chi(\alpha) \int_{\mathbb{T}^d} du\, (T_{|t-s|}H)(u)\, G(u) \tag{1.7}$$

for every $0 \le s \le t$ and H, G in $\mathscr{H}_{k_0}$. Here $\chi(\alpha)$ stands for the static compressibility given by $\chi(\alpha) = \mathbf{Var}_{\nu_\alpha}[\eta(0)] = \alpha(1-\alpha)$. Then, the sequence Q^N converges weakly to the probability measure Q.

Formally, Y_t is the solution of the stochastic differential equation

$$dY_t = \mathscr{A} Y_t dt + dB_t^\nabla,$$

where B_t^∇ is a mean zero Gaussian field with covariances given by

$$E_Q\left[B_t^\nabla(H)B_s^\nabla(G)\right] = 2\chi(\alpha)(s\wedge t)\int_{\mathbb{T}^d} (\nabla H)\cdot D\,(\nabla G).$$

2. Diffusive behavior of the asymmetric simple exclusion process

We recall in this section some results concerning the diffusive behavior of the asymmetric simple exclusion process. For a cylinder function h, denote by $\tilde{h}$: $[0,1] \to \mathbb{R}$ the function defined by $\tilde{h}(\alpha) = E_{\nu_\alpha}[h]$. For $0 < \alpha < 1$, denote by $\mathscr{G}_\alpha$ the set of cylinder functions h such that

$$E_{\nu_\alpha}[h] = \frac{d}{d\beta} E_{\nu_\beta}[h]\bigg|_{\beta=\alpha} = 0.$$

Notice Lh and L^*h belong to $\mathscr{G}_\alpha$ for every cylinder function h. Here L stands for the infinite volume generator of the asymmetric simple exclusion process and L^* for its adjoint. With the notation just introduced, the previous identities require the function $\tilde{h}$ to vanish at α as well as its derivative. In the exclusion set-up, all cylinder functions h can be written as linear combinations of functions of type $\Psi_A = \prod_{x \in A}[\eta(x) - \alpha]$ for finite subsets A of $\mathbb{Z}^d$:

$$h = \sum_A C_A \Psi_A + C_\phi .$$

In this formula the summation is carried over all finite subsets A of $\mathbb{Z}^d$ and, since h is a cylinder function, all but a finite number of coefficients C_A vanish. With this representation, h belongs to $\mathcal{G}_\alpha$ if and only if $C_\phi = 0$ and $\sum_{z \in \mathbb{Z}^d} C_{\{z\}} = 0$.

Fix a cylinder function h in $\mathcal{G}_\alpha$ and an integer K large enough for $\Lambda_K = \{-K, \dots, K\}^d$ to contain the support of h. Denote by $\tilde{h}_{\Lambda_K}$ the conditional expectation of h given the total number of particles in Λ_K:

$$\tilde{h}_{\Lambda_K}(M/|\Lambda_K|) = E_{\nu_\alpha}\left[h \mid \sum_{x \in \Lambda_K} \eta(x) = M \right].$$

By the local central limit theorem (cf. Lemma A2.2.2 [KL]), there exists a finite constant $C(h)$ such that

$$\sup_M \left| \tilde{h}_{\Lambda_K}(M/|\Lambda_K|) - \tilde{h}(M/|\Lambda_K|) \right| \le C(h) K^{-d}$$

for all K large enough. In particular,

$$E_{\nu_\alpha}[\tilde{h}^2_{\Lambda_K}] \le C(h) K^{-2d}. \tag{2.1}$$

Indeed, the left hand side is bounded above by $2E_{\nu_\alpha}[(\tilde{h}_{\Lambda_K} - \tilde{h})^2] + 2E_{\nu_\alpha}[\tilde{h}^2]$. The first term, by the local central limit theorem, is bounded above by $C(h)K^{-2d}$. The second one, since $\tilde{h}(\alpha) = \tilde{h}'(\alpha) = 0$, is equal to $2E_{\nu_\alpha}[(\tilde{h}(\eta^K(0)) - \tilde{h}(\alpha) - \tilde{h}'(\alpha)(\eta^K(0) - \alpha))^2]$. In this formula, $\eta^K(0)$ stands for the empirical density of particles in the cube Λ_K: $\eta^K(0) = |\Lambda_K|^{-1} \sum_{|x| \le K} \eta(x)$. This expression is bounded above by $2\|\tilde{h}''\|_\infty^2 E_{\nu_\alpha}[(\eta^K(0) - \alpha)^4]$. This proves (2.1) because ν_α is a product measure.

To distinguish between statements concerning the process on the lattice $\mathbb{Z}^d$ from statements concerning the process on the finite volume $\mathbb{T}_N^d$, we denote in this section the Bernoulli product measure on $\mathcal{E}_N$ by ν_α^N and by ν_α the Bernoulli product measure on $\{0, 1\}^{\mathbb{Z}^d}$.

Denote by $\mathbb{T}_{N,*}^d$ (resp. $\mathbb{Z}_*^d$) the set of unoriented bonds of $\mathbb{T}_N^d$ (resp. $\mathbb{Z}^d$). By a bond we understand a pair of sites $\{x, y\}$ such that $|x - y| = 1$. Unoriented means that we don't distinguish between (x, y) and (y, x). Denote by D_N the Dirichlet form associated to the generator of the asymmetric simple exclusion process L_N: $D_N(f) = E_{\nu_\alpha}[f(-L_N)f]$ for all positive functions $f: \mathcal{E}_N \to \mathbb{R}_+$. A simple computation shows that

$$D_N(f) = (1/2) \sum_{b \in \mathbb{T}_{N,*}^d} \int (\nabla_b f)^2 \, d\nu_\alpha^N, \tag{2.2}$$

where the summation is carried out over all unoriented bonds b of $\mathbb{T}^d_{N,*}$ and $(\nabla_b f)(\eta) = f(\eta^b) - f(\eta)$. Notice that each pair $\{x, y\}$ appears once and only once in the summation. We denote below by $D_b(f)$ the piece of the Dirichlet form corresponding to the bond b:

$$D_b(f) = (1/2) \int (\nabla_b f)^2 \, d\nu^N_\alpha . \tag{2.3}$$

For a bond $\{x, y\}$, denote by $W_{x,y}$ the instantaneous current over the bond $\{x, y\}$. This is the rate at which a particle jumps from x to y minus the rate at which a particle jumps from y to x. A simple computation gives that $W_{x,x+e_i}$ is equal to $p_i\eta(x)[1-\eta(x+e_i)]-q_i\eta(x+e_i)[1-\eta(x)]$. Observe that the current in the asymmetric case has positive mean with respect to all invariant states ν_α. The current minus its mean $W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}]$ can be decomposed as

$$\begin{aligned} W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}] = -\gamma_i w_i &+ [p_i(1-\alpha) + q_i\alpha][\eta(0) - \alpha] \\ &-[q_i(1-\alpha) + p_i\alpha][\eta(e_i) - \alpha], \end{aligned}$$

where $w_i = [\eta(e_i) - \alpha][\eta(0) - \alpha]$. Let $b_i = q_i(1-\alpha) + p_i\alpha$ and notice that $p_i(1-\alpha) + q_i\alpha = v_i + b_i$. With this notation we can rewrite the normalized current as $-\gamma_i w_i + [v_i + b_i][\eta(0) - \alpha] - b_i[\eta(e_i) - \alpha]$ and obtain that

$$W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}] = -\gamma_i w_i - b_i[\eta(e_i) - \eta(0)] + v_i[\eta(0) - \alpha]. \tag{2.4}$$

This formula for the normalized current will be needed later.

We are now in a position to state some results concerning the diffusive behavior of the asymmetric simple exclusion process. The following integration by part formula is proven in [EMY].

Lemma 2.1. *Suppose h is a cylinder function in $\mathcal{G}_\alpha$. Then there exist cylinder functions Φ_b, $b \in \mathbb{Z}^d_*$, such that for any cylinder function f*

$$\int hf \, d\nu_\alpha = \sum_{b\in\mathbb{Z}^d_*} \int \Phi_b \nabla_b f \, d\nu_\alpha$$

and

$$\sum_{b\in\mathbb{Z}^d_*} |b|^{d+(1/2)} \int (\Phi_b)^2 \, d\nu_\alpha < \infty$$

This statement has a finite volume version. Consider a cylinder function $h \in \mathcal{G}_\alpha$ and denote by s_h the size of the support of h, i.e. the smallest integer l for which the support of h is contained in Λ_l. For each $N \geq s_h$, denote

$$\tilde{h}_N = E_{\nu^N_\alpha}\left[h | \sum_{x\in T^d_N} \eta(x) \right]$$

For each $N \geq s_h$, there exists a collection of cylinder functions $\{\Phi_b^N, b \in \mathbb{T}_{N,*}^d\}$ such that

$$\int hf\, d\nu_\alpha^N = \sum_{b \in \mathbb{T}_{N,*}^d} \int \Phi_b^N \nabla_b f\, d\nu_\alpha^N. \tag{2.5}$$

Moreover there exists a constant $C(h)$ independent of N such that

$$\sum_{b \in \mathbb{T}_{N,*}^d} |b|^{d+(1/2)} \int (\Phi_b^N)^2\, d\nu_\alpha^N < C(h). \tag{2.6}$$

We define the finite volume variances of cylinder functions belonging to $\mathscr{G}_\alpha$. For a positive integer ℓ, denote by Λ_ℓ the cube $\{-\ell, \ldots, \ell\}^d$. For $0 \leq M \leq |\Lambda_\ell|$, let $\nu_{\ell,M}$ be the canonical measure on $\{0,1\}^{\Lambda_\ell}$ concentrated on configurations with M particles: $\nu_{\ell,M}(\cdot) = \nu_\alpha^\ell(\cdot | \sum_{|x| \leq \ell} \eta(x) = M)$.

Let g be a local function in $\mathscr{G}_\alpha$. For each $\ell > s_g$ and $0 \leq M \leq |\Lambda_\ell|$, define the finite volume variance $V_\ell(g, M)$ of g with respect to $\nu_{\ell,M}$ as

$$V_\ell(g, M) = \frac{1}{|\Lambda_\ell|} \left\langle \sum_{|x| \leq \ell_g} \{\tau_x g - \tilde{g}_{\Lambda_\ell}\}, (-L_\ell^s)^{-1} \sum_{|x| \leq \ell_g} \{\tau_x g - \tilde{g}_{\Lambda_\ell}\} \right\rangle_{\nu_{\ell,M}}. \tag{2.7}$$

In this formula, ℓ_g stands for $\ell - s_g$ so that $\tau_x g$ is measurable with respect to $\{\eta(z),\ z \in \Lambda_\ell\}$ for $|x| \leq \ell_g$, $\langle \cdot, \cdot \rangle_{\nu_{\ell,M}}$ stands for the inner product in $L^2(\nu_{\ell,M})$ and L_ℓ^s is the generator of the *symmetric* simple exclusion process restricted to the cube Λ_ℓ:

$$(L_\ell^s f)(\eta) = (1/2) \sum_{x \in \Lambda_\ell} \sum_{\substack{y \in \Lambda_\ell \\ |y-x|=1}} \eta(x)[1 - \eta(y)][f(\eta^{x,y}) - f(\eta)].$$

Define the infinite volume variance of g by

$$\mathbf{V}_\alpha(g) = \limsup_{\ell \to \infty} E_{\nu_\alpha} \left[V_\ell \left(g, \sum_{|x| \leq \ell} \eta(x) \right) \right]. \tag{2.8}$$

We need the following results. The first one is due to Varadhan [V] and the proof can be found in Chap 7 of [KL]. The second one follows from Theorem 5.9 of [EMY] and Lemma 5.2, Lemma 5.3, Corollary 6.2 and Theorem 7.1 of [LOY2]. The third one is Lemma 7.3 of [LOY2] with w_i and LH_m^i in place of w_i^* and $L^* H_m^i$.

Theorem 2.2. *For each cylinder function h in $\mathscr{G}_\alpha$, the finite volume variance of h converges uniformly in $[0, 1]$ to the infinite volume variance: Fix α in $[0, 1]$ and consider a sequence M_ℓ such that $M_\ell / |\Lambda_\ell|$ converges to α as $\ell \uparrow \infty$. Then,*

$$\lim_{\ell \to \infty} V_\ell(h, M_\ell) = \mathbf{V}_\alpha(h).$$

Theorem 2.3. *Fix α in $(0, 1)$. There exists a sequence of local functions $\{h_{i,m},\ 1 \le i \le d,\ m \ge 1\}$ in $\mathscr{G}_\alpha$ such that*

$$\lim_{m\to\infty} \mathbf{V}_\alpha \left(\gamma_i w_i - \sum_{j=1}^{d} [D_{ij}(\alpha) - (1/2)\delta_{ij}][\eta(e_j) - \eta(0)] + L h_{i,m} \right) = 0.$$

D is bounded below by $(1/2)I$ and $D(\cdot)$ is continuous on $(0, 1)$.

For a cylinder function h, denote by Γ_h the formal sum $\sum_{z\in\mathbb{Z}^d} \tau_z h$. Although Γ_h is formal, $\nabla_{x,x+e_i}\Gamma_h$ is well defined since all but a finite number of terms vanish because h is a local function.

Theorem 2.4. *Consider the matrix $D(\alpha)$ and the sequences $h_{i,m}$ given by the previous theorem. Then, for every $\theta \in \mathbb{R}^d$,*

$$\lim_{m\to\infty} \sum_{i=1}^{d} E_{\nu_\alpha} \left[\left\{ \nabla_{0,e_i} \sum_{j=1}^{d} \theta_j \Gamma_{h_{j,m}} \right\}^2 \right] = 4\alpha(1-\alpha)\theta \cdot D\theta - 2\alpha(1-\alpha)||\theta||^2.$$

3. Central limit theorem for the density field

We prove in this section Theorem 1.1. Recall the definition of the cylinder functions $h_{i,m}$ defined in Theorem 2.3. For each $m \ge 1$ define the modified density field $Z_t^{N,m}\colon \mathscr{H}_k \to \mathbb{R}$ by

$$Z_t^{N,m}(G) = Y_t^N(G)\ -\ N^{-1-d/2} \sum_{x\in\mathbb{T}_N^d} \sum_{i=1}^{d} (\partial_{u_i} G)(x/N - vtN)\tau_x h_{i,m}.$$

Recall that Q^N denotes the probability measure on the path space $D([0, T], \mathscr{H}_{-k_0})$ induced by the process Y_t^N and the product measure ν_α.

For each smooth function $G\colon \mathbb{T}^d \to \mathbb{R}$, define the martingales $M_t^{1,G} = M_t^{1,G,N}$, $M_t^{2,G} = M_t^{2,G,N}$ by

$$M_t^{1,G} = Z_t^{N,m}(G) - Z_0^{N,m}(G) - \int_0^t \Gamma_1^N(s)\, ds$$
$$M_t^{2,G} = \left(M_t^{1,G}\right)^2 - \int_0^t \Gamma_2^N(s)\, ds,$$

where

$$\Gamma_1^N(s) = (\partial_s + N^2 L_N) Z_s^{N,m}(G),$$
$$\Gamma_2^N(s) = N^2 L_N Z_s^{N,m}(G)^2 - 2N^2 Z_s^{N,m}(G) L_N Z_s^{N,m}(G).$$

Recall the definition of the differential operator $\mathscr{A}$, $\mathscr{B}$ introduced in section 1. We shall prove that there exists a random variable $A_{N,m}$, which converges to 0 in

$L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$ and then $m \uparrow \infty$, and a random variable A_N, which converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$, such that

$$M_t^{1,G} = Y_t^N(G) - Y_0^N(G) - \int_0^t ds\, Y_s^N(\mathscr{A}G) + A_{N,m} \tag{3.1}$$

and

$$M_t^{2,G} = (M_t^{1,G})^2 - \|\mathscr{B}G\|_0^2\, t + A_N. \tag{3.2}$$

To compute $\Gamma_1^N(s)$, for $1 \le i \le d$, denote by $\partial_{u_i}^N$ the discrete derivative in the i-th direction: $(\partial_{u_i}^N H)(u) = N[H(u + N^{-1}e_i) - H(u)]$ and by G_s the function G translated by $-vsN$: $G_s(u) = G(u - vsN)$. Since $L_N\eta(x) = \sum_{1\le j\le d}\{W_{x-e_j,x} - W_{x,x+e_j}\}$, a summation by parts yields that $\Gamma_1^N(s)$ is equal to

$$\begin{aligned} &N^{1-d/2}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d(\partial_{u_i}^N G_s)(x/N)\Big\{W_{x,x+e_i} - E_{\nu_\alpha}[W_{0,e_i}]\Big\}\\ &-N^{1-d/2}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d(\partial_{u_i} G_s)(x/N)\Big\{v_i[\eta_s(x) - \alpha] + \tau_x L h_{i,m}\Big\}\\ &+N^{-d/2}\sum_{x\in\mathbb{T}_N^d}\sum_{i,j=1}^d(\partial^2_{u_i,u_j} G_s)(x/N)v_j\tau_x h_{i,m}. \end{aligned} \tag{3.3}$$

In this first sum we were allowed to introduce the constant $E_{\nu_\alpha}[W_{0,e_i}]$ because the summation of $(\partial_{u_i}^N G)$ over $\mathbb{T}_N^d$ vanishes. In the second sum, we first commuted the translation τ_x with the generator L_N and then replaced L_N with the infinite volume generator L because $h_{i,m}$ is a cylinder function.

Since $h_{i,m}$ belongs to $\mathscr{G}_\alpha$ for each i, m, it follows from Theorem 4.2 that the time integral of the third term of (3.3) converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$. We now turn to the first two terms. Recall from (2.4) the decomposition of the normalized current $W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}]$ as

$$W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}] = -\gamma_i w_i - b_i[\eta(e_i) - \eta(0)] + v_i[\eta(0) - \alpha].$$

In particular, if we had a continuous derivative in the first sum of (3.3), the terms with $v_i[\eta(0) - \alpha]$ would cancel and it would remain $b_i[\eta(0) - \eta(e_i)]$. This last expression allows a second summation by parts which cancels another factor N.

Rewrite the first two terms of (3.3) as

$$\begin{aligned} &N^{1-d/2}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d\Big\{(\partial_{u_i}^N G_s)(x/N) - (\partial_{u_i} G_s)(x/N)\Big\}\,\tau_x\Big\{W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}]\Big\}\\ &-N^{1-d/2}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d(\partial_{u_i} G_s)(x/N)\tau_x \end{aligned}$$

$$\times \left\{ \gamma_i w_i - \sum_{j=1}^{d} [D_{i,j} - (1/2)\delta_{i,j}][\eta(e_j) - \eta(0)] + L h_{i,m} \right\}$$
$$+ N^{-d/2} \sum_{x \in \mathbb{T}_N^d} \sum_{i,j=1}^{d} (\partial_{u_j}^{N,-} \partial_{u_i} G_s)(x/N) \big\{ D_{i,j} + [b_j - 1/2]\delta_{i,j} \big\} [\eta(x) - \alpha]. \tag{3.4}$$

In this formula, $\partial_{u_j}^{N,-}$ stands for the discrete derivative in the j-th direction defined by $(\partial_{u_j}^{N,-} H)(u) = H(u) - H(u - N^{-1} e_j)$. The first term of this expression can be written as

$$(1/2) N^{-d/2} \sum_{x \in \mathbb{T}_N^d} \sum_{i=1}^{d} (\partial_{u_i}^2 G_s)(x/N) \tau_x$$
$$\times \Big\{ -\gamma_i w_i + [b_i + v_i][\eta(0) - \alpha] - b_i [\eta(e_i) - \alpha] \Big\} + R_N,$$

where R_N is an expression whose time integral, in virtue of Lemma 4.1, converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$. Hereafter R_N (resp. $R_{N,m}$) stands for a term, that might be different from line to line, whose time integral converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$ (resp. as $N \uparrow \infty$ and then $m \uparrow \infty$). By similar reasons and an integration by parts, the time integral of $N^{-d/2} \sum_x (\partial_{u_i}^2 G_s)(x/N) \tau_x [\eta(0) - \eta(e_i)]$ converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$. Finally, by Theorem 4.2 the time integral of $N^{-d/2} \sum_{x,i} (\partial_{u_i}^2 G_s)(x/N) \tau_x w_i$ converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$. Therefore, the first term in (3.4) is equal to

$$(1/2) N^{-d/2} \sum_{x \in \mathbb{T}_N^d} \sum_{i=1}^{d} (\partial_{u_i}^2 G_s)(x/N) v_i [\eta(x) - \alpha] + R_N.$$

It follows from Theorem 4.2, Theorem 2.2 and Theorem 2.3 that the time integral of the second term in (3.4) converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$ and then $m \uparrow \infty$.

By Lemma 4.1 below, in the third term of (3.4), we may replace the discrete derivative by the continuous one, paying the price of a remainder whose time integral converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$. Since $b_j + (v_j/2) = 1/2$, up to this point, we showed that Γ_1^N is equal to

$$N^{-d/2} \sum_{x \in \mathbb{T}_N^d} \sum_{i,j=1}^{d} (\partial_{u_i,u_j}^2 G_s)(x/N) D_{i,j}(\alpha) [\eta(x) - \alpha] + R_{N,m}.$$

By definition of the differential operator $\mathscr{A}$, we may write the previous expression as

$Y_s^N(\mathscr{A} G) + R_{N,m}$. This proves (3.1) because the difference between $Z_t^{N,m}$ and Y_t^N converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N \uparrow \infty$.

We now turn to the martingale $M_t^{2,G}$. A simple computation shows that Γ_2^N is equal to the sum of two similar expressions. The first one comes from jumps in the positive direction (i.e. from a site x to a site $x+e_i$), while the second one corresponds to jumps in the negative direction. The first expression is given by

$$N^{-d} \sum_{x\in\mathbb{T}_N^d} \sum_{i=1}^d p_i\, \eta(x)[1-\eta(x+e_i)] \\ \times \left\{ (\partial_{u_i}^N G_s)(x/N) + \sum_{z\in\mathbb{T}_N^d} \sum_{j=1}^d (\partial_{u_j} G_s)(z/N) \nabla_{x,x+e_i} \tau_z h_{j,m} \right\}^2 .$$

We may replace in the previous formula the discrete derivative by the continuous one paying a price $O(N^{-1})$. On the other hand, since $h_{j,m}$ is a local function, $\nabla_{x,x+e_i}\tau_z h_{j,m}$ vanishes for $|z-x|$ large enough so that the summation over z is a finite one. Moreover, we may replace $(\partial_{u_j} G_s)(z/N)$ by $(\partial_{u_j} G_s)(x/N)$ paying a price of order $O(N^{-1})$. Therefore, Γ_2^N is the sum of two terms, the first one being

$$N^{-d} \sum_{x\in\mathbb{T}_N^d} \sum_{i=1}^d p_i \eta(x)[1-\eta(x+e_i)] \\ \times \left\{ (\partial_{u_i} G_s)(x/N) + \sum_{j=1}^d (\partial_{u_j} G_s)(x/N) \nabla_{x,x+e_i} \Gamma_{h_{j,m}} \right\}^2 + O(N^{-1}).$$

From this explicit formula for Γ_2^N, it is not difficult to check that the time integral of $\Gamma_2^N(s) - E_{\nu_\alpha}[\Gamma_2^N(s)]$ converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N\uparrow\infty$. On the other hand, as $N\uparrow\infty$, the expectation of the previous expression with respect to ν_α converges to

$$\sum_{i=1}^d p_i \int_{\mathbb{T}^d} du\, E_{\nu_\alpha}\left[\eta(0)[1-\eta(e_i)] \left\{ (\partial_{u_i} G)(u) + \sum_{j=1}^d (\partial_{u_j} G)(u) \nabla_{0,e_i} \Gamma_{h_{j,m}} \right\}^2 \right].$$

Notice that the index s disappeared when taking the limit in N. We now develop the square. The first contribution is $\alpha(1-\alpha)\sum_i p_i (\partial_{u_i} G)^2$. Adding this expression with the one corresponding to the jumps in the opposite direction, we obtain $\alpha(1-\alpha)\|\nabla G\|^2$ because $p_i + q_i = 1$ for all i. The second contribution comes from the cross terms and is equal to

$$2\sum_{i,j=1}^d p_i \int_{\mathbb{T}^d} du\, (\partial_{u_i} G)(u)(\partial_{u_j} G)(u) E_{\nu_\alpha}\Big[\eta(0)[1-\eta(e_i)]\nabla_{0,e_i}\Gamma_{h_{j,m}}\Big].$$

Recall the definition of $\Gamma_{h_{j,m}}$. Performing a change of variables $\xi = \tau_z \eta$, we obtain that the previous expectation is equal to

$$\sum_{z\in\mathbb{Z}^d} E_{\nu_\alpha}\Big[\eta(z)[1-\eta(z+e_i)]\nabla_{z,z+e_i}h\Big].$$

Since ν_α is an invariant measure for the asymmetric simple exclusion process, multiplying this expression by p_i and adding to it the piece that corresponds to jumps in the negative direction, we obtain a term that vanishes.

The third contribution is

$$\begin{aligned}
&\sum_{i=1}^d p_i \int_{\mathbb{T}^d} du\, E_{\nu_\alpha}\left[\eta(0)[1-\eta(e_i)]\left\{\sum_{j=1}^d (\partial_{u_i}G)(u)\nabla_{0,e_i}\Gamma_{h_{j,m}}\right\}^2\right]\\
&\quad=\frac{1}{2}\sum_{i=1}^d p_i\int_{\mathbb{T}^d} du\, E_{\nu_\alpha}\left[\Big\{\eta(0)[1-\eta(e_i)]+\eta(e_i)[1-\eta(0)]\Big\}\right.\\
&\qquad\times\left.\left\{\sum_{j=1}^d(\partial_{u_i}G)(u)\nabla_{0,e_i}\Gamma_{h_{j,m}}\right\}^2\right].
\end{aligned}$$

We performed the change of variables $\xi = \eta^{0,e_i}$ to obtain the previous identity. Notice that we may remove the expression inside the first parenthesis in the previous expectation because for every function h, $\nabla_{0,e_i}h$ vanishes unless $\eta(0)[1-\eta(e_i)]+\eta(e_i)[1-\eta(0)]=1$. Adding this expression with the one corresponding to jumps in the opposite direction, we obtain that the third term is equal to

$$(1/2)\sum_{i=1}^d\int_{\mathbb{T}^d} du\, E_{\nu_\alpha}\left[\left\{\sum_{j=1}^d(\partial_{u_i}G)(u)\nabla_{0,e_i}\Gamma_{h_{j,m}}\right\}^2\right].$$

By Theorem 2.4, as $m\uparrow\infty$, this expression converges to

$$2\alpha(1-\alpha)\int_{\mathbb{T}^d} du \sum_{i,j=1}^d(\partial_{u_i}G)[D_{i,j}(\alpha)-(1/2)\delta_{i,j}](\partial_{u_j}G).$$

We have therefore proved that the time integral of $\Gamma_2^N(s)$ is equal to

$$2\alpha(1-\alpha)t\int_{\mathbb{T}^d} du\sum_{i,j=1}^d(\partial_{u_i}G)D_{i,j}(\alpha)(\partial_{u_j}G)+\int_0^t ds\, R_N(s),$$

where the second term of the right hand side converges to 0 in $L^2(\mathbb{P}_{\nu_\alpha})$ as $N\uparrow\infty$. Recall that we denoted by $\sigma(\alpha)$ the square root of the matrix $D(\alpha)$ and that $\mathscr{B}$ stands for the operator $\sqrt{2\alpha(1-\alpha)}\,\sigma\,\nabla$. The previous sum can be written as $\|\mathscr{B}G\|_0^2$. This proves (3.2). We are now in a position to prove the main result of this article.

Proof of Theorem 1.1. The proof is divided in two steps. We first consider the question of tightness. We then prove that all limit points solve a martingale problem and invoke Holley–Stroock [HS] theory of generalized Ornstein–Uhlenbeck processes to conclude the uniqueness of limit points.

We prove in section 5 that the sequence Q^N is tight. For $t \geq 0$, denote by $\mathscr{F}_t$ the σ-algebra generated by $Y_s(H)$ for $s \leq t$ and H in $\mathscr{H}_{k_0}$. Let Q^* be a limit point of the sequence Q^N. Fix G in $\mathscr{H}_{k_0}$. It follows from (3.1) and (3.2) that Q^*-almost surely

$$M_t^{1,G} = Y_t(G) - Y_0(G) - \int_0^t ds\, Y_s(\mathscr{A}G)$$

and

$$M_t^{2,G} = (M_t^{1,G})^2 - \|\mathscr{B}G\|_0^2\, t$$

are martingales. The proof of this statement is standard. It is easy to adapt the arguments presented in the proof of Proposition 11.2.3 of [KL] to our context. On the other hand, a simple computation, relying on the fact that ν_α is a product measure, shows that Q^* restricted to $\mathscr{F}_0$ is a Gaussian field with covariance given by

$$E_{Q^*}[Y_0(G)Y_0(H)] = \chi(\alpha)\langle H, G\rangle,$$

where $\chi(\alpha)$ is the static compressibility defined in section 1. To conclude the proof of the theorem, it remains to recall Holley–Stroock [HS] theory of generalized Ornstein–Uhlenbeck processes. □

4. The Boltzmann–Gibbs principle

We prove in this section the Boltzmann–Gibbs principle which is one of the main ingredients in the proof of the equilibrium fluctuations of the asymmetric exclusion process. It states that the local fields are projected on the density field and was first proved by Brox and Rost in [BR]. We start with a simple result that was used several times in the previous section.

Lemma 4.1. *Let h be a mean zero local function and $G\colon \mathbb{T}^d \to \mathbb{R}$ a continuous function. Then, there exists a finite constant C depending only on h such that for all $t \geq 0$ and all $N \geq 1$,*

$$\mathbb{E}_{\nu_\alpha}\left[\sup_{0\leq t\leq T}\Big(\int_0^t ds\, N^{-d/2}\sum_{x\in\mathbb{T}_N^d} G_s(x/N)\tau_x h(\eta_s)\Big)^2\right] \leq CT^2\|G\|_\infty^2.$$

Proof. By Schwarz inequality and since ν_α is an invariant measure, the expectation is bounded above by

$$T\int_0^T ds\, E_{\nu_\alpha}\left[\left(N^{-d/2}\sum_{x\in\mathbb{T}_N^d} G_s(x/N)\tau_x h(\eta)\right)^2\right].$$

Since h is a mean zero local function, developing the square, we obtain that this expression is less than or equal to

$$CT \int_0^T ds\, N^{-d} \sum_{x\in\mathbb{T}_N^d} G_s(x/N)^2$$

for some constant that depends on h only. It is now easy to conclude the proof of the lemma. □

Theorem 4.2 (the Boltzmann–Gibbs principle). *There exists a finite universal constant C_0 such that for all cylinder functions h in $\mathscr{G}_\alpha$ and all smooth function $G\colon \mathbb{T}^d \to \mathbb{R}$,*

$$\begin{aligned}\limsup_{N\to\infty} \mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in\mathbb{T}_N^d} G_s(x/N)\tau_x h(\eta_s)\,ds\right)^2\right]&\\ \le C_0 T\|G\|_0^2\,\mathbf{V}_\alpha(h).&\end{aligned}\tag{4.1}$$

In this formula $\mathbf{V}_\alpha(h)$ stands for the infinite volume variance of h defined in (2.8).

The proof of the Boltzmann–Gibbs principle relies on the following general estimate. Let X_t be a Markov process on a finite state space $\mathscr{E}$ with generator $\mathscr{L}$ and invariant measure π. Denote by $\langle\cdot,\cdot\rangle_\pi$ the inner product in $L^2(\pi)$ and by $\mathscr{L}^*$ (resp. $\mathscr{L}^s$) the adjoint (resp. symmetric part) of $\mathscr{L}$ with respect to π. Given a mean zero function f on $\mathscr{E}$, we denote $\|f\|_{-1}^2 = \langle f(-\mathscr{L}^s)^{-1} f\rangle_\pi$, that can also be written in the variational form

$$\|f\|_{-1}^2 = \sup_g \left\{2\langle f, g\rangle - \langle g, (-\mathscr{L}^s)g\rangle_\pi\right\}$$

where the supremum is taken over all functions g on $\mathscr{E}$.

Lemma 4.3. *Fix $T > 0$. There exists a finite universal constant C_0 such that for every smooth mean zero function $f\colon [0,T]\times\mathscr{E}\to\mathbb{R}$,*

$$\mathbb{E}_\pi\left[\sup_{0\le t\le T}\left(\int_0^t f(s,X_s)\,ds\right)^2\right] \le C_0\int_0^T ds\,\|f(s,\cdot)\|_{-1}^2. \tag{4.2}$$

In this lemma, it is understood that $f(\cdot,x)$ is a smooth function for every x in $\mathscr{E}$ and that $f(s,\cdot)$ has π-mean zero for all $s\ge 0$.

Proof. Fix $T > 0$ and a function $f\colon[0,T]\times\mathscr{E}\to\mathbb{R}$. For each $0\le t\le T$, let $h_t\colon\mathscr{E}\to\mathbb{R}$ be such that $\mathscr{L}^s h_t = f(t,\cdot)$. There exists such a solution because $f(t,\cdot)$ has mean zero and π is ergodic.

For each $s\ge 0$, denote by $\mathscr{F}_s$ the σ-algebra generated by $\{X_r,\ 0\le r\le s\}$. Define the $(\mathbb{P}_\pi,\mathscr{F}_t)$ mean zero martingale M_t by

$$M_t = h(t,X_t) - h(0,X_0) - \int_0^t ds\,(\mathscr{L}+\partial_s)h(s,X_s).$$

In the same way, denote by $\mathscr{F}_t^-$ the backward filtration generated by $\{X_s, s \geq t\}$. Recall that $\mathscr{L}^*$ stands for the adjoint of the generator $\mathscr{L}$ with respect to the invariant measure π. It is easy to check that the process $\{M_t^-, \ 0 \leq t \leq T\}$ defined by

$$M_t^- = h(T-t, X_{T-t}) - h(T, X_T) - \int_0^t ds \, (\mathscr{L}^* - \partial_s) h(T-s, X_{T-s})$$

is a $(\mathbb{P}_\pi, \mathscr{F}_t^-)$ martingale, called the backward martingale.

Let $Z_t = M_T^- - M_{T-t}^-$. A change of variables permits to write Z_t as

$$Z_t = h(0, X_0) - h(t, X_t) - \int_0^t ds \, (\mathscr{L}^* - \partial_s) h(s, X_s).$$

In particular, since $\mathscr{L}^s h_t = f(t, \cdot)$, $M_t + Z_t = 2\int_0^t f(s, X_s) ds$. Therefore,

$$\mathbb{E}_\pi \left[\sup_{0 \leq t \leq T} \left(\int_0^t f(s, X_s) \, ds \right)^2 \right] = (1/4) \ \mathbb{E}_\pi \left[\sup_{0 \leq t \leq T} \left(M_t + M_T^- - M_{T-t}^- \right)^2 \right].$$

Since $(a_1 + a_2 + a_3)^2 \leq 3a_1^2 + 3a_2^2 + 3a_3^2$, by Doob's inequality, the previous expression is bounded above by $(15/4)\mathbb{E}_\pi[(M_T^-)^2] + 3\mathbb{E}_\pi[(M_T)^2]$. The variances of these martingales are equal to $2\int_0^T ds \| f(s, \cdot) \|_{-1}^2$ so that

$$\mathbb{E}_\pi \left[\sup_{0 \leq t \leq T} \left(\int_0^t f(s, X_s) \, ds \right)^2 \right] \leq 14 \int_0^T ds \, \| f(s, \cdot) \|_{-1}^2.$$

This concludes the proof of the lemma. □

In the proof of the Boltzmann–Gibbs principle, we shall need the following estimate whose proof relies mostly on the integration by parts formula (2.5).

Lemma 4.4. *Fix a cylinder function h in $\mathscr{G}_\alpha$. There exists a finite constant C_0 depending only on h and T such that for every subset B of $\mathbb{T}_N^d$, every smooth function $G\colon \mathbb{T}^d \to \mathbb{R}$ and every N sufficiently large,*

$$\begin{aligned} &\mathbb{E}_{\nu_\alpha} \left[\sup_{0 \leq t \leq T} \left(\int_0^t N^{1-d/2} \sum_{x \in B} G_s(x/N) \tau_x h(\eta_s) \, ds \right)^2 \right] \\ &\quad \leq \frac{C_0(1+T)}{N^d} \int_0^T ds \sum_{x \in B} G_s(x/N)^2. \end{aligned} \tag{4.3}$$

Proof. Recall that $\tilde{h}_N$ denotes the conditional expectation of h given $N^{-d} \sum_{x \in \mathbb{T}_N^d} \eta(x)$. We add and subtract $\tilde{h}_N(\eta_s)$ inside the time integral of (4.3). Applying Schwarz inequality, we decompose the expectation in two terms. In the first one appears the time integral of $h(\eta_s) - \tilde{h}_N(\eta_s)$ and in the second one the time integral of $\tilde{h}_N(\eta_s)$. We consider each one separately.

By Lemma 4.3, the first expectation is bounded by

$$14\int_0^T \left\| N^{1-d/2}\sum_{x\in B} G_s(x/N)\tau_x\{h-\tilde h_N\}\right\|_{-1,N}^2 \, ds.$$

Here we have denoted $\|f\|_{-1,N} = \langle f, (-N^2 L_N^s)^{-1} f\rangle$. The norm inside the time integral above can be expressed as

$$\sup_f \left\{ 2N^{1-d/2}\sum_{x\in B} G_s(x/N)\int \tau_x\{h-\tilde h_N\}\, f\, d\nu_\alpha - N^2\langle f, (-L_N)f\rangle_\alpha \right\}.$$

In this formula the supremum is carried over all functions f of $\{0,1\}^{\mathbb{T}_N^d}$ and $\langle\cdot,\cdot\rangle_\alpha$ stands for the inner product of $L^2(\nu_\alpha)$. Since ν_α is translation invariant, by the integration by parts formula (2.5), (2.6), there exists a collection of cylinder functions $\{\Phi_b^N,\ b\in\mathbb{T}_{N,*}^d\}$, indexed by the bonds of $\mathbb{T}_N^d$, such that

$$\int \tau_x\{h-\tilde h_N\}\, f\, d\nu_\alpha = \int \{h-\tilde h_N\}\,\tau_{-x} f\, d\nu_\alpha = \sum_{b\in\mathbb{T}_{N,*}^d}\int \Phi_b^N\, \nabla_b \tau_{-x} f\, d\nu_\alpha.$$

In particular, by Schwarz inequality, $2N^{1-d/2}\sum_{x\in B} G_s(x/N)\int \tau_x\{h-\tilde h_N\} f d\nu_\alpha$ is bounded above by

$$\frac{A}{N^d}\sum_{x\in B} G_s(x/N)^2 \sum_{b\in\mathbb{T}_{N,*}^d} |b|^{d+(1/2)} E_{\nu_\alpha}\big[(\Phi_b^N)^2\big]$$
$$+\frac{N^2}{A}\sum_{x\in B}\sum_{b\in\mathbb{T}_{N,*}^d} |b|^{-d-(1/2)} E_{\nu_\alpha}\big[(\nabla_b\tau_{-x} f)^2\big]$$

for every $A>0$. By (2.6) the first term is bounded above by

$$C_1 A N^{-d}\sum_{x\in B} G_s(x/N)^2$$

for some finite constant C_1 that depends only on h. On the other hand, since $\nabla_b\tau_{-x} f = \tau_{-x}\nabla_{b+x} f$ and since ν_α is translation invariant, the second term is equal to

$$\frac{N^2}{A}\sum_{x\in B}\sum_{b\in\mathbb{T}_{N,*}^d} |b|^{-d-(1/2)} E_{\nu_\alpha}\big[(\nabla_{b+x} f)^2\big].$$

The sum over x of the expectations is bounded above by a multiple of the Dirichlet form of f. Since $|z|^{-d-(1/2)}$ is summable in $\mathbb{Z}^d$, the second term is bounded above by $C_2 A^{-1} N^2 < f, (-L_N) f > \alpha$. Choosing $A = C_2$, we obtain that the first expectation in the decomposition performed in the beginning of the proof is bounded above by the time integral of $C_3 N^{-d}\sum_{x\in B} G_s(x/N)^2$.

The second term in the decomposition is just

$$\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in B} G_s(x/N)\tilde{h}_N(\eta_s)\,ds\right)^2\right].$$

Denote $\sum_{x\in B} G_s(x/N)$ by $\tilde{G}_s$. By Schwarz inequality and since ν_α is an invariant state, this expectation is bounded above by $TN^{2-d}\int_0^T ds\, \tilde{G}_s^2\, E_{\nu_\alpha}[\tilde{h}_N^2]$. Since h belongs to $\mathscr{G}_\alpha$, by (2.1), $E_{\nu_\alpha}[\tilde{h}_N^2]$ is bounded above by CN^{-2d} for some finite constant C that depends on h only. By Schwarz inequality, the second term in the decomposition is bounded by $CTN^{2(1-d)}\int_0^T ds\sum_{x\in B} G_s(x/N)^2$. This concludes the proof of the lemma. □

We turn now to the proof of the Boltzmann–Gibbs principle.

Proof of Theorem 4.2. Fix a cylinder function h in $\mathscr{G}_\alpha$. Denote by ℓ_h the smallest integer ℓ such that the support of h is contained in $\{-\ell,\dots,\ell\}^d$.

Consider a sequence $K = K_N$ such that $N^{2/d} \ll K_N \ll N$. We first decompose $\mathbb{T}_N^d$ into disjoint subcubes of length K. Let $\Omega_N = \{0,\dots,N-1\}^d$. For each x in $\mathbb{Z}^d$, denote by B_x the cube $Kx + \Omega_K$. Notice that B_x is contained in Ω_N if and only if $|x| \le [N/K] - 1$ and $x_i \ge 0$, $1 \le i \le d$. Here and below $[a]$ stands for the integer part of a. Let A_K be defined as

$$\Omega_N = \bigcup_{x\in\mathscr{I}_K} B_x \;\cup\; A_K,$$

where $\mathscr{I}_K$ is the set of sites x such that $|x| \le [N/K] - 1$ and $x_i \ge 0$, $1 \le i \le d$. It follows from Lemma 4.4 that

$$\limsup_{N\to\infty}\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in A_K} G_s(x/N)\tau_x h(\eta_s)\,ds\right)^2\right] = 0$$

provided K_N/N converges to 0 as $N \uparrow \infty$.

For each x in $\mathscr{I}_K$, denote by B_x^0 the set of sites y that are at a distance greater than ℓ_h from the boundary of B_x: $B_x^0 = \{y \in B_x,\ d(y, B_x^c) > \ell_h\}$. In particular, for each y in B_x^0, the support of $\tau_y h$ belongs to B_x. Denote by E_K the set of all points that are not interior points: $E_K = \cup_{x\in\mathscr{I}_K}(B_x - B_x^0)$. In view of Lemma 4.4, since $K_N \uparrow \infty$ as $N \uparrow \infty$, in order to prove the theorem, we just need to show that

$$\limsup_{N\to\infty}\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in\mathscr{I}_K}\sum_{y\in B_x^0} G_s(y/N)\tau_y h(\eta_s)\,ds\right)^2\right]$$
$$\le C_0 T\|G\|_0^2\,\mathbf{V}_\alpha(h).$$

Recall that B_x is the cube $Kx + \Omega_K$. To replace in the previous sum $G_s(y/N)$ by $G_s(Kx/N)$, we add and subtract this expression and apply Schwarz inequality to decompose the expectation in two pieces. The first one is just

$$\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in\mathscr{I}_K}\sum_{y\in B_x^0}\{G_s(y/N)-G_s(Kx/N)\}\tau_y h(\eta_s)\,ds\right)^2\right].$$

By Lemma 4.4 this expression vanishes as $N\uparrow\infty$ because G is a smooth function and $K\ll N$. Therefore, in order to prove the theorem, we just need to show that

$$\limsup_{N\to\infty}\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in\mathscr{I}_K}G_s(Kx/N)\sum_{y\in B_x^0}\tau_y h(\eta_s)\,ds\right)^2\right]$$
$$\le C_0 T\|G\|_0^2\,\mathbf{V}_\alpha(h).$$

For each x in $\mathscr{I}_K$, denote by $\tilde h_x$ the conditional expectation of h given $|B_x|^{-1}\sum_{y\in B_x}\eta(y)$. We add and subtract $\tilde h_x(\eta_s)$ inside the time integral of the previous expression. By Schwarz inequality, the expectation of the previous formula is bounded by

$$\begin{aligned}&2\,\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in\mathscr{I}_K}G_s(Kx/N)\sum_{y\in B_x^0}[\tau_y h(\eta_s)-\tilde h_x(\eta_s)]\,ds\right)^2\right]\\&+2\,\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}\left(\int_0^t N^{1-d/2}\sum_{x\in\mathscr{I}_K}G_s(Kx/N)|B_x^0|\tilde h_x(\eta_s)\,ds\right)^2\right].\end{aligned}\tag{4.4}$$

By Schwarz inequality, the second term in (4.4) is bounded above by

$$2TN^{2-d}|B_0|^2\int_0^T ds\,E_{\nu_\alpha}\left[\left(\sum_{x\in\mathscr{I}_K}G_s(Kx/N)\tilde h_x(\eta)\right)^2\right].$$

Since ν_α is a product measure and since the supports of the functions $\tilde h_x$ are disjoint, the previous formula is equal to

$$2TN^{2-d}|B_0|^2\int_0^T\sum_{x\in\mathscr{I}_K}G_s(Kx/N)^2E_{\nu_\alpha}\left[\left(\tilde h_x(\eta)\right)^2\right]ds.$$

Since the cylinder function h belongs to $\mathscr{G}_\alpha$, by (2.1), $E_{\nu_\alpha}[(\tilde h_x(\eta))^2]$ is bounded by $C(h)K^{-2d}$. Therefore, the previous expression is bounded by $C(h,G)T^2N^2K^{-d}$ because $|\mathscr{I}_K|\le(N/K)^d$. This expression vanishes in the limit as $N\uparrow\infty$ because $N^{2/d}\ll K_N$.

We now turn to the first term in (4.4). By Lemma 4.3, this expectation is bounded above by

$$C N^{-d} \int_0^T ds \left\langle \sum_{x \in \mathscr{I}_K} G_s(xK/N) \sum_{y \in B_x^0} [\tau_y h - \tilde{h}_x], (-L_N^s)^{-1} \sum_{x \in \mathscr{I}_K} G_s(xK/N) \right.$$
$$\left. \sum_{y \in B_x^0} [\tau_y h - \tilde{h}_x] \right\rangle_\alpha$$

for some finite constant C. Since for each fixed x, $\sum_{y \in B_x^0} \tau_y h(\eta)$ is measurable with respect to $\{\eta(z),\ z \in B_x\}$, the previous expression is bounded above by

$$C_0 N^{-d} \int_0^T ds \sum_{x \in \mathscr{I}_K} G_s(xK/N)^2 \left\langle \sum_{y \in B_x^0} [\tau_y h - \tilde{h}_x], (-L_{B_x}^s)^{-1} \sum_{y \in B_x^0} [\tau_y h - \tilde{h}_x] \right\rangle_{\nu_\alpha} .$$

In this formula, $L_{B_x}^s$ stands for the generator of the nearest neighbor symmetric simple exclusion process restricted to B_x. Since the process and the invariant measure are translation invariant, this expression is bounded by

$$C_0 K^{-d} T (\|G\|_0^2 + o_N(1)) \left\langle \sum_{y \in B_0^0} [\tau_y h - \tilde{h}_0], (-L_{B_0}^s)^{-1} \sum_{y \in B_0^0} [\tau_y h - \tilde{h}_0] \right\rangle_{\nu_\alpha} .$$

The term $o_N(1)$, which converges to 0 as $N \uparrow \infty$, appeared because we replaced the Riemann sum by the integral. The previous inner product is exactly the finite volume variance defined in section 2 that converges, in virtue of Theorem 2.2, to $\mathbf{V}_\alpha(h)$ as $N \uparrow \infty$. This concludes the proof of the the Boltzmann–Gibbs principle. □

5. Tightness

We prove in this section that the sequence of probability measures Q^N is tight. For $\delta > 0$ and a path Y in $D([0,T], \mathscr{H}_{-k})$, define the uniform modulus of continuity $w_\delta(Y)$ by

$$w_\delta(Y) = \sup_{\substack{|s-t| \le \delta \\ 0 \le s,t \le T}} \|Y_t - Y_s\|_{-k} .$$

Lemma 11.3.2 in [KL] states that a sequence of probability measures Q^N on $D([0,T], \mathscr{H}_{-k})$ is tight provided

$$\lim_{A \to \infty} \limsup_{N \to \infty} Q^N \left[\sup_{0 \le t \le T} \|Y_t\|_{-k} > A \right] = 0 \tag{5.1}$$

and

$$\lim_{\delta \to 0} \limsup_{N \to \infty} Q^N [w_\delta(Y) \ge \varepsilon] = 0 \tag{5.2}$$

for every $\varepsilon > 0$. The next result is a key estimate in the proof of (5.1), (5.2).

Lemma 5.1. *Fix a smooth function* $G\colon \mathbb{T}^d \to \mathbb{R}$. *There exist a finite constant* C, *depending only on* d, α *and* T *such that for all* $N \geq 1$

$$\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T} |Y_t^N(G)|^2\right] \le C\,\left\{\|G\|_0^2 + \|G'\|_0^2 + \|G''\|_0^2\right\}.$$

In this formula $\|G'\|_0$, $\|G''\|_0$ *stand for* $\sum_{1\le i\le d}\|\partial_{u_i}G\|_0$, $\sum_{1\le i,j\le d}\|\partial^2_{u_i,u_j}G\|_0$.

Proof. Rewrite $Y_t^N(G)$ as $M_t^G + Y_0^N(G) + \int_0^t \Gamma_1^G(s)ds$. We estimate each term separately. By definition of Y^N, $E_{\nu_\alpha}[Y_0^N(G)^2]$ is equal to

$$E_{\nu_\alpha}\left[\left(N^{-d/2}\sum_{x\in\mathbb{T}_N^d} G(x/N)[\eta(x)-\alpha]\right)^2\right].$$

Since the measure ν_α is product, this expression is equal to

$$\chi(\alpha)N^{-d}\sum_{x\in\mathbb{T}_N^d} G(x/N)^2,$$

where $\chi(\alpha)$ is equal to $\alpha(1-\alpha)$.

We now turn to the martingale term. By Doob's inequality,

$$\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}(M_t^G)^2\right] \le 4\,\mathbb{E}_{\nu_\alpha}\left[(M_T^G)^2\right] = 4\,\mathbb{E}_{\nu_\alpha}\left[\int_0^T \Gamma_2^G(s)\,ds\right].$$

A straightforward computation gives that

$$\Gamma_2^G(s) = N^{-d}\sum_{\substack{x\in\mathbb{T}_N^d\\ |y|=1}} p(y)\eta_s(x)[1-\eta_s(x+y)]\left(N\left[G_s((x+y)/N) - G_s(x/N)\right]\right)^2.$$

Therefore, the previous expectation is equal to

$$4T\chi(\alpha)N^{-d}\sum_{i=1}^d\sum_{x\in\mathbb{T}_N^d}\left((\partial_{u_i}^N G_s)(x/N)\right)^2$$

because ν_α is an invariant measure.

To estimate the third term, recall that $\Gamma_1^G(s)$ is given by

$$-N^{1-d/2}\sum_{x,i}(\partial_{u_i}G_s)(x/N)v_i[\eta_s(x)-\alpha]$$
$$+N^{1-d/2}\sum_{x,i}(\partial_{u_i}^N G_s)(x/N)\left\{W_{x,x+e_i}(s) - E_{\nu_\alpha}[W_{0,e_i}]\right\}.$$

Since $W_{0,e_i} - E_{\nu_\alpha}[W_{0,e_i}]$ is equal to $-\gamma_i w_i + (b_i + v_i)\eta(0) - b_i\eta(e_i) - \alpha v_i$, we may rewrite the previous expression as

$$\begin{aligned} &N^{1-d/2}\sum_{x,i}(\partial^N_{u_i} G_s - \partial_{u_i} G_s)(x/N)\left\{W_{x,x+e_i}(s) - E_{\nu_\alpha}[W_{0,e_i}]\right\} \\ &-N^{1-d/2}\sum_{x,i}(\partial_{u_i} G_s)(x/N)\gamma_i \tau_x w_i(s) \\ &+N^{-d/2}\sum_{x,i}(\partial^{N,-}_{u_i}\partial_{u_i} G_s)(x/N) b_i[\eta(x) - \alpha]. \end{aligned} \tag{5.3}$$

In this formula $\partial^{N,-}_{u_i}$ stands for the discrete derivative defined by

$$(\partial^{N,-}_{u_i} H)(x/N) = N[H(x/N) - H((x-e_i)/N)].$$

Denote these three expressions by $A_j(s)$, $j = 1, 2, 3$.

We now estimate $\mathbb{E}_{\nu_\alpha}[\sup_{0\le t\le T} | \int_0^t A_j(s)ds|^2]$ for each j. For $j = 1$, by Schwarz inequality, since ν_α is a product invariant measure, the expectation is bounded above by

$$CTN^{-d}\sum_{i=1}^d \mathbf{Var}_{\nu_\alpha}(W_{0,e_i})\int_0^T ds \sum_{x\in\mathbb{T}^d_N}\left(N[\partial^N_{u_i} G_s - \partial_{u_i} G_s](x/N)\right)^2$$

for some finite constant C. This expression is less than or equal to $C(\alpha, d)T^2\|G''\|_0^2$. By a similar argument, the third term is bounded above by $C(\alpha, d)T^2\|G''\|_0^2$. Finally, by the Boltzmann-Gibbs principle (Theorem 4.2), the second term is bounded above by $C(\alpha)T\|G'\|_0^2$. To conclude the proof of the lemma it remains to collect all estimates. □

Corollary 5.2. *For* $k > 2 + (d/2)$,

(a) $\limsup_{N\to\infty}\mathbb{E}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\|Y_t\|^2_{-k}\Big] < \infty$

(b) $\lim_{n\to\infty}\limsup_{N\to\infty}\mathbb{E}_{\nu_\alpha}\Big[\sup_{0\le t\le T}\sum_{|z|\ge n}\gamma_z^{-k}|Y_t(\psi_z)|^2\Big] = 0.$

Proof. The first expression is bounded above by

$$\sum_{z\in\mathbb{Z}^d}\gamma_z^{-k}\,\mathbb{E}_{\nu_\alpha}\left[\sup_{0\le t\le T}|Y_t(\psi_z)|^2\right].$$

By the previous lemma and the explicit formula for ψ_z, the previous sum is less than or equal to

$$C\sum_{z\in\mathbb{Z}^d}\gamma_z^{-k}(1 + \|z\|^4)$$

for all $N \ge 1$ and for some finite constant C depending only on α, d and T. By definition of γ_z, this expression is equal to

$$C \sum_{z \in \mathbb{Z}^d} \frac{1 + \|z\|^4}{(1 + 4\pi^2 \|z\|^2)^k} \le C \sum_{z \in \mathbb{Z}^d} \frac{1}{(1 + 4\pi^2 \|z\|^2)^{k-2}} .$$

This estimate proves the first statement. The second claim follows by the same argument. □

It follows from (5.1), (5.2) and Corollary 5.2 (a) that in order to prove that the sequence Q^N is tight, we only have to show that for every $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_{\nu_\alpha} \Big[w_\delta(Y) > \varepsilon \Big] = 0.$$

In view of part (b) of the previous corollary, this result follows from the following lemma.

Lemma 5.3. *Fix $k > 2 + (d/2)$. For every positive integer n and every $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{\substack{0 \le |s-t| \le \delta \\ 0 \le s,t \le T}} \sum_{|z| \le n} \gamma_z^{-k} |Y_t(\psi_z) - Y_s(\psi_z)|^2 > \varepsilon \right] = 0.$$

Proof. To prove this lemma we just have to show that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{\substack{0 \le |s-t| \le \delta \\ 0 \le s,t \le T}} |Y_t(\psi_z) - Y_s(\psi_z)| > \varepsilon \right] = 0 \tag{5.4}$$

for every z in $\mathbb{Z}^d$ and $\varepsilon > 0$. Fix z in $\mathbb{Z}^d$ and recall the definition of $Z_t^{N,m}(\psi_z)$ given at the beginning of section 3. We prove below in Lemma 5.4 that the difference $Y_t^N(\psi_z) - Z_t^{N,m}(\psi_z)$ is negligible in the sense that

$$\limsup_{N \to \infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{0 \le t \le T} |Y_t^N(\psi_z) - Z_t^{N,m}(\psi_z)| > \varepsilon \right] = 0$$

for all $m \ge 1$. Therefore, in order to prove the lemma, we just need to show that

$$\lim_{m \to \infty} \lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{\substack{0 \le |s-t| \le \delta \\ 0 \le s,t \le T}} |Z_t^{N,m}(\psi_z) - Z_s^{N,m}(\psi_z)| > \varepsilon \right] = 0$$

for every z in $\mathbb{Z}^d$ and $\varepsilon > 0$.

Fix a smooth function $G \colon \mathbb{T}^d \to \mathbb{R}$ and rewrite $Z_t^{N,m}(G)$ as $Z_0^{N,m}(G) + M_t^G + \int_0^t \Gamma_1^G(s) ds$, where M_t^G is a martingale. We first claim that for every $m \ge 1$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{\substack{0 \le |s-t| \le \delta \\ 0 \le s,t \le T}} |M_t^G - M_s^G| > \varepsilon \right] = 0 \tag{5.5}$$

for every $\varepsilon > 0$. Indeed, denote by $w'_\delta(M^G)$ the modified modulus of continuity defined as

$$w'_\delta(M^G) = \inf_{\{t_i\}} \max_{0\le i<r} \sup_{t_i\le s<t<t_{i+1}} |M_t^G - M_s^G|,$$

where the first infimum is taken over all partitions of $[0, T]$ such that

$$\begin{cases} 0 = t_0 < t_1 < \cdots < t_r = T, \\ t_{i+1} - t_i > \delta \;\; 0 \le i < r. \end{cases}$$

Since $\sup_t |M_t^G - M_{t-}^G| = \sup_t |Z_t^{N,m}(G) - Z_{t-}^{N,m}(G)|$ is bounded by $C(G)N^{-d/2}$ and

$$w_\delta(M^G) \le 2w'_\delta(M^G) + \sup_{0\le t\le T} |M_t^G - M_{t-}^G|,$$

in order to prove the lemma we just need to show that

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}_{\nu_\alpha}\Big[w'_\delta(M^G) > \varepsilon\Big] = 0 \tag{5.6}$$

for every $\varepsilon > 0$.

By Aldous criterion for tightness (cf. [KL], Proposition 4.1.6), to prove (5.6) it is enough to check that for every $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{\substack{\tau\in\mathfrak{T}_T \\ 0\le\theta\le\delta}} \mathbb{P}_{\nu_\alpha}\Big[|M_{\tau+\theta}^G - M_\tau^G| > \varepsilon\Big] = 0,$$

where $\mathfrak{T}_T$ stands for all stopping times bounded by T. By Chebychev inequality, the last probability is less than or equal to

$$\frac{1}{\varepsilon^2}\,\mathbb{E}_{\nu_\alpha}\Big[(M_{\tau+\theta}^G - M_\tau^G)^2\Big] = \frac{1}{\varepsilon^2}\,\mathbb{E}_{\nu_\alpha}\Big[\int_\tau^{\tau+\theta} \Gamma_2^G(s)\,ds\Big]$$

because M_t^G is a martingale and τ a bounded stopping time. By the explicit formula for Γ_2^G presented in section 3, this expression is bounded by $C\delta\|G'\|_\infty^2$ for all $N \ge 1$ and for some finite constant C that depends only on the dimension d. This proves (5.6) and thus (5.5).

To conclude the proof of the lemma, it remains to show that

$$\lim_{m\to\infty} \lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}_{\nu_\alpha}\left[\sup_{\substack{0\le|s-t|\le\delta \\ 0\le s,t\le T}} \left|\int_s^t dr\;\Gamma_1^G(r)\right| > \varepsilon\right] = 0. \tag{5.7}$$

Recall the explicit formula (4.3) for $\Gamma_1^G(r)$. We first estimate the third term of (4.3), which we denote by I_0. Notice first that

$$\mathbb{P}_{\nu_\alpha}\left[\sup_{\substack{0\le|s-t|\le\delta \\ 0\le s,t\le T}} \left|\int_s^t dr\; I_0(r)\right| > \varepsilon\right] \le 2\mathbb{P}_{\nu_\alpha}\left[\sup_{0\le t\le T} \left|\int_0^t dr\; I_0(r)\right| > \varepsilon/2\right].$$

The right hand side of this inequality vanishes as $N \uparrow \infty$ in virtue of the Boltzmann–Gibbs principle.

We turn now to the first two terms of (4.3). In (3.4) we decomposed these terms as the sum of three expressions. Denote the k-th line of (3.4) by $I_k(s)$. For $k = 1, 3$, by Schwarz inequality, the left hand side of the previous formula with I_k in place of I_0 is bounded above by

$$\frac{\delta}{\varepsilon^2} \mathbb{E}_{\nu_\alpha} \left[\int_0^T dr \, I_k^2(r) \right].$$

It is easy to show that this expression vanishes as $N \uparrow \infty$ and then $\delta \downarrow 0$. Finally, the arguments used to estimate I_0 together with Theorem 2.3 show that

$$\lim_{m\to\infty} \limsup_{N\to\infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{\substack{0\le|s-t|\le\delta \\ 0\le s,t\le T}} \left| \int_s^t dr \, I_2(r) \right| > \varepsilon \right] = 0.$$

This concludes the proof of the lemma. □

Lemma 5.4. *For each z in $\mathbb{Z}^d$ and $m \ge 1$,*

$$\limsup_{N\to\infty} \mathbb{P}_{\nu_\alpha} \left[\sup_{0\le t\le T} |Y_t^N(\psi_z) - Z_t^{N,m}(\psi_z)| > \varepsilon \right] = 0.$$

Proof. Fix a smooth function $G\colon \mathbb{T}^d \to \mathbb{R}$ and a cylinder function h belonging to $\mathscr{G}_\alpha$. By definition of $Z_t^{N,m}$, the probability appearing in the statement of the lemma is equal to

$$\mathbb{P}_{\nu_\alpha} \left[\sup_{0\le t\le T} \left| N^{-d/2} \sum_{x\in\mathbb{T}_N^d} G_t(x/N) \tau_x h(\eta_t) \right| > \varepsilon N \right].$$

Denote by $\tilde{\mathbb{P}}_{\nu_\alpha}$ the probability on the path space induced by the Markov process η_t and the probability measure ν_α. The difference between $\tilde{\mathbb{P}}_{\nu_\alpha}$ and $\mathbb{P}_{\nu_\alpha}$ is that the process is speeded up by N^2 in the latter probability. With this notation, we may rewrite the previous expression as

$$\tilde{\mathbb{P}}_{\nu_\alpha} \left[\sup_{0\le t\le TN^2} \left| N^{-d/2} \sum_{x\in\mathbb{T}_N^d} G((x - vt)/N) \tau_x h(\eta_t) \right| > \varepsilon N \right].$$

Since the measure ν_α is stationary, the previous probability is bounded above by

$$TN^2 \sup_G \tilde{\mathbb{P}}_{\nu_\alpha} \left[\sup_{0\le t\le 1} \left| N^{-d/2} \sum_{x\in\mathbb{T}_N^d} G((x - vt)/N) \tau_x h(\eta_t) \right| > \varepsilon N \right].$$

In this formula, the supremum is carried over all smooth functions G which are bounded, as well as their first derivative, by a constant C_0. By Chebychev inequality, this expression is bounded above by

$$\frac{T}{\varepsilon^4 N^2} \sup_G \tilde{\mathbb{E}}_{\nu_\alpha} \left[\sup_{0 \le t \le 1} \left(N^{-d/2} \sum_{x \in \mathbb{T}_N^d} G((x - vt)/N) \tau_x h(\eta_t) \right)^4 \right]. \tag{5.8}$$

Denote by $F(t, \eta_t)$ the expression $N^{-d/2} \sum_{x \in \mathbb{T}_N^d} G((x - vt)/N) \tau_x h(\eta_t)$. As before, decompose $F(t, \eta_t)$ as the sum of $F(0, \eta_0) + M_t + \int_0^t \Gamma_1(s) ds$, where M_t is a martingale. Since h is a mean-zero cylinder function, it follows from Schwarz inequality that

$$\sup_G \left\{ \tilde{\mathbb{E}}_{\nu_\alpha} \left[F(0, \cdot)^4 \right] + \tilde{\mathbb{E}}_{\nu_\alpha} \left[\sup_{0 \le t \le 1} \left(\int_0^t \Gamma_1(s) ds \right)^4 \right] \right\} \le C$$

for some finite constant C that depends only on the bound on the smooth functions G and on the cylinder function h.

To estimate the martingale part, denote by $[M, M]_t$ its square bracket. A simple computation shows that

$$[M, M]_t = \sum_{s \le t} \left[F(s, \eta_s) - F(s, \eta_{s-}) \right]^2.$$

Since h is a cylinder function, $[M, M]_t$ is bounded above by a constant, which depends on N, times a Poisson process that dominates the total number of jumps of the Markov process η_t. In particular, $[M, M]_t$ has finite exponential moments. By Doob's and Burkholder's [B] inequality,

$$\tilde{\mathbb{E}}_{\nu_\alpha} \left[\sup_{0 \le t \le 1} M_t^4 \right] \le C \, \tilde{\mathbb{E}}_{\nu_\alpha} \left[[M, M]_1^2 \right]$$

for some universal constant C. Recall the definition of $\Gamma_2(s)$ and denote by Z_t the martingale $[M]_t - \int_0^t \Gamma_2(s) ds$. The right hand side of the previous expression is bounded above by

$$2C \, \tilde{\mathbb{E}}_{\nu_\alpha} \left[Z_1^2 \right] + 2C \, \tilde{\mathbb{E}}_{\nu_\alpha} \left[\left(\int_0^t \Gamma_2(s) \, ds \right)^2 \right].$$

A straightforward computation shows that the second term is bounded above by a constant that depends only on the bounds on the smooth function G and on the cylinder function h. On the other hand, applying once more Burkholder's inequality, we have that the first term is bounded above by $C \tilde{\mathbb{E}}_{\nu_\alpha} [\, [Z, Z]_1^2 \,]$. Notice that the square bracket of the martingale Z is given by

$$[Z, Z]_t = \sum_{s \le t} \left[F(s, \eta_s) - F(s, \eta_{s-}) \right]^4.$$

Since $F(s, \eta_s) - F(s, \eta_{s-})$ is bounded above by $C(G, h) N^{-d/2}$ and the total number of jumps is bounded by a Poisson process of rate $C N^d$, $\tilde{\mathbb{E}}_{\nu_\alpha}[[Z, Z]_1]$ is bounded by $C N^{-d}$ for some finite constant depending only on G and h. This shows that (5.8) is bounded above by $C(\varepsilon, T, C_0, h) N^{-2}$ and concludes the proof of the lemma. □

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