Diffusive behavior of conserved quantities in systems with thermal and mechanical boundary forcing: hydrodynamic limits and non-equilibrium stationary states.

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In many systems energy is not the only *macroscopic* conserved quantity and the interplay between extra conserved quantities and energy has a deep impact on the thermal properties of the system, in particular when all these conserved quantities evolve in the macroscopic diffusive scale.

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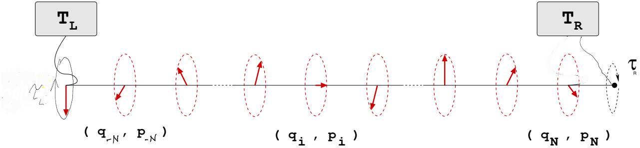
- **Non stationary behaviour**:
- **Stationary non-equilibrium states**: induced by boundary forces and thermostats. Interesting phenomena: *uphill diffusion, non monotonous temperature profiles, negative linear response*.

Rotors Chain



$$r_i = q_i - q_{i-1} \in \mathbb{S}^1, \quad U(r) = 1 - \cos(2\pi r), \quad \mathcal{H} = \sum_{i=-N+1}^{N} \left(\frac{p_i^2}{2} + U(r_i) \right) + \frac{p_{-N}^2}{2} = \sum_{i=-N}^{N} e_i$$

$$\begin{aligned} dr_i &= (p_i - p_{i-1})dt, & i = -N+1, \dots, N \\ dp_i &= (U'(r_{i+1}) - U'(r_i))\, dt & i = -N, \dots, N-1 \\ dp_{-N} &= (\tau_L + U'(r_{-N}))\, dt - \gamma p_{-N} dt + \sqrt{2\gamma T_L} dw_L(t) \\ dp_N &= (\tau_R - U'(r_N))\, dt - \gamma p_N dt + \sqrt{2\gamma T_R} dw_R(t) \end{aligned}$$

Conserved quantities, currents, equilibrium states

Two conserved quantities

$$p_i, \qquad e_i = \left(\frac{p_i^2}{2} + U(r_i)\right),$$

microscopic currents

$$\dot{p}_i = j^p_{i-1} - j^p_i, \quad j^p_i = -U'(r_i), \quad j^p_N = -\tau_{\mathrm{R}} + \gamma p_N - \sqrt{2\gamma T_{\mathrm{R}}}\,\dot{w}_{\mathrm{R}}, \quad j^p_{-N-1} = \tau_{\mathrm{L}} + \dots$$

$$\dot{e}_i = j^e_{i-1} - j^e_i, \quad j^e_i = -p_i U'(r_{i+1}), \quad j^e_N = \gamma(p_N^2 - T_{\mathrm{R}}) - \sqrt{2\gamma T_{\mathrm{R}}}\,p_N \dot{w}_{\mathrm{R}}, \; j^e_{-N-1} = \dots$$

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Equilibrium distribution for $T_R = T_L = T = \beta^{-1}$ and $\tau_R = \tau_L = \gamma \bar{p}$

$$d\mu_{\beta,\bar{p}} = \prod_{i=-N}^{N} \frac{e^{-\beta e_i + \beta \bar{p} p_i}}{Z_{\beta,\bar{p}}} \prod_{i=-N+1}^{N} dr_i \, dp_i \, dp_{-N}$$

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For $T_R \neq T_L$ or $\tau_R \neq \tau_L$, even the existence of the *non-equilibrium stationary state* is an open problem.
($N = 2, 3, 4$: Cuneo, Eckmann, Poquet).

Linear Response: Onsager Matrix (formal argument)

Start with small gradients in temperature and momentum:

$$\beta_i = \frac{1}{2}(T_{\mathrm{R}}^{-1} - T_{\mathrm{L}}^{-1})\frac{i}{N} + \frac{1}{2}(T_{\mathrm{R}}^{-1} + T_{\mathrm{L}}^{-1})$$

$$\beta_i \bar{p}_i = \frac{1}{2\gamma}(T_{\mathrm{R}}^{-1}\tau_{\mathrm{R}} - T_{\mathrm{L}}^{-1}\tau_{\mathrm{L}})\frac{i}{N} + \frac{1}{2\gamma}(T_{\mathrm{R}}^{-1}\tau_{\mathrm{R}} + T_{\mathrm{L}}^{-1}\tau_{\mathrm{L}})$$

$$\delta(\beta) = \frac{1}{2}(T_{\mathrm{R}}^{-1} - T_{\mathrm{L}}^{-1}), \qquad \delta(\beta\bar{p}) = \frac{1}{2\gamma}(T_{\mathrm{R}}^{-1}\tau_{\mathrm{R}} - T_{\mathrm{L}}^{-1}\tau_{\mathrm{L}}),$$

$$d\widetilde{\nu}_N = \prod_{i=-N}^{N} \frac{e^{-\beta_i e_i + \beta_i \bar{p}_i p_i}}{Z_{\beta_i,\bar{p}_i}} \prod_{i=-N+1}^{N} dr_i \, dp_i \, dp_{-N}$$

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$$N\langle j^{p}_{0,1}(t)\rangle_{\widetilde{\nu}_N} = K^{p,p}_N(t)\delta(\beta\bar{p}) + K^{p,e}_N(t)\delta(\beta) + o(|\delta(\beta)|, |\delta(\beta\bar{p})|),$$

$$N\langle j^{e}_{0,1}(t)\rangle_{\widetilde{\nu}_N} = K^{e,p}_N(t)\delta(\beta\bar{p}) + K^{e,e}_N(t)\delta(\beta) + o(|\delta(\beta)|, |\delta(\beta\bar{p})|),$$

Onsager Matrix and symmetries

$$\lim_{t\to\infty} \lim_{N\to\infty} K_N^{a,b}(t) = K^{a,b}(\beta, \bar{p})$$

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K^{p,p}(\beta,\bar{p}) &= -\int_0^\infty \sum_i \langle j_0^p(0) j_i^p(t)\rangle_{\beta,\bar{p}} dt \\
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K^{p,p}(\beta,\bar{p}) &= -\int_0^\infty \sum_i \langle j_0^p(0) j_i^p(t)\rangle_{\beta,\bar{p}} dt = K^{p,p}(\beta) \leq 0. \\
K^{e,e}(\beta,\bar{p}) &= \int_0^\infty \sum_i \langle j_0^e(0) j_i^e(t)\rangle_{\beta,\bar{p}} dt = K^{e,e}(\beta,0) - \bar{p}^2 K^{p,p}(\beta) \geq 0, \\
K^{e,p}(\beta,\bar{p}) &= -\int_0^\infty \sum_i \langle j_0^e(0) j_i^p(t)\rangle_{\beta,\bar{p}} dt = \bar{p} K^{p,p}(\beta) \\
K^{p,e}(\beta,\bar{p}) &= \int_0^\infty \sum_i \langle j_0^p(0) j_i^e(t)\rangle_{\beta,\bar{p}} dt = -\bar{p} K^{p,p}(\beta)
\end{aligned}$$

Transport coefficients as function of $T = \beta^{-1}$

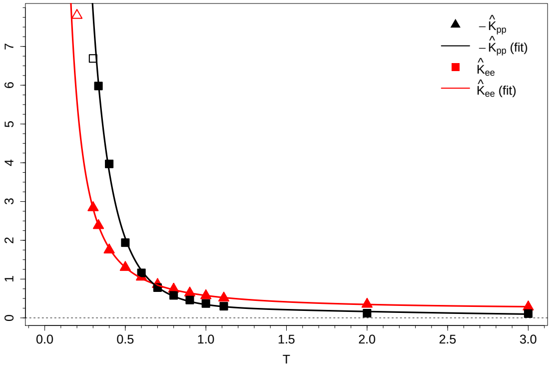

Figure: Onsager coefficients $-\hat{K}^{p,p}$ and $\hat{K}^{e,e}$ as functions of the temperature T.

Macroscopic conservation laws

$$\partial_t p(t,x) = -\partial_x J^p(t,x) \qquad \partial_t e(t,x) = -\partial_x J^e(t,x)$$

$$J^p(t,x) = K^{p,p}(\beta,p)\partial_x(\beta(t,x)p(t,x)) + K^{p,e}(\beta,p)\partial_x\beta(t,x)$$

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In order to close the equations, use the thermodynamic relations:

$$e(t,x) = u(t,x) + \frac{p^2(t,x)}{2}$$

$$T = \beta^{-1}, \qquad u(T) = -\frac{d}{d\beta}\log\iint e^{-\beta[U(r)+p^2/2]}dr\,dp, \qquad C_v(T) = u'(T),$$

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Momentum diffusivity and thermal conductivity

$$D^p(T) = -\frac{1}{T}K^{p,p}\left(\frac{1}{T}\right), \qquad \kappa(T) = \frac{1}{T^2}K^{e,e}\left(\frac{1}{T}\right)$$

Non-stationary macroscopic evolution of $p - T$ profiles

After taking into account all these relation we obtain the closed equations:

$$\partial_t p = \partial_x \left[D^p(T) \partial_x p \right]$$
$$C_v(T) \partial_t T = \partial_x \left[\kappa(T) \partial_x T \right] + D^p(T) \left[\partial_x p \right]^2 .$$

$$T(t,-1) = T_L, \quad T(t,1) = T_R, \quad p(t,-1) = \tau_L/\gamma, \quad p(t,1) = \tau_R/\gamma$$

Gradients of p rise the temperature locally. This is the transfer of *mechanical energy* into *thermal energy*.

Thermodynamic Entropy dissipation (non-stationary)

$$S(u) = \inf_{\beta} \left\{ \beta u + \log \iint e^{-\beta[U(r)+p^2/2]} dr\, dp \right\}, \qquad S'(u) = \beta(u),$$

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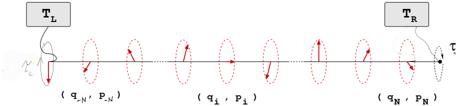
$$e(t,x) = u(t,x) + \frac{p^2(t,x)}{2}$$

$$\frac{d}{dt} \int_{-1}^{1} S(u(t,x))dx = \int_{-1}^{1} \left[\frac{D^p(T)}{T} (\partial_x p)^2 + \frac{\kappa(T)}{T^2} (\partial_x T)^2 \right] dx + \frac{J^Q(t,1)}{T_R} - \frac{J^Q(t,-1)}{T_L}$$

$$J^Q(t,x) = \kappa(T(t,x))\partial_x T(t,x) \qquad \text{heat flux}$$

Rotors: Non-equilibrium Stationary State



Currents of conserved quantities are constant in space, and we expect that:

$$\lim_{N\to\infty} N \left\langle j^a_{i,i+1} \right\rangle_{N,\mathrm{ss}} = J^a \left(T_{\mathrm{L}}, T_{\mathrm{R}}, \tau_{\mathrm{L}}, \tau_{\mathrm{R}} \right), \qquad a \in \{p, e\},$$

$$\lim_{N\to\infty} \left\langle p_{[Nx]} \right\rangle_{N,\mathrm{ss}} = p_{\mathrm{ss}}(x), \qquad \lim_{N\to\infty} \left\langle e_{[Nx]} \right\rangle_{N,\mathrm{ss}} = e_{\mathrm{ss}}(x)$$

$$\lim_{N\to\infty} \left\langle p^2_{[Nx]} \right\rangle_{N,\mathrm{ss}} - \left\langle p_{[Nx]} \right\rangle^2_{N,\mathrm{ss}} = T_{\mathrm{ss}}(x)$$

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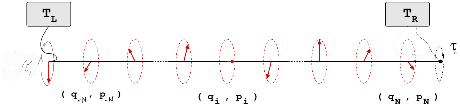

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$$-J^p = D^p[T_\mathrm{ss}(x)]\partial_x p_\mathrm{ss}(x)$$

$$-J^e = D^p[T_\mathrm{ss}(x)]\partial_x \left(\frac{p_\mathrm{ss}(x)^2}{2}\right) + \kappa[T_\mathrm{ss}(x)]\partial_x T_\mathrm{ss}(x)$$

$$T_{ss}(-1) = T_L, \quad T_\mathrm{ss}(1) = T_R, \quad p_\mathrm{ss}(-1) = \frac{\tau_R}{\gamma}, \quad p_\mathrm{ss}(1) = \frac{\tau_R}{\gamma},$$

Rotors: stationary profiles

$$-J^p = D^p[T_{ss}]\partial_x p_{ss}$$

$$-J^e = D^p[T_{ss}]\partial_x\left(\frac{p_{ss}^2}{2}\right) + \kappa[T_{ss}]\partial_x T_{ss} = -J^M(x) - J^Q(x)$$

$$= -\text{mechanical energy current} - \text{thermal energy current}$$

$$T_{ss}(-1) = T_L, \quad T_{ss}(1) = T_R, \quad p_{ss}(-1) = \frac{\tau_R}{\gamma}, \quad p_{ss}(1) = \frac{\tau_R}{\gamma},$$

$$p_{ss}(x)J^p - J^e = \kappa[T_{ss}(x)]\partial_x T_{ss}(x) := -J^Q(x)$$

$$\underset{N\to\infty}{\sim} -N\left\langle\left(p_{[Nx]} - p_{ss}(x)\right)U'(r_{[Nx]})\right\rangle_{N,ss}$$

the *heat current* $J^Q(x)$ is a linear function of $p_{ss}(x)$.

NESS: Entropy production

$$\sigma_N := \left(T_{\mathrm{L}}^{-1} - T_{\mathrm{R}}^{-1}\right)\langle j_i^e\rangle_{N,\mathrm{ss}} - \gamma^{-1}\left(T_{\mathrm{L}}^{-1}\tau_{\mathrm{R}} - T_{\mathrm{R}}^{-1}\tau_{\mathrm{L}}\right)\langle j_i^p\rangle_{N,\mathrm{ss}} > 0.$$

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$$= \frac{J^Q(1)}{T_{\mathrm{R}}} - \frac{J^Q(-1)}{T_{\mathrm{L}}} \qquad J^Q(x) = -\kappa(T_{\mathrm{ss}}(x))\partial_x T_{\mathrm{ss}}(x)$$

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$$= \frac{J^Q(1)}{T_{\mathrm{R}}} - \frac{J^Q(-1)}{T_{\mathrm{L}}} \qquad J^Q(x) = -\kappa(T_{\mathrm{ss}}(x))\partial_x T_{\mathrm{ss}}(x)$$

$$= \int_{-1}^{1} \partial_x \left(\frac{J^Q(x)}{T_{\mathrm{ss}}(x)} \right) dx$$

$$= \int_{-1}^{1} \left[\frac{\kappa(T_{\mathrm{ss}}(x))}{T_{\mathrm{ss}}^2(x)} \left(\partial_x T_{\mathrm{ss}}(x)\right)^2 + \frac{D^p(T_{\mathrm{ss}}(x))}{T_{\mathrm{ss}}(x)} \left(\partial_x p_{\mathrm{ss}}(x)\right)^2 \right] dx.$$

NESS : uphill diffusion

it can happen that

$$\text{sign} J^e = \text{sign}(T_{\mathrm{R}} - T_{\mathrm{L}})$$

This is a obvious phenomena, since

$$-J^e = D^p[T_{\mathrm{ss}}]\partial_x \left(\frac{p_{\mathrm{ss}}^2}{2}\right) + \kappa[T_{\mathrm{ss}}]\partial_x T_{\mathrm{ss}} = -J^M(x) - J^Q(x)$$

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But a bound follows from the positivity of the entropy production:

$$(T_{\rm L} - T_{\rm R})\, J^e \geq -|T_{\rm L} p_{\rm R} - T_{\rm R} p_{\rm L}|$$

NESS: Stationary profiles

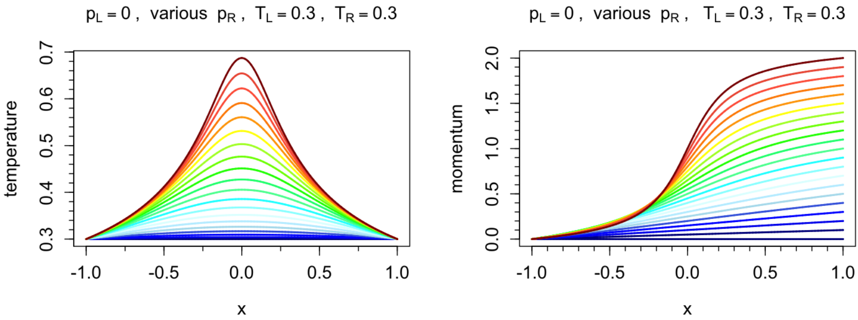


Figure: $T_{\rm L} = T_{\rm R} = 0.3$ (no thermal forcing), $p_{\rm L} = 0$ and values of $p_{\rm R}$ varying from 0 to 2

NESS: Stationary profiles

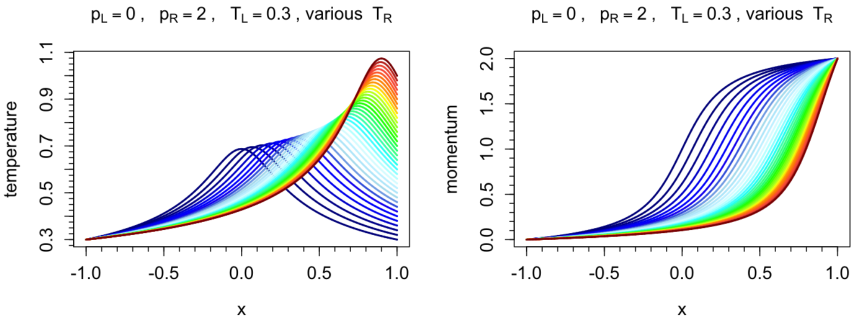
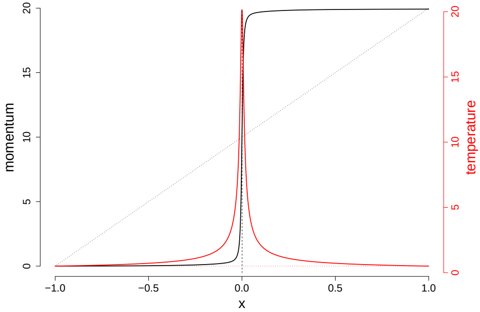


Figure: Note the different behaviors of $\partial_x T_{\mathrm{ss}}\big|_{x=-1}$ as T_{R} increases: this derivative is decreasing for large values of p_{R}.

NESS: low boundary temperatures

For $T_{\mathrm{R}}, T_{\mathrm{L}} \to 0$, the temperature profile spikes:



This behaviour can be proven analitically and it is due to the non-linear decrease behaviour of $\kappa(T)$ and $D^p(T)$.

Negative linear response (negative thermal conductivity)

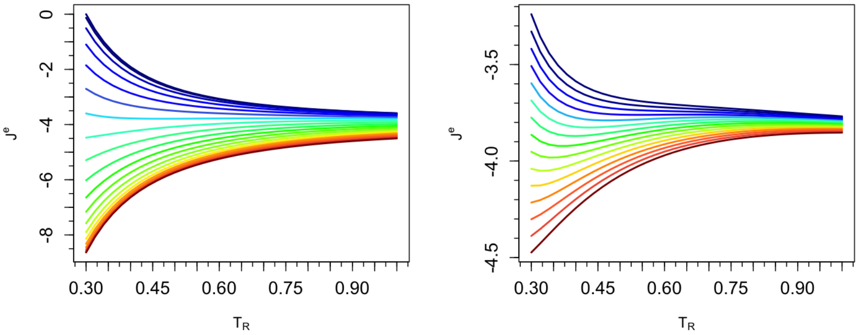


Figure: Energy current J^e as a function of T_{R} at $T_{\mathrm{L}} = 0.3$. Left: $J^e(T_{\mathrm{R}})$ for different values of p_{R}, from $p_{\mathrm{R}} = 0$ (dark-blue line) to $p_{\mathrm{R}} = 2$ (dark-red line), for values of p_{R} incremented by 0.1. Right: zoom on the range of values $p_{\mathrm{R}} \in [0.56, 0.7]$, increments of 0.01. These plots show how the response of the system to an increase of $\Delta T = T_{\mathrm{R}} - T_{\mathrm{L}} \geq 0$ depends on the value of p_{R}. In the right panel, we clearly see the emergence of a minimum in J^e for values of $p_{\mathrm{R}} \in (0.6, 0.66)$. Thermal conductivity is as normal up to a temperature T_{R} corresponding to the minimum of J^e, and becomes negative at larger T_{R}.

References for the Rotors chain

- O. V. Gendelman and A. V. Savin. *Normal heat conductivity of the one-dimensional lattice with periodic potential of nearest-neighbor interaction*. Phys. Rev. Lett., 84:23812384, 2000.
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Other models and rigorous results

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contrexample

The *Garrido-Lebowitz* model of hard discs with energy conserving velocity randomization. The presence of *mechanical energy* is important for such behaviour.

Harmonic chain with velocity random flip

- T. Komorowski. S. Olla, M. Simon, *Hydrodynamic limit for a chain with thermal and mechanical boundary forces*, arXiv:2004.08623, 2020
- T. Komorowski. S. Olla, M. Simon, *An open microscopic model of heat conduction: evolution and non-equilibrium stationary states*, Comm. Math. Sci., 18, 3, 2020

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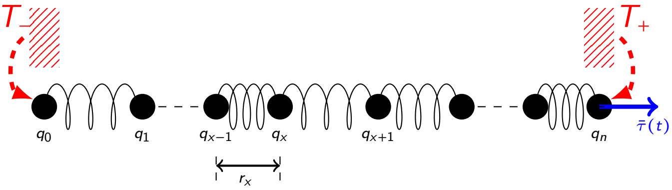
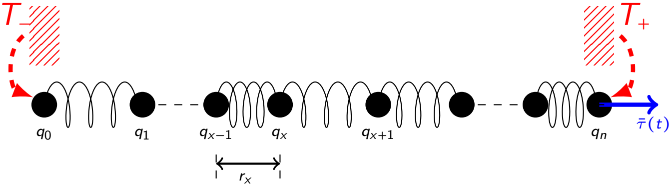


Chain of n harmonic springs,

- with random velocity sign flip: each particle wait an independent exponential time of rate $\tilde\gamma$ and then change sign of its velocity
- in contact with two Langevin thermostats at the boundaries,
- pulled on one side by a force $\bar\tau$.

Harmonic chain with velocity random flip



$r_i = q_i - q_{i-1}, i = 1, \dots, N.$

$$(\mathbf{r}, \mathbf{p}) = (r_1, \dots, r_N, p_0, \dots, p_N) \in \mathbb{R}^N \times \mathbb{R}^{N+1}.$$

$$\mathcal{H}_n := \sum_i \left\{ \frac{p_i^2}{2} + \frac{r_i^2}{2} \right\} + \frac{p_0^2}{2}.$$

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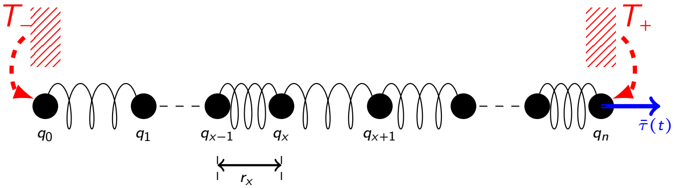

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Two (locally) conserved quantities:

$$r_i \quad \text{volume}, \qquad e_i = \left(\frac{p_i^2}{2} + \frac{r_i^2}{2} \right)$$

Conserved quantities and currents

Theorem

$$r_{[Nx]}(N^2 t) \longrightarrow r(t,x), \qquad e_{[Nx]}(N^2 t) \longrightarrow e(t,x) \qquad \textit{weakly in} [0,1],$$

solution of

$$\partial_t r(t,x) = \frac{1}{2\tilde{\gamma}}\, \partial_x^2 r(t,x)$$

$$\partial_t e(t,x) = \frac{1}{4\tilde{\gamma}} \partial_x^2 \left(e(t,x) + \frac{r(t,u)^2}{2} \right)$$

$$r(t,0) = 0, \quad r(t,1) = \bar{\tau}(t), \quad e(t,0) = \frac{T_-}{2}, \quad e(t,1) = \frac{T_+ + \bar{\tau}(t)^2}{2},$$

Evolution of temperature profile

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$$e(t,x) = \frac{T(t,x)}{2} + \frac{r(t,x)^2}{2} = \text{Thermal energy} + \text{Mechanical energy}$$

$$\partial_t T(t,x) = \frac{1}{4\tilde{\gamma}} \partial_x^2 T(t,x) + \frac{1}{2\tilde{\gamma}} \left(\partial_x r(t,x)\right)^2$$

$$T(t,0) = T_-, \quad T(t,1) = T_+.$$

NESS: stationary profiles

$$
\begin{aligned}
0 &= \partial_x^2 r_{\mathrm{ss}}(x), \\
0 &= \partial_x^2 T_{\mathrm{ss}}(x) + 2\left(\partial_x r_{\mathrm{ss}}(x)\right)^2 \\
r_{\mathrm{ss}}(0) &= 0, \quad r_{\mathrm{ss}}(1) = \bar{\tau}, \quad T_{\mathrm{ss}}(0) = T_-, \quad T_{\mathrm{ss}}(1) = T_+.
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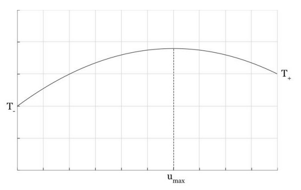
\end{aligned}
$$

$$
\begin{aligned}
r_{\rm ss}(x) &= \bar\tau x & x \in [0,1]. \\
T_{\rm ss}(x) &= 2\bar\tau^2 x(1-x) + (T_+ - T_-)x + T_-, & x \in [0,1].
\end{aligned}
$$



NESS: Stationary current

stationary energy current is given by

$$J^e_{\mathrm{ss}} = -\frac{1}{4\tilde{\gamma}}(T_+ - T_-) - \frac{\bar{\tau}^2}{\tilde{\gamma}}.$$

If $\bar{\tau}^2$ is large enough we can have (*uphill diffusion*).

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$$J^Q(x) = -\frac{1}{4\tilde{\gamma}}\partial_x T_{ss}(x) = -\frac{\bar{\tau}^2}{4\tilde{\gamma}}(1 - 2x) - \frac{1}{4\tilde{\gamma}}(T_+ - T_-)$$

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No negative linear response, no spiking of temperature profile. These depends on the non-linear decrease of $\kappa(T)$ in the rotors chain.

Non-Acoustic Chains: Beam Dynamics

T. Komorowski, S. Olla, *Diffusive propagation of energy in a non-acoustic chain*, Arch. Ration. Mech. Appl. 223 (1) (2017)..

$$\mathcal{H} := \sum_i \left(\frac{p_i^2}{2} + \frac{(q_{i+1} + q_{i-1} - 2q_i)^2}{2} \right).$$

Hamiltonian dynamics plus random exchanges of velocities of n.n. particles.

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Hamiltonian dynamics plus random exchanges of velocities of n.n. particles.
3 conserved quantities that evolve in diffusive time scale.

$$\sum_i k_i, \qquad k_i = q_{i+1} + q_{i-1} - 2q_i \qquad \textit{bending or curvature}$$

$$\sum_i p_i,$$

$$\sum_i e_i, \qquad e_i = \frac{(q_{i+1} + q_{i-1} - 2q_i)^2}{2}$$

Non-Acoustic chain: macroscopic equations

Hydrodynamic limit (diffusive scaling):

Bernoulli beam equation + heat equation:

$$\partial_t k = -\partial_x^2 p$$

$$\partial_t p = \partial_x^2 k + \gamma \partial_x^2 p$$

$$\partial_t T = \frac{1}{\gamma} \partial_x^2 T + \gamma \left(\partial_x p\right)^2$$

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Open question: What about non-linear interaction with potential

$$V\left(q_{i+1} + q_{i-1} - 2q_i\right)$$

and deterministic hamiltonian dynamics?

A different class of models: one dimensional point particles with random flip of velocities sign, no mechanical energy

This is a one dimensional version of the Garrido-Lebowitz model. Particles move on the segment $[-N, N]$, with random change of sign of their velocities. We can ignore the deterministic collisions, but add some other random interaction to destroy all other conservation laws, except energy and density.

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At the boundaries particles reflects or a destroyed and recreated with given densities $\rho_{\mathrm{L}}, \rho_{\mathrm{R}}$ and temperatures $T_{\mathrm{L}}, T_{\mathrm{R}}$.

$$\frac{1}{N}\sum_i G\left(\frac{q_i(N^2 t)}{N}\right) \longrightarrow \int_{-1}^{1} G(y)\rho(t,y)dy$$

$$\frac{1}{N}\sum_i G\left(\frac{q_i(N^2 t)}{N}\right) p_i(N^2 t)^2 \longrightarrow \int_{-1}^{1} G(y)\rho(t,y)T(t,y)dy$$

$$\partial_t \rho = \frac{1}{2}\partial_{yy}\left(\rho T\right), \qquad \partial_t\left(\rho T\right) = \frac{3}{2}\partial_{yy}\left(\rho T^2\right),$$

$$\rho(t,-1) = \rho_{\mathrm{L}},\ \rho(t,1) = \rho_{\mathrm{R}}, \quad T(t,-1) = T_{\mathrm{L}},\ T(t,1) = T_{\mathrm{R}}$$

Garrido-Lebowitz model

- Garrido, P.L. and Lebowitz J.L. , *Diffusion equations from kinetic models with non-conserved momentum*, Nonlinearity 31 54 (2018).
- Esposito, R., Garrido, P.L., Lebowitz, J.L. and Marra, R., *Diffusive limit for a Boltzmann-like equation with non-conserved momentum*, Nonlinearity in press (2019).
- Garrido, P.L., Lebowitz, *Heat Conduction in a hard disc system with non-conserved momentum*, arXiv:1911.03178, 2019.

Hydrodynamic limit from the Boltzmann equation

$$\partial_t F + v \cdot \nabla F = Q_B(F, F) + \alpha Q_d(F)$$

$$Q_d(F) = \int_{S^2} d\omega \left[F(v - 2(v \cdot \omega)\omega) - F(v) \right] |v \cdot \omega|$$

$$F^\epsilon(y, v, t) = F(\epsilon^{-1} y, v, \epsilon^{-2} t)$$

On the mathematical proof of the macroscopic equation

- Usual *relative entropy* techniques are not sufficient with energy conserving models.

On the mathematical proof of the macroscopic equation

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- *Wigner distributions*

$$\hat{\psi}(t,k) = \hat{r}(t,k) + i\hat{p}(t,k)$$

$$\widehat{W}_N(t,\eta,k) = \frac{1}{2N}\left\langle \hat{\psi}\left(N^2 t, k + \frac{\eta}{N}\right)^* \hat{\psi}(N^2 t, k)\right\rangle$$

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$$W_N(t,x,k) \underset{N\to\infty}{\longrightarrow} \frac{1}{2}T(t,x)dk + \frac{1}{2}r(t,x)^2\delta_0(k)$$

= thermal energy + mechanical energy

$T(t,x), r(t,x)$ solution of the macroscopic equations with boundary conditions.