Fluctuations in the weakly asymmetric exclusion process with open boundary conditions

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Abstract

We investigate the fluctuations around the average density profile in the weakly asymmetric exclusion process with open boundaries in the steady state. We show that these fluctuations are given, in the macroscopic limit, by a centered Gaussian field and we compute explicitly its covariance function. We use two approaches. The first method is dynamical and based on fluctuations around the hydrodynamic limit. We prove that the density fluctuations evolve macroscopically according to an autonomous stochastic equation, and we search for the stationary distribution of this evolution. The second approach, which is based on a representation of the steady state as a sum over paths, allows one to write the density fluctuations in the steady state as a sum over two independent processes, one of which is the derivative of a Brownian motion, the other one being related to a random path in a potential.

Key words: Exclusion process, stationary non-equilibrium states, fluctuations

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1 Introduction

Non equilibrium systems such as systems in contact with two thermostats at unequal temperatures or with two reservoirs at unequal densities are known to exhibit long range correlations in their steady state [1]. These long range correlations have been calculated from the microscopic dynamics only in very few cases: mainly in the case of the symmetric exclusion process [2] and of the asymmetric exclusion process [3].

In the present paper we focus on the weakly asymmetric exclusion process (WASEP) (for which the bias scales as the inverse of the system size). We show that the fluctuations of density are Gaussian and that, as for the symmetric case, the direct calculation of the two point function by fluctuating hydrodynamics agrees with the expression derived by expanding the large deviation function around the average density profile. We also show that, as for the asymmetric case [3], the correlation functions can be expressed in terms of two independent random processes.

The asymmetric exclusion process describes the stochastic evolution of a system of particles on a one dimensional lattice of L sites. Each site of the lattice is occupied by at most one particle at a given time. Thus a configuration of the system is specified by a series of occupation numbers $\{\eta_i\}_{i=1,\dots,L}$, where $\eta_i = 1$ if site i is occupied by a particle, $\eta_i = 0$ if site i is empty.

At any given time, each particle independently attempts to jump to one of its neighboring sites on the lattice with rates which depend on the direction of the jump. The jump succeeds only if the target site is empty, otherwise the particle doesn't move. We choose to scale the time such that the hopping rate to a right neighbor is 1, and we note by q the hopping rate in the opposite direction. Physically, the asymmetry between left and right hopping rates mimics the effect of some external field acting on the particles.

The first and last sites of the lattice, respectively labeled 1 and L, are in contact with reservoirs of density respectively ρ_a and ρ_b. This can be achieved by adding, to the left of site 1, a site 0 whose probability of being occupied by a particle is kept constant to ρ_a independently of the rest of the system, and, to the right of site L, a site $L+1$ whose probability of being occupied by a particle is kept constant to ρ_b independently of the rest of the system. Another equivalent way of expressing the action of the left reservoir is to say that particles are added to the site 1 at rate $\alpha = \rho_a$ when the first site is empty, and when it is occupied, the particle is removed from the system at $\gamma = q(1-\rho_a)$ (i.e. a particle at site 1 attempts to jump to the left reservoir (site 0) with a rate q and the jump succeeds with a rate $1-\rho_a$). In the same way, if site L is occupied, the particle is removed at rate $\beta = 1 - \rho_b$, and if it is empty, a particle is added at rate $\gamma = q\rho_b$.

The generator $\mathbb{L}$ of the dynamics, acting on a given function g of the configuration of the system $\eta = \{\eta_i\}_{i=1,\dots,L}$ is given by

$$\begin{aligned}
\mathbb{L}g(\eta) &= \sum_{i=1}^{L-1} \eta_i(1-\eta_{i+1})\big[g(\eta^{i,i+1}) - g(\eta)\big] \\
&+ \sum_{i=2}^{L} q\eta_i(1-\eta_{i-1})\big[g(\eta^{i-1,i}) - g(\eta)\big] \\
&+ \big[\rho_a(1-\eta_1) + q(1-\rho_a)\eta_1\big]\big[g(\eta^1) - g(\eta)\big] \\
&+ \big[q\rho_b(1-\eta_L) + (1-\rho_b)\eta_L\big]\big[g(\eta^L) - g(\eta)\big] \ ,
\end{aligned}$$

where $\eta^{i,i+1}$ is the configuration obtained from η by exchanging the occupation numbers of sites i and $i+1$ and η^i is obtained from η by changing the occupation number of site i.

In the following, we consider the weakly asymmetric exclusion process, where the asymmetry q scales with the system size L by

$$q = 1 - \frac{\lambda}{L} \quad .$$

Of particular interest are the macroscopic properties of the system in the large L limit. We focus here on the macroscopic density profiles $\{\rho(x)\}$, $0 \le x \le 1$, obtained by rescaling by L^{-1} and smoothening the microscopic density profiles $\{\eta_i\}_{i=1,L}$. To do that, we may for example divide the system in mesoscopic boxes of size L_k, with $1 \ll L_k \ll L$. The macroscopic density $\rho_L(\frac{i}{L})$ is then simply the number of particles in the box containing the site i, divided by the size of the box. A more mathematical approach consists in defining, for each size L, the following distribution acting on smooth test function

$$\rho_L(t,x) = \frac{1}{L}\sum_{i=1}^{L} \delta(x - i/L)\eta_i(L^2 t) \tag{1.1}$$

In spite of the stochastic evolution of the system at the microscopic scale, it is known that, as $L \to \infty$, the macroscopic profile ρ_L converges almost surely to a deterministic evolution $\rho^H(t,x)$ given by the solution of the hydrodynamic equations of the process [6, 7, 1]. In the case of the weakly asymmetric exclusion process, it has been proved [8, 9] that this is given by the viscous Burgers equation:

$$\begin{cases} \partial_t \rho^H = \partial_x^2 \rho^H - \lambda(1-2\rho^H)\partial_x \rho^H \ , \\ \rho^H(t,0) = \rho_a, \quad \rho^H(t,1) = \rho_b \ , \\ \rho^H(0,x) = \rho_0(x) \end{cases} \tag{1.2}$$

where $\rho_0(x)$ is the macroscopic limit profile at time $t = 0$.

At a finer scale, the density profile ρ_L has random fluctuations of order $L^{-1/2}$ around the hydrodynamic trajectory ρ^H. One defines the density fluctuation field by

$$\xi_L(t,x) = \sqrt{L}\big[\rho_L(t,x) - \rho^H(t,x)\big] \quad . \tag{1.3}$$

It is known [8, 10] that these fields have a well defined large L limit $\xi(x,t)$ which is a generalized Ornstein-Uhlenbeck process in the case of WASEP on an infinite lattice (for such system, the parameter L is not any more related to the size of the lattice, which is infinite, but only to the rescaling of time and space, as well as to the asymmetry rate).

For a finite system of size L, after a long time, the system eventually reaches a steady state where the properties of the system become time-independent. Except for particular values of the reservoir densities and of the asymmetry parameter q for which detailed balance is satisfied [11], this stationary state is a non-equilibrium steady state, which differs from an equilibrium state by the presence of a non-zero, site-independent average current j

$$j = \langle \eta_i(1-\eta_{i+1})\rangle - q\langle \eta_{i+1}(1-\eta_i)\rangle \tag{1.4}$$

(where the brackets $\langle . \rangle$ stands for the average with respect to the steady state probability denoted by μ^L).

The system presents almost surely an average macroscopic profile $\{\bar{\rho}(x)\}$ such that, for any site i, one has in the large L limit:

$$\langle \eta_i \rangle \simeq \bar{\rho}(\frac{i}{L}) \quad .$$

Putting $\partial_t \rho$ to 0 into (1.2) gives the following equation for the steady state average profile $\{\bar{\rho}(x)\}$:

$$\left\{ \begin{array}{l} \bar{\rho}'(x) = \lambda \bar{\rho}(x)(1-\bar{\rho}(x)) - J \ , \\ \bar{\rho}(0) = \rho_a \ , \\ \bar{\rho}(1) = \rho_b \ , \end{array} \right. \tag{1.5}$$

where J is an integration constant solution of

$$\int_{\rho_a}^{\rho_b} \frac{d\rho}{\lambda\rho(1-\rho) - J} = 1 \quad .$$

J is related to the steady state average current j (1.4) by

$$J = \lim_{L\to\infty} Lj \quad .$$

The steady state fluctuation field will be noted ξ^{st}:

$$\xi^{st}(x) = \lim_{L\to\infty} \sqrt{L}\big[\rho_L(x) - \bar{\rho}(x)\big]$$

with $\rho_L(x)$ defined as in (1.1).

In the stationary state the macroscopic limit of the density profile coincides with the deterministic solution of the stationary equation (1.5) with the steady current J determined by the asymmetry parameter λ and the boundary conditions ρ_a, ρ_b. Actually, there exist [11] explicit expressions for $\bar{\rho}$, but we will not use them in this paper. Without loss of generality we assume that

$$\rho_a \ < \ \rho_b$$

so that we have always $\bar{\rho}'(x) \geq 0$. The current J can be positive or negative depending on λ.

Our main result is that the macroscopic fluctuation field $\xi^{st}(x)$ is a centered Gaussian field on $[0,1]$ with covariance given by

$$\big\langle \xi^{st}(x)\xi^{st}(y)\big\rangle = \chi(\bar{\rho}(x))\delta(x-y) + f(x,y) \tag{1.6}$$

where

$$\chi(\rho) = \rho(1-\rho) \tag{1.7}$$

and $f(x,y)$ is given by

$$f(x,y) = \frac{J\bar{\rho}'(x)\bar{\rho}'(y)\int_0^{x\wedge y}\frac{du}{\bar{\rho}'(u)}\int_{x\vee y}^1\frac{dv}{\bar{\rho}'(v)}}{\int_0^1\frac{du}{\bar{\rho}'(u)}} \ , \tag{1.8}$$

where $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$.

In terms of the two points correlation function, this result implies that when $x \neq y$

$$L\left(\left\langle \eta_{[Lx]}\eta_{[Ly]}\right\rangle - \left\langle \eta_{[Lx]}\right\rangle\left\langle \eta_{[Ly]}\right\rangle\right) \underset{L\to\infty}{\longrightarrow} f(x,y) \tag{1.9}$$

Some comments are in order.

- In the symmetric case ($\lambda = 0$) we have $\bar{\rho}'(x) = \rho_b - \rho_a$ and $J = -(\rho_b - \rho_a)$, so that

  $$f(x,y) = -(\rho_b-\rho_a)^2(x\wedge y)[1-(x\vee y)] = -(\rho_b-\rho_a)^2(-\Delta)^{-1}(x,y) \ ,$$

  in agreement with the result of Spohn on the 2-point correlation function [2].

- In the equilibrium case $J=0$, the stationary fluctuation field is just white noise on $[0,1]$, corresponding to the fact that the stationary state will be given by a product measure.
- The sign of $f(x,y)$ depends on the sign of the current J. If λ is positive and large enough, $J>0$ (the weak shock regime) and $f(x,y)\geq 0$. In the other cases J and f are negative.

We derive the result (1.6,1.8) by two different methods. The first approach is dynamical: we search for the stationary solutions of the macroscopic fluctuations process. The second approach is static and based on a representation (valid only when $J(\rho_a-\rho_b)>0$, see [11]) of the weights in the steady state as sums over paths which was used in [11] to calculate the large deviation function of the stationary measure. A priori there is no reason that the correlation functions are simply related to the large deviation functional of the density profile: density fluctuations describe variations of density of order $\frac{1}{\sqrt{L}}$ whereas the large deviation functional describes variations of order 1. For the symmetric exclusion process, it was however shown [5] that the expression of correlation functions obtained by a direct calculation can be recovered by expanding the large deviation functional around the average profile. On the other hand, for the asymmetric exclusion process, the large deviation functional is non-analytic close to the average profile and there is no simple connection between the large deviations and the two point correlations [4]. One outcome of the present work is that the fluctuations of the density are still given, for the WASEP, as the expansion of the large deviation function around the average profile.

Both methods rely on distinct representations of the kernel f. In the dynamical approach we proceed as follows. Let $\mathcal{L}$ be the differential operator $\Delta+\lambda(1-2\bar\rho(x))\nabla$ with Dirichlet boundary conditions on $[0,1]$. Since $\bar\rho$ is the solution of (1.5), it is easy to show that $\mathcal{L}$ can be extended to a negative self-adjoint operator on $L^2(\bar\rho'(x)dx)$. The kernel of the inverse operator $(-\mathcal{L})^{-1}(x,y)$ can be calculated explicitly (cf. Section 2) and shown to satisfy

$$f(x,y)=J\bar\rho'(x)\bar\rho'(y)(-\mathcal{L})^{-1}(x,y)\ . \tag{1.10}$$

In the static approach, one shows that the Gaussian field $\xi^{st}(x)$ can be decomposed as a sum of two **independent** processes

$$\xi^{st}(x)=\sqrt{\frac{\chi(\bar\rho(x))}{2}}B'(x)+\frac{1}{2}Y'(x)\ , \tag{1.11}$$

where $B(x)$ is a standard Brownian motion such that

$$\left\langle [B(x)-B(x')]^2\right\rangle=|x-x'| \tag{1.12}$$

(i.e. $B'(x)$ is a standard δ-correlated white noise), while $Y(x)$ is a centered Gaussian process whose distribution is formally given by

$$d\mathbb{Q}(\{Y\}) \propto \exp\left\{-\int_0^1 dx\left(\frac{-J\bar\rho'(x)Y(x)^2}{2\chi(\bar\rho(x))^2}+\frac{Y'(x)^2}{4\chi(\bar\rho(x))}\right)\right\}\mathcal{D}[\{Y\}] \tag{1.13}$$

where $\mathcal{D}\big[\{Y\}\big]$ is the standard Feynman measure. Writing the distribution of the centered Gaussian process $Y'(x)$ as

$$d\mathbb{Q}(\{Y'\}) \propto \exp\left\{-\frac{1}{2}\int_0^1\int_0^1 Y'(u)T(u,v)Y'(v)\,du\,dv\right\}\mathcal{D}[\{Y'\}]\,, \tag{1.14}$$

we show in section 4 that its covariance can be written as

$$\langle Y'(x)Y'(y)\rangle = T^{-1}(x,y) = 2\chi(\bar\rho(x))\delta(x-y)+4f(x,y) \tag{1.15}$$

where $f(x,y)$ can be explicitly calculated and shown to be given by (1.8). The covariance (1.6) of the fluctuation field $\xi^{st}(x)$ follows from (1.11) and (1.15).

2 Dynamical approach

Consider the time dependent fluctuation field $\xi_L(t,x)$ defined by (1.3). Under proper assumptions on the initial distribution, it is proved in [12] that the law of ξ_L converges, as $L\to\infty$, to the solution $\xi(t,x)$ of the linear stochastic partial differential equation:

$$\partial_t\xi = \partial_x^2\xi - \lambda\,\partial_x\{[1-2\rho^H(t,x)]\xi\} - \partial_x\left(\sqrt{2\chi(\rho^H(t,x))}\;W(t,x)\right)\,, \tag{2.1}$$

where $W(t,x)$ is the standard space time white noise, i.e., the Gaussian process on $\mathbb{R}_+\times(0,1)$ with covariance

$$\langle W(t,q)W(t',q')\rangle = \delta(q-q')\delta(t-t')\,,$$

and χ is given by (1.7).
Since $\xi(t,x)$ and $W(t,x)$ are distributions-valued processes, equation (2.1) should be interpreted in the weak form: for any smooth test function G with compact support in $(0,1)$, denoting by $\xi_t(G)=\int_0^1 G(x)\xi(t,x)dx$,

$$\begin{aligned}\xi_t(G)-\xi_0(G) &= \int_0^t \xi_s(G'')\,ds \;+\; \lambda\int_0^t \xi_s\big([1-2\rho^H(s,\cdot)]G'\big)\,ds \\ &+ \int_0^t\int_0^1 G'(x)\sqrt{2\chi(\rho^H(s,x))}\;W(s,x)dsdx\,.\end{aligned} \tag{2.2}$$

To investigate the asymptotic behavior of ξ_t, consider the weakly asymmetric exclusion process starting from a local Gibbs state associated to the steady state density profile $\bar\rho$. In this case, the solution of the hydrodynamic equation (1.2) is constant in time and equal to $\bar\rho$. In particular, the density fluctuation field ξ_L converges, as $L \to \infty$, to the Ornstein-Uhlenbeck process (2.2) with $\bar\rho(\cdot)$ in place of $\rho^H(s,\cdot)$. Let $\mathcal{L}$ be the differential operator defined by

$$\mathcal{L} = \Delta + \lambda(1 - 2\bar\rho(x))\nabla \tag{2.3}$$

with Dirichlet conditions at the boundary. An elementary computation shows that for any smooth function G with compact support on $(0,1)$

$$\xi_t(G) = \xi_0(e^{t\mathcal{L}}G) + \int_0^t \int_0^1 \left(\nabla e^{(t-s)\mathcal{L}} G\right)(x) \sqrt{2\chi(\bar\rho(x))}\, W(s,x) ds dx\ . \tag{2.4}$$

Since we imposed Dirichlet boundary conditions for $\mathcal{L}$, $e^{t\mathcal{L}}G$ converges to 0 as $t \to \infty$. The second term of the right hand side of (2.4) is a Gaussian variable with zero mean and variance given by

$$2 \int_0^t ds \int_0^1 dx\ \left[\nabla \left(e^{(t-s)\mathcal{L}} G\right)(x)\right]^2 \chi(\bar\rho(x))\ .$$

Integrating by parts the previous expression with respect to x, recalling the explicit formula (2.3) for the operator $\mathcal{L}$ and equation (1.5) for the stationary density profile, we obtain that the previous integral is equal to

$$\begin{aligned} &\int_0^1 G(x)^2 \chi(\bar\rho(x))\, dx \ - \int_0^1 (e^{t\mathcal{L}}G)(x)^2 \chi(\bar\rho(x))\, dx \\ &\quad + 2J \int_0^t ds \int_0^1 dx\ \left(e^{(t-s)\mathcal{L}} G(x)\right)^2\ \bar\rho'(x)\ . \end{aligned}$$

Since $e^{t\mathcal{L}}G \to 0$ as $t \to \infty$, the second term of the previous equation vanishes as $t \to \infty$. On the other hand, since $\bar\rho$ is solution of the stationary equation (1.5), the operator $\mathcal{L}$ is symmetric in $L^2(\bar\rho'(x)dx)$. In particular, the last term of the previous equation can be rewritten as

$$2J \int_0^t ds \int_0^1 dx\ G(x)\, [e^{2(t-s)\mathcal{L}} G](x)\, \bar\rho'(x)$$

which converges, as $t \to \infty$, to

$$J \int_0^1 G(x)[(-\mathcal{L})^{-1} G](x)\ \bar\rho'(x)\ dx\ .$$

In conclusion, the generalized Ornstein-Uhlenbeck process ξ_t converges, as $t \to \infty$, to a Gaussian field ξ^{st} with covariance given by

$$\begin{aligned} &\langle \xi^{st}(G), \xi^{st}(F) \rangle \\ &\quad = \int_0^1 G(x)F(x)\chi(\bar\rho(x))\, dx \;+\; J \int_0^1 G(x)[(-\mathcal{L})^{-1}F](x)\, \bar\rho'(x)\, dx \, . \end{aligned} \tag{2.5}$$

This Gaussian field is also the stationary state for the Ornstein-Uhlenbeck process (2.2) and the limit of the density fluctuation field under the stationary measure.

An elementary computation permits to rewrite the covariance as follows

$$\begin{aligned} &\langle \xi^{st}(G), \xi^{st}(F) \rangle \;= \\ &\quad \int_0^1 G(x)\, F(x)\, \chi(\bar\rho(x))\, dx \;+\; \int_0^1 dx \int_0^1 dy\, F(x)\, f(x,y)\, G(y)\;, \end{aligned} \tag{2.6}$$

where f is given by (1.8). Indeed, for $F = G$, the second term on the right hand side of (2.5) can be written as

$$J \sup_H \Big\{ 2 \int_0^1 dx\, G(x)\, H(x)\, \bar\rho'(x) \;-\; \int_0^1 dx\, H(x)\, (-\mathcal{L}H)(x)\, \bar\rho'(x) \Big\}\;,$$

where the supremum is carried over all smooth functions H vanishing at the boundary. Since $\bar\rho$ is the solution of the equation (1.5), an integration by parts gives that

$$\int_0^1 dx\, H(x)\, (-\mathcal{L}H)(x)\, \bar\rho'(x) = \int_0^1 dx\, [H'(x)]^2 \bar\rho'(x)\;.$$

On the other hand, since H vanishes at the boundary, an integration by parts permits to rewrite the linear term as

$$2 \int_0^1 dx\, G(x)\, H(x)\, \bar\rho'(x) = -2 \int_0^1 dx \Big(\int_0^x dy\, \bar\rho'(y) G(y) - A \Big) H'(x)$$

for any constant A. In view of the previous two expressions, it is not difficult to show that the supremum in H is equal to

$$\int_0^1 dx\, \frac{1}{\bar\rho'(x)} \Big(\int_0^x dy\, \bar\rho'(y) G(y) - A \Big)^2 \tag{2.7}$$

with A given by

$$A \;=\; \frac{\int_0^1 dx\, \frac{1}{\bar\rho'(x)} \int_0^x dy\, \bar\rho'(y) G(y)}{\int_0^1 dx\, \frac{1}{\bar\rho'(x)}}\;.$$

It remains to develop the square and to recall the factor J in front of the variational formula to obtain the second term on the right hand side of (2.6) with f given by (1.8).

Notice that in the symmetric case (when $q = 1$, i.e. $\lambda = 0$), $\mathcal{L}$ is the usual Laplacian Δ, $\bar{\rho}'(x) = \rho_b - \rho_a = -J$, and (2.5) becomes

$$\left\langle \xi^{st}(G), \xi^{st}(F) \right\rangle = \int_0^1 G(x)F(x)\chi(\bar{\rho}(x))\, dx - (\rho_b - \rho_a)^2 \int_0^1 G(x)[(-\Delta)^{-1}F](x)\, dx$$

in accordance with the covariance formula obtained by Spohn in [2].

3 Static Approach

The idea of the derivation is similar to the one in the totally asymmetric case [3]. In [11], under the assumption that

$$J(\rho_a - \rho_b) \ \geq \ 0\ ,$$

(which holds, for example, when $\rho_a < \rho_b$ and $q > 1$), it was showed that the probability $d\mathbb{P}\big(\{\rho(x)\}\big)$ of observing a given macroscopic profile $\rho(x)$ in the steady state can be written in the large L limit as

$$d\mathbb{P}(\{\rho\}) \sim \mathcal{D}[\{\rho\}] \int_{\{y\}} e^{-L\mathcal{G}(\{\rho\},\{y\})} \mathcal{D}[\{y\}] \tag{3.1}$$

where the sum is over all positive continuous functions $\{y(x), 0 \leq x \leq 1\}$. We will call such a function $\{y(x)\}$ a path. The expression (3.1) is written there as a path integral, but it is nothing more than the large L limit of a sum over discrete paths $y(i/L) = \frac{n_i}{L}$, and the measure (3.1) can be simply thought as the weight of these discrete paths.

The function $\mathcal{G}(\{\rho(x)\}, \{y(x)\})$ was given in [11], equation (3.22):

$$\begin{aligned}
\mathcal{G}(\{\rho(x)\};\{y(x)\}) = {} & -K_\lambda(\rho_a, \rho_b) + y(0)\log\frac{\rho_a}{1-\rho_a} + y(1)\log\frac{1-\rho_b}{\rho_b} \\
& + \int_0^1 dx \left[-\log\frac{1-e^{-\lambda y}}{\lambda} + \rho\log\rho + (1-\rho)\log(1-\rho) \right. \\
& \left. + (1-\rho+y')\log(1-\rho+y') + (\rho-y')\log(\rho-y') \right]
\end{aligned}$$

where $K_\lambda(\rho_a, \rho_b)$ is a normalization constant. The quantity $e^{-L\mathcal{G}(\{\rho\},\{y\})}\mathcal{D}[\{y\}]\mathcal{D}[\{\rho\}]$ can be thought as the joint probability of the profile $\{\rho\}$ and the path $\{y\}$.

One can rewrite (3.1) as

$$d\mathbb{P}(\{\rho\}) = \mathcal{D}[\{\rho\}] \int_{\{y\}} r(\{\rho\}|\{y\}) d\mathbb{Q}(\{y\})$$

where $d\mathbb{Q}(\{y\})$ is the probability measure of the positive walk $\{y\}$

$$d\mathbb{Q}(\{y\}) = \mathcal{D}[\{y\}] \int_{\{\rho\}} e^{-L\mathcal{G}(\{\rho\},\{y\})} \mathcal{D}[\{\rho\}] \ \sim \ \mathcal{D}[\{y\}] e^{-L\mathcal{Q}(\{y\})}$$

with $\mathcal{Q}(\{y\}) = \inf_{\{\rho(x)\}} \mathcal{G}(\{\rho\},\{y\})$, i.e.

$$\begin{aligned} \mathcal{Q}(\{y\}) = -K_\lambda(\rho_a,\rho_b) - \log 4 + y(0) \log \frac{\rho_a}{1-\rho_a} + y(1) \log \frac{1-\rho_b}{\rho_b} \\ + \int_0^1 dx \left[- \log \frac{1-e^{-\lambda y}}{\lambda} \right. \\ \left. + (1+y') \log (1+y') + (1-y') \log (1-y') \right] \end{aligned} \quad (3.2)$$

(as for a given y, the optimal profile is $\rho_y = \frac{1+y'}{2}$) and $r(\{\rho\}|\{y\})$ is the conditional probability of the profile ρ given the path $\{y\}$ given by the Radon-Nikodym derivative

$$r(\{\rho\}|\{y\}) = \frac{e^{-L\mathcal{G}(\{\rho\},\{y\})} \mathcal{D}[\{y\}]}{d\mathbb{Q}(\{y\})}$$

so

$$\begin{aligned} \frac{\log \left(r(\{\rho\}|\{y\}) \right)}{L} = - \log 4 + \int_0^1 dx \, \{ -\rho \log \rho - (1-\rho) \log (1-\rho) \\ + (1+y') \log (1+y') + (1-y') \log (1-y') \\ -(1-\rho+y') \log (1-\rho+y') - (\rho - y') \log (\rho - y') \} \end{aligned}$$

The fluctuations of $\rho(x)$ in (3.1) have thus two contributions: one coming from the choice of the path $y(x)$, and the other one from the randomness of the profile $\rho(x)$ once the path $y(x)$ is chosen. We shall see that for small fluctuations, these two contributions are uncorrelated, leading to (1.11).

The optimal path $y_{\text{opt}}(x)$, which maximizes the expression (3.2) of $d\mathbb{Q}(\{y\})$, is solution of

$$\begin{aligned} y'_{\text{opt}}(0) &= 2\rho_a - 1 \\ y'_{\text{opt}}(1) &= 2\rho_b - 1 \\ \frac{2y''_{\text{opt}}}{1 - y'^2_{\text{opt}}} &= -\frac{\lambda e^{-\lambda y_{\text{opt}}}}{1 - e^{-\lambda y_{\text{opt}}}} \quad . \end{aligned} \quad (3.3)$$

By expanding (3.2) up to the second order, we get for a path y close to y_{opt}:

$$y(x) = y_{\mathrm{opt}}(x) + \delta y(x)$$

that its probability measure is given by

$$d\mathbb{Q}(y) \sim \exp\left(-L\int_0^1 dx\left[\frac{\lambda^2}{2}\frac{e^{-\lambda y_{\mathrm{opt}}}(\delta y)^2}{(1-e^{-\lambda y_{\mathrm{opt}}})^2} + \frac{(\delta y')^2}{1-y'^2_{\mathrm{opt}}}\right]\right)\mathcal{D}[\{y\}]$$

As $y_{\mathrm{opt}}(x) > 0$ and $\delta y \sim \frac{1}{\sqrt{L}}$, the condition that

$$y(x) = y_{\mathrm{opt}}(x) + \delta y(x) > 0$$

is automatically satisfied.

The optimal density profile $\rho_y(x)$ for a given $\{y\}$ (i.e. the one which maximizes $r(\{\rho\}|\{y\}))$ is given by

$$\rho_y = \frac{1+y'}{2}. \tag{3.4}$$

Given the fluctuation $\delta y(x) = y(x) - y_{\mathrm{opt}}(x)$ of the walk $\{y\}$, the probability of a small fluctuation of density $\delta b(x) = \rho(x) - \rho_y(x)$ around ρ_y is obtained by expanding $r(\{\rho\}|\{y\})$ up to the second order in δb and δy:

$$r(\{\rho\}|\{y\}) \sim \exp\left(-L\int_0^1 dx(\delta b)^2\frac{4}{1-y'^2_{\mathrm{opt}}}\right) \quad . \tag{3.5}$$

As $r(\{\rho\}|\{y\})$ does not depend of δy at order $(\delta b)^2$, the choice of $\delta b = \rho - \rho_y$ is independent of the choice of the fluctuation of the path δy.
The total density fluctuation $\delta\rho$ is then given by

$$\delta\rho = \delta b + \rho_y - \bar{\rho} \tag{3.6}$$

$$= \delta b + \frac{\delta y'}{2} \tag{3.7}$$

where we used (3.4). Thus, by rewriting in (3.7) $\delta y = \frac{Y}{\sqrt{L}}$ and $\delta b = \dot{B}\frac{\sqrt{\bar{\rho}(1-\bar{\rho})}}{\sqrt{2L}}$, and using (3.3), (3.4), (3.8), and the fact that

$$\bar{\rho} = \frac{1+y'_{\mathrm{opt}}}{2} \quad , \tag{3.8}$$

one gets (1.11).

We now make the link between the small fluctuations and the large deviation functional of the density. From (3.7), we see that the probability of a fluctuation $\delta\rho$ of order $\frac{1}{\sqrt{L}}$ around the optimal profile $\bar{\rho}$ is given by the sum over all path fluctuations δy of the probability of having a path fluctuation δy and a density fluctuation $\delta b = \delta\rho - \frac{\delta y'}{2}$ around ρ_y, i.e.

$$d\mathbb{P}(\{\delta\rho\}) \sim \mathcal{D}[\{\delta\rho\}] \int_{\{\delta y\}} \mathcal{D}[\{\delta y\}] e^{-L\int_0^1 dx \left[\frac{\lambda^2}{2} \frac{e^{-\lambda y_{\text{opt}}} (\delta y)^2}{(1-e^{-\lambda y_{\text{opt}}})^2} + \frac{(\delta y')^2 + 4\left(\delta\rho - \frac{\delta y'}{2}\right)^2}{1-y_{\text{opt}}'^2} \right]} . \tag{3.9}$$

In fact, this expression follows directly from (3.1). Integrating it over δy leads to a gaussian form for the density fluctuations $\delta\rho$ of order $\frac{1}{\sqrt{L}}$ which is another way of writing (3.7). On the other hand, considering (3.9) for $\delta\rho$ small but of order 1 (i.e. of order L^0 in L), and performing a saddle point evaluation over δy leads to leading order in $\delta\rho$ to a deviation functional quadratic in $\delta\rho$ which is identical to the gaussian. The fact that this quadratic form of the large deviation functional (for $\delta\rho$ small but of order 1 in L) is equivalent to the expression of the gaussian fluctuations (for $\delta\rho \sim \frac{1}{\sqrt{L}}$) shows that for the WASEP the fluctuations of order $\frac{1}{\sqrt{L}}$ can be calculated by expanding the large deviation functional to leading order around the most likely profile. Mathematically, this is simply due to the fact that the saddle point calculation is exact when one deals with gaussian variables (here the δy).

4 Derivation of (1.15)

Let us define

$$a(x) = \frac{-\bar{\rho}'(x) J}{2\chi(\bar{\rho}(x))^2} .$$

Writing $Y(x) = Y(0) + \int_0^x Y'(s) ds$ and performing the Gaussian integral over $Y(0)$ in (1.13), one obtains the following expression for $T(u,v)$ in (1.14):

$$T(u,v) = \frac{\delta(u-v)}{2\chi(\bar{\rho}(x))} + 2\frac{\int_{u\vee v}^1 a(z)dz \int_0^{u\wedge v} a(z')dz'}{\int_0^1 a(z)dz} \tag{4.1}$$

We show now that when one writes $T^{-1}(x,y)$ as in (1.15), one gets expression (1.8) for $f(x,y)$.
Firstly, the symmetry of $T^{-1}(x,y)$ implies the symmetry of $f(x,y)$:

$$f(x,y) = f(y,x) .$$

Then, by definition of the inverse of an operator, we have

$$\delta(u-v) = \int_0^1 T(u,t)T^{-1}(t,v)dt \quad . \tag{4.2}$$

Inserting (4.1) and (1.15) in (4.2) gives the following integral equation for $f(x,y)$:

$$\frac{f(x,y)}{4\chi(\bar\rho(x))} + \frac{1}{\int_0^1 a(t)dt}\left[\int_x^1 dt \int_t^1 a(z)dz \int_0^x a(z')dz' f(t,y)\right.$$
$$+ \int_0^x dt \int_0^t a(z)dz \int_x^1 a(z')dz' f(t,y) + \frac{\Theta(y-x)\chi(\bar\rho(y))}{2}\int_y^1 a(z)dz \int_0^x a(z')dz'$$
$$\left. + \frac{\Theta(x-y)\chi(\bar\rho(y))}{2}\int_x^1 a(z)dz \int_0^y a(z')dz'\right] = 0 \tag{4.3}$$

(4.3) implies that for $0 < y < 1$, one has

$$f(0,y) = f(1,y) = 0 \quad . \tag{4.4}$$

Furthermore, when one applies the operator $\partial_x\{\frac{\partial_x(.)}{a(x)}\}$ to (4.3), one gets the following differential equation:

$$\partial_x \frac{\partial_x \left\{\frac{f(x,y)}{4\chi(\bar\rho(x))}\right\}}{a(x)} - f(x,y) - \delta(x-y)\frac{\chi(\bar\rho(x))}{2} = 0 \tag{4.5}$$

This equation can be written as

$$\partial_x^2 f(x,y) + \lambda\partial_x\left\{(2\bar\rho(x)-1)f(x,y)\right\} + \delta(x-y)J\bar\rho'(x) = 0 \tag{4.6}$$

The $\delta(x-y)$ function simply means that the $\partial_x f(x,y)$ is discontinuous in $x=y$, i.e.

$$\partial_1 f(x^-,x) - \partial_1 f(x^+,x) = J\bar\rho'(x) \tag{4.7}$$

This differential equation with boundary condition (4.4) is equivalent to equation (1.10) and its solution is given by (1.8) (using equation (1.5) and the symmetry of $f(x,y)$) .

5 Conclusion

We have proved that the density fluctuations in the stationary state of the weakly asymmetric exclusion process with open boundaries are distributed like a Gaussian field. We have obtained a simple expression (1.6), (1.8) of the two point function of the weakly asymmetric exclusion process which extends the result of Spohn [2]. This correlation function is long ranged and is a signature of the non locality of the large deviation function. As for the symmetric case, expanding the large deviation functional of the density around the optimal profile leads to the right expression. Our results have also some similarity with those of the totally asymmetric case (TASEP) [3]. There too, the fluctuations of density can be written as a sum of two terms. However for the TASEP one of the two processes (the Brownian excursion) is non Gaussian. It is to be noted that this Brownian excursion is not the large λ limit of the process $Y(x)$ (1.13), and, more generally, the correlation function (1.6) does not converge in the large λ limit to the correlation function of the totally asymmetric case, meaning that the large L limit and the large λ limit can't be inverted.

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