Commun. Math. Phys. 189, 481 – 496 (1997)

Communications in
Mathematical Physics

Reversibility in Infinite Hamiltonian Systems with Conservative Noise*

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Received: 26 September 1996 / Accepted: 3 January 1997

In Memoriam Roland Dobrushin

Abstract: The set of stationary measures of an infinite Hamiltonian system with noise is investigated. The model consists of particles moving in $\mathbb{R}^3$ with bounded velocities and subject to a noise that does not violate the classical laws of conservation, see [OVY]. Following [LO] we assume that the noise has also a finite radius of interaction, and prove that translation invariant stationary states of finite specific entropy are reversible with respect to the stochastic component of the evolution. Therefore the results of [LO] imply that such invariant measures are superpositions of Gibbs states.

0. Introduction

Let Ω denote the space of locally finite configurations $\omega = (q_\alpha, p_a)_{\alpha\in I}$ indexed by a countable set I, that is $q_\alpha, p_\alpha \in \mathbb{R}^3$ are the position and momentum of particle $\alpha \in I$; the set $\{q_\alpha\}_{\alpha\in I}$ has no limit points in $\mathbb{R}^3$ by assumption. The classical dynamics of the system is governed by a formal Hamiltonian $\mathcal{H}$,

$$\mathcal{H}(\omega) = \sum_{\alpha\in I} \phi(p_\alpha) + \frac{1}{2}\sum_{\alpha\in I}\sum_{\beta\neq\alpha} V(q_\alpha - q_\beta)\,,$$

where the kinetic energy $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ is strictly convex with bounded derivatives, and $V : \mathbb{R}^3 \mapsto \mathbb{R}$ is a symmetric and superstable pair potential of finite range. The associated Liouville operator will be denoted by L,

$$L\varphi = \sum_{\alpha\in I} \quad \langle \frac{\partial\mathcal{H}}{\partial p_\alpha}, \frac{\partial\varphi}{\partial q_\alpha}\rangle - \langle \frac{\partial\mathcal{H}}{\partial q_\alpha}, \frac{\partial\varphi}{\partial p_\alpha}\rangle\,,$$

* Work partially supported by grants CIPA-CT92-4016 and CHRX-CT94-0460 of the Commission of the European Community, and by grant T 16665 of the Hungarian NSF. Two of us (J.F. and C.L.) acknowledge hospitality of the Ervin Schrödinger Institute.

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^3$. Almost nothing is known on the ergodic properties of such infinite systems. In fact, very few results are available even for finite systems of this type (e.g., [KSS, DL, BLPS, LW]). To ensure proper ergodic behavior of the system we add some noise, whereby obtaining stochastic equations of motion; these equations read

$$\begin{aligned} dq_\alpha &= \phi'(p_\alpha)\, dt\,, \\ dp_\alpha &= -\sum_{\beta \neq \alpha} V'(q_\alpha - q_\beta)\, dt + b_\alpha(\omega)\, dt + \sum_{\theta=1}^{d} \sum_{\beta \neq \alpha} \sigma^\theta_{\alpha,\beta}(\omega)\, dw^\theta_{\alpha,\beta}\,, \end{aligned} \tag{0.1}$$

where $w^\theta_{\alpha,\beta}$ is a family of independent one-dimensional Wiener processes for $\theta = 1, 2, ..., d$ and $\alpha \neq \beta$ such that $w^\theta_{\alpha,\beta} = -w^\theta_{\beta,\alpha}$; ϕ' and V' denote the gradient of ϕ and V, respectively. The coefficients $b_\alpha, \sigma^\theta_{\alpha,\beta} : \Omega \mapsto \mathbb{R}^3$ are smooth local functions to be specified in the next section in such a way that total energy and momentum are both preserved by the randomized evolution (0.1). In addition, any Gibbs state $I\!P$ with energy $\mathcal{H}$ will be a reversible measure for the stochastic part of the evolution:

$$\int \varphi(\omega)\widehat{L}\psi(\omega)\, I\!P\,(d\omega) = \int \psi(\omega)\widehat{L}\varphi(\omega)\, I\!P\,(d\omega) \tag{0.2}$$

for all smooth local functions $\varphi, \psi : \Omega \mapsto \mathbb{R}$, where

$$\widehat{L}\psi = \sum_{\alpha \in I} \langle b_\alpha, \frac{\partial \psi}{\partial p_\alpha} \rangle + \frac{1}{4} \sum_{\theta=1}^{d} \sum_{\alpha \in I} \sum_{\beta \neq \alpha} \langle \sigma^\theta_{\alpha,\beta}, (D^2_{\alpha,\beta}\psi)\sigma^\theta_{\alpha,\beta} \rangle\,, \tag{0.3}$$

and $D^2_{\alpha,\beta}\psi$ is the matrix of second derivatives obtained by applying $D_{\alpha,\beta} = \partial/\partial_{p_\alpha} - \partial/\partial_{p_\beta}$ twice to ψ. Since the Liouville operator is antisymmetric with respect to Gibbs distributions, the full generator, $\widetilde{L} = L + \widehat{L}$ also satisfies the stationary Kolmogorov equation, $\int \widetilde{L}\psi\, dI\!P = 0$ for a wide class of test functions ψ and any Gibbs state $I\!P$.

The converse problem is much more complex. In our basic reference [LO] it is shown that if a translation invariant measure Q with finite specific entropy satisfies the stationary Kolmogorov equation and (0.2), together with some other technical conditions, then Q enjoys the Gibbs property. Let us remark that finiteness of specific entropy is a fairly natural and effective condition in the theory of hydrodynamics limits (see [OVY]). On the contrary, condition (0.2) looks rather restrictive and, at least in general, not particularly natural. The main purpose of this paper is to show that condition (0.2) of reversibility is superfluous (i.e., it follows from the stationarity of the measure).[1] To obtain such a result we are forced to prove the existence of a semigroup defined by (0.1); its regularity (locality) will play a crucial role in the argument. This problem may not seem to be a very difficult one since $\phi'(p_\alpha)$, the velocity of particle α, is bounded by assumption. However, the evolution must be defined for a very large set $\overline{\Omega} \subset \Omega$ of initial configurations: we need $Q(\overline{\Omega}) = 1$ for any probability measure Q of finite specific entropy. On the other hand, to obtain the necessary regularity properties of the dynamics we have to restrict the configuration space by excluding extremely high values of particle density. We shall see that the desired construction fails unless the dimension of the space is less than four, cf. [FD] and [S].

[1] Note that in applications to hydrodynamics the reversibility (0.2) is insured by construction (see [OVY], Lemma 4.4), hence the present paper does not add to hydrodynamics type problems for which the results in [LO] suffice. The focus here is on the classification of stationary translation invariant measures.

1. Notations and Results

Configurations can be interpreted as σ-finite integer valued measures on $\mathbb{R}^3 \times \mathbb{R}^3$; sometimes we write $\omega = (q,p)$ with $q = (q_\alpha)_{\alpha\in I}$ and $p = (p_\alpha)_{\alpha\in I}$, and if $\Lambda \subset \mathbb{R}^3$, then ω_Λ denotes the restriction of ω to Λ, i.e. $\omega_\Lambda = (q_\alpha, p_\alpha)_{q_a\in\Lambda}$, while $|\omega_\Lambda|$ is the cardinality of this set. The centered cubic box of side $r > 0$ will be denoted by Λ_r. Referring to functionals $\omega(\varphi) := \sum_{\alpha\in I}\varphi(q_\alpha, p_\alpha)$, where $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ is continuous with compact support, we equip Ω with the associated weak topology and Borel structure, and $C_0(\Omega)$ denotes the space of cylinder (local) functions $\Psi(\omega) = f(\omega(\varphi_1), \omega(\varphi_2), \ldots, \omega(\varphi_n))$ such that $f \in C(\mathbb{R}^n)$.

Since all sets $\Sigma(\delta)$, defined for an increasing sequence $\delta = (\delta_1, \delta_2, \ldots)$ such that $\delta_1 \geq 1$ by

$$\Sigma(\delta) := \{\omega \in \Omega \,:\, |\omega_{\Lambda_n}| \leq \delta_n \text{ and } |p_\alpha| \leq \delta_n \text{ if } q_\alpha \in \Lambda_n\}$$

are compact, we need not worry too much about topology. Indeed, in the forthcoming considerations we always do have an a priori bound allowing us to restrict calculations to some compact $\Sigma(\delta)$. In these situations all reasonable topologies coincide, moreover any continuous function can be uniformly approximated by elements of $C_0(\Omega)$ in view of the Stone-Weierstrass Theorem.

Interaction. We consider a repelling pair potential V of finite range such that $V(x) = V(-x)$ is twice continuously differentiable, $V(0) > 0$ but $V(x) = 0$ if $|x| > R_0$, finally $\langle x, V'(x)\rangle \leq 0$ for all $x \in \mathbb{R}^3$. These conditions imply that V is superstable, see [R]: for each cubic box or ball $\Lambda \subset \mathbb{R}^3$ there exist some constants $A_\Lambda \geq 0$ and $B_\Lambda > 0$ such that, for any configuration, we have

$$\sum_{\alpha: q_\alpha\in\Lambda}\sum_{\beta\neq\alpha} V(q_\alpha - q_\beta) \geq B_\Lambda|\omega_\Lambda|^2 - A_\Lambda|\omega_\Lambda|. \tag{1.1}$$

Kinetic energy. We assume that ϕ has bounded second derivatives, and velocities are also bounded, i.e. $|\phi'(y)| \leq \bar{c} < +\infty$ for all $y \in \mathbb{R}^3$. To define Gibbs measures we need a lower bound: $\liminf_{|y|\to\infty}\phi(y)/|y| \geq \underline{c} > 0$. When results of [LO] are applied an extra technical condition on ϕ is needed. For simplicity one can consider the case in which $\phi(y) = \sum_{i=1}^3 \phi_0(y_i)$ for $y = (y_1, y_2, y_3)$, where $\phi_0 \in C^\infty(\mathbb{R})$ is strictly convex and

$$\frac{1}{2}\frac{d^2}{du^2}(\phi_0''(u))^2 = \phi_0'''(u)^2 + \phi_0^{iv}(u)\phi''(u) \neq 0$$

apart from, at most, finitely many points (see [LO], Sect. Two, "Condition on the Noise" for the general condition that ϕ must satisfy). Notice that if $\frac{d^2}{du^2}(\phi_0''(u))^2 = 0$ for each u and if we require the natural condition $\phi_0(u) = \phi_0(-u)$, then ϕ_0'' is a constant, which is the classical case of a quadratic kinetic energy function.

Stochastic perturbation of classical dynamics. There are several ways to select the coefficients of the stochastic perturbation. We set

$$b_\alpha(\omega) = \sum_{\beta\neq\alpha}\gamma_{\alpha,\beta}(q)F(p_\alpha, p_\beta) \text{ and } \sigma^\theta_{\alpha,\beta}(\omega) = \sqrt{\gamma_{\alpha\beta}(q)}\,G_\theta(p_\alpha,\, p_\beta)\,, \tag{1.2}$$

where, as in [LO],[2] $\gamma_{\alpha,\beta}(q) = \gamma_{\beta,\alpha}(q) \geq 0$ is continuously differentiable, $\gamma_{\alpha,\beta}(q) > 0$ if $|q_\alpha - q_\beta| < R_1$ and it is zero for $|q_\alpha - q_\beta| \geq R_1$, i.e. $R_1 > R_0$ is the radius of stochastic interaction. The functions $F, G_\theta : \mathbb{R}^6 \mapsto \mathbb{R}^3$ are infinitely differentiable and bounded together with their derivatives; they are chosen in such a way that the stochastic interaction also preserves the total momentum and energy of an interacting couple of particles. Moreover, $\{G_\theta\}_{\theta=1}^d$ spans, at each point, all $\mathbb{R}^3$. It is natural to assume that $\gamma_{\alpha,\beta}$ depends only on the interparticle distances, and it does not depend on a coordinate q_δ if $|q_\alpha - q_\delta| > R_2$ or $|q_\beta - q_\delta| > R_2$, where $R_2 > 2R_1$ is a constant. Therefore the stochastic interaction is also translation invariant and has a finite range $R_3 := R_1 + R_2$. A new feature of the present model is that $\gamma_{\alpha,\beta}$ vanishes when the number of particles near q_α or q_β tends to infinity. For convenience, we set $\gamma_{\alpha,\beta}(q) = \sigma(q_\alpha - q_\beta)\Theta_{\alpha,\beta}(q)$, where

$$\Theta_{\alpha,\beta}(q) := \Big(1 + \sum_{\delta\in I} \chi(q_\alpha - q_\delta) + \sum_{\delta\in I} \chi(q_\beta - q_\delta)\Big)^{-1}. \tag{1.3}$$

In (1.3) $\sigma, \chi : \mathbb{R}^3 \mapsto [0, 1]$ are twice continuously differentiable, $\sigma(x) = \sigma(-x) > 0$ if $|x| < R_1$ and it is zero for $|x| \geq R_1$. Similarly, $\chi(x) = \chi(-x) > 0$ if $|x| \leq 2R_1$ and $\chi(x) = 0$ if $|x| > R_2$ with some $R_2 > 2R_1$. A technical condition, $|\chi'(x)| \leq K\chi(x)^{1-\kappa}$, where $0 < \kappa < 1/9$, will be exploited in Lemma 2.4.

Since $w^\theta_{\alpha,\beta} = -w^\theta_{\beta,\alpha}$, the condition $F(p_\alpha, p_\beta) = -F(p_\beta, p_\alpha)$ of antisymmetry of F clearly implies the conservation of total momentum. For convenience, we choose

$$\begin{aligned} F(p_\alpha,\, p_\beta) \quad &:= \quad \frac{1}{2}\sum_{\theta=1}^{d} \langle G_\theta(p_\alpha,\, p_\beta), D_{\alpha,\beta}\rangle G_\theta(p_\alpha,\, p_\beta) \quad \text{and} \\ X^\theta_{\alpha,\beta}\varphi \quad &:= \quad \frac{1}{\sqrt{2}} \langle G_\theta(p_\alpha,\, p_\beta), D_{\alpha,\beta}\varphi\rangle\,, \end{aligned}$$

then the formal generator $\widehat{L}$ of the random component of our process becomes[3]

$$\widehat{L}\varphi = \frac{1}{2}\sum_{\theta=1}^{d}\sum_{\alpha\in I}\sum_{\beta\neq\alpha} \gamma_{\alpha,\beta}(q) X^\theta_{\alpha,\beta}\big(X^\theta_{\alpha,\beta}\varphi\big). \tag{1.4}$$

In this case the orthogonality relations

$$\langle G_\theta(p_\alpha,\, p_\beta), \phi'(p_\alpha) - \phi'(p_\beta)\rangle = 0 \tag{1.5}$$

imply the formal conservation of energy, see [LO]. To have conservation of phase volume it is also assumed that

$$\langle D_{\alpha,\beta}, G_\theta(p_\alpha, p_\beta)\rangle = 0\,, \tag{1.6}$$

i.e. the operators $\gamma_{\alpha,\beta} X^\theta_{\alpha,\beta} X^\theta_{\alpha,\beta}$ are symmetric with respect to Lebesgue measure $dp_\alpha dp_\beta$. As a consequence we shall see that the conservation laws imply the reversibility of Gibbs states with respect to $\widehat{L}$. For an explicit example of F and G_θ, see the Appendix of [LO].

[2] In fact, in [LO], the functions $\gamma_{\alpha\beta}$ depend only on the variables q_α and q_β; yet all that is done there applies without changes to the situation described here.

[3] The future requirements (1.5) and (1.6) imply that ${X^\theta_{\alpha,\beta}}^* = -X^\theta_{\alpha,\beta}$, where the adjoint is taken with respect to the measure defined by the kinetic energy, (see[LO]).

Gibbs measures. Let $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be a set of real parameters with $\lambda_4 > 0$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 < \underline{c}^2$, and denote by Π the distribution of a Poisson process of unit intensity in $\mathbb{R}^3$. A probability measure $I\!P$ on Ω is called a Gibbs state for $\mathcal{H}$ with parameters λ if its conditional distributions given the configuration outside of any cubic box $\Lambda \subset \mathbb{R}^3$ can be represented as

$$I\!P\,[d\omega_\Lambda|\omega_{\Lambda^c}] = \frac{1}{Z_\Lambda} \exp\left[\lambda_0|\omega_\Lambda| + \sum_{\alpha=1}^{n}\sum_{i=1}^{3}\lambda_i p_\alpha^i - \lambda_4 \mathcal{H}_\Lambda(\omega_\Lambda, \omega_{\Lambda^c})\right] \Pi(dq_\Lambda)\, dp_\Lambda\,,$$

where Z_Λ is the normalization, and a natural decomposition $\omega_\Lambda = (q_\Lambda, p_\Lambda)$ is used, see [D]. The local Hamiltonian, $\mathcal{H}_\Lambda$ is defined as

$$\mathcal{H}_\Lambda(\omega_\Lambda, \omega_{\Lambda^c}) = \sum_{q_\alpha \in \omega_\Lambda}\left[\phi(p_\alpha) + \frac{1}{2}\sum_{q_\beta \in \omega_\Lambda;\ \alpha \neq \beta} V(q_\alpha - q_\beta) + \sum_{q_\beta \in \omega_{\Lambda^c}} V(q_\alpha - q_\beta)\right]\,,$$

the set of such measures will be denoted by $\mathcal{P}_\lambda$, see [R] for the existence of Gibbs states for superstable interactions.

Relative entropy. Let Q and P be probability measures on Ω, and for any $\Lambda \subset \mathbb{R}^3$ denote $\mathcal{F}_\Lambda$ the set of continuous and bounded functions $\psi : \Omega \mapsto \mathbb{R}$ such that $\psi(\omega) = \psi(\omega_\Lambda)$ for all $\omega \in \Omega$. The entropy of Q in Λ , relative to P , is defined by

$$H_\Lambda(Q|P) \;=\; \sup_{\psi \in \mathcal{F}_\Lambda}\left\{I\!E^{\,Q}(\psi) - \log I\!E^{\,P}(e^\psi)\right\}\,, \tag{1.7}$$

where $I\!E^{\,Q}$ denotes the expectation with respect to the probability measure Q . If $\Lambda = \mathbb{R}^3$ then the subscript Λ of H_Λ will be omitted; for properties of H_Λ, see, for example [OVY]. As a reference measure a distinguished, translation invariant, Gibbs state $P = I\!P$ will be chosen. We say that Q has finite specific entropy if there exists a constant C such that $H_\Lambda(Q|I\!P) \le C(1 + |\Lambda|)$ for any cubic box Λ . If Q is translation invariant with finite specific entropy, then the particle density $\rho = \rho(\omega)$ is Q-a.s. defined as the following limit taken along any increasing sequence of cubic boxes, see [LO],

$$\rho(\omega) = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1}|\omega_\Lambda|.$$

Main results of [LO] for the system under consideration can be summarized as follows.

Theorem 1.1. *Suppose that Q is a translation invariant probability measure on Ω with finite specific entropy, and let $\rho_c := 3/(4\pi R_1^3)$. If*

(i) $Q[\rho(\omega) > \rho'] \;=\; 1$ *for some* $\rho' > \rho_c$,

(ii) *Q is invariant with respect to $\widetilde{L} = L + \widehat{L}$ in the sense that, for any smooth local function ψ we have $I\!E^{\,Q}(\widetilde{L}\psi) = 0$,*

(iii) *Q is reversible with respect to $\widehat{L}$, i.e. $I\!E^{\,Q}(\psi\widehat{L}\varphi) \;=\; I\!E^{\,Q}(\varphi\widehat{L}\psi)$ for any pair φ, ψ of smooth local functions,*

then Q is a convex combination of Gibbs states.

Statement of the result. Notice that the theorem above is stated without any reference to the existence of the infinite dynamics; properties (ii) and (iii) of invariance are purely formal. However, the extraction of such local information as reversibility is usually based on a method of Liapunov functions, namely entropy and its rate of change are compared, so the first step of our argument is intrinsically related to the evolution.

Theorem 1.2. *Under the conditions on the stochastic dynamics listed above, there exists an explicitly defined set $\bar{\Omega} \subseteq \Omega$ such that $Q(\bar{\Omega}) = 1$ for each Q with finite specific entropy. Moreover, for each $\omega_0 \in \bar{\Omega}$ we have a unique strong solution $\omega(t)$, $t \geq 0$ to (0.1) such that $\omega(0) = \omega_0$ and $\omega(t) \in \bar{\Omega}$ a.s. The solution is a measurable function of the initial configuration, and every Gibbs state $\mathbb{P} \in \mathcal{P}_\lambda$ with $\lambda_4 > 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is a stationary measure for the random evolution.*

This theorem is proven in the next section; solutions are defined by a limiting procedure starting from finite systems. The restriction on the parameters of a Gibbs measure in the last statement could have been removed by elaborating some technical details, but we do not need such a general assertion.

Having constructed the infinite evolution we can consider stationary measures instead of simply measures formally invariant as in Theorem 1.1 (ii).[4]

Theorem 1.3. *Every translation invariant stationary measure with finite specific entropy is reversible with respect to the stochastic part $\widehat{L}$ of the generator; that is, condition (iii) in Theorem 1.1 holds.*

The proof of Theorem 1.3 is the content of Sect. 3. Combining the above results we get the final result of the paper:

Theorem 1.4. *Let Q be a translation invariant stationary measure with finite specific entropy, then condition (i) of Theorem 1.1 implies that Q is a superposition of Gibbs states.*

2. Infinite Dynamics

We start this section by describing the set of allowed initial configurations. Although the definition is a bit technical our choice boils down to configurations for which the energy in a box does not grow too fast with respect to the size of the box. The exact meaning of this construction will become more clear later on when the desired a priori bounds for a family of partial dynamics and the requirements for the existence of a unique limiting dynamics are discussed.

Initial conditions. Let $\mathcal{H}_m(\omega, r)$ denote the total energy of $\omega \in \Omega$ in a ball $B_m(r) \subset \mathbb{R}^3$ of center m and radius $r \leq \infty$, i.e. $\mathcal{H}_m(\omega, r) := \mathcal{H}(\omega_{B_m(r)})$; the number of points of q in $B_{q_\alpha}(r)$ will be denoted as $N_\alpha(q, r)$. For $\kappa \in (0, 1/9)$, see (1.3), and $r \geq R_3 = R_1 + R_2$, define

$$\bar{\mathcal{H}}_{\kappa,r}(\omega) := \sup_{|m| \leq r} \frac{\mathcal{H}_m(\omega, R_3)}{1 + |m|^{3+2\kappa}}\,, \qquad \bar{\Omega}_{\kappa,r}(h) = \{\omega \in \Omega \;:\; \bar{\mathcal{H}}_{\kappa,r}(\omega) \leq h\}.$$

[4] Notice that, since the infinite dynamics satisfies Eqs. (0.1), if Q is stationary, then it satisfies (ii) of Theorem 1.1.

Let $\mathcal{Q}_r(k)$ be the set of Borel probability measures on $\bar{\Omega}_{\kappa,r}(h)$ such that $Q(\mathcal{H}_m(\omega, R_3)) \leq k$ for all $|m| \leq r$.

Now the set of all allowed configurations is defined as

$$\bar{\Omega}_{\kappa,\infty} = \bigcup_{h>0} \bar{\Omega}_{\kappa,\infty}(h) = \{\omega \in \Omega \ : \ \bar{\mathcal{H}}_{\kappa,\infty}(\omega) < \infty\}.$$

Remember that the level sets $\bar{\Omega}_{\kappa,\infty}(h)$ are compact in the weak topology of Ω and, in view of (1.1), $N_\alpha(q, R_3) = O(\sqrt{h}L^{3/2+\kappa})$ for $q_\alpha \in B_m(L - R_3)$ and $(q,p) \in \bar{\Omega}_{\kappa,L}(h)$. We shall see that the initial condition for the existence and uniqueness of the limiting dynamics could have been formulated in terms of N_α only, but a preservation of bounds on kinetic energy will be needed when we prove locality of the dynamics.

Lemma 2.1. *If $\kappa > 0$ then for any fixed $k > 0$ we have*

$$\lim_{h\to\infty} \inf_{r\geq R_3} \inf_{Q\in\mathcal{Q}_r(k)} Q(\bar{\Omega}_{\kappa,r}(h)) = 1,$$

that is $Q(\bar{\Omega}_{\kappa,\infty}) = 1$ if $Q \in \mathcal{Q}_\infty := \bigcup_{k>0} \mathcal{Q}_\infty(k)$.

Proof. This statement is a direct consequence of the Markov inequality. In fact we have some universal $v > 0$ such that (by $v\mathbb{Z}^3$ we denote the tridimensional cubic lattice of size v)

$$\begin{aligned} Q(\bar{\Omega}_{\kappa,r}(h)^c) &\leq \sum_{m\in B_0(r)\cap v\mathbb{Z}^3} Q[\mathcal{H}_m(\omega,\ R_3) > vh(1 + |m|^{3+2\kappa})] \\ &\leq \sum_{m\in B_0(r)\cap v\mathbb{Z}^3} \frac{Q(\mathcal{H}_m)}{h(1 + |m|^{3+2\kappa})v} \leq \frac{k}{vh} \sum_{m\in v\mathbb{Z}^3} \frac{1}{(1 + |m|^{3+2\kappa})}, \end{aligned}$$

which proves the statement for any $\kappa > 0$. □

Observe that the entropy condition $H_\Lambda(Q|I\!P) \leq C(1 + |\Lambda|)$ implies $Q \in \mathcal{Q}_\infty$ via (1.7) and (1.1), see Lemma 3.1 of [LO].

Local dynamics. There are several ways to define a family of partial dynamics, the advantage of the following construction consists in its direct relation to Gibbs states. Let $a : \mathbb{R}^3 \mapsto [0,1]$ be twice continuously differentiable with compact support. We assume also $|a'(x)| \leq 1$ for all $x \in \mathbb{R}^3$. We interpret a as a smooth version of the indicator function of a ball, its concrete shape is not very important. For every such cutoff a and inverse temperature $\lambda_4 > 0$ we consider a system of stochastic differential equations,

$$\begin{aligned} dq_\alpha &= -\frac{1}{\lambda_4} e^{\lambda_4 \mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})} \frac{\partial}{\partial p_\alpha} \left(a(q_\alpha) e^{-\lambda_4 \mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})} \right) dt\,, \\ dp_\alpha &= \frac{1}{\lambda_4} e^{\lambda_4 \mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})} \frac{\partial}{\partial q_\alpha} \left(a(q_\alpha) e^{-\lambda_4 \mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})} \right)\ dt \\ &\quad + a(q_\alpha) \sum_{\beta\neq\alpha} \gamma_{\alpha,\beta}(q) a(q_\beta) F(p_\alpha, p_\beta)\, dt \\ &\quad + \sqrt{a(q_\alpha)} \sum_{\theta=1}^{d} \sum_{\beta\neq\alpha} \sqrt{a(q_\beta)\gamma_{\alpha,\beta}(q)} G_\theta(p_\alpha, p_\beta)\, dw^\theta_{\alpha,\beta}\,, \end{aligned} \tag{2.1}$$

where it is assumed that $\Lambda \subset \mathbb{R}^3$ is bounded and contains the support of a in its interior; in such a situation the equations above do not depend on the particular choice of Λ. Notice that in a region where $a = 1$ our particles follow the original equations of motion, while they are frozen outside of the support of a, i.e. $\dot{q}_\alpha = \dot{p}_\alpha = 0$. Particles approaching the boundary of the support of a slow down, thus we have a smooth transition between moving and frozen particles, see [F1] for a similar construction. This means that we essentially have a finite dimensional diffusion. Let $P^t_{\lambda_4,a}$ denote the Markov semigroup induced by partial dynamics (2.1), i.e. $P^t_{\lambda_4,a}\psi(\omega) := \mathbb{E}_w(\psi(\omega(t)))$, where $\omega(t)$ is the solution with initial condition $\omega(0) = \omega$, $\psi : \Omega \mapsto \mathbb{R}$ is continuous and bounded, while $\mathbb{E}_w$ denotes the expectation with respect to the joint distribution of our Wiener processes.

By a direct application of the Ito lemma we see that the (formal) generator of $P^t_{\lambda_4,a}$ decomposes as $\widetilde{L}_{\lambda_4,a} = L_{\lambda_4,a} + \widehat{L}_a$, where

$$\begin{aligned} L_{\lambda_4,a}\psi &= -\frac{1}{\lambda_4}\sum_{\alpha\in I} e^{\lambda_4\mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})}\frac{\partial}{\partial p_\alpha}\left(a(q_\alpha)e^{-\lambda_4\mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})}\right)\frac{\partial\psi}{\partial q_\alpha} \\ &\quad +\frac{1}{\lambda_4}\sum_{\alpha\in I} e^{\lambda_4\mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})}\frac{\partial}{\partial q_\alpha}\left(a(q_\alpha)e^{-\lambda_4\mathcal{H}_\Lambda(\omega_\Lambda,\omega_{\Lambda^c})}\right)\frac{\partial\psi}{\partial p_\alpha}, \qquad (2.2) \\ \widehat{L}_a\psi &= \frac{1}{2}\sum_{\theta=1}^{d}\sum_{\alpha\in I}\sum_{\beta\neq\alpha}\gamma_{\alpha,\beta}(q)a(q_\alpha)a(q_\beta)X^\theta_{\alpha,\beta}(X^\theta_{\alpha,\beta}\psi). \end{aligned}$$

Since the coefficients of (2.1) are bounded smooth functions, we have a differentiable dependence of solutions on initial values. Therefore a class $\mathcal{D}_a$ of twice continuously differentiable functions forms a common core of $L_{\lambda_4,a}$ and $\widehat{L}_a$, e.g. in the space of continuous and bounded functions. An extension to the space $L^2(\mathbb{P}_\lambda)$ of square integrable functions with respect to a distinguished Gibbs state $\mathbb{P}_\lambda$ follows by the next lemma.

Lemma 2.2. *Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$ while $\lambda_4 > 0$, then every Gibbs state $\mathbb{P} \in \mathcal{P}_\lambda$ satisfies*

$$\mathbb{E}^{\mathbb{P}}(\psi_1 L_{\lambda_4,a}\psi_2) = -\mathbb{E}^{\mathbb{P}}(\psi_2 L_{\lambda_4,a}\psi_1) \text{ and } \mathbb{E}^{\mathbb{P}}(\psi_1\widehat{L}_a\psi_2) = \mathbb{E}^{\mathbb{P}}(\psi_2\widehat{L}_a\psi_1)$$

for $\psi_1, \psi_2 \in \mathcal{D}_a$, consequently $\mathbb{P}$ is a stationary measure of the process $P^t_{\lambda_4,a}$ for each cutoff a.

Proof. Both symmetry relations follow from the definition of $\mathbb{P}$ by integrating by parts. The property of reversibility is a direct consequence of (1.6). Integration by parts with respect to positions is possible because of the presence of the cutoff a. □

Since (2.1) violates the law of momentum conservation in regions where a is not a constant, Lemma 2.2 is not true for general Gibbs measures.

Construction of the infinite dynamics. First we derive an a priori bound for local dynamics; we show that the set of initial conditions is preserved for all $t > 0$ and the related bound does not depend on the particular choice of the cutoff function a.

Lemma 2.3. *There exists a constant c_1 depending only on λ_4 and on the parameters of the infinite system (0.1) such that*

$$\mathbb{E}_w(\mathcal{H}_m(\omega_a(t), R_3)) \leq (c_1 + c_1 t)(1 + \mathcal{H}_m(\omega_a(0), R_3 + \overline{c}t)$$

for all m, t and a, where $\omega_a(t)$ is any solution to (2.1).

Proof. Since all velocities are bounded by $\bar{c}$, we have $|q_\alpha(t) - q_\alpha(0)| \le \bar{c}t$, whence $N_\alpha(q(t), r) \le N_\alpha(q(0), r + \bar{c}t)$, which yields an explicit deterministic bound for the potential energy via superstability (1.1). On the other hand,

$$|b_\alpha(\omega)| + \sum_{\theta=1}^{d} \sum_{\beta \neq \alpha} |\sigma^\theta_{\alpha,\beta}(\omega)|^2 \le c_1' N_\alpha(q, R_1),$$

and the same bound holds true for the corresponding coefficients of (2.1). From the stochastic equations by the Schwarz inequality we get

$$\mathbb{E}_w \left(|p_\alpha(t) - p_\alpha(0)|^2\right) \le c_1'' t(1+t) N_\alpha(q(0), R_1 + \bar{c}t)^2, \tag{2.3}$$

which completes the proof. Indeed, as $\phi(y) \le \phi(0) + \bar{c}|y|$, taking the square root of both sides and summing for $\alpha \in I$ such that $|q_\alpha(0) - m| \le R_3 + \bar{c}t$, we get a bound for $\mathcal{H}_m(\omega(t), R_3)$; the square of the number of points at time zero is estimated again by superstability. □

To prove the existence of limiting solutions when the cutoff is removed we have to compare different partial solutions. Let A_L denote the set of twice continuously differentiable $a : \mathbb{R}^3 \mapsto [0,1]$ with compact support such that $|a'(x)| \le 1$ for all x and $a(x) = 1$ if $|x| \le L$. For $a, \bar{a} \in A_L$ let $\omega(t) = (p(t), q(t))$ and $\bar{\omega}(t) = (\bar{q}(t), \bar{p}(t))$ denote the corresponding solutions to (2.1) with a common initial value $\omega(0) = \bar{\omega}(0) = (\xi, \eta)$. Supposing $|x| \le L - 2\bar{c}t$, for $|x - \xi_\alpha| \le R_0 + 2\bar{c}t$ we get $\partial_t |q_\alpha - \bar{q}_\alpha| \le K_0 |p_\alpha - \bar{p}_\alpha|$, whence

$$|q_\alpha(t) - \bar{q}_\alpha(t)|^2 \le 2K_0 \int_0^t |q_\alpha(s) - \bar{q}_\alpha(s)||p_\alpha(s) - \bar{p}_\alpha(s)|\, ds; \tag{2.4}$$

here and in what follows, K_0, K_1, K_1'... denote constants depending only on the coefficients of the infinite system. The case of the momentum variables is more complex. If $a = \bar{a} = 1$ can be assumed as before, then by Ito's formula we get

$$\mathbb{E}_w \left(|p_\alpha(t) - \bar{p}_\alpha(t)|^2\right) = \mathbb{E}_w \left(\int_0^t \left(J_{\alpha,1}(s) + J_{\alpha,2}(s) + J_{\alpha,3}(s)\right) ds \right),$$

where

$$\begin{aligned}
J_{\alpha,1} &= -2 \sum_{\alpha \neq \beta} \langle p_\alpha - \bar{p}_\alpha, V'(q_\alpha - q_\beta) - V'(\bar{q}_\alpha - \bar{q}_\beta) \rangle, \\
J_{\alpha,2} &= 2 \sum_{\beta \neq \alpha} \langle p_\alpha - \bar{p}_\alpha, \gamma_{\alpha,\beta}(q) F(p_\alpha, p_\beta) - \gamma_{\alpha,\beta}(\bar{q}) F(\bar{p}_\alpha, \bar{p}_\beta) \rangle, \\
J_{\alpha,3} &= \sum_{\theta=1}^{d} \sum_{\beta \neq \alpha} |\sqrt{\gamma_{\alpha,\beta}(q)} G_\theta(p_\alpha, p_\beta) - \sqrt{\gamma_{\alpha\beta}(\bar{q})} G_\theta(\bar{p}_\alpha, \bar{p}_\beta)|^2.
\end{aligned}$$

Introduce now Γ_i for $i = 1, 2$ and $r, t \ge 0$ by

$$\Gamma_1(\xi, \eta, a, \bar{a}; r, t) := \sum_{\alpha : |\xi_\alpha| \le r} |q_\alpha(t) - \bar{q}_a(t)|^2;$$

in the definition of Γ_2 the variables q_α and $\bar{q}_a$ should be replaced by p_α and $\bar{p}_\alpha$, respectively. Our main tool is the following:

Lemma 2.4. *Suppose that* $\kappa < 1/9$, $2\bar{c}T < R_1$. *For all* $r \geq R_3$, $t \leq T$, $h > 0$ *and* $i = 1, 2$ *we have*

$$\lim_{L\to\infty} \sup_{a,\bar{a}\in A_L} \sup_{(\xi,\eta)\in\bar{\Omega}_{\kappa,L}(h)} I\!E_w\left(\Gamma_i(\xi,\eta,a,\bar{a};r,t)\right) = 0\,,$$

and the convergence is uniform on the time interval $[0, T]$.

Proof. The idea of the proof is to define and to estimate a “distance" (based on Γ_i) among different partial dynamics in boxes of radius $r < L$. This will lead us to Eq. (2.9) in which such a distance in a given box is related to the distance in a larger box, the result will easily follow.

Let $\bar{N} = \bar{N}_{L,T} := \max N_\alpha(\xi, R_3 + 2\bar{c}T)$ for all $\alpha \in I$ such that $|\xi_a| + R_3 + 2\bar{c}T \leq L$. Suppose $r + R_3 + 2\bar{c}T < L$, $|\xi_\alpha| \leq r$, and remember that $|q_\delta(t) - \xi_\delta| \leq \bar{c}t$ is always true. The uniform Lipschitz continuity of V' , σ , F and G_θ shall also be used without any further reference in the following calculations. Let $\tilde{\gamma} = \tilde{\gamma}_{\alpha,\beta}(t)$ denote any matrix such that $0 \leq \tilde{\gamma}_{\alpha,\beta}(t) \leq 1$ if $t \leq T$, moreover $\tilde{\gamma}_{\alpha,\beta}(t) = 0$ whenever $|\xi_\alpha - \xi_\beta| \geq R_3 + 2\bar{c}t$. For J_1 we get

$$J_{\alpha,1}(t) \leq K_1|p_\alpha - \bar{p}_\alpha| \sum_{\beta\in I} \tilde{\gamma}_{\alpha,\beta}(t)\big(|q_\alpha - \bar{q}_\alpha| + |q_\beta - \bar{q}_\beta|\big) =: K_1\tilde{J}_{\alpha,1}(t). \tag{2.5}$$

In the case of J_2 the pattern $|ax - by| \leq \min\{|a|,|b|\}|x-y| + |a-b|\max\{|x|,|y|\}$ is used several times to derive

$$\begin{aligned} J_{\alpha,2}(t) \quad &\leq \quad K_2\tilde{J}_{\alpha,1}(t) + K_2|p_\alpha - \bar{p}_\alpha| \sum_{\beta\neq\alpha} \gamma_{\alpha,\beta}(q)\big(|p_\alpha - \bar{p}_\alpha| + |p_\beta - \bar{p}_\beta|\big) \\ &\quad + K_2|p_\alpha - \bar{p}_\alpha|\sigma(q_\alpha - q_\beta) \sum_{\beta\neq\alpha}\sum_{\delta\in I} |\partial_\delta \Theta_{\alpha,\beta}(\tilde{q}^{\alpha,\beta})||q_\delta - \bar{q}_\delta|\,, \end{aligned}$$

where $\partial_\delta := \partial/\partial q_\delta$ and $\tilde{q}^{\alpha,\beta}$ is an intermediate configuration on the line segment connecting q and $\bar{q}$. Observe that by Hölder’s inequality

$$\begin{aligned} \sum_{\delta\in I} |\partial_\delta \Theta_{\alpha,\beta}(\tilde{q})| \quad &\leq \quad K\Theta^2_{\alpha,\beta}(\tilde{q}) \sum_{\delta\in I}\big(\chi(\tilde{q}_\alpha - \tilde{q}_\delta)^{1-\kappa} + \chi(\tilde{q}_\beta - \tilde{q}_\delta)^{1-\kappa}\big) \\ &\leq \quad K\Theta_{\alpha,\beta}(\tilde{q})\big(N_\alpha(\tilde{q}, R_2)^\kappa + N_\beta(\tilde{q}, R_2)^\kappa\big)\,. \end{aligned} \tag{2.6}$$

On the other hand, $\sigma(q_\alpha - q_\beta) > 0$ implies $|\tilde{q}^{\alpha,\beta}_\alpha - \tilde{q}^{\alpha,\beta}_\beta| \leq R_1 + 2\bar{c}t \leq 2R_1$, i.e. $N_\alpha(q, R_1) \leq N_\alpha(\tilde{q}^{\alpha,\beta}, 2R_1)$. This means that $N_\alpha(q, R_1) \leq 1/\Theta_{\alpha,\beta}(\tilde{q}^{\alpha,\beta})$, consequently

$$\begin{aligned} J_{\alpha,2}(t) \quad &\leq \quad K_2'\bar{N}^\kappa_{L,T}\tilde{J}_{\alpha,1}(t) + K_2'\tilde{J}_{\alpha,2}(t)\,; \\ \tilde{J}_{\alpha,2}(t) \quad &:= \quad \sum_{\beta\neq\alpha} \gamma_{\alpha,\beta}(q)\big(|p_\alpha - \bar{p}_\alpha|^2 + |p_\beta - \bar{p}_\beta|^2\big)\,. \end{aligned} \tag{2.7}$$

In a similar way we get

$$\begin{aligned} J_{\alpha,3}(t) \quad &\leq \quad K_3\tilde{J}_{\alpha,2}(t) + K_3\tilde{J}_{\alpha,3}(t) \\ &\quad + K_3 \sum_{\beta\neq\alpha} \frac{\sigma(q_\alpha - q_\beta)}{\Theta_{\alpha,\beta}(\tilde{q}^{\alpha,\beta})}\Big(\sum_{\delta\in I} |\partial_\delta\Theta_{\alpha,\beta}(\tilde{q}^{\alpha,\beta})||q_\delta - \bar{q}_\delta|\Big)^2\,, \\ &\quad \tilde{J}_{\alpha,3}(t) := \sum_{\delta\in I} \tilde{\gamma}_{\alpha,\beta}(t)\big(|q_\alpha - \bar{q}_\alpha|^2 + |p_\delta - \bar{p}_\delta|^2\big)\,, \end{aligned}$$

whence by the Cauchy inequality and (2.6)

$$J_{\alpha,3}(t) \leq K_3' \tilde{J}_{\alpha,2} + K_3' \bar{N}_{L,T}^{2\kappa} \tilde{J}_{\alpha,3}(t). \tag{2.8}$$

Introduce now $d(r,t) := I\!\!E_w\,(\Gamma_2(\xi,\eta,a,\bar{a};r,t)) + \bar{N}_{L,T} I\!\!E_w\,(\Gamma_1(\xi,\eta,a,\bar{a};r,t))$ for $t < T$. Comparing (2.4), (2.5)–(2.8) and using the elementary inequality

$$2|p_\alpha - \bar{p}_\alpha||q_\delta - \bar{q}_\delta| \leq \bar{N}^{-1/2}|p_\alpha - \bar{p}_\alpha|^2 + \bar{N}^{1/2}|q_\delta - \bar{q}_\delta|^2\,,$$

we obtain, by a direct calculation,

$$d(r,t) \leq K_4 \bar{N}_{L,T}^{1/2+\kappa} \int_0^t d(r + R_3 + 2\bar{c}T, s)\,ds\,, \tag{2.9}$$

which completes the proof by a standard iteration procedure. Indeed, we get

$$d(r,t) \leq \frac{T^{\ell+1}}{\ell!} \left(K_4 \bar{N}_{L,T}^{1/2+\kappa}\right)^\ell \sup_{t<T} d(L,t)\,, \tag{2.10}$$

where ℓ, the number of allowed iterations is at least $c_T L$ with $c_T > 0$ depending only on R_3 and T, while $\bar{N}_{L,T} = O(\sqrt{h} L^{3/2+\kappa})$. Using $|q_\alpha(t) - \xi_\alpha| \leq \bar{c}t$ and the second a priori bound (2.3), we see that the right-hand side of (2.10) vanishes as $L \to +\infty$ because $\ell! = O\big((\ell/e)^\ell\big)$ and $(1/2+\kappa)(3/2+\kappa) < 1$ by hypothesis. □

Now we are in a position to prove the existence and uniqueness of limiting solutions to (0.1). Let us consider a sequence of partial solutions $\omega_n = \omega_n(t)$, $n \in \mathbb{N}$ of (2.1) with a common initial value $\omega_n(0) = (\xi,\eta) \in \bar{\Omega}_{\kappa,\infty}$; the corresponding cutoff $a_n : \mathbb{R}^3 \mapsto \mathbb{R}$ is assumed to be a decreasing smooth function of $|x|$ such that $a_n(x) = 1$ if $|x| \leq n$ and $a_n(x) = 0$ if $|x| > n+1$. In view of Lemma 2.4 ω_n converges in probability to some limit $\omega(t)$ for each $t < T = R_1/2\bar{c}$. It is easy to verify that the limit satisfies the infinite system (0.1); the uniqueness of limiting solutions follows again by Lemma 2.4. Since T does not depend on the initial configuration $(\xi,\eta) \in \bar{\Omega}_{\kappa,\infty}$, the construction extends to all times.

Properties of the infinite dynamics. Let P_n^t denote the Markov semigroup induced by the partial dynamics ω_n. Since $\omega_n(t)$ is a continuous function of the initial data, it is well defined by $P_n^t\psi(\xi,\eta) := I\!\!E_w(\psi(\omega_n(t)))$ if $\omega_n(0) = (\xi,\eta)$ for any measurable and bounded $\psi : \Omega_{\kappa,\infty} \mapsto \mathbb{R}$. As a limit of measurable functions, the limiting solution $\omega(t)$ is again a jointly measurable function of (ξ,η) and the random element representing the Wiener processes $w_{\alpha,\beta}^\theta$, the limiting semigroup, P^t can be defined in the same way. If the initial configuration is distributed by $Q \in \mathcal{Q}_\infty$, then QP_n^t and QP^t denote the evolved measure at time $t > 0$. In view of Lemma 2.1 and Lemma 2.3 we know that $QP^t \in \mathcal{Q}_\infty$, too. While P_n^t has fairly good regularity properties, semigroup theory does not apply directly to the limiting case. Nevertheless, all we need in the next section is summarized as follows.

Lemma 2.5. *Suppose that $\psi : \bar{\Omega}_{\kappa,\infty} \mapsto \mathbb{R}$ is a continuous and bounded local function, i.e. $\psi(\omega) := \psi(\omega_{B_0(r)})$ for some $r > 0$, then*

$$\lim_{\ell\to\infty} \sup_{n>\ell+r} \sup_{Q\in\mathcal{Q}_n(k)} |QP_n^t\psi - QP^t\psi| = 0$$

for all $t, k > 0$, and the convergence is uniform on compact time intervals.

Proof. The a priori bound of Lemma 2.3 extends immediately to the limiting dynamics, thus for any $\varepsilon, T > 0$ we have some $\overline{k} > k$ and $h > \overline{k}$ such that $Q(\overline{\Omega}_{\kappa,r}(h)) \geq 1 - \varepsilon$, $QP_n^t(\overline{\Omega}_{\kappa,r}(h)) \geq 1-\varepsilon$, and $QP^t(\overline{\Omega}_{\kappa,r}(h)) \geq 1-\varepsilon$ whenever $t < T$, $n > r + R_3 + 2\overline{c}T$ and $Q \in \mathcal{Q}_n(k)$. Since $\overline{\Omega}_{\kappa,r}(\overline{k})$ is compact, there exists also an $\varepsilon' > 0$ such that $|q_\alpha - \overline{q}_\alpha| + |p_\alpha - \overline{p}_\alpha| < \varepsilon'$, for all $\alpha \in I$ with $|q_\alpha|, |\overline{q}_\alpha| \leq r$, implies $|\psi(\omega) - \psi(\overline{\omega})| \leq \varepsilon$ for $\omega, \overline{\omega} \in \Omega_{\kappa,r}(\overline{k})$. Therefore the statement follows from Lemma 2.4 and Chebishev inequality by a 3ε argument. □

The final statement of Theorem 1.2 on stationarity of certain Gibbs states is now a direct consequence of Lemma 2.2.

3. An Entropy Argument

In this section we extend a familiar argument by Holley [H] to the present more complex situation.

For a probability measure Q on Ω, let $H(Q|I\!P_\lambda)$ denote the entropy relative to a distinguished Gibbs state $I\!P_\lambda$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0$, as defined by (1.7) with $\Lambda = \mathbb{R}^3$. The family of partial dynamics (2.1) has been chosen such that $I\!P_\lambda$ is a common stationary measure of each local dynamics $P_n^t = P_{\lambda_4,a_n}$ introduced in Sect. 2. Therefore P_n^t is a strongly continuous semigroup in $L^2(I\!P_\lambda)$, and smooth cylinder functions form a core for its generator $\widetilde{L}_n = L_n + \widehat{L}_n$, see (2.2). Remember that $L_n := L_{\lambda_4,a_n}$, the Hamiltonian part, is antisymmetric in $L^2(I\!P_\lambda)$ while the symmetric (reversible) component is just $\widehat{L}_n := \widehat{L}_{a_n}$.

If $\mathcal{G}$ is a generator in $L^2(I\!P_\lambda)$, then the corresponding Donsker-Varadhan rate function is defined as

$$D(Q|\mathcal{G}) = \sup_\psi \Big\{ -\int \frac{\mathcal{G}\psi}{\psi} dQ \;:\; \psi \in \text{Dom}\,\mathcal{G},\ \inf \psi > 0 \Big\}.$$

If $\mathcal{G}$ is self-adjoint and $\mathcal{G} < 0$, then we can apply the following result due to Donsker and Varadhan (cf. [DV], Theorem 5).

Theorem 3.1. *$D(Q|\mathcal{G}) < +\infty$ if and only if $Q << I\!P_\lambda$ and $g := \sqrt{dQ/dI\!P_\lambda} \in$ Dom $\sqrt{-\mathcal{G}}$; moreover*

$$D(Q|\mathcal{G}) = \int \left(\sqrt{-\mathcal{G}}g\right)^2 dI\!P_\lambda. \tag{3.1}$$

Our main tool consists of the following entropy inequality.

Proposition 3.2. *Let $\bar{Q}_n^t := (1/t)\int_0^t QP_n^s ds$. If $H(Q|I\!P_\lambda) < \infty$, then*

$$H(QP_n^t|I\!P_\lambda) + 2tD(\bar{Q}_n^t|\hat{L}_n) \leq H(Q|I\!P_\lambda). \tag{3.2}$$

Proof. Let P_n^{*t} be the adjoint semigroup with respect to $I\!P_\lambda$, which is again a diffusion with formal generator $\tilde{L}_n^* = -L_n + \hat{L}_n$. Both forward and backward diffusion are essentially finite dimensional with smooth coefficients, thus twice continuously differentiable functions form a common core $\mathcal{D}_n$ of L_n and L_n^*. This suffices to justify the following computations. Observe first that, as an easy consequence of Jensen's inequality, we have

$$H(Q'P_n^\tau|Q''P_n^\tau) \leq H(Q'|Q'') \tag{3.3}$$

for any two measures Q', Q''. For any strictly positive $\psi \in \mathcal{D}_n$ with $I\!P_\lambda(\psi) = 1$ define Q'' by $dQ'' = \psi dI\!P_\lambda$. Since

$$\frac{dQ'' P_n^\tau}{dI\!P_\lambda} = P_n^{*\tau}\psi ,$$

we have

$$H(Q'|Q'') = H(Q'|I\!P_\lambda) - Q'(\log\psi)$$

and

$$H(Q' P_n^\tau | Q'' P_n^\tau) = H(Q' P_n^\tau | I\!P_\lambda) - Q' P_n^\tau(\log P_n^{*\tau}\psi).$$

Accordingly, by (3.3),

$$H(Q'|I\!P_\lambda) - H(Q' P_n^\tau | I\!P_\lambda) \geq Q'(\log\psi) - Q' P_n^\tau(\log P_n^{*\tau}\psi),$$

whence, by the concavity of the logarithm and the inequality $\log(x+1) \leq x$,

$$H(Q'|I\!P_\lambda) - H(Q' P_n^\tau | I\!P_\lambda) \geq Q'(\log\psi) - Q'(\log P_n^\tau P_n^{*\tau}\psi) \geq \int \frac{\psi - P_n^\tau P_n^{*\tau}\psi}{\psi} dQ'.$$

Remembering that P_n^t and P_n^{*t} are both Feller semigroup and ψ belongs to the common core of $\tilde{L}_n$ and $\tilde{L}_n^*$, we have, for small τ,

$$\psi - P_n^\tau P_n^{*\tau}\psi = \psi - P_n^\tau\psi + \psi - P_n^{*\tau}\psi + P_n^\tau\left(\psi - P_n^{*\tau}\psi\right) - \left(\psi - P_n^{*\tau}\psi\right) = -2\tau\hat{L}_n\psi + o(\tau).$$

Therefore, by dividing the given interval $[0, t]$ into m small pieces, with $\tau = t/m$ and $Q' = QP_n^{\frac{it}{m}}$, we get

$$\begin{aligned} H(Q|I\!P_\lambda) - H(QP_n^t|I\!P_\lambda) &= \lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\left[H(QP_n^{\frac{i}{m}t}|I\!P_\lambda) - H(QP_n^{\frac{i+1}{m}t}|I\!P_\lambda)\right] \\ &\geq -2\int_0^t ds \int \frac{\hat{L}_n\psi}{\psi}\, dQP_n^s. \end{aligned}$$

By taking the supremum over all ψ considered we conclude the proof. □

Observe now that if $D(Q|\hat{L}_n) < \infty$, then by Theorem 3.1 it can be written as a sum, namely, if $g = \sqrt{dQ/dI\!P_\lambda}$, [5]

$$D(Q|\hat{L}_n) = \frac{1}{2}\int \sum_{\theta,\alpha,\beta} a_n(q_\alpha)a_n(q_\beta)\gamma_{\alpha\beta}(q)(X_{\alpha\beta}^\theta g)^2\, dI\!P_\lambda.$$

Let $a_{n,1}(x), a_{n,2}(x), \ldots, a_{n,j}(x)$ be smooth non-negative functions with compact support, and assume that their supports are disjoint. Furthermore, assume that $a_n(x) \geq a_{n,1}(x) + \cdots + a_{n,j}(x)$, then

$$D(Q|\hat{L}_n) \geq D(Q|\hat{L}_{a_{n,1}}) + \cdots + D(Q|\hat{L}_{a_{n,j}}).$$

Therefore, from (3.2), we have

[5] To see this, since $g \in \text{Dom}\sqrt{-\hat{L}_n}$ (hence $g \in \text{Dom}\sqrt{a(q_\alpha)a(q_\beta)\gamma_{\alpha\beta}(q)}X_{\alpha\beta}$)), one can approximate it by local smooth functions, then use the closability of the Dirichlet form D.

$$H(QP_n^t|I\!P_\lambda) + 2t\sum_{i=1}^{j} D(\bar{Q}_n^t|\hat{L}_{a_{n,i}}) \le H(Q|I\!P_\lambda).$$

Thus we can choose strictly positive and smooth functions $\psi_0, \psi_1, \ldots, \psi_j$ such that

$$QP_n^t(\psi_0) - \log I\!P_\lambda(e^{\psi_0}) - 2t\sum_{i=1}^{j} \bar{Q}_n^t\left(\frac{\hat{L}_{a_{n,i}}\psi_i}{\psi_i}\right) \le H(Q|I\!P_\lambda).$$

This inequality extends by continuity to the infinite dynamics (cf. Lemma 2.5 and note that $Q \in \mathcal{Q}_\infty$)

$$QP^t(\psi_0) - \log I\!P_\lambda(e^{\psi_0}) - 2t\sum_{i=1}^{j} \bar{Q}^t\left(\frac{\hat{L}_{a_{n,i}}\psi_i}{\psi_i}\right) \le H(Q|I\!P_\lambda). \tag{3.4}$$

Now we are in a position to take the thermodynamic limit and conclude the main result of this section.

Proposition 3.3. *If Q_* is a translation invariant stationary measure of the infinite system (0.1), and Q_* has finite specific entropy with respect to $I\!P_\lambda$, then $D(Q_*|\widehat{L}_{\bar{a}}) = 0$ for all smooth functions $\bar{a} \le 1$ of compact support.*

Proof. We are going to use (3.4) with $Q = Q_{*n}$, where Q_{*n} is defined by

$$Q_{*n}(\psi) \;=\; \int I\!P_\lambda(\psi|\mathcal{F}_{\Lambda_n})\, dQ_*\,,$$

and Λ_n denotes the centered cubic box of size n. Of course, $H(Q_{*n}|I\!P_\lambda) = H_{\Lambda_n}(Q_*|I\!P_\lambda)$, thus

$$\bar{H}(Q_*|I\!P_\lambda) := \lim_{n\to\infty}\frac{H(Q_{*n}|I\!P_\lambda)}{|\Lambda_n|} = \sup_{\psi}\left(Q_*(\psi) - \bar{F}(\psi)\right)\,, \tag{3.5}$$

where ψ are the local, bounded and continuous functions; in addition,

$$\bar{F}(\psi) := \lim_{n\to\infty}\frac{1}{|\Lambda_n|}\log\int\exp\left(\sum_{k\in\Lambda_n\cap\mathbb{Z}^3}\mathbf{s}^k\psi\right)dI\!P_\lambda, \tag{3.6}$$

and $\mathbf{s}^k$ denotes the shift in $\mathbb{R}^3$ by $k \in \mathbb{R}^3$, i.e. $\mathbf{s}^k\psi(p,q) = \psi(p,\mathbf{s}^k q)$ and $\mathbf{s}^k q = \{q_\alpha + k\}$ if $q = \{q_\alpha\}$. The proof of the existence of (3.5) and (3.6) can be found in [OVY].

Now we set

$$\psi_0 = \sum_{k\in\Lambda_n\cap\mathbb{Z}^3}\mathbf{s}^k\psi$$

for some local bounded continuous function ψ.

Without loss of generality we can suppose n so large that Λ_n contains the support of $\bar{a}$, and define $a_{n,i}(x) = \bar{a}(x+k_i)$, $k_i \in J_n$, and J_n is a discrete subset of Λ_n such that the $a_{n,i}$ have the disjoint supports contained in Λ_n, and $\frac{|J_n|}{n^3} \ge \bar{J}_0$, for some fixed constant $\bar{J}_0$.[6] Correspondingly we choose $\psi_i = \mathbf{s}^{k_i}\bar{\psi}$, $\mathbf{k}_i \in J_n$, for a local bounded continuous function $\bar{\psi}$.

[6] This can be done in such away to ensure that $a_n \ge \sum_{i=1}^{j} a_{n,i}$.

Substituting in (3.4) and dividing by $|\Lambda_n|$, it remains to prove that

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{k\in\Lambda_n\cap\mathbb{Z}^3} Q_{*n} P^t(\mathbf{s}^k\psi) = Q_*(\psi) \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{1}{|J_n|} \sum_{k_i\in J_n} \bar{Q}^t_{*n}\left(\mathbf{s}^{k_i}\frac{\widehat{L}_{\bar{a}}\bar{\psi}}{\psi}\right) = Q_*\left(\frac{\widehat{L}_{\bar{a}}\bar{\psi}}{\psi}\right). \tag{3.8}$$

Indeed, then (3.4), (3.5), (3.6), (3.7) and (3.8) imply

$$Q_*(\psi) - \bar{F}(\psi) - 2tJ_0 Q_*\left(\frac{\hat{L}_{\bar{a}}\bar{\psi}}{\psi}\right) \leq \bar{H}(Q_*|I\!P_\lambda),$$

and taking the supremum over all ψ and $\bar{\psi}$ considered we obtain the wanted result.

To prove (3.7), observe first that the rate of convergence in Lemma 2.5 depends only on the magnitude and the modulus of continuity of the underlying function. In the present situation all functions are translates of each other, thus the convergence is uniform on such functions. Therefore, for $k \in \Lambda_{n-\sqrt{n}}$ we approximate P^t with the local dynamics $\mathbf{s}^k P^t_{\sqrt{n}}$ in the ball $B_k(\sqrt{n})$, otherwise we use simply the uniform bound of ψ. The proof of (3.8) is similar. □

As it is well known, see [DV], $D(Q_*|\widehat{L}_{\bar{a}}) = 0$ implies the reversibility of Q_* with respect to $\widehat{L}_{\bar{a}}$, which completes the proof of Theorem 1.3, whereby proving Theorem 1.4 as well, by a direct argument.

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Communicated by J.L. Lebowitz