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Fluctuations for $\nabla\phi$ interface model on a wall

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Abstract

We consider $\nabla\phi$ interface model on a hard wall. The hydrodynamic large-scale space-time limit for this model is discussed with periodic boundary by Funaki et al. (2000, preprint). This paper studies fluctuations of the height variables around the hydrodynamic limit in equilibrium in one dimension imposing Dirichlet boundary conditions. The fluctuation is non-Gaussian when the macroscopic interface is attached to the wall, while it is asymptotically Gaussian when the macroscopic interface stays away from the wall. Our basic method is the penalization. Namely, we substitute in the dynamics the reflection at the wall by strong drift for the interface when it goes down beyond the wall and show the fluctuation result for such massive $\nabla\phi$ interface model. Then, this is applied to prove the fluctuation for the $\nabla\phi$ interface model on the wall.

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1. Introduction and main result

This paper concerns fluctuations of an interface on a wall. System is defined on the one-dimensional lattice $\Gamma \equiv \Gamma_N := \{1, 2, \ldots, N-1\}$ and the location of the interface at time t is represented by height variables $\phi_t = \{\phi_t(x); x \in \Gamma\}$ measured from the wall Γ, see Funaki and Spohn (1997) and Funaki et al. (2000) for details. Since the interface stays over the wall, the height variables $\phi_t(x)$ are always non-negative. The dynamics of ϕ_t is determined by the stochastic differential equation (SDE) of Skorohod type:

$$\begin{aligned} \mathrm{d}\phi_t(x) = &-\{V'(\phi_t(x)-\phi_t(x-1)) + V'(\phi_t(x)-\phi_t(x+1))\}\,\mathrm{d}t \\ &+ \sqrt{2}\,\mathrm{d}w_t(x) + \mathrm{d}\ell_t(x), \quad x \in \Gamma, \end{aligned} \tag{1.1}$$

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subject to the conditions:

$$\begin{aligned}&\text{(i)}\quad \phi_t(x)\geqslant 0,\\&\text{(ii)}\quad \ell_t(x)\text{ is non-decreasing in } t,\\&\text{(iii)}\quad \int_0^\infty \phi_t(x)\,\mathrm{d}\ell_t(x)=0,\end{aligned}\qquad(1.2)$$

for every $x\in\Gamma$. Condition (iii) means that $\ell_t(x)$ increases only when the interface touches the wall, i.e., $\phi_t(x)=0$. We impose boundary conditions at $\partial\Gamma\equiv\partial\Gamma_N=\{0,N\}$:

$$\phi_t(0)=\phi_t(N)=0. \qquad (1.3)$$

In Eq. (1.1), the potential V satisfies the following conditions:

$$\begin{aligned}&\text{(i)}\quad V\in C^2(\mathbb{R}),\\&\text{(ii)}\quad \text{(symmetry)},\quad V(-\eta)=V(\eta),\quad \eta\in\mathbb{R},\\&\text{(iii)}\quad \text{(strict convexity)},\quad c_-\leqslant V''(\eta)\leqslant c_+,\ \eta\in\mathbb{R},\ \text{for some } c_-,c_+>0.\end{aligned}\qquad(1.4)$$

Since V' is Lipschitz continuous, the SDE (1.1) subject to the conditions (1.2)–(1.3) has a unique solution ϕ_t for every initial data $\phi_0\in[0,\infty)^\Gamma$; see Tanaka (1979) for the existence and uniqueness for the SDEs of Skorohod type on a convex domain and Lions and Sznitman (1984) for more general domains.

The dynamics ϕ_t constructed as above is stationary under the Gibbs measure associated with the Hamiltonian $H_N(\phi)\equiv\sum_{x=1}^N V(\phi(x)-\phi(x-1))$ conditioned for all $\phi(x)$ to be non-negative. Let $\mu\equiv\mu^N$ be the probability measure on $\mathbb{R}^\Gamma$ defined by

$$\mathrm{d}\mu\equiv\mathrm{d}\mu^N:=Z_N^{-1}\exp\left\{-\sum_{x=1}^N V(\phi(x)-\phi(x-1))\right\}\prod_{x\in\Gamma}\mathrm{d}\phi(x), \qquad (1.5)$$

where $\phi(0)=\phi(N)=0$, Z_N is a normalization, and let $\mu^+\equiv\mu^{N,+}$ be its conditional probability:

$$\mu^+\equiv\mu^{N,+}:=\mu^N(\cdot\,|\phi(x)\geqslant 0,\ x\in\Gamma). \qquad (1.6)$$

Then, $\mu^{N,+}$ is the unique (tempered) stationary probability measure for the SDE (1.1)–(1.3), see Proposition 2.3 below. The related problem of entropic repulsion is discussed by several authors including Deuschel and Giacomin (2000).

The macroscopic height variables are defined from the microscopic ones ϕ_t by rescaling them in space and time as

$$h^N(t,\theta):=\frac{1}{N}\sum_{x\in\Gamma}\phi_{N^2t}(x)1_{[x/N-1/2N,x/N+1/2N)}(\theta)\ \geqslant 0,\quad \theta\in[0,1], \qquad (1.7)$$

see Funaki and Spohn (1997) for the physical background of such rescaling. The companion paper by Funaki et al. (2000) discusses the SDE (1.1), (1.2) added weakly perturbed drift as a macroscopic external force on the d-dimensional lattice under periodic boundary conditions, i.e., Γ is taken as $\Gamma=\Gamma_N^d:=(\mathbb{Z}/N\mathbb{Z})^d$. It is proved that $h^N(t,\theta)$ defined similarly to (1.7) for $\theta\in(\mathbb{R}/\mathbb{Z})^d$ converges to $h(t,\theta)$ which is a solution of certain non-linear partial differential equation with reflection at the wall.

In this paper, we shall assume that ϕ_t is stationary, namely ϕ_0 is distributed by $\mu^{N,+}$. Then, since its mean behaves as $E^{\mu^{N,+}}[\phi(x)] = O(\sqrt{N})$ (cf. Section 3), the macroscopic height variable $h^N(t,\theta)$ converges to 0 as $N \to \infty$. The main purpose of this paper is to study the asymptotic behavior of the fluctuation field of $h^N(t,\theta)$ around its limit 0:

$$\Phi^N(t,\theta) := \sqrt{N} h^N(t,\theta) = \frac{1}{\sqrt{N}} \sum_{x \in \Gamma} \phi_{N^2 t}(x) 1_{[x/N - 1/2N, x/N + 1/2N)}(\theta). \tag{1.8}$$

We are now at the position to formulate our main result. To be precise, we introduce the Sobolev space on the interval $[0,1]$. Consider on $L^2([0,1])$ the positive symmetric linear operator $-\Delta$ with Dirichlet boundary conditions. Then we can denote by $H^\alpha([0,1]), \alpha \in \mathbb{R}$ the Hilbert space obtained as the completion of $C_0^\infty([0,1])$ endowed with the inner product

$$\langle f, g \rangle_\alpha = \int_0^1 f(\theta)(-\Delta)^\alpha g(\theta)\, d\theta.$$

The corresponding norm in $H^\alpha([0,1])$ can be written as

$$\|f\|_\alpha^2 = \sum_{k=1}^{\infty} (\pi k)^{2\alpha} |\hat{f}(k)|^2$$

with the corresponding Fourier coefficient $\hat{f}(k) := \langle f, h_k \rangle$, where $h_k(\theta) = \sqrt{2}\sin(\pi k \theta)$ and $\langle f, h_k \rangle$ denotes the inner product of $L^2([0,1])$.

Theorem 1.1. *Define* $\{\Phi^N(t,\theta);\ \theta \in [0,1]\}$ *by* (1.8) *from the solution* $\{\phi_t(x); x \in \Gamma\}$ *of the SDE* (1.1)–(1.3) *with initial distribution* $\mu^{N,+}$. *Then, as* $N \to \infty$, $\Phi^N(t,\theta)$ *weakly converges to* $\Phi(t,\theta)$ *in* $C([0,T], H^{-\alpha}([0,1])) \cap L^2_w(Q_T)$, $Q_T = [0,T] \times [0,1]$ *for every* $T > 0$ *and* $\alpha > \frac{1}{2}$, *where* L^2_w *denotes the* L^2*-space endowed with the weak topology. The limit* $\Phi(t,\theta)$ *is a unique weak stationary solution of the stochastic partial differential equation* (*SPDE*) *with reflection of the type of Nualart and Pardoux* (1992):

$$\frac{\partial \Phi}{\partial t}(t,\theta) = q \frac{\partial^2 \Phi}{\partial \theta^2}(t,\theta) + \sqrt{2}\dot{B}(t,\theta) + \xi(t,\theta), \quad \theta \in [0,1], \tag{1.9}$$

satisfying the conditions

$$\Phi(t,\theta) \geqslant 0, \quad \int_Q \Phi(t,\theta)\, \xi(dt\, d\theta) = 0, \tag{1.10}$$

$$\Phi(t,0) = \Phi(t,1) = 0, \tag{1.11}$$

where ξ *is random measure on* $Q = [0,\infty) \times [0,1]$ *and* $\dot{B}(t,\theta)$ *is the space–time white noise. The positive constant* q *is determined by*

$$q^{-1} = \langle \eta^2 \rangle_\nu, \quad \nu(d\eta) = \left(\int_{\mathbb{R}} e^{-V(\eta)}\, d\eta \right)^{-1} e^{-V(\eta)}\, d\eta. \tag{1.12}$$

In Section 2, weakly massive dynamics is introduced by replacing the reflection terms $\ell_t(x)$ in (1.1) with penalizations. The fluctuation limit for the penalized dynamics is stated in Section 4.1 and proved in Section 6 based on the Boltzmann–Gibbs principle established in Section 5. This gives by the comparison argument the lower bound for the fluctuation limit for the original dynamics defined by the SDE (1.1)–(1.3), see Section 4.2. The upper bound is also necessary to complete the proof of Theorem 1.1 but, since the dynamics is in equilibrium, static upper bound is sufficient and shown in Section 3. Section 7 discusses the case where the macroscopic interface $h(t,\theta):=\lim_{N\to\infty} h^N(t,\theta)$ stays away from the wall and it is proved that the limit of the fluctuation field is Gaussian, see Theorem 7.1.

One of the key ideas in our approach is the comparison theorem which corresponds to the maximum principle in the PDE theory. The convexity condition on the potential V is crucial for such property, see Proposition 2.1. In this respect, our technique is rather restrictive. For instance, in modeling the surface diffusion, one needs to introduce the microscopic dynamics which conserve the total height variables $\sum_x \phi_t(x)$ and after taking the hydrodymanic scaling limit the macroscopic evolution is expected to be described by a partial differential equation of fourth order, see Spohn (1993). The comparison argument does not work for such model. The extension of our result to other SOS-type models with discrete height variables might be difficult, as well.

2. Penalized SDE and energy estimates

The proof of Theorem 1.1 relies on the penalty method and comparison argument.

2.1. Penalized SDE

We replace the local times $\mathrm{d}\ell_t(x)$ appearing in the SDE (1.1) with penalizations $-U_\varepsilon'(\phi_t(x))\,\mathrm{d}t$, where the function U_ε, $\varepsilon>0$ is defined by

$$U_\varepsilon(z)=0 \ \text{ for } z\geqslant 0 \quad \text{and} \quad U_\varepsilon(z)=(2\varepsilon)^{-1}z^2 \ \text{ for } z\leqslant 0. \tag{2.1}$$

Namely, we consider the penalized SDE:

$$\begin{aligned} \mathrm{d}\phi_t^\varepsilon(x) &= -\{V'(\phi_t^\varepsilon(x)-\phi_t^\varepsilon(x-1))+V'(\phi_t^\varepsilon(x)-\phi_t^\varepsilon(x+1))\}\,\mathrm{d}t \\ &\quad +\sqrt{2}\,\mathrm{d}w_t(x)-U_\varepsilon'(\phi_t^\varepsilon(x))\,\mathrm{d}t, \quad x\in\Gamma, \\ \phi_t^\varepsilon(0) &= \phi_t^\varepsilon(N)=0. \end{aligned} \tag{2.2}$$

The solution $\phi_t^\varepsilon=\{\phi_t^\varepsilon(x);x\in\Gamma\}$ of the penalized SDE can take negative values. For two microscopic height variables $\phi=\{\phi(x);x\in\Gamma\}$ and $\bar\phi=\{\bar\phi(x);x\in\Gamma\}$, we write $\phi\geqslant\bar\phi$ if $\phi(x)\geqslant\bar\phi(x)$ holds for every $x\in\Gamma$. Then, the following comparison theorem holds for the solutions ϕ_t^ε of the penalized SDE (2.2) and ϕ_t of the SDE (1.1)–(1.3) of Skorohod type, see Propositions 4.1 and 4.2 of Funaki et al. (2000). The convexity condition $V''\geqslant 0$ which follows from the condition (1.4) (iii) plays a crucial role for the proof.

Proposition 2.1. (i) *Let* ϕ_t^ε *and* $\bar\phi_t^\varepsilon$, $\varepsilon > 0$ *be two solutions of* (2.2). *If* $\phi_0^\varepsilon \geqslant \bar\phi_0^\varepsilon$ *holds at* $t = 0$, *then* $\phi_t^\varepsilon \geqslant \bar\phi_t^\varepsilon$ (*a.s.*) *hold for all* $t > 0$. *Similar result holds for two solutions* ϕ_t *and* $\bar\phi_t$ *of* (1.1)–(1.3).
(ii) *Assume* $0 < \varepsilon < \bar\varepsilon$ *and* $\phi_0^\varepsilon \geqslant \phi_0^{\bar\varepsilon}$. *Then,* $\phi_t^\varepsilon \geqslant \phi_t^{\bar\varepsilon}$ (*a.s.*) *hold for all* $t > 0$.
(iii) *Assume* $\phi_0 = \phi_0^\varepsilon$ *for all* $\varepsilon > 0$. *Then,* $\phi_t(x) = \sup_{\varepsilon>0} \phi_t^\varepsilon(x)$ (*a.s.*) *hold for all* $t > 0$.

Remark 2.1. For the proof of Theorem 1.1, we replace $\mathrm{d}\ell_t(x)$ with $-N^{-1}U_\varepsilon'(N^{-1}\phi_t(x))\,\mathrm{d}t$ rather than $-U_\varepsilon'(\phi_t(x))\,\mathrm{d}t$, see Section 4. Note that $N^{-1}U_\varepsilon'(N^{-1}z) = U_{\varepsilon N^2}'(z)$.

2.2. Energy estimates

We denote $\bar\Gamma := \Gamma_N \cup \partial\Gamma_N \equiv \{0,1,2,\ldots,N\}$. Let $\overline{\Gamma^*} = \{b = \{x, x-1\} \subset \bar\Gamma\} \cong \{1,2,\ldots,N\}$ and set $\nabla\phi(b) = \phi(x) - \phi(x-1)$ for $b = \{x, x-1\} \in \overline{\Gamma^*}$.

Proposition 2.2. (i) *Let* ϕ_t^ε *and* $\bar\phi_t^\varepsilon$, $\varepsilon > 0$ *be two solutions of* (2.2) *and set* $\tilde\phi_t^\varepsilon := \phi_t^\varepsilon - \bar\phi_t^\varepsilon$. *Then, we have*

$$\sum_{x\in\Gamma} (\tilde\phi_t^\varepsilon(x))^2 + 2c_- \int_0^t \sum_{b\in\overline{\Gamma^*}} (\nabla\tilde\phi_s^\varepsilon(b))^2\,\mathrm{d}s \leqslant \sum_{x\in\Gamma} (\tilde\phi_0^\varepsilon(x))^2.$$

(ii) *The same estimate holds for solutions of the SDE* (1.1)–(1.3) *of Skorohod type.*

Proof. Similar to Lemma 2.3 of Funaki and Spohn (1997), noting that $\tilde\phi_t^\varepsilon(0) = \tilde\phi_t^\varepsilon(N) = 0$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x\in\Gamma}(\tilde\phi_t^\varepsilon(x))^2 &= -2\sum_{b\in\overline{\Gamma^*}} \nabla\tilde\phi_t^\varepsilon(b)\{V'(\nabla\phi_t^\varepsilon(b)) - V'(\nabla\bar\phi_t^\varepsilon(b))\}\\ &\quad -2\sum_{x\in\Gamma}\tilde\phi_t^\varepsilon(x)\{U_\varepsilon'(\phi_t^\varepsilon(x)) - U_\varepsilon'(\bar\phi_t^\varepsilon(x))\}. \end{aligned}\tag{2.3}$$

The first term on the right-hand side is bounded from above by

$$-2c_- \sum_{b\in\overline{\Gamma^*}} (\nabla\tilde\phi_t^\varepsilon(b))^2,$$

while the second term is non-positive since U_ε is convex. Therefore, we get the assertion (i). For the difference of two solutions (ϕ_t, ℓ_t) and $(\bar\phi_t, \bar\ell_t)$ of the SDE (1.1)–(1.3), we have similar equality to (2.3) with second term on the right-hand side replaced by

$$2\int_0^t \sum_{x\in\Gamma} (\phi_s(x) - \bar\phi_s(x))(\mathrm{d}\ell_s(x) - \mathrm{d}\bar\ell_s(x)),$$

written in integrated form. However, this term is equal to

$$-2\int_0^t \sum_{x\in\Gamma} (\bar\phi_s(x)\,\mathrm{d}\,\ell_s(x) + \phi_s(x)\,\mathrm{d}\bar\ell_s(x)),$$

which is again non-positive and accordingly assertion (ii) is shown. □

2.3. Stationarity of $\mu^{N,+}$

It is easy to show that the probability measure

$$\mathrm{d}\mu_\varepsilon^N := Z_{N,\varepsilon}^{-1} \exp\left\{ -\sum_{b\in\overline{\Gamma^*}} V(\nabla\phi(b)) - \sum_{x\in\Gamma} U_\varepsilon(\phi(x)) \right\} \prod_{x\in\Gamma} \mathrm{d}\phi(x) \tag{2.4}$$

with $\phi(0)=\phi(N)=0$ is reversible under the dynamics defined by the penalized SDE (2.2), where $Z_{N,\varepsilon}$ is a normalization. Based on this fact and using the energy estimate, we can show the following result for the SDE (1.1)–(1.3) of Skorohod type.

Proposition 2.3. (i) *The probability measure* $\mu^+ = \mu^{N,+}$ *is reversible under the dynamics defined by the SDE* (1.1)–(1.3).

(ii) *The tempered stationary probability measure for* (1.1)–(1.3) *is unique, where temperedness means* L^2*-integrability.*

Proof. It follows from the definition that μ_ε^N weakly converges to $\mu^{N,+}$ as $\varepsilon\downarrow 0$ and therefore, one can construct on a proper probability space a sequence of random variables $\{\phi_0^\varepsilon\}_{\varepsilon>0}$ and ϕ_0 such that ϕ_0^ε are μ_ε^N-distributed, ϕ_0 is $\mu^{N,+}$-distributed and ϕ_0^ε converge to ϕ_0 a.s. as $\varepsilon\downarrow 0$. Let ϕ_t^ε and ϕ_t be solutions of the SDEs (2.2) and (1.1)–(1.3) with initial data ϕ_0^ε and ϕ_0, respectively. Then, from Propositions 2.1(iii) and 2.2(i), we see $\phi_t = \lim_{\varepsilon\downarrow 0}\phi_t^\varepsilon$, a.s., and this proves the reversibility of $\mu^{N,+}$ under the SDE (1.1)–(1.3).

To show the uniqueness, let μ and $\bar{\mu}$ be two tempered stationary probability measures for (1.1)–(1.3), and let ϕ_t and $\bar{\phi}_t$ be two solutions of (1.1)–(1.3) realized on a common probability space with common Brownian motions and having initial data ϕ_0 and $\bar{\phi}_0$ distributed by μ and $\bar{\mu}$, respectively. Then, by Proposition 2.2(ii), we have

$$\frac{1}{t}\int_0^t \sum_{b\in\overline{\Gamma^*}} E[(\nabla\phi_s(b) - \nabla\bar{\phi}_s(b))^2]\,\mathrm{d}s \leqslant \frac{1}{2c_- t}\sum_{x\in\Gamma} E[(\phi_0(x)-\bar{\phi}_0(x))^2],$$

which goes to 0 as $t\to\infty$; note that the temperedness assumption on μ and $\bar{\mu}$ ensures finiteness of the expectation on the right-hand side. This proves that the distributions of $\{\nabla\phi(b); b\in\overline{\Gamma^*}\}$ are the same under μ and $\bar{\mu}$, which implies $\mu=\bar{\mu}$. $\square$

3. Static upper bound

In this section, we give an upper bound on the limit distribution of stationary measures $\{\mu^{N,+}\}$ of the SDE (1.1)–(1.3) as $N\to\infty$. For any probability measures μ on $\mathbb{R}^\Gamma$, we denote the distribution of

$$\Phi^N(\theta) := \frac{1}{\sqrt{N}}\sum_{x\in\Gamma}\phi(x)1_{[x/N-1/2N,x/N+1/2N)}(\theta), \quad \theta\in[0,1] \tag{3.1}$$

on the space $L^2([0,1])$ by $\hat{\mu}$, where $\{\phi(x); x\in\Gamma\}\in\mathbb{R}^\Gamma$ is μ-distributed. For each $a\geqslant 0$, let $\{h(\theta);\theta\in[0,1]\}$ be the pinned three-dimensional Bessel process with time parameter θ starting at $q^{1/2}a$ and reaching $q^{1/2}a$ (i.e., $h(0)=h(1)=q^{1/2}a$) and let $\nu_{a,a}$ be the distribution of $\{q^{-1/2}h(\theta);\theta\in[0,1]\}$; q is the constant determined by (1.12).

In general, for a topological space $\mathscr{E}$, $\mathscr{P}(\mathscr{E})$ stands for the family of all Borel probability measures on $\mathscr{E}$. Further assuming that the space $\mathscr{E}$ is endowed with semiorder $\geqslant$, we denote by $\mu \geqslant v$ for μ and $v \in \mathscr{P}(\mathscr{E})$ if $E^{\mu}[F] \geqslant E^{v}[F]$ holds for every increasing function F on $\mathscr{E}$. Here, we call $F : \mathscr{E} \to \mathbb{R}$ increasing if $e \geqslant \bar{e}$ implies $F(e) \geqslant F(\bar{e})$ for $e, \bar{e} \in \mathscr{E}$. In the following, we shall take $\mathscr{E} = \mathbb{R}^{\Gamma}$ with semiorder $\phi \geqslant \bar{\phi}$ introduced in Section 2.1 and $\mathscr{E} = L^2(Q_T)$ with that given in Section 4.2 below.

Proposition 3.1. (i) *The family* $\{\hat{\mu}^{N,+}\}_N$ *of probability measures on* $L^2_w([0,1])$ *is tight, where* $L^2_w([0,1])$ *stands for the space* $L^2([0,1])$ *equipped with the weak topology.*
(ii) *The inequality* $v_{0,0} \geqslant v$ *holds for an arbitrary limit* v *of* $\{\hat{\mu}^{N,+}\}_N$ *as* $N \to \infty$.

Remark 3.1. Combining this result with the dynamic lower bound, we see that $\hat{\mu}^{N,+}$ itself weakly converges to $v_{0,0}$ as $N \to \infty$.

Proof. *Step* 1: For $a \geqslant 0$, we define $\mu^{N,a} \in \mathscr{P}(\mathbb{R}^{\Gamma})$ in a similar manner to (1.5) but with Dirichlet boundary conditions $\phi(0) = \phi(N) = a\sqrt{N}$, and define $\mu^{N,a,+}$ by the conditional probability of $\mu^{N,a}$ on the event $\Omega^+ = \{\phi \in \mathbb{R}^{\Gamma}; \phi(x) \geqslant 0, x \in \Gamma\}$. Then, we have

$$\mu^{N,a,+} \geqslant \mu^{N,0,+} \; (=\mu^{N,+}), \quad a > 0. \tag{3.2}$$

Indeed, we can give a dynamical proof for (3.2). Let ϕ_t and $\bar{\phi}_t$ be the solutions of the SDE (1.1), (1.2) of Skorohod type with a- and 0-boundary conditions, respectively, with deterministic initial data satisfying $\phi_0 \geqslant \bar{\phi}_0$. Then, $\phi_t \geqslant \bar{\phi}_t$ hold for all $t \geqslant 0$ by the comparison theorem. However, the time averages of distributions of ϕ_t (or $\bar{\phi}_t$) over the long interval $[0, T]$ weakly converges to $\mu^{N,a,+}$ (or $\mu^{N,0,+}$, respectively) as $T \to \infty$. This can be seen based on the coupling argument with the help of the energy estimate derived in Proposition 2.2 (ii). Accordingly, we get (3.2).

Step 2: The invariance principle for $\mu^{N,a}$ shows that $\hat{\mu}^{N,a}$ weakly converges to the distribution $\mu_{a,a}$ of pinned Brownian motion multiplied by $q^{-1/2}$ starting at a and reaching a as $N \to \infty$. However, if $a > 0$, the event Ω^+ has uniformly positive measure under $\mu^{N,a}$, i.e. $\inf_N \mu^{N,a}(\Omega^+) > 0$. Therefore, we see that

$$\hat{\mu}^{N,a,+} \Rightarrow v_{a,a} \quad (N \to \infty), \tag{3.3}$$

note $v_{a,a}$ is the conditional distribution on Ω^+ of $\mu_{a,a}$.

Step 3: In particular, $\{\hat{\mu}^{N,a,+}\}_N$ is tight on $L^2([0,1])$ (with strong topology) and therefore the assertion (i) follows from (3.2). Let v be an arbitrary weak limit of $\{\hat{\mu}^{N,+}\}_N$. Then, from (3.2) and (3.3), we have $v_{a,a} \geqslant v$ for all $a > 0$ so that $v_{0,0} \geqslant v$ by letting $a \downarrow 0$. □

4. Dynamic lower bound and proof of Theorem 1.1

4.1. Scaling limit for interface dynamics with weakly massive term

We consider the SDE (2.2) with $U'_{\varepsilon}(\phi_t(x))$ replaced by $N^{-1}U'_{\varepsilon}(N^{-1}\phi_t(x))$ and study the fluctuation limit for the solutions. This will give the lower bound for the fluctuation limit of the SDE (1.1)–(1.3) of Skorohod type. Since $\varepsilon > 0$ is fixed at this stage, we study a slightly more general SDE:

$$\begin{aligned} \mathrm{d}\phi_t(x) = &-\{V'(\phi_t(x)-\phi_t(x-1)) + V'(\phi_t(x)-\phi_t(x+1))\}\,\mathrm{d}t \\ &+\sqrt{2}\,\mathrm{d}w_t(x) - N^{-1}U'(N^{-1}\phi_t(x))\,\mathrm{d}t, \quad x \in \Gamma, \\ \phi_t(0) = &\ \phi_t(N) = 0. \end{aligned} \tag{4.1}$$

The potential U is assumed to satisfy

$$\begin{aligned} &U \in C^1(\mathbb{R}) \cap C^3(\mathbb{R}\setminus\{0\}), \quad U(z) \geqslant 0, \quad U(0) = U'(0) = 0, \\ &0 \leqslant U''(z) \leqslant c \ (z \neq 0) \quad \text{for some } c > 0, \\ &\text{the limits } \alpha_\pm := U''(\pm 0) \quad \text{exist and } \sup_{z\neq 0} |U'''(z)| < \infty. \end{aligned} \tag{4.2}$$

The function U may not be C^2. The unique tempered stationary probability measure for (4.1) is given by the formula (2.4) with $U_\varepsilon(z)$ replaced by $U(N^{-1}z)$

$$\mu^{N,U}(\mathrm{d}\phi) := Z_{N,U}^{-1} \exp\{-H_{N,U;0,0}(\phi)\} \prod_{x\in\Gamma} \mathrm{d}\phi(x), \tag{4.3}$$

where $Z_{N,U}$ is a normalization and

$$H_{N,U;0,0}(\phi) = \sum_{b\in\overline{\Gamma^*}} V(\nabla\phi(b)) + \sum_{x\in\Gamma} U(N^{-1}\phi(x)), \quad \phi = \{\phi(x); x \in \Gamma\}$$

with $\phi(0) = \phi(N) = 0$. The fluctuation limit for weakly massive dynamics (4.1) is formulated as follows. The proof will be established in Sections 5 and 6.

Theorem 4.1. *Let $\phi_t = \{\phi_t(x); x \in \Gamma\}$ be the stationary solution of the SDE* (4.1), *i.e., we take $\mu^{N,U}$ as its initial distribution. Then, the family of distributions of the fluctuation fields $\{\Phi^N(t,\theta); (t,\theta) \in Q_T\}$, $Q_T = [0,T] \times [0,1]$ defined by* (1.8) *is tight on the space $C([0,T], H^{-\alpha}([0,1])) \cap L^2_w(Q_T)$ for $\alpha > \frac{1}{2}$, and every limit $\Phi(t,\theta)$ satisfies*

$$E[\|\Phi\|^2_{L^2(Q_T)}] < \infty \tag{4.4}$$

and is characterized by the SPDE:

$$\begin{aligned} &\frac{\partial\Phi}{\partial t}(t,\theta) = q\,\frac{\partial^2\Phi}{\partial\theta^2}(t,\theta) + \sqrt{2}\dot{B}(t,\theta) - \alpha_+\Phi^+(t,\theta) + \alpha_-\Phi^-(t,\theta), \quad \theta \in [0,1], \\ &\Phi(t,0) = \Phi(t,1) = 0, \end{aligned} \tag{4.5}$$

where $\Phi^\pm$ stand for the positive and negative parts of Φ, respectively.

In this theorem, we call $\Phi = \Phi(t,\theta)$ a weak solution of the SPDE (4.5) if $\Phi(t,\theta) \in C([0,T], \mathscr{D}'((0,1))) \cap L^2(Q_T)$ (a.s.) and satisfies

$$\langle\Phi(t),J\rangle = \langle\Phi(0),J\rangle + q\int_0^t \langle\Phi(s),J''\rangle\,\mathrm{d}s + \sqrt{2}B_t(J) + \int_0^t \langle f(\Phi(s)),J\rangle\,\mathrm{d}s, \quad \text{a.s.} \tag{4.6}$$

for every $t \in [0,T]$ and test function $J = J(\theta) \in C_b^2((0,1)) \cap C([0,1])$ satisfying $J(0)=J(1)=0$, where $\mathscr{D}'((0,1))$ is the topological dual space of $\mathscr{D}((0,1)) = C_0^\infty((0,1))$, $\langle \Phi, J\rangle$ is the inner product of $L^2([0,1])$ and

$$f(z) = -\alpha_+ z^+ + \alpha_- z^-, \quad z \in \mathbb{R}. \tag{4.7}$$

The following lemma makes a remark on the uniqueness of weak solutions of the SPDE (4.5).

Lemma 4.2. *For each initial data* $\Phi(0) \in L^1([0,1])$, *the weak solution of the SPDE* (4.5) *satisfying*

$$E[\|\Phi\|_{L^1(Q_T)}] < \infty \tag{4.8}$$

is unique. Moreover, if $\Phi(0) \in C([0,1])$ *and* $\Phi(0,0) = \Phi(0,1) = 0$, *then* Φ *admits a version such that*

$$\Phi(t,\theta) \in C(Q_T), \quad \Phi(t,0) = \Phi(t,1) = 0. \tag{4.9}$$

Proof. The proof goes quite similar to Theorem 3.2 of Walsh (1986, p. 313) or Theorem 4.2 of Iwata (1987) so that we only outline it. Note that Walsh treats the case of Neumann boundary conditions (and solutions in the class of L^p-functions), while Iwata discusses the solutions in the class of continuous functions.

First, one shows that $\Phi(t,\theta)$ is the so-called mild solution of the SPDE (4.5), i.e., it satisfies

$$\begin{aligned}\Phi(t,\theta) = {} & \int_0^1 p(t,\theta,\theta')\Phi(0,\theta')\,\mathrm{d}\theta' + \sqrt{2}\int_0^t\int_0^1 p(t-s,\theta,\theta')\,\mathrm{d}B_s(\theta')\,\mathrm{d}\theta' \\ & + \int_0^t\int_0^1 p(t-s,\theta,\theta') f(\Phi(s,\theta'))\,\mathrm{d}s\,\mathrm{d}\theta',\end{aligned}$$

where $p(t,\theta,\theta')$, $t > 0$, $\theta,\theta' \in [0,1]$ is the fundamental solution of the heat equation having diffusion constant q and Dirichlet 0-boundary conditions. Therefore, if $\Phi^1(t)$ and $\Phi^2(t)$ are two solutions with common initial data $\Phi(0)$, then we have

$$\Phi^1(t,\theta) - \Phi^2(t,\theta) = \int_0^t\int_0^1 p(t-s,\theta,\theta')\{f(\Phi^1(s,\theta')) - f(\Phi^2(s,\theta'))\}\,\mathrm{d}s\,\mathrm{d}\theta'.$$

This proves $\Phi^1(t,\theta) = \Phi^2(t,\theta)$ a.e. (t,θ); in fact, take the norms in the space $L^1([0,1])$ of both sides and then apply Gronwall's lemma. The existence of solutions Φ of the SPDE (4.5) satisfying the regularity condition (4.9) is known and therefore such version always exists, see Funaki (1983). □

4.2. Dynamic lower bound

Let P^N be the distribution on the space $L_T^2 := L^2(Q_T)$ of $\{\Phi^N(t,\theta); (t,\theta) \in Q_T\}$ which is determined from stationary solutions of the SDE (1.1)–(1.3) of Skorohod type and let $P^{N,\varepsilon}$ be that determined from stationary solution of the penalized SDE (4.1) taking $U(z) = U_\varepsilon(z)$. The semiorder "$\geqslant$" can be naturally introduced on the space L_T^2: We say $\Phi \geqslant \bar{\Phi}$ for Φ and $\bar{\Phi} \in L_T^2$ iff $\Phi(t,\theta) \geqslant \bar{\Phi}(t,\theta)$ holds for a.e.-$(t,\theta) \in Q_T$.

Lemma 4.3. *For every $\varepsilon > 0$, we have $P^N \geqslant P^{N,\varepsilon}$, recall Section 3 for inequalities for probability measures.*

Proof. Let $\mu^{N,\varepsilon}$ be the measure defined by (4.3) with $U = U_\varepsilon$. Then, $\mu^{N,\varepsilon}$ is stationary for $P^{N,\varepsilon}$ and $\mu^{N,+} \geqslant \mu^{N,\varepsilon}$ holds for every $\varepsilon > 0$. Indeed, to show such inequality, we first notice that $\mu^{N,\varepsilon} \geqslant \mu^{N,\bar{\varepsilon}}$ holds if $0 < \varepsilon < \bar{\varepsilon}$, which follows from Proposition 2.1(ii) and coupling argument, and then take the limit $\varepsilon \downarrow 0$. This inequality is for initial distributions. For dynamics, one can use Proposition 2.1(ii) and (iii), and obtain the conclusion. $\square$

Let P^ε be the distribution on the space L_T^2 of unique weak stationary solution of the SPDE:

$$\frac{\partial \Phi}{\partial t}(t,\theta) = q\,\frac{\partial^2 \Phi}{\partial \theta^2}(t,\theta) + \sqrt{2}\dot{B}(t,\theta) + \frac{1}{\varepsilon}\Phi^-(t,\theta), \quad \theta \in [0,1],$$

$$\Phi(t,0) = \Phi(t,1) = 0, \tag{4.10}$$

Then, the next theorem is immediate from Theorem 4.1; we take $U = U_\varepsilon$.

Theorem 4.4. *For each $\varepsilon > 0$, $P^{N,\varepsilon}$ weakly converges to P^ε as $N \to \infty$.*

Let $\mathbb{P}$ be the distribution on the space L_T^2 of unique stationary solution of the SPDE (1.9)–(1.11) with reflection of Nualart–Pardoux type. The following theorem gives the lower bound for an arbitrary limit of $\{P^N\}_N$ as $N \to \infty$.

Theorem 4.5. (i) *The family $\{P^N\}_N$ of probability measures on $C([0,T],H^{-\alpha}([0,1])) \cap L_w^2(Q_T)$ is tight for $\alpha > \frac{1}{2}$.*
(ii) *The inequality $\hat{P} \geqslant \mathbb{P}$ holds for an arbitrary limit $\hat{P}$ of $\{P^N\}_N$ as $N \to \infty$.*

Proof. (i) In Section 6, we prove that $\{P^{N,\varepsilon}\}_{N,\varepsilon}$ is tight in $C([0,T],H^{-\alpha}([0,1])) \cap L_w^2(Q_T)$ for $\alpha > \frac{1}{2}$ (cf. Remark 6.1(i)).
(ii) Taking the limit $N \to \infty$ in the inequality $P^N \geqslant P^{N,\varepsilon}$ shown by Lemma 4.3, from Theorem 4.4, we have $\hat{P} \geqslant P^\varepsilon$ for every $\varepsilon > 0$ and for arbitrary limit $\hat{P}$ of $\{P^N\}_N$. However, Nualart and Pardoux (1992, p. 83) shows that P^ε weakly converges to $\mathbb{P}$ as $\varepsilon \downarrow 0$ and this implies the second assertion. $\square$

4.3. Proof of Theorem 1.1

We first notice the following property of the distribution $\mathbb{P}$ of Nualart–Pardoux process.

Proposition 4.6. $\mathbb{P} \circ \Phi_t^{-1} = \nu_{0,0}$ *for every $t > 0$; i.e., unique tempered stationary probability measure for the SPDE* (1.9)–(1.11) *is* $\nu_{0,0}$.

Proof. *Step* 1: We use the coupling argument again to show the uniqueness of tempered stationary probability measures. Let μ_1 and μ_2 be two tempered stationary probability measures and let $(\Phi^1(t,\theta), \xi^1(\mathrm{d}t\,\mathrm{d}\theta))$ and $(\Phi^2(t,\theta), \xi^2(\mathrm{d}t\,\mathrm{d}\theta))$ be solutions of

(1.9)–(1.11) with common space-time white-noise $\dot{B}(t,\theta)$ having initial distributions μ_1 and μ_2, respectively. Then, since

$$\int_{Q_T} (\Phi^1(t,\theta) - \Phi^2(t,\theta))(\xi^1(\mathrm{d}t\,\mathrm{d}\theta) - \xi^2(\mathrm{d}t\,\mathrm{d}\theta)) \leqslant 0,$$

we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi^1(t) - \Phi^2(t)\|^2_{L^2([0,1])} &\leqslant -2q\|\nabla(\Phi^1(t) - \Phi^2(t))\|^2_{L^2([0,1])}\\ &\leqslant -2\pi^2 q\|\Phi^1(t) - \Phi^2(t)\|^2_{L^2([0,1])}\end{aligned}$$

and this concludes $\mu_1 = \mu_2$; recall that $\Phi^i(t)$ are μ_i-distributed for $i = 1,2$ and every $t \geqslant 0$.

Step 2: It is shown by Otobe (1998) that, for $a > 0$, $v_{a,a}$ is a stationary measure for the SPDE (1.9), (1.10) imposed a-boundary conditions:

$$\Phi(t,0) = \Phi(t,1) = a, \quad t \geqslant 0. \tag{4.11}$$

Moreover, a similar argument as in Step 1 proves that $v_{a,a}$ is a unique tempered stationary probability measure for such SPDE. To conclude the proof of the proposition, we shall let $a \downarrow 0$ and show that $v_{0,0}$ is stationary for (1.9)–(1.11) (i.e., 0-boundary conditions). To this end, we prepare the next lemma.

Lemma 4.7. *Let* $(\Phi^a(t,\theta), \xi^a(\mathrm{d}t\,\mathrm{d}\theta))$, $a \geqslant 0$ *be the solutions of the SPDE* (1.9), (1.10) *imposed a-boundary conditions* (4.11) *having initial data* $\Phi^a(\theta)$ *which satisfy* $\Phi^a(0) = \Phi^a(1) = a$ *and are decreasing as* $a \downarrow 0$: $\Phi^a(\theta) \geqslant \Phi^{a'}(\theta)$ *if* $a > a' \geqslant 0$. *Then, we have* $\Phi^a(t,\theta) \geqslant \Phi^{a'}(t,\theta)$ *and* $\xi^a(\mathrm{d}t\,\mathrm{d}\theta) \leqslant \xi^{a'}(\mathrm{d}t\,\mathrm{d}\theta)$ (*a.s.*) (*i.e.*, $\xi^{a'} - \xi^a$ *is a non-negative measure*) *for every* $a > a' \geqslant 0$. *In addition, if* $\Phi^0(\theta) = \lim_{a\downarrow 0} \Phi^a(\theta)$ *holds, then we have* $\Phi^0(t,\theta) = \lim_{a\downarrow 0} \Phi^a(t,\theta)$ (*a.s.*).

Proof. Consider the penalized SPDE (4.10) with a-boundary conditions (4.11) in place of 0-boundary conditions and denote the solutions with initial data $\Phi^a(\theta)$ by $\Phi^{a,\varepsilon}(t,\theta)$. Then, the comparison theorem for such SPDEs shows that $\Phi^{a,\varepsilon}(t,\theta) \geqslant \Phi^{a',\varepsilon}(t,\theta)$, $a > a' \geqslant 0$ for every $\varepsilon > 0$ (cf. Nualart and Pardoux, 1992). This, in particular, implies that $\xi^{a,\varepsilon}(\mathrm{d}t\,\mathrm{d}\theta) \leqslant \xi^{a',\varepsilon}(\mathrm{d}t\,\mathrm{d}\theta)$, where $\xi^{a,\varepsilon}(\mathrm{d}t\,\mathrm{d}\theta) := (1/\varepsilon)(\Phi^{a,\varepsilon}(t,\theta))^- \,\mathrm{d}t\,\mathrm{d}\theta$. Therefore, letting $\varepsilon \downarrow 0$, we have the first assertion of the lemma. Note that $\xi^{a,\varepsilon}(\mathrm{d}t\,\mathrm{d}\theta)$ converges to $\xi^a(\mathrm{d}t\,\mathrm{d}\theta)$ as $\varepsilon \downarrow 0$, see Nualart and Pardoux.

From the first assertion, the limit $\bar{\Phi}(t,\theta) := \lim_{a\downarrow 0} \Phi^a(t,\theta) \geqslant 0$ and the weak limit $\bar{\xi}(\mathrm{d}t\,\mathrm{d}\theta) := \lim_{a\downarrow 0} \xi^a(\mathrm{d}t\,\mathrm{d}\theta)$ exist (a.s.). One can show that $(\bar{\Phi}, \bar{\xi})$ is a solution of the SPDE (1.9)–(1.11). Indeed, from $\int_Q \Phi^a(t,\theta)\xi^a(\mathrm{d}t\,\mathrm{d}\theta) = 0$ and $0 \leqslant \bar{\Phi} \leqslant \Phi^a$, we have $\int_Q \bar{\Phi}(t,\theta)\xi^a(\mathrm{d}t\,\mathrm{d}\theta) = 0$ which implies $\int_Q \bar{\Phi}(t,\theta)\bar{\xi}(\mathrm{d}t\,\mathrm{d}\theta) = 0$ by letting $a \downarrow 0$. The weak form corresponding to Eq. (1.9) can be easily checked by letting $a \downarrow 0$ again. Therefore, the uniqueness of solutions of (1.9)–(1.11) proves the conclusion. □

We continue the proof of Proposition 4.6. As $a \downarrow 0$, $v_{a,a}$ weakly converges to $v_{0,0}$. In fact, this is easily seen by recalling the fact that $v_{a,a}$ is the distribution of the distance from the origin of the three-dimensional pinned Brownian motion. Furthermore, if $a > a' > 0$, $\nu_{a,a} \geqslant \nu_{a',a'}$ holds. Indeed, to see this, take two solutions satisfying $\Phi^a(t,\theta) \geqslant \Phi^{a'}(t,\theta)$ as in Lemma 4.7 and denote the distribution of $\Phi^a(t,\cdot)$ by μ_t^a. Then, from the uniqueness of stationary measures, the Cesàro mean $(1/T)\int_0^T \mu_t^a \, \mathrm{d}t$ of $\{\mu_t^a; t \in [0,T]\}$ always converges to $\nu_{a,a}$ as $T \to \infty$ and, since $\mu_t^a \geqslant \mu_t^{a'}$, we have $\nu_{a,a} \geqslant \nu_{a',a'}$. Therefore, on a proper probability space, one can construct a family of random functions $\{\Phi^a(\theta); a \geqslant 0\}$ such that the distribution of each $\Phi^a(\cdot)$ is given by $\nu_{a,a}$ and $\Phi^a(\theta)$ monotone decreasingly converges to $\Phi^0(\theta)$ as $a \downarrow 0$ (a.s.).

Now, consider the solutions $\Phi^a(t,\theta)$ of the SPDE (1.9), (1.10) imposed a-boundary conditions (4.11) with initial data $\Phi^a(\theta)$ constructed as above. The noise $\dot{B}(t,\theta)$ is common and independent of the initial data $\{\Phi^a(\theta); a \geqslant 0\}$. Then, $\Phi^a(t,\theta)$ is stationary for $a > 0$. Finally, let $a \downarrow 0$ for $\Phi^a(t,\theta)$ and we see from Lemma 4.7 that $\nu_{0,0}$ is stationary for (1.9)–(1.11). This completes the proof of Proposition 4.6. □

We are at the position to complete *the proof of Theorem* 1.1. Let $\hat{P}$ be an arbitrary limit of $\{P^N\}_N$. Then, from Theorem 4.5(ii) and Proposition 4.6

$$\hat{P} \circ \Phi_t^{-1} \geqslant \mathbb{P} \circ \Phi_t^{-1} = \nu_{0,0}.$$

On the other hand, since $P^N \circ \Phi_t^{-1} = \hat{\mu}^{N,+}$, Proposition 3.1(ii) shows

$$\nu_{0,0} \geqslant \hat{P} \circ \Phi_t^{-1}$$

and therefore we obtain

$$\hat{P} \circ \Phi_t^{-1} = \mathbb{P} \circ \Phi_t^{-1} \tag{4.12}$$

for each $t \geqslant 0$. However, since Theorem 4.5(ii) claims $\hat{P} \geqslant \mathbb{P}$, equality (4.12) for every one-dimensional marginal distribution implies the equality $\hat{P} = \mathbb{P}$ for distributions on the path space L_T^2. This proves that P^N itself converges to $\mathbb{P}$ as $N \to \infty$. □

5. Boltzmann–Gibbs principle for weakly massive dynamics

The main step for the proof of Theorem 4.1 is to establish the so-called Boltzmann–Gibbs principle, by which one can replace the "non-linear Laplacian" (i.e. the first term on the right-hand side of (4.1)) with "linear Laplacian" under the large space-time scaling limit. In Section 5.1, the Boltzmann–Gibbs principle is formulated for the height difference process η_t^N in equilibrium. After giving several uniform a priori bounds (Section 5.2) and equivalence of ensemble (Section 5.4) for equilibrium measures, the proof of the Boltzmann–Gibbs principle will be established in Section 5.5 based on the localization scheme introduced in Section 5.3.

5.1. Boltzmann–Gibbs principle

Let $\phi_t = \{\phi_t(x); x \in \Gamma\}$ be the solution of the SDE (4.1). The associated height difference process $\eta_t = \{\eta_t(b); b \in \overline{\Gamma^*}\}$ is then defined by $\eta_t(b) := \phi_t(x) - \phi_t(x-1)$ for the bond $b = \{x, x-1\} \in \overline{\Gamma^*}$. Identifying the bond $b = \{x, x-1\}$ with $x \in \{1, 2, \ldots, N\}$, we shall simply denote $\eta_t(b)$ by $\eta_t(x)$. Note that the process η_t has a conservation law, $\sum_{x=1}^N \eta_t(x)=0$. The time-changed processes ϕ_{N^2t} and η_{N^2t} of ϕ_t and η_t are denoted by ϕ_t^N and η_t^N, respectively. Then, the Boltzmann–Gibbs principle for η_t^N in equilibrium can be stated as follows. Recall that $q>0$ is the constant defined by (1.12).

Theorem 5.1. *For every* $J\in C^1([0,1])$,

$$\lim_{N\to\infty} E^{\mu^{N,U}}\left[\left|\int_0^t \frac{1}{\sqrt{N}}\sum_{x=1}^N J(x/N)G(\eta_s^N(x))\,\mathrm{d}s\right|^2\right]=0, \tag{5.1}$$

where

$$G(\eta)=V'(\eta)-q\eta, \quad \eta\in\mathbb{R}.$$

The result does not depend on the massive perturbation U. As we shall see, such weakness of U is guaranteed by condition (4.2) together with the fact that the height variables $\phi(x)$ behave like $\mathrm{O}(\sqrt{N})$ under equilibrium.

5.2. *A priori bounds for* $\mu^{N,U}$

We prepare several a priori bounds for $\mu^{N,U}$. Recall that the measures $\mu^{N,U}$ and $\mu^N\equiv\mu^{N,0}$ are defined by (4.3) and (1.5), respectively.

Lemma 5.2. (i) *There exist* C_1 *and* $C_2>0$ *such that*

$$E^{\mu^{N,U}}[\exp\{\alpha\phi(x)\}]\leqslant C_1\mathrm{e}^{C_2\alpha^2N}, \quad E^{\mu^{N,U}}[\exp\{\alpha(\phi(x+K)-\phi(x))\}]\leqslant C_1\mathrm{e}^{C_2\alpha^2K},$$

for every $\alpha\in\mathbb{R}$ *and* $x\in\Gamma$.

(ii) *For every* $p\geqslant 1$, *there exists* $C=C_p>0$ *such that*

$$E^{\mu^{N,U}}[|\phi(x)|^p]\leqslant CN^{p/2}, \quad E^{\mu^{N,U}}[|\phi(x+K)-\phi(x)|^p]\leqslant CK^{p/2}.$$

(iii) *We have*

$$\sup_{N\in\mathbb{N}}\max_{1\leqslant x\leqslant N} E^{\mu^{N,U}}[G(\eta(x))^2]<\infty.$$

Proof. Since the potential V is strictly convex, noting that $\langle\phi(x)\rangle_{\mu^N}=0$, the assertion (i) and therefore (ii) for $\mu^N\equiv\mu^{N,0}$ in place of $\mu^{N,U}$ are consequences of Brascamp–Lieb inequality, see Lemma 2.9 of Deuschel et al. (2000) for instance. Now set

$$X:=\sum_{x\in\Gamma} U(N^{-1}\phi(x))\geqslant 0.$$

Then, we have $\mathrm{d}\mu^{N,U}=\tilde{Z}_{N,U}^{-1}\mathrm{e}^{-X}\,\mathrm{d}\mu^N$ with a normalization $\tilde{Z}_{N,U}$. However, the condition (4.2) on U implies

$$0\leqslant\langle X\rangle_{\mu^N}\leqslant\frac{1}{2N^2}\|U''\|_\infty\sum_{x\in\Gamma}\langle\phi(x)^2\rangle_{\mu^N}\leqslant\bar{C}$$

for some $\bar{C}>0$; we have used (ii) with $p=2$ for μ^N. Accordingly, $\tilde{Z}_{N,U}$ is uniformly positive in N:

$$\tilde{Z}_{N,U}=\langle\mathrm{e}^{-X}\rangle_{\mu^N}\geqslant\mathrm{e}^{-2\bar{C}}\{1-\mu^N(X>2\bar{C})\}\geqslant\mathrm{e}^{-2\bar{C}}/2>0.$$

Thus, the assertion (i) for $\mu^{N,U}$ follows from that for μ^N. The assertion (ii) is an immediate consequence of (i) and the assertion (iii) follows from the second inequality in (ii) with $K=1$ and $p=2$, since $V'(\eta)$ is linearly growing. □

5.3. Localization scheme

We follow Kipnis and Landim (1999, p. 292) for the proof of Theorem 5.1. For each $x \in \Gamma$, introduce a differential operator L_x by

$$L_x = \frac{\partial^2}{\partial\phi(x)^2} - \{V'(\phi(x)-\phi(x-1)) + V'(\phi(x)-\phi(x+1)) \\ +N^{-1}U'(N^{-1}\phi(x))\}\frac{\partial}{\partial\phi(x)}$$

and for the box $B=[x_-,x_+]\cap\mathbb{Z}$, $1\leqslant x_- < x_+ \leqslant N$ and $\phi_-,\phi_+ \in \mathbb{R}$

$$L_B \equiv L_{B;\phi_-,\phi_+} := \sum_{x\in \overset{\circ}{B}} L_x, \quad \overset{\circ}{B} := B\setminus\{x_+\}. \tag{5.2}$$

The operator L_B acts on functions of $\{\phi(x); x\in \overset{\circ}{B}\}$ and $\phi(x_- - 1)=\phi_-$, $\phi(x_+)=\phi_+$ on the right-hand side of (5.2). The variables $\{\phi(x); x \in \overset{\circ}{B}\}$ and $\{\eta(x); x\in B\}$ satisfying $\sum_{y\in B}\eta(y) = \phi_+ - \phi_-$ correspond to each other by the relations

$$\eta(x) := \phi(x)-\phi(x-1), \quad x\in B$$

and

$$\phi(x) := \sum_{y=x_-}^{x} \eta(y) + \phi_-, \quad x \in \overset{\circ}{B}.$$

In this way, the operator L_B naturally acts on functions of $\{\eta(x); x\in B\}$. We denote such operator by $\tilde{L}_B \equiv \tilde{L}_{B;\phi_-,\phi_+}$. The unique reversible probability measure for L_B is denoted by $\mu^{N,U}_{B;\phi_-,\phi_+} \in \mathscr{P}(\mathbb{R}^{\overset{\circ}{B}})$; namely

$$\mu^{N,U}_{B;\phi_-,\phi_+}(\mathrm{d}\phi) = Z^{-1}_{B,U;\phi_-,\phi_+} \exp\{-H_{B,U;\phi_-,\phi_+}(\phi)\} \prod_{x\in \overset{\circ}{B}} \mathrm{d}\phi(x) \tag{5.3}$$

for $\phi = \{\phi(x); x\in \overset{\circ}{B}\}$, where $Z_{B,U;\phi_-,\phi_+}$ is a normalization and

$$H_{B,U;\phi_-,\phi_+}(\phi) = \sum_{x\in B} V(\phi(x)-\phi(x-1)) + \sum_{x\in \overset{\circ}{B}} U(N^{-1}\phi(x))$$

with $\phi(x_- - 1) = \phi_-$, $\phi(x_+)=\phi_+$. Note that $\mu^{N,U}_{B;\phi_-,\phi_+}$ coincides with the conditional probability measure $\mu^{N,U}(\cdot|\sigma\{\phi(x), x\notin \overset{\circ}{B}\})$ of $\mu^{N,U}$. Since $\mu^{N,U}$ enjoys the Markov property, this conditional probability is $\sigma\{\phi(x_- - 1), \phi(x_+)\}$-measurable. The image measure of $\mu^{N,U}_{B;\phi_-,\phi_+}(\mathrm{d}\phi)$ under the map $\{\eta(x) := \phi(x)-\phi(x-1); x\in B\}$ is denoted by $\tilde{\mu}^{N,U}_{B;\phi_-,\phi_+}(\mathrm{d}\eta)$.

We divide the total set $\overline{\Gamma^*} \cong \{1,2,\dots,N\}$ into M small boxes with size K; namely, let $M := [(N-1)/K]+1$ and consider the boxes $B_i = \{x_{i-1}+1,\dots,x_i\}, 1\leqslant i\leqslant M$ in

$\overline{\Gamma^*}$, where $x_i = iK$ for $0 \leqslant i \leqslant M-1$ and $x_M = N$ (we assume $N = MK$ for simplicity). Set

$$G_i(\eta) := \sum_{x \in B_i} G(\eta(x)),$$

$$F_i(\eta) \equiv F_i(\phi(x_{i-1}), \phi(x_i)) := E^{\tilde{\mu}^{N,U}_{B_i;\phi(x_{i-1}),\phi(x_i)}}[G_i],$$

for $1 \leqslant i \leqslant M$; recall that $\phi(x_i) := \sum_{y=1}^{x_i} \eta(y)$ are regarded as functions of η-variables.

Lemma 5.3. *The conclusion of Theorem* 5.1 *follows from*

$$\lim_{K\to\infty} \overline{\lim_{M\to\infty}}\, E^{\mu^{N,U}}\left[\frac{1}{N}\left\{\sum_{i=1}^{M} J(x_i/N) F_i(\eta)\right\}^2\right] = 0. \tag{5.4}$$

Proof. Take $J \in C^1([0,1])$ and then we have from Lemma 5.2(iii) and the stationarity of the process η_s^N

$$\lim_{K\to\infty} \overline{\lim_{M\to\infty}}\, E^{\mu^{N,U}}\left[\left|\int_0^t \frac{1}{\sqrt{N}} \sum_{x=1}^{N} \{J(x/N) - J(x_{i(x)}/N)\} G(\eta_s^N(x))\,\mathrm{d}s\right|^2\right] = 0, \tag{5.5}$$

where $i(x)$, $x \in \overline{\Gamma^*}$ denotes the number i for which $x \in B_i$ holds. On the other hand, using the bound (1.2) of Kipnis and Landim (1999, p. 294) and by similar computations there, we have

$$\begin{aligned} &E^{\mu^{N,U}}\left[\left|\int_0^t \frac{1}{\sqrt{N}} \sum_{i=1}^{M} J(x_i/N) \tilde{L}_{B_i} H_i(\eta_s^N)\,\mathrm{d}s\right|^2\right] \\ &\quad \leqslant 20tM\|J\|_\infty^2 N^{-3} \max_{1\leqslant i\leqslant M} \langle -\tilde{L}_{B_i} H_i, H_i\rangle_{\mu^{N,U}}, \end{aligned} \tag{5.6}$$

for arbitrary functions $\{H_i(\eta); 1 \leqslant i \leqslant M\}$. If we take

$$H_i(\eta) = \tilde{L}_{B_i}^{-1}\{G_i(\eta) - F_i(\eta)\},$$

since the operator $-\tilde{L}_{B_i}$ on the space $L^2(\tilde{\mu}^{N,U}_{B_i;\phi(x_{i-1}),\phi(x_i)})$ has a spectral gap larger than CK^{-2} with certain universal constant $C > 0$ (indeed the logarithmic Sobolev inequality can be shown by means of Bakry and Emery's result, since the potential functions V and U are convex), we see

$$\begin{aligned} \langle -\tilde{L}_{B_i} H_i, H_i\rangle_{\mu^{N,U}} &\leqslant C^{-1}K^2\|G_i - F_i\|^2_{L^2(\mu^{N,U})} \\ &\leqslant 4C^{-1}K^2\|G_i\|^2_{L^2(\mu^{N,U})} \leqslant 4C^{-1}K^3 \sum_{x\in B_i} E^{\mu^{N,U}}[G(\eta(x))^2]. \end{aligned}$$

Hence, from (5.6) and Lemma 5.2(iii), we obtain

$$\lim_{K\to\infty} \overline{\lim_{M\to\infty}}\, E^{\mu^{N,U}}\left[\left|\int_0^t \frac{1}{\sqrt{N}} \sum_{i=1}^{M} J(x_i/N)\{G_i(\eta_s^N) - F_i(\eta_s^N)\}\,\mathrm{d}s\right|^2\right] = 0. \tag{5.7}$$

Combining (5.5) and (5.7), (5.1) can be proved once we have

$$\lim_{K\to\infty} \overline{\lim_{M\to\infty}}\, E^{\mu^{N,U}}\left[\left|\int_0^t \frac{1}{\sqrt{N}} \sum_{i=1}^{M} J(x_i/N) F_i(\eta_s^N)\,\mathrm{d}s\right|^2\right] = 0. \tag{5.8}$$

However, (5.8) is an easy consequence of (5.4) by using Schwarz's inequality and noting the stationarity of η_s^N under $\mu^{N,U}$. □

5.4. Equivalence of ensemble

To show the static property (5.4), we need to establish the equivalence of ensemble for $\tilde{\mu}^{N,U}_{B;\phi_-,\phi_+}$ (see Proposition 5.5 below). To this end, we first refer to the result for $\tilde{\mu}^{N,0}_{B;\phi_-,\phi_+}$; the probability measure $\tilde{\mu}^{N,U}_{B;\phi_-,\phi_+}$ with $U \equiv 0$. Since $\tilde{\mu}^{N,0}_{B;\phi_-,\phi_+}$ is determined depending only on the size $K := |B|$ of the set B and $u = (\phi_+ - \phi_-)/K$, we denote it simply by $\tilde{\mu}_{K,u}$ and regard it as a measure on $\mathbb{R}^{[1,K]\cap\mathbb{Z}} \equiv \{\eta = (\eta(x); 1 \leqslant x \leqslant K)\}$. Let $\hat{\nu}_\lambda \in \mathscr{P}(\mathbb{R})$, $\lambda \in \mathbb{R}$ be the probability measure defined by

$$\hat{\nu}_\lambda(\mathrm{d}\eta) = \hat{Z}_\lambda^{-1} \mathrm{e}^{-V(\eta)+\lambda\eta}\,\mathrm{d}\eta, \quad \hat{Z}_\lambda = \int_{\mathbb{R}} \mathrm{e}^{-V(\eta)+\lambda\eta}\,\mathrm{d}\eta,$$

and set $\nu_u := \hat{\nu}_{\lambda(u)} \in \mathscr{P}(\mathbb{R})$, $u \in \mathbb{R}$ with a function $\lambda = \lambda(u)$ introduced by the relation $E^{\hat{\nu}_\lambda}[\eta] = u$.

Proposition 5.4 (Equivalence of ensemble for $\tilde{\mu}_{K,u}$). *For each* $u_0 > 0$, *there exists* $C > 0$ *such that*

$$|\langle \Psi(\eta(x)) \rangle_{K,u} - \langle \Psi \rangle_u| \leqslant \frac{C}{K} \left\{ \sup_{|\lambda| \leqslant \lambda(u_0)+1, |\varepsilon| \leqslant 1/K} \langle \Psi; \Psi \rangle_{\lambda,\varepsilon\Psi} + 1 \right\}, \quad |u| \leqslant u_0$$

for every at most linearly growing function Ψ *on* $\mathbb{R}$ *and every* $1 \leqslant x \leqslant K$. *Here* $\langle\cdot\rangle_{K,u}$ *and* $\langle\cdot\rangle_u$ *denote averages under* $\tilde{\mu}_{K,u}$ *and* ν_u, *respectively*; $\langle\cdot;\cdot\rangle_{\lambda,\varepsilon\Psi}$ *denotes the truncated correlation function under* $\hat{\nu}_{\lambda,\varepsilon} \in \mathscr{P}(\mathbb{R})$ *which is defined similarly to* $\hat{\nu}_\lambda$ *but with* $-V(\eta) + \lambda\eta$ *replaced by* $-V(\eta) + \varepsilon\Psi(\eta) + \lambda\eta$.

Proof. See Lemma 3.1 and Corollary 3.2 of Chang and Yau (1992). Note that $\tilde{\mu}_{K,u}$ is symmetric, i.e., invariant under permutations of coordinates $\{x; 1 \leqslant x \leqslant K\}$. □

This result is generalized to the weakly massive case and the speed of convergence can be estimated as follows. Since $\tilde{\mu}^{N,U}_{B;\phi_-,\phi_+}$ depends only on $K = |B|$, $u = (\phi_+ - \phi_-)/K$ and ϕ_-, we denote it by $\tilde{\mu}^{N,U}_{K,u,\phi_-}$ and regard as a measure on $\mathbb{R}^{[1,K]\cap\mathbb{Z}}$.

Proposition 5.5 (Equivalence of ensemble for $\tilde{\mu}^{N,U}_{K,u,\phi_-}$). *For each* u_0 *and* $h_0 > 0$, *there exists* $C > 0$ *such that*

$$\begin{aligned} |\langle \Psi(\eta(x)) \rangle^{N,U}_{K,u,\phi_-} - \langle \Psi \rangle_u| \leqslant{}& \frac{C}{K} \left\{ \sup_{|\lambda| \leqslant \lambda(u_0)+1, |\varepsilon| \leqslant 1/K} \langle \Psi; \Psi \rangle_{\lambda,\varepsilon\Psi} + 1 \right\} \\ &+ \frac{C}{N^2}(|\phi_-|K^{3/2} + |u|K^{5/2} + K^2)\langle \Psi(\eta(x)); \Psi(\eta(x)) \rangle_{K,u}^{1/2} \end{aligned}$$

for every at most linearly growing function Ψ *on* $\mathbb{R}$ *and every* $1 \leqslant x \leqslant K$, *if* $|u| \leqslant u_0, |\phi_-| \leqslant h_0 N$ *and* $K = o(N)$. *Here* $\langle\cdot\rangle^{N,U}_{K,u,\phi_-}$ *and* $\langle\cdot;\cdot\rangle_{K,u}$ *denote the average under* $\tilde{\mu}^{N,U}_{K,u,\phi_-}$ *and the truncated correlation function under* $\tilde{\mu}_{K,u}$, *respectively.*

Proof. The measure $\tilde{\mu}^{N,U}_{K,u,\phi_-}$ has a representation:

$$\tilde{\mu}^{N,U}_{K,u,\phi_-}(\mathrm{d}\eta) = \tilde{Z}^{-1}\mathrm{e}^{-X}\tilde{\mu}_{K,u}(\mathrm{d}\eta), \quad \eta = \{\eta(x);\ 1 \leqslant x \leqslant K\},$$

with a normalization $\tilde{Z}$ and a random variable

$$X := \sum_{x=1}^{K-1}\{U(N^{-1}\phi(x)) - U(N^{-1}\langle\phi(x)\rangle_{K,u})\},$$

where $\phi(x) \equiv \phi(x,\eta) = \phi_- + \sum_{y=1}^{x}\eta(y)$ and $\langle\phi(x)\rangle_{K,u} = \phi_- + xu$. Hence, we see

$$\langle\Psi\rangle^{N,U}_{K,u,\phi_-} = \langle\Psi\rangle_{K,u} + \tilde{Z}^{-1}\langle\Psi(\mathrm{e}^{-X} - \tilde{Z})\rangle_{K,u},$$

where $\Psi = \Psi(\eta(\bar{x}))$ for some $1 \leqslant \bar{x} \leqslant K$. One can apply Proposition 5.4 for the first term on the right-hand side. For the second term, we observe that

$$\tilde{Z} = \langle\mathrm{e}^{-X}\rangle_{K,u} \geqslant \langle 1 - X\rangle_{K,u} \geqslant 1 - CK^2/N^2 > \tfrac{1}{2}$$

for large N; recall that $K = \mathrm{o}(N)$. Here the second inequality in the above line is shown based on the following two bounds:

$$\left|X - \frac{1}{N}\sum_{x=1}^{K-1}U'(N^{-1}\langle\phi(x)\rangle_{K,u})\tilde{\phi}(x)\right| \leqslant \frac{1}{2N^2}\|U''\|_\infty\sum_{x=1}^{K-1}\tilde{\phi}(x)^2, \tag{5.9}$$

by the Taylor expansion, and

$$\langle\tilde{\phi}(x)^2\rangle_{K,u} \leqslant CK,$$

by Brascamp–Lieb inequality, where $\tilde{\phi}(x) := \phi(x) - \langle\phi(x)\rangle_{K,u}$. Applying Schwarz's inequality for the term $\langle\Psi(\mathrm{e}^{-X} - \tilde{Z})\rangle_{K,u}$, the proof of the proposition can be concluded once

$$\langle(\mathrm{e}^{-X} - \tilde{Z})^2\rangle^{1/2}_{K,u} \leqslant \frac{C}{N^2}(|\phi_-|K^{3/2} + |u|K^{5/2} + K^2) \tag{5.10}$$

is shown. However,

$$\begin{aligned}\langle(\mathrm{e}^{-X} - \tilde{Z})^2\rangle_{K,u} &= \langle\mathrm{e}^{-2X}\rangle_{K,u} - \langle\mathrm{e}^{-X}\rangle^2_{K,u}\\ &\leqslant (1 - 2\langle X\rangle_{K,u} + 2\langle\mathrm{e}^{-2X^*}X^2\rangle_{K,u}) - (1 - \langle X\rangle_{K,u})^2\\ &\leqslant 2\langle\mathrm{e}^{-2X^*}X^2\rangle_{K,u} \leqslant 2\langle\mathrm{e}^{-4X^*}\rangle^{1/2}_{K,u}\langle X^4\rangle^{1/2}_{K,u},\end{aligned}$$

where X^* is a certain random variable taking values between 0 and X. Since $X > 0$ implies $X^* > 0$ and since $U'' \geqslant 0$,

$$\begin{aligned}\langle\mathrm{e}^{-4X^*}\rangle_{K,u} &\leqslant \langle\mathrm{e}^{-4X}\rangle_{K,u} + 1\\ &\leqslant \left\langle\exp\left\{-4N^{-1}\sum_{x=1}^{K-1}U'(N^{-1}\langle\phi(x)\rangle_{K,u})\tilde{\phi}(x)\right\}\right\rangle_{K,u} + 1\end{aligned}$$

which is bounded uniformly in K, u and N by Brascamp–Lieb inequality. Finally, recalling that

$$|U'(N^{-1}\langle\phi(x)\rangle_{K,u})| \leqslant \|U''\|_\infty N^{-1}(|\phi_-| + |u|K),$$

we have from (5.9)

$$\begin{aligned}\langle X^4\rangle_{K,u} &\leqslant C\left\{\frac{(|\phi_-|+|u|K)K^{3/2}}{N^2}\right\}^4 \left\langle\left(\frac{1}{K}\sum_{x=1}^{K-1}\bar\phi(x/K)\right)^4\right\rangle_{K,u}\\ &\quad + C\left(\frac{K^2}{N^2}\right)^4 \left\langle\left(\frac{1}{K}\sum_{x=1}^{K-1}\bar\phi(x/K)^2\right)^4\right\rangle_{K,u}\\ &\leqslant \frac{C'}{N^8}(|\phi_-|K^{3/2}+|u|K^{5/2}+K^2)^4,\end{aligned}$$

where $\bar\phi_\theta := K^{-1/2}\tilde\phi_{[K\theta]}$ for $\theta\in[0,1]$. This concludes the proof of (5.10). □

5.5. Proof of Theorem 5.1

We finally complete the proof of (5.4). The next lemma is a consequence of the equivalence of ensemble.

Lemma 5.6. *There exists* $C>0$ *such that*

$$E^{\mu^{N,U}}[F_i^2]\leqslant C\left(1+\frac{K^5}{N^2}\right),\quad 1\leqslant i\leqslant M, \tag{5.11}$$

$$E^{\mu^{N,U}}[F_iF_j]\leqslant C\left(1+\frac{K^{5/2}}{N}\right)KN^{-7/12},\quad 1\leqslant i<j\leqslant M. \tag{5.12}$$

The last bound can be improved as

$$E^{\mu^{N,U}}[F_iF_j]\leqslant CK^2N^{-7/6},\quad 1\leqslant i<j\leqslant M, \tag{5.13}$$

if i *and* j *satisfy* $N^{3/5}\leqslant x_i\leqslant x_j\leqslant N-N^{3/5}$ *or* $x_j-x_i\geqslant N^{3/5}$.

Proof. *Step* 1: For $0\leqslant x_-^*<x_+^*\leqslant N$, set $B^*=[x_-^*,x_+^*]\cap\mathbb{Z}$, $\tilde K = x_+^*-x_-^*$,

$$F_x^*\equiv F_x^*(x_-^*,x_+^*):=E^{\tilde\mu^{N,U}_{B^*;\phi(x_-^*),\phi(x_+^*)}}[G(\eta(x))],\quad x\in B^*$$

and

$$A(x_-^*,x_+^*;\delta):=\{|\phi(x_+^*)-\phi(x_-^*)|\leqslant\tilde K^{1/2+\delta}, |\phi(x_-^*)|\leqslant N^{1/2+\delta}\},\quad 0<\delta\leqslant\tfrac{1}{2}.$$

Then, since $\lambda(u)=\langle V'(\eta)\rangle_u$, $\lambda(0)=0$ by symmetry of V and $q=\lambda'(0)$, we have

$$|\langle G\rangle_u| = |\langle V'(\eta)\rangle_u - qu|\leqslant\tfrac{1}{2}|u|^2\sup_{|v|\leqslant|u|}|\lambda''(v)|$$

and therefore, from Proposition 5.5 with $\Psi=G$ and $K=\tilde K$

$$|F_x^*|\leqslant C\{\tilde K^{-2}(\phi(x_+^*)-\phi(x_-^*))^2+\tilde K^{-1}+N^{-3/2+\delta}\tilde K^{3/2}+N^{-2}\tilde K^{2+\delta}\}, \tag{5.14}$$

on the set $A(x_-^*,x_+^*;\delta)$; the first term on the right-hand side is the contribution of $|\langle G\rangle_u|$ with $u=(\phi(x_+^*)-\phi(x_-^*))/\tilde K$. On the other hand, noting $\delta>0$, by Lemma 5.2(ii) and Chebyshev's inequality,

$$\mu^{N,U}(A(x_-^*,x_+^*;\delta)^c)\leqslant C_p\tilde K^{-p} \tag{5.15}$$

for some $C_p>0$ and every $p\geqslant 1$.

Step 2: We first take $\tilde{K}=K$ and $\delta=\frac{1}{2}$ to show (5.11). With this choice, (5.14) implies

$$|F_i(\eta)| = \left|\sum_{x\in B_i} F_x^*(x_{i-1},x_i)\right|$$

$$\leqslant C\{K^{-1}(\phi(x_i)-\phi(x_{i-1}))^2+1+N^{-1}K^{5/2}\}, \tag{5.16}$$

on the set $A_i := A(x_{i-1},x_i;\frac{1}{2})$. Therefore,

$$\begin{aligned} E^{\mu^{N,U}}[F_i^2] &= E^{\mu^{N,U}}[F_i^2,A_i]+E^{\mu^{N,U}}[F_i^2,A_i^c] \\ &\leqslant C\{K^{-2}E^{\mu^{N,U}}[(\phi(x_i)-\phi(x_{i-1}))^4]+1+N^{-2}K^5\} \\ &\quad + E^{\mu^{N,U}}[F_i^4]^{1/2}\mu^{N,U}(A_i^c)^{1/2}, \end{aligned}$$

which proves (5.11) since $E^{\mu^{N,U}}[(\phi(x_i)-\phi(x_{i-1}))^4]\leqslant CK^2$ and $E^{\mu^{N,U}}[F_i^4]\leqslant CK^4$. Next, let us give the proof of (5.12) and (5.13). Choose $x_{i-1}^*,x_i^*\in\mathbb{Z}$ such that $0\leqslant x_{i-1}^*\leqslant x_{i-1} < x_i\leqslant x_i^*\leqslant N$ and $x_i^*-x_{i-1}^*=N^{3/5}$, and set

$$F_i^* := E^{\mu^{N,U}}[F_i\,|\,\phi(x_{i-1}^*),\phi(x_i^*)].$$

Then, (5.14) taking $\tilde{K}=N^{3/5}$ and $\delta=\frac{1}{72}$ shows

$$|F_i^*|\leqslant CKN^{-7/12}, \tag{5.17}$$

on the set $A_i^* := A(x_{i-1}^*,x_i^*;\frac{1}{72})$. Similar arguments are possible for F_j with x_{j-1}^*, x_j^* and F_j^* defined similarly. Now, in general case, one can choose four points x_{i-1}^*, x_i^*, x_{j-1}^*, x_j^* in such a way that $0\leqslant x_{i-1}^*\leqslant x_{i-1} < x_i\leqslant x_i^*\leqslant x_{j-1}^*\leqslant x_{j-1} < x_j\leqslant x_j^*\leqslant N$ and $x_i^*-x_{i-1}^*=N^{3/5}$ or $x_j^*-x_{j-1}^*=N^{3/5}$ hold. The Markov property of $\mu^{N,U}$ implies

$$E^{\mu^{N,U}}[F_iF_j]=E^{\mu^{N,U}}[F_i^*F_j] \quad (\text{or} = E^{\mu^{N,U}}[F_iF_j^*])$$

and therefore, from (5.16) and (5.17), we obtain (5.12) with the help of similar cut-off argument used above. Under the additional assumption for (5.13), we can choose four points such that both $x_i^*-x_{i-1}^*=N^{3/5}$ and $x_j^*-x_{j-1}^*=N^{3/5}$ hold. Hence, in this case, we can apply the bound (5.17) both for F_i^* and F_j^*, and obtain (5.13) from $E^{\mu^{N,U}}[F_iF_j]=E^{\mu^{N,U}}[F_i^*F_j^*]$. □

Proof of Eq. (5.4). The expectation in (5.4) is expanded as

$$\frac{1}{N}\sum_{i,j=1}^{M} J(x_i/N)J(x_j/N)E^{\mu^{N,U}}[F_iF_j].$$

Divide the sum into those in the following three regions: $S_1=\{i=j\}$, $S_2=\{i<j; x_i, N-x_j\geqslant N^{3/5}$ or $x_j-x_i\geqslant N^{3/5}\}$, $S_3=\{i<j;(i,j)\notin S_2\}$ and use (5.11) for S_1, (5.13) for S_2, (5.12) for S_3, respectively. Then, the absolute value of the above sum is bounded by

$$C'\frac{\|J\|_\infty^2}{N}\left\{M\left(1+\frac{K^5}{N^2}\right)+M^2K^2N^{-7/6}+(K^{-1}N^{3/5})^2\left(1+\frac{K^{5/2}}{N}\right)KN^{-7/12}\right\}$$

which tends to 0 as $N\to\infty$ and then $K\to\infty$, recall $N=MK$. This concludes the proof of (5.4). □

6. Proof of Theorem 4.1

Recall that $\Gamma = \Gamma_N \equiv \{1, 2, \dots, N-1\}$ and $\overline{\Gamma^*} \cong \{1, 2, \dots, N\}$ index the sets of sites and bonds, respectively. We begin with showing the tightness of the family of distributions of the fluctuation fields $\{\Phi^N(t,\theta)\}$ defined as in Theorem 4.1. Set $(\mathbb{L}^2_{T,w})^3 := \{L^2_w(Q_T)\}^3$ and $(\mathbb{L}^2)^3 := \{L^2([0,1])\}^3$, where $L^2_w(Q_T)$ and $L^2([0,1])$ are L^2-spaces endowed with the weak and strong topologies, respectively. The coordinate functions of these spaces are denoted by $\Psi(t,\theta) \equiv (\Psi_1(t,\theta), \Psi_2(t,\theta), \Psi_3(t,\theta)) \in (\mathbb{L}^2_{T,w})^3$ and $\Psi(\theta) \equiv (\Psi_1(\theta), \Psi_2(\theta), \Psi_3(\theta)) \in (\mathbb{L}^2)^3$, respectively.

Proposition 6.1. (i) *The family of distributions of* $\{\Phi^N(t,\theta)\}_{N\geqslant 1}$ *on the space* $C([0,T], H^{-\alpha}([0,1])) \cap L^2_w(Q_T)$ *is tight for* $\alpha > \frac{1}{2}$.
(ii) *The families of distributions of positive and negative parts* $\{\Phi^{N,\pm}(t,\theta)\}_{N\geqslant 1}$ *of* $\{\Phi^N(t,\theta)\}_{N\geqslant 1}$ *on the space* $L^2_w(Q_T)$ *are tight.*
(iii) *Let* $P^{N,U}$ *be the joint distribution of* $\{(\Phi^N(t,\theta), \Phi^{N,+}(t,\theta), \Phi^{N,-}(t,\theta)); (t,\theta) \in Q_T\}$ *on the space* $(\mathbb{L}^2_{T,w})^3$ *and let* $P^U \in \mathscr{P}((\mathbb{L}^2_{T,w})^3)$ *be an arbitrary limit of* $\{P^{N,U}\}_N$ *as* $N \to \infty$. *Then,*

$$\Psi_2(t,\theta) = \Psi_1^+(t,\theta), \quad \Psi_3(t,\theta) = \Psi_1^-(t,\theta), \quad \text{a.e. } (t,\theta) \in Q_T,$$

holds for P^U*-a.s.* $\Psi \in (\mathbb{L}^2_{T,w})^3$.

Proof. The tightness of $\Phi^N(t,\theta)$ and $\Phi^{N,\pm}(t,\theta)$ on the space $L^2_w(Q_T)$ is shown from

$$E^{\mu^{N,U}}[\|\Phi^N(t,\cdot)\|^2_{L^2([0,1])}] = N^{-2} \sum_{x\in\Gamma} E^{\mu^{N,U}}[\phi(x)^2] \leqslant C, \quad t \in [0,T], \tag{6.1}$$

which follows from Lemma 5.2(ii) by noting the stationarity of $\Phi^N(t,\cdot)$.

The tightness of $\Phi^N(t,\theta)$ on the space $C([0,T], H^{-\alpha}([0,1]))$ can be studied with a method similar to the one used by Giacomin et al. (2001). Observe that

$$E^{\mu^{N,U}}\left[\sup_{0\leqslant t\leqslant T} \|\Phi^N(t,\cdot)\|^2_{-\alpha}\right] \leqslant \sum_{k=1}^{\infty} \frac{1}{(\pi k)^{2\alpha}} E^{\mu^{N,U}}\left[\sup_{0\leqslant t\leqslant T} |\hat{\Phi}^N(t,k)|^2\right].$$

Recall the general bound for reversible processes (cf. formulas 5.40–5.42 of Giacomin et al., 2001)

$$E^{\mu^{N,U}}\left[\sup_{0\leqslant t\leqslant T} |G(\phi(t))|^2\right] \leqslant C_1 E^{\mu^{N,U}}[G(\phi)^2] + C_2 N^2 T E^{\mu^{N,U}}\left[\sum_{x\in\Gamma}\left(\frac{\partial G}{\partial \phi(x)}\right)^2\right].$$

We apply this bound to the function $G = \hat{\Phi}^N(k)$ which is the Fourier coefficient of $\Phi^N(\theta)$ defined in (3.1) and viewed as a function of $\phi \in \mathbb{R}^\Gamma$. Using (6.1) we have

$$E^{\mu^{N,U}}[|\hat{\Phi}^N(k)|^2] = E^{\mu^{N,U}}[\langle \Phi^N, h_k\rangle^2] \leqslant E^{\mu^{N,U}}[\|\Phi^N\|^2_{L^2([0,1])}] \leqslant C.$$

It is easy to see by explicit computation that

$$N^2 \sum_{x\in\Gamma}\left(\frac{\partial \hat{\Phi}^N(k)}{\partial \phi(x)}\right)^2 = \frac{1}{N}\sum_{x\in\Gamma} \bar{h}_k(x/N)^2 \leqslant 2,$$

where we denote for $h = h(\theta)$ and $x \in \Gamma$

$$\bar{h}(x/N) := N\langle 1_{[x/N-1/2N, x/N+1/2N)}, h\rangle. \tag{6.2}$$

Then, putting all these together, we have shown that for every $\alpha > \frac{1}{2}$

$$E^{\mu^{N,U}}\left[\sup_{0\leqslant t\leqslant T}\|\Phi^N(t,\cdot)\|_{-\alpha}^2\right]\leqslant C_3.$$

We can now control the modulus of continuity by observing

$$\begin{aligned}
&E^{\mu^{N,U}}\left[\sup_{\substack{|t-s|\leqslant\delta\\0\leqslant s<t\leqslant T}}\|\Phi^N(t,\cdot)-\Phi^N(s,\cdot)\|_{-\alpha}^2\right]\\
&\quad\leqslant 4\sum_{k=R+1}^{\infty}\frac{1}{(\pi k)^{2\alpha}}E^{\mu^{N,U}}\left[\sup_{0\leqslant t\leqslant T}|\hat{\Phi}^N(t,k)|^2\right]\\
&\quad\quad+\sum_{k=1}^{R}\frac{1}{(\pi k)^{2\alpha}}E^{\mu^{N,U}}\left[\sup_{\substack{|t-s|\leqslant\delta\\0\leqslant s<t\leqslant T}}|\hat{\Phi}^N(t,k)-\hat{\Phi}^N(s,k)|^2\right].
\end{aligned}$$

The first term on the right-hand side will converge to 0 as $R\to\infty$ uniformly in N. Then it is enough to show that for each k

$$\lim_{\delta\downarrow 0}E^{\mu^{N,U}}\left[\sup_{\substack{|t-s|\leqslant\delta\\0\leqslant s<t\leqslant T}}|\hat{\Phi}^N(t,k)-\hat{\Phi}^N(s,k)|\right]=0. \tag{6.3}$$

From the SDE (4.1) we have

$$\hat{\Phi}^N(t,k)-\hat{\Phi}^N(s,k)=\int_s^t b_k^N(\phi_\tau^N)\,\mathrm{d}\tau+\sqrt{2}(m_k^N(t)-m_k^N(s)),$$

where $m_k^N(t)$ is a martingale given explicitly by

$$m_k^N(t)=N^{-1/2}\sum_{x\in\Gamma}\bar{h}_k(x/N)w_t(x)$$

and

$$b_k^N(\phi)=-N^{1/2}\sum_{x\in\Gamma}\left(\frac{\partial H_{N,U;0,0}}{\partial\phi(x)}\right)(\phi)\bar{h}_k(x/N).$$

Since $m_k^N(t)$ has bounded quadratic variation (uniform in N), its modulus of continuity is controlled by standard martingale estimates.

From the Garsia–Rodemich–Rumsey inequality one obtains

$$\begin{aligned}
&\sup_{\substack{|t-s|\leqslant\delta\\0\leqslant s<t\leqslant T}}\left|\int_s^t b_k^N(\phi_\tau^N)\,\mathrm{d}\tau\right|\\
&\quad\leqslant C_4\delta\log\delta^{-1}\left[\log\left(4\int_0^T\int_0^T\exp\left\{\left|\int_s^t b_k^N(\phi_\tau^N)\mathrm{d}\tau\right|\Big/\sqrt{t-s}\right\}\mathrm{d}s\,\mathrm{d}t\right)+C_5\right],
\end{aligned}$$

where the constants C_4, C_5 do not depend on b_k^N, cf. the proof of Lemma 11.3.10 of Kipnis and Landim (1999, p. 306). Using Feynman–Kac formula one can show that

$$\begin{aligned}
&\log E^{\mu^{N,U}}(\mathrm{e}^{\beta|\int_s^t b_k^N(\phi_\tau^N)\,\mathrm{d}\tau|}) \\
&\quad \leqslant (t-s)\sup_f \left\{ \beta\langle b_k^N f^2\rangle_{\mu^{N,U}} - N^2 \sum_{x\in\Gamma} \left\langle \left(\frac{\partial f}{\partial\phi(x)}\right)^2 \right\rangle_{\mu^{N,U}} \right\} \\
&\quad = (t-s)\sup_f \sum_{x\in\Gamma} \left[-\beta N^{1/2}\bar{h}_k(x/N) \left\langle 2f\frac{\partial f}{\partial\phi(x)} \right\rangle_{\mu^{N,U}} - N^2 \left\langle \left(\frac{\partial f}{\partial\phi(x)}\right)^2 \right\rangle_{\mu^{N,U}} \right] \\
&\quad \leqslant (t-s)\frac{\beta^2}{N}\sum_{x\in\Gamma}\bar{h}_k(x/N)^2 \leqslant 2(t-s)\beta^2,
\end{aligned}$$

where $\sup_f$ are taken over all $f=f(\phi)>0$ satisfying $\langle f^2\rangle_{\mu^{N,U}}=1$. Then, by stationarity, the estimates above and Jensen's inequality, we have

$$E^{\mu^{N,U}}\left[\sup_{\substack{|t-s|\leqslant\delta \\ 0\leqslant s<t\leqslant T}} \left| \int_s^t b_k^N(\phi_\tau^N)\,\mathrm{d}\tau \right|^2 \right] \leqslant C_6\delta\log\delta^{-1},$$

which completes the proof of (6.3). Assertion (i) is therefore concluded.

Finally we prove (iii). Denoting by $(\Omega,\mathscr{F},P)$ the probability space on which the fluctuation fields $\{\Phi^N(t,\theta)=\Phi^N(t,\theta;\omega);(t,\theta)\in Q_T\}$ are defined, let $\bar{P}^{N,U}\in\mathscr{P}((\mathbb{L}^2)^3)$ be the distribution of $\bar{\Phi}^N(t,\cdot;\omega)\equiv(\Phi^N(t,\cdot;\omega),\Phi^{N,+}(t,\cdot;\omega),\Phi^{N,-}(t,\cdot;\omega))\in(\mathbb{L}^2)^3$ realized on the probability space $([0,T]\times\Omega,\mathrm{d}t\,\mathrm{d}P/T)$. Then, the family $\{\bar{P}^{N,U}\}_N$ is tight on $(\mathbb{L}^2)^3$; recall that this space is equipped with the strong topology. In fact, this follows from the stationarity of $\Phi^N(t,\cdot)$ and the second inequality in Lemma 5.2(ii) which gives uniform bound on Hölder norms. Now assume that $P^{N',U}$ weakly converges to P^U on $(\mathbb{L}^2_{T,w})^3$ as $N'\to\infty$. Then, one can find further subsequence $\{N''\}$ of $\{N'\}$ and $\bar{P}^U\in\mathscr{P}((\mathbb{L}^2)^3)$ such that $\bar{P}^{N'',U}$ weakly converges to $\bar{P}^U$ on $(\mathbb{L}^2)^3$ as $N''\to\infty$. Consider a function

$$F(\Psi):=\|\Psi_1^+-\Psi_2\|_{L^2([0,1])}\wedge 1,\quad \Psi=(\Psi_1,\Psi_2,\Psi_3)\in(\mathbb{L}^2)^3.$$

Then, since $F\in C_b((\mathbb{L}^2)^3)$, we have

$$E^{\bar{P}^U}[F(\Psi)]=\lim_{N''\to\infty}E^{\bar{P}^{N'',U}}[F(\Psi)]=0.$$

However, since

$$\frac{1}{T}E^{P^U}\left[\int_0^T |\langle\Psi_1^+(s)-\Psi_2(s),J\rangle|\,\mathrm{d}s\right]=E^{\bar{P}^U}[|\langle\Psi_1^+-\Psi_2,J\rangle|]=0$$

for every $J\in C([0,1])$, we get $\Psi_2=\Psi_1^+$, P^U-a.s. Similarly, we have $\Psi_3=\Psi_1^-$, P^U-a.s. □

Remark 6.1. (i) Observe that the bounds that are involved in the proof above are independent from U. It follows that the family of distribution $\{P^{N,\varepsilon}\}_{N,\varepsilon}$ introduced in Section 4.2 is tight in $C([0,T],H^{-\alpha}([0,1]))\cap L^2_w(Q_T)$ for $\alpha>\frac{1}{2}$.
(ii) Since $E^{\mu^{N,U}}[\|\Phi^N(t,\cdot)\|^2_{L^2((\mathbb{R}/\mathbb{Z})^d)}]=\mathrm{O}(N^{d-1})$ in d-dimension (e.g., on the periodic lattice Γ^d_N), the tightness in L^2-space can be shown only when $d=1$.

Now we turn to *the proof of Theorem* 4.1. The process $\Phi^N(t,\theta)$ satisfies

$$\langle\Phi^N(t),J\rangle=\langle\Phi^N(0),J\rangle+\sqrt{2}m^N_t(J)+\int_0^t\{b^{N,1}_s(J)+b^{N,2}_s(J)\}\,\mathrm{d}s, \tag{6.4}$$

for every test function $J\in C^2([0,1])$ such that $J(0)=J(1)=0$, where

$$m^N_t(J)=N^{-1/2}\sum_{x\in\Gamma}\bar{J}(x/N)w_t(x),$$

$$b^{N,1}_t(J)=-N^{1/2}\sum_{x\in\overline{\Gamma^*}}V'(\eta^N_t(x))\{\bar{J}(x/N)-\bar{J}((x-1)/N)\},$$

$$b^{N,2}_t(J)=-N^{-1/2}\sum_{x\in\Gamma}U'(N^{-1}\phi^N_t(x))\bar{J}(x/N)$$

and $\bar{J}(x/N)$ is defined from $J(\theta)$ by (6.2) for $x\in\Gamma$ and $\bar{J}(x/N):=0$ for $x\in\partial\Gamma=\{0,N\}$; recall that $\phi^N_t(x)=\phi_{N^2t}(x)$ and $\eta^N_t(x)=\phi^N_t(x)-\phi^N_t(x-1)$.

The asymptotic behaviors of two terms $\int_0^t b^{N,i}_s(J)\,\mathrm{d}s, i=1,2$ are given by the following two lemmas, respectively.

Lemma 6.2. *For every* $J\in C^2([0,1])$ *satisfying* $J(0)=J(1)=0$,

$$\lim_{N\to\infty}E^{\mu^{N,U}}\left[\left|\int_0^t\{b^{N,1}_s(J)-q\langle\Phi^N(s),J''\rangle\}\,\mathrm{d}s\right|^2\right]=0. \tag{6.5}$$

Proof. Set

$$\tilde{b}^{N,1}_t(J):=-N^{-1/2}\sum_{x\in\overline{\Gamma^*}}V'(\eta^N_t(x))J'(x/N),$$

and

$$\bar{b}^{N,1}_t(J):=-N^{-1/2}\sum_{x\in\overline{\Gamma^*}}q\eta^N_t(x)J'(x/N).$$

Then, we have

$$\lim_{N\to\infty}E^{\mu^{N,U}}\left[\left|\int_0^t\{b^{N,1}_s(J)-\tilde{b}^{N,1}_s(J)\}\,\mathrm{d}s\right|^2\right]=0, \tag{6.6}$$

since

$$|b^{N,1}_s(J)-\tilde{b}^{N,1}_s(J)|\leqslant 2^{-1}N^{-3/2}\|J''\|_\infty\sum_{x\in\overline{\Gamma^*}}|V'(\eta^N_s(x))|$$

and Lemma 5.2(ii) gives $\sup_{N,x} E^{\mu^{N,U}}[\eta(x)^2] < \infty$. However, the Boltzmann–Gibbs principle (Theorem 5.1) enables us to replace $\tilde{b}_s^{N,1}(J)$ further with $\bar{b}_s^{N,1}(J)$ in (6.6). Noting that $\phi_t^N(0) = \phi_t^N(N) = 0$, we have

$$\bar{b}_s^{N,1}(J) = qN^{-3/2} \sum_{x\in\Gamma} \phi_s^N(x)\{J''(x/N) + R_N(x)\},$$

with error terms $R_N(x)$ defined by

$$R_N(x) = N\{J'((x+1)/N) - J'(x/N)\} - J''(x/N)$$

which satisfies

$$\lim_{N\to\infty} \sup_x |R_N(x)| = 0.$$

Since Lemma 5.2(ii) implies

$$E^{\mu^{N,U}}[\phi(x)^2] \leqslant CN,$$

the error terms are negligible as $N \to \infty$ and we obtain the conclusion. □

Lemma 6.3.

$$\lim_{N\to\infty} E^{\mu^{N,U}}\left[\left|\int_0^t \{b_s^{N,2}(J) + \langle \alpha_+\Phi^{N,+}(s) - \alpha_-\Phi^{N,-}(s), J\rangle\}\,\mathrm{d}s\right|^2\right] = 0. \tag{6.7}$$

Proof. From condition (4.2) on U, we have

$$U'(z) = \alpha_+ z^+ - \alpha_- z^- + R(z)$$

with an error term $R(z)$ satisfying

$$|R(z)| \leqslant Cz^2, \quad C = \|U'''\|_\infty/2.$$

Hence, the expectation in (6.7) is rewritten as

$$E^{\mu^{N,U}}\left[\left|\int_0^t N^{-1/2}\sum_{x\in\Gamma} R(N^{-1}\phi_s^N(x))\bar{J}(x/N)\,\mathrm{d}s\right|^2\right].$$

By Schwarz's inequality and the stationarity of ϕ_s^N, this is bounded from above by

$$t^2\|J\|_\infty^2 C^2 \sum_{x\in\Gamma} N^{-4} E^{\mu^{N,U}}[\phi(x)^4]$$

which tends to 0 as $N \to \infty$; use Lemma 5.2(ii) with $p = 4$. □

Proposition 6.1, Lemmas 6.2 and 6.3 conclude the proof of Theorem 4.1 by a standard argument (cf. Kipnis and Landim, 1999, Chapter 11).

7. Fluctuations of interfaces away from the wall

So far we have studied the case with 0-boundary conditions. In this case, as we have seen, the macroscopic interface is attached to the wall in the sense that $h^N(t,\theta)$

converges to 0 as $N \to \infty$, and the fluctuation is non-Gaussian. This section discusses the case where the macroscopic interface stays away from the wall. Contrastively in such case, the limit of the fluctuation field is Gaussian.

For the SDE (1.1), (1.2), we impose the boundary conditions

$$\phi_t(0) = aN, \quad \phi_t(N) = \bar{a}N \tag{7.1}$$

with $a, \bar{a} > 0$ in place of (1.3). Let $\mu \equiv \mu^N_{aN,\bar{a}N}$ be the probability measure on $\mathbb{R}^\Gamma$ defined by the formula (1.5) with boundary conditions $\phi(0) = aN$ and $\phi(N) = \bar{a}N$, and let $\mu^+ \equiv \mu^{N,+}_{aN,\bar{a}N}$ be its conditional probability defined by (1.6) with μ^N replaced with $\mu^N_{aN,\bar{a}N}$. Then, $\mu^{N,+}_{aN,\bar{a}N}$ is stationary for the SDE (1.1), (1.2) with (7.1). Let $\phi_t = \{\phi_t(x); x \in \Gamma\}$ be its stationary solution and we define the macroscopic height variable $h^N(t,\theta)$ by the formula (1.7). Then, one can show that $h^N(t,\theta)$ converges as $N \to \infty$ to the linear profile $h(\theta)$ defined by

$$h(\theta) := a + (\bar{a} - a)\theta, \quad \theta \in [0,1]. \tag{7.2}$$

Let $\Phi^N(t,\theta)$ be the fluctuation field of $h^N(t,\theta)$ around the limit $h(\theta)$:

$$\Phi^N(t,\theta) := \sqrt{N}(h^N(t,\theta) - h(\theta)), \quad \theta \in [0,1]. \tag{7.3}$$

We determine the positive constant q_u for $u \in \mathbb{R}$ by

$$q_u^{-1} = \langle (\eta - \langle \eta \rangle_{\nu_u})^2 \rangle_{\nu_u},$$

where ν_u is the probability measure on $\mathbb{R}$ defined in Section 5.4.

Theorem 7.1. *As $N \to \infty$, the fluctuation field $\Phi^N(t,\theta)$ weakly converges to $\Phi(t,\theta)$ in $C([0,T], H^{-\alpha}([0,1])) \cap L^2_w(Q_T)$, $Q_T = [0,T] \times [0,1]$ for $\alpha > \frac{1}{2}$. The limit $\Phi(t,\theta)$ is a unique weak stationary solution of the SPDE*:

$$\begin{aligned} &\frac{\partial \Phi}{\partial t}(t,\theta) = q_{\bar{a}-a} \frac{\partial^2 \Phi}{\partial \theta^2}(t,\theta) + \sqrt{2}\dot{B}(t,\theta), \quad \theta \in [0,1], \\ &\Phi(t,0) = \Phi(t,1) = 0. \end{aligned} \tag{7.4}$$

The proof goes quite similarly to that of Theorem 1.1. We only outline it in the following.

Step 1 (cf. Proposition 3.1): Denote by $\hat{\mu}^{N,+}_{aN,\bar{a}N}$ the distribution of

$$\Phi^N(\theta) := \sqrt{N}(h^N(\theta) - h(\theta)) = \sqrt{N}\left\{\frac{1}{N}\sum_{x \in \Gamma} \phi(x) 1_{[x/N - 1/2N, x/N + 1/2N)}(\theta) - h(\theta)\right\}$$

with $\{\phi(x); x \in \Gamma\}$ distributed under $\mu^{N,+}_{aN,\bar{a}N}$. Then, one can show that

$$\hat{\mu}^{N,+}_{aN,\bar{a}N} \Rightarrow \mu_{0,0} \quad (N \to \infty),$$

weakly on $L_w([0,1])$, where $\mu_{0,0}$ denotes the distribution of pinned Brownian motion multiplied by $q_{\bar{a}-a}^{-1/2}$ starting at 0 and reaching 0.

Step 2 (cf. Theorem 4.1): Let us consider the SDE (4.1) with the boundary conditions (7.1) instead of the 0-boundary conditions having U which satisfies (4.2) and

$U(z) = 0$ for $z \geqslant 0$. Then the limit of the fluctuation field $\Phi^N(t,\theta)$ defined from the solution of this SDE satisfies the SPDE (7.4). In fact, under the present situation, $\alpha_\pm$ should be replaced with $U''(h(\theta))$ which is equal to 0, since $h(\theta) > 0$. The diffusion coefficient $q_{\bar{a}-a}$ is obtained through the Boltzmann–Gibbs principle (cf. Theorem 5.1), which tells that $V'(\eta_t^N(x))$ can be replaced with the equilibrium average

$$\langle V'\rangle(\overline{(\eta_t^N)}^K(x))$$

with an error $o(1/\sqrt{N})$ as $N \to \infty$ and $K \to \infty$ under the space–time average, where we denote $\langle\cdot\rangle(u) = \langle\cdot\rangle_u$ and

$$\overline{(\eta_t^N)}^K(x) := \frac{1}{2K+1} \sum_{y:|y-x|\leqslant K} \eta_t^N(y).$$

By the Taylor's expansion, this quantity can be further approximated by

$$\langle V'\rangle_{\bar{a}-a} + \frac{\mathrm{d}}{\mathrm{d}u}\langle V'\rangle_u\bigg|_{u=\bar{a}-a} \overline{(\eta_t^N)}^K(x).$$

Noting the identity

$$\frac{\mathrm{d}}{\mathrm{d}u}\langle V'\rangle_u = q_u,$$

the SPDE (7.4) can be derived.

Step 3: Step 2 gives the dynamic lower bound for the fluctuation field defined by the SDE (1.1), (1.2) with (7.1) and the proof of Theorem 7.1 can be completed similarly to Section 4.3.

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