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Invariant measures for passive tracer dynamics in Ornstein–Uhlenbeck flows ☆

Tomasz Komorowski[a,*], Stefano Olla[b]

[a] Institute of Mathematics, PAN, Warsaw, Institute of Mathematics, UMCS, Lublin PL. Marii Curie-Skłodowskiej 1., 20-031 Lublin, Poland

[b] Ceremade, UMR CNRS 7534, Université de Paris IX-Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris Cedex 16, France

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Abstract

Let $\mathbf{V}(t,\mathbf{x})$, $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d$ be a time–space stationary d-dimensional Markovian and Gaussian random field given over a probability space $\mathscr{T}_0 := (\Omega,\mathscr{V},\mathbb{P})$. Consider a diffusion with a random drift given by the stochastic differential equation $\mathrm{d}\mathbf{x}(t) = \mathbf{V}(t,\mathbf{x}(t))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}\mathbf{w}(t)$, $\mathbf{x}(0) = \mathbf{0}$, where $\mathbf{w}(\cdot)$ is a standard d-dimensional Brownian motion defined over another probability space $\mathscr{T}_1 := (\Sigma,\mathscr{W},\mathbb{W})$. The so-called Lagrangian process, i.e. the process describing the velocity at the position of the moving particle, $\eta(t) := \mathbf{V}(t,\mathbf{x}(t))$, $t\geqslant 0$ is considered over the product probability space $\mathscr{T}_0\otimes\mathscr{T}_1$. It is well known, see e.g. (Lumley, Méchanique de la Turbulence. Coll. Int du CNRS á Marseille. Ed. du CNRS, Paris; Port and Stone, J. Appl. Probab. 13 (1976) 499), that $\eta(\cdot)$ is stationary when the realizations of the drift are incompressible. We consider the case of fields with compressible realizations and show that there exists a probability measure, absolutely continuous with respect to $\mathbb{P}\otimes\mathbb{W}$, under which the Lagrangian process is stationary, provided that the velocity field $\mathbf{V}$ decorrelates sufficiently fast in time. Our result includes also the case $\kappa = 0$, i.e. motions in a random field.

We prove that in the case of positive molecular diffusivity κ the absolutely continuous invariant measure is unique and in fact is equivalent to $\mathbb{P}\otimes\mathbb{W}$. We formulate sufficient conditions on the spectrum of $\mathbf{V}$ that allow to claim ergodicity of the invariant measure in the case of random motions ($\kappa = 0$).

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* Corresponding author. Tel.: +48-81-743-1369; fax: +48-81-537-5102.

E-mail addresses: komorow@golem.umcs.lublin.pl (T. Komorowski), olla@ceremade.dauphine.fr (S. Olla).

URLs: http://golem.umcs.lublin.pl/~komorow, http://www.ceremade.dauphine.fr/~olla

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1. Introduction

Turbulent transport of a passive tracer is often modeled by a stochastic differential equation with a random drift

$$\mathrm{d}\mathbf{x}(t) = \mathbf{V}(t,\mathbf{x}(t))\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}\mathbf{w}(t), \qquad t \geqslant 0,$$
$$\mathbf{x}(0) = \mathbf{0}. \tag{1.1}$$

$\mathbf{V}: \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is assumed to be a d-dimensional, time-space stationary, random field over a certain probability space $\mathscr{T}_0 := (\Omega, \mathscr{V}, \mathbb{P})$ and $\mathbf{w}(\cdot)$ is a standard d-dimensional Brownian motion, given over another probability space $\mathscr{T}_1 := (\Sigma, \mathscr{W}, \mathbb{W})$. The tracer particle trajectory $\mathbf{x}(\cdot)$ is considered as a stochastic process over the probability space $\mathscr{T}_0 \otimes \mathscr{T}_1 := (\Omega \times \Sigma, \mathscr{V} \otimes \mathscr{W}, \mathbb{P} \otimes \mathbb{W})$. The parameter $\kappa \geqslant 0$, also called *the molecular diffusivity*, models the strength of the intrinsic diffusive dispersion of the medium. In the special case when $\kappa = 0$ the motion of the tracer is described by an ordinary differential equation.

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{V}(t,\mathbf{x}(t)), \qquad t \geqslant 0,$$
$$\mathbf{x}(0) = \mathbf{0}, \tag{1.2}$$

and, under suitable assumptions guaranteeing the existence and uniqueness of solutions of (1.2), the trajectory process is defined over $\mathscr{T}_0$.

One of the central questions appearing in the asymptotic analysis of the turbulent transport is *the stationarity of the Lagrangian velocity process*

$$\eta_t := \mathbf{V}(t,\mathbf{x}(t)), \qquad t \geqslant 0. \tag{1.3}$$

This issue is crucial for an application of the ergodic theory tools in homogenization that leads further to establishing the law of large numbers, or central limit theorem for the tracer trajectory.

Only in few particular cases the existence of an invariant measure for the Lagrangian velocity is proven. For example, when $\mathbf{V}$ is incompressible, i.e. $\nabla_{\mathbf{x}} \cdot \mathbf{V}(t,\mathbf{x}) := \sum_{i=1}^d \partial_{x_i} V_i(t,\mathbf{x}) \equiv 0$, process $(\eta_t)_{t\geqslant 0}$ is stationary under $\mathbb{P} \otimes \mathbb{W}$, see Lumley (1962) and Port and Stone (1976).

In the case when $\mathbf{V}$ is a gradient of a stationary and steady (time independent) potential and $\kappa > 0$, the corresponding Gibbs measure gives rise to an ergodic, invariant measure on $(\Omega \times \Sigma, \mathscr{V} \otimes \mathscr{W})$, see Olla (1994). This measure is absolutely continuous w.r.t. (in fact equivalent to) $\mathbb{P} \otimes \mathbb{W}$.

There are also some other special instances when the invariant measure for $(\eta_t)_{t\geqslant 0}$ can be constructed, mainly by reducing the problem to the case when the phase space of $\mathbf{V}(t,\cdot)$, $t \geqslant 0$ is of finite dimension. This is, for example, the case for spatially periodic velocity fields. For a review of the existing literature on the subject a reader is referred to Zirbel (2001).

Recently, some general results concerning the existence of absolutely continuous invariant measures for fields decorrelating fast in time, or space have been established (Komorowski, 2002; Komorowski and Krupa, 2002a; Komorowski and Krupa, 2002b).

In particular, in Komorowski (2002) it has been shown that if the molecular diffusivity is strictly positive, $\mathbf{V}$ is a centered Gaussian field that decorrelates at finite time and satisfies some additional regularity properties then there exists a measure $\mathbb{P}_*$ on $(\Omega \times \Sigma, \mathscr{V} \otimes \mathscr{W})$ that is equivalent to $\mathbb{P} \otimes \mathbb{W}$ and the Lagrangian process $(\eta_t)_{t \geqslant 0}$ is stationary and ergodic under $\mathbb{P}_*$. Such a measure has been called a *regular*, *invariant measure*. The aforementioned result has been obtained using a certain factoring property of the field that allows to decompose it into the part that is determined by the past, up to a certain moment of time, and independent of it the "renewal part". This factoring lead to a definition of a linear operator preserving densities w.r.t. $\mathbb{P}$. The key observation made was the existence of an invariant density for this operator. This density was used to construct $\mathbb{P}_*$. An analogous result can be also obtained by this method in the non-gaussian case, see Komorowski and Krupa (2002a).

In the present paper we shall consider Gaussian, Markovian fields that possess a spectral gap property. More specifically, suppose that:

(V1) $\mathbf{V} : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a zero mean, Gaussian field over the probability space $\mathscr{T}_0$

(V2) the co-variance matrix of the field is given by

$$\begin{aligned} \mathbf{R}(t-s, \mathbf{x}-\mathbf{y}) &:= \mathbb{E}[\mathbf{V}(t,\mathbf{x}) \otimes \mathbf{V}(s,\mathbf{y})] \\ &= \int_{\mathbb{R}^d} \cos((\mathbf{x}-\mathbf{y}) \cdot \mathbf{k})\, \mathrm{e}^{-r(\mathbf{k})|t-s|} \hat{\Gamma}(\mathbf{k})\, \mathrm{d}\mathbf{k}, \\ & \qquad (t,\mathbf{x}),(s,\mathbf{y}) \in \mathbb{R} \times \mathbb{R}^d. \end{aligned} \tag{1.4}$$

Here $\mathbb{E}$ is the expectation operator corresponding to measure $\mathbb{P}$. $\hat{\Gamma}(\cdot)$ is a certain Borel measurable function taking values in the space $\mathscr{S}_+(d)$ consisting of all $d \times d$, real, symmetric, positive matrices. We assume that it is even (i.e. $\Gamma(-\mathbf{k}) = \Gamma(\mathbf{k})$, $\mathbf{k} \in \mathbb{R}^d$) and

$$\int_{\mathbb{R}^d} (1 + |\mathbf{k}|^2)^m\, \hat{\Gamma}(\mathbf{k})\, \mathrm{d}\mathbf{k} < +\infty, \qquad \forall m \geqslant 0. \tag{1.5}$$

(V3) The function $r : \mathbb{R}^d \to [0, +\infty)$ is continuous, even and satisfies $r(\cdot) \geqslant a$ for some $a > 0$.

It is well known that, thanks to (1.5), such a random field possesses a modification that is $\mathbb{P}$ a.s. jointly continuous in $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ and C^∞ smooth in $\mathbf{x}$ for any fixed $t \in \mathbb{R}^d$.

We shall denote

$$V_* := \{\mathbb{E}[|\mathbf{V}(0,\mathbf{0})|^2 + |\nabla_{\mathbf{x}} \mathbf{V}(0,\mathbf{0})|^2]\}^{1/2} = \left[\int_{\mathbb{R}^d} (1 + |\mathbf{k}|^2)\, \mathrm{tr}\, \hat{\Gamma}(\mathbf{k})\, \mathrm{d}\mathbf{k}\right]^{1/2}. \tag{1.6}$$

Our main objectives are the following results.

Theorem 1.1. *Let* $\mathbf{V}$ *be a field satisfying* (V1)–(V3) *and* $\kappa > 0$. *Then, there is a constant* $C > 0$ *depending only on* V_* *and such that for any* $a \geqslant C$ *there exists a measure* $\mathbb{P}_*$ *defined on the measurable space* $(\Omega \times \Sigma, \mathscr{V} \otimes \mathscr{W})$ *that is equivalent to*

$\mathbb{P} \otimes \mathbb{W}$. *In addition the Lagrangian process* $(\eta_t)_{t \geqslant 0}$ *is stationary and ergodic under* $\mathbb{P}_*$ (*i.e., using our terminology*, $\mathbb{P}_*$ *is regular*).

Theorem 1.2. *Suppose that* $\mathbf{V}$ *satisfies* (V1)–(V3) *and* $\kappa = 0$. *Then, there is a constant* $C > 0$ *depending only on* V_* *and such that for any* $a \geqslant C$ *there exists a measure* $\mathbb{Q}_*$ *defined on the measurable space* $(\Omega, \mathscr{V})$ *that is absolutely continuous with respect to* $\mathbb{P}$ *and such that the Lagrangian process* $(\eta_t)_{t \geqslant 0}$ *is stationary under* $\mathbb{Q}_*$.

Remark 1.3. At the expense of further complication of the notation we could generalize our results to cover the case of stationary Gaussian fields with the covariance matrix given by

$$\mathbf{R}(t-s, \mathbf{x}-\mathbf{y}) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(\mathbf{x}-\mathbf{y}) \cdot \mathbf{k}} \, \mathrm{e}^{-r(\mathbf{k})|t-s|} \hat{\Gamma}(\mathrm{d}\mathbf{k}), \qquad (t, \mathbf{x}), (s, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.7}$$

where $\hat{\Gamma}(\cdot)$ is a complex hermitian matrix valued Borel measure that is no longer invariant under the reflection $\mathbf{k} \mapsto -\mathbf{k}$ but satisfies $\hat{\Gamma}(-\mathrm{d}\mathbf{k}) = \hat{\Gamma}^*(\mathrm{d}\mathbf{k})$ (because the field $\mathbf{V}$ is real valued) and (1.5). Here * denotes the complex matrix conjugation.

Ergodicity of the invariant measure in case of vanishing molecular diffusivity is a delicate matter, that requires further investigations. In the present paper we consider only the fields with an ultraviolet cut-off on their spatial spectrum. Then, we are able to prove the following.

Theorem 1.4. *Suppose that* $\kappa = 0$ *and the field* $\mathbf{V}$ *satisfies, besides the assumptions* (V1)–(V3), *also the following condition*:

(A) *the spatial structure measure of* $\mathbf{V}(0, \cdot)$ *is of compact support, i.e. there exists* $K > 0$ *such that*

$$\operatorname{supp} \hat{\Gamma}(\cdot) \subseteq B_K(\mathbf{0}),$$

where $B_r(\mathbf{x})$ *denotes the ball of radius* $r > 0$ *centered at* $\mathbf{x}$ *in the Euclidean space* $\mathbb{R}^d$.

Then, we can additionally claim the ergodicity of the invariant measure $\mathbb{Q}_*$ *whose existence is stated in Theorem* 1.2.

One significant, in our view, aspect of this and preceding theorems is the fact that they admit the case of motions in a random field, i.e. when $\kappa = 0$. It is, according to our knowledge, one of the very few existing results concerning the stationarity of the Lagrangian velocity process for random motions.

The method of the proof of the above results differs substantially from the approach taken in Komorowski (2000), Komorowski and Krupa (2002a) and Komorowski and Krupa (2002b). In the present paper we consider an infinite dimensional, time stationary Ornstein–Uhlenbeck process $(V(t, \cdot))_{t \geqslant 0}$ whose state space $\mathbb{H}^m_\rho$, see Section 2.1 for its definition, is an appropriate functional space containing all spatial realizations of the given velocity field. The Lagrangian process introduced in (1.3) can be then identified, see (3.1), with a functional of a certain $\mathbb{H}_\rho^m$-valued Markov process $(\mathscr{Z}_t)_{t\geqslant 0}$. This type of a process, called also *an environment*, or *Lagrangian canonical process*, is frequently used in the homogenization theory of random media, see e.g. Olla (1994, 2000).

In what follows we show the existence of an invariant measure for this infinite dimensional process, see Theorems 3.2 and 3.8 below. These results, as we demonstrate in Section 3, imply the conclusions of Theorems 1.1 and 1.2.

To prove Theorems 3.2 and 3.8, we consider the equation for an invariant density formulated via the formal adjoint to the L^2-generator of the process. This equation is interpreted in terms of finite dimensional approximations. The invariant density for the finite dimensional problem, when $\kappa > 0$, can be obtained by a quite simple perturbative argument. In addition, we notice, see (4.28), that the L^2-estimates of the invariant densities do not depend on the molecular diffusivity hence the result extends also to the case of $\kappa = 0$. Since we are able to control the L^2 norms of the densities of the finite dimensional approximations, see Proposition 4.3, we can conclude tightness of the family of such measures, see Section 4.2 below. In addition, any limiting measure of this family is absolutely continuous w.r.t. the law of $\mathbf{V}(0,\cdot)$.

In Section 5.1 we present a simple argument, which shows that for $\kappa > 0$ the invariant measure is in fact equivalent to the law of $\mathbf{V}(0,\cdot)$ thus, gives rise to a probability measure $\mathbb{P}_*$ on $(\Omega \times \Sigma, \mathscr{V} \otimes \mathscr{W})$ that is equivalent to $\mathbb{P} \otimes \mathbb{W}$. In addition, it is ergodic under the dynamics of the Markovian process $(\mathscr{Z}_t)_{t\geqslant 0}$. This fact implies also the uniqueness of a stationary measure for the Lagrangian process in the class of measures absolutely continuous w.r.t. $\mathbb{P} \otimes \mathbb{W}$. Section 5.2 is devoted to the proof of Theorem 1.4.

2. Preliminaries on Ornstein–Uhlenbeck processes

2.1. Homogeneous Gaussian measures on Hilbert spaces

To give an appropriate functional setting we introduce $\mathbb{H}_\rho^m$—the Hilbert space of d-dimensional vector fields that is the completion of $\mathscr{S}_d := \mathscr{S}(\mathbb{R}^d;\mathbb{R}^d)$ with respect to the norm

$$\|\varphi\|^2_{\mathbb{H}_\rho^m} := \int_{\mathbb{R}^d} (|\varphi(\mathbf{x})|^2 + |\nabla_{\mathbf{x}}\varphi(\mathbf{x})|^2 + \cdots + |\nabla_{\mathbf{x}}^m\varphi(\mathbf{x})|^2)\vartheta_\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

for any positive integer m, $\varphi \in \mathscr{S}_d$ and the weight function $\vartheta_\rho(\mathbf{x}) := (1 + |\mathbf{x}|^2)^{-\rho}$, where $\rho > d/2$. We shall also assume that $m > d/2 + 1$ so any $\varphi \in \mathbb{H}_\rho^m$ is of C^1 class of regularity. In the particular cases when $m = 0$, or $\rho = 0$, $m = 0$ we shall write $\mathbb{L}_\rho^2$, $\mathbb{L}^2$ instead of the respective $\mathbb{H}_\rho^0$, or $\mathbb{H}_0^0$ spaces. We shall also denote by $C_b(\mathbb{H}_\rho^m)$ the space of all bounded and continuous functions on $\mathbb{H}_\rho^m$.

On $\mathbb{H}_\rho^m$ we have a group of transformations $\tau_{\mathbf{x}} : \mathbb{H}_\rho^m \to \mathbb{H}_\rho^m$, given by $\tau_{\mathbf{x}}\varphi(\cdot) := \varphi(\cdot + \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$. Let μ be a Gaussian, spatially homogeneous measure of zero mean and with the covariance given by

$$\int_{\mathbb{H}_\rho^m} \langle \varphi_1, \varphi \rangle \langle \varphi_2, \varphi \rangle \mu(\mathrm{d}\varphi) = \int_{\mathbb{R}^d} \hat{\Gamma}(\mathbf{k}) \hat{\varphi}_1(\mathbf{k}) \cdot \hat{\varphi}_2(\mathbf{k}) \, \mathrm{d}\mathbf{k} \quad \forall \varphi_1, \varphi_2 \in \mathscr{S}_d.$$

Here $\hat{\Gamma}(\cdot)$ is an $\mathscr{S}_+(d)$-valued function satisfying (1.5) and

$$\langle \psi, \varphi \rangle := \int_{\mathbb{R}^d} \psi(\mathbf{x}) \cdot \varphi(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \psi \in \mathscr{S}_d, \ \varphi \in \mathbb{H}_\rho^m.$$

$\hat{\varphi}$ denotes the Fourier transform of a given function φ.

We denote by $\mathscr{T}_2$ the probability triple $(\mathbb{H}_\rho^m, \mathscr{B}(\mathbb{H}_\rho^m), \mu)$ and we shorthand $L^p := L^p(\mathscr{T}_2)$, $1 \leqslant p \leqslant +\infty$. Let also L_d^2 be the space of all d-dimensional random vectors with L^2 integrable components.

$S_0(\mathbf{x}; \varphi) := \varphi(\mathbf{x})$, $(\mathbf{x}, \varphi) \in \mathbb{R}^d \times \mathbb{H}_\rho^m$ defines a Gaussian homogeneous random over $\mathscr{T}_2$. According to the Spectral Theorem for homogeneous random fields, see e.g. Adler (1981, Theorem 2.4.1, p. 30), we can find a spectral measure $\hat{S}_0$, i.e. Borel L_d^2-valued measure with orthogonal increments such that

$$S_0(\mathbf{x}) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\mathbf{x} \cdot \mathbf{k}} \hat{S}_0(\mathrm{d}\mathbf{k}). \tag{2.1}$$

S_0 is a real valued random field, therefore we have

$$\hat{S}_0(-A) = \hat{S}_0^*(A), \quad A \in \mathscr{B}(\mathbb{R}^d). \tag{2.2}$$

Here * denotes the complex conjugate. Also, because S_0 is Gaussian, the family of r.v.-s: $\operatorname{Re} \hat{S}_0(A_1), \ldots, \operatorname{Re} \hat{S}_0(A_n)$, $\operatorname{Im} \hat{S}_0(A_1), \ldots, \operatorname{Im} \hat{S}_0(A_n)$ is jointly Gaussian for any $A_1, \ldots, A_n \in \mathscr{B}(\mathbb{R}^d)$.

We define a strongly continuous group of isometries $U^{\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^d$ on L^p, $p \in [1, +\infty)$ given by $U^{\mathbf{x}} F = F \circ \tau_{\mathbf{x}}$ and set

$$D_j F := \partial_{x_j | \mathbf{x} = \mathbf{0}} U^{\mathbf{x}} F, \quad j = 1, \ldots, d \tag{2.3}$$

for $F \in L^p$, such that the partial derivative on the right hand side of (2.3) exists in the L^p sense. For any $m \geqslant 1$, $p \in [1, +\infty]$ we let $W^{m,p}$ be the space consisting of F, such that $D_1^{i_1} \cdots D_d^{i_d} F$ exist in the L^p space, when $i_1 + \cdots + i_d \leqslant n$. It is equipped with the norm $\|F\|_{m,p}^p := \sum_{i_1 + \cdots + i_d \leqslant m} \|D_1^{i_1} \cdots D_d^{i_d} F\|_{L^p}^p$. Here $\|\cdot\|_{L^p}$ denotes the respective L^p norm. In case when $F = (F_1, \ldots, F_d)$ is a random vector we define also $\|F\|_{L_d^p}^p := \sum_{i=1}^d \|F_i\|_{L^p}^p$. We shall denote $\nabla F := (D_1 F, \ldots, D_d F)$ for $F \in W^{1,2}$, and $\Delta F := \sum_{i=1}^d D_i^2 F$ for $F \in W^{2,2}$.

Let $\mathscr{C}_b^m$ denote the space of those elements $F \in L^2$ such that $\mathbf{x} \mapsto F(\tau_{\mathbf{x}}(\varphi))$ is of class $C^2(\mathbb{R}^d)$ with m derivatives bounded by a deterministic constant for μ-a.s. φ.

Let

$$\mathscr{R} := \left\{ \varphi \in \mathbb{H}_\rho^m \colon \sup_{\mathbf{x} \in \mathbb{R}^d} |\varphi(\mathbf{x})| (1 + |\mathbf{x}|)^{-1} < +\infty, \varphi \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \right\}. \tag{2.4}$$

It can be shown, see e.g. Adler (1990), that $\mu(\mathscr{R}) = 1$.

Let $\mathscr{P}_n$ be the L^2 closure of the linear space spanned by the monomials $\langle\varphi_1,\cdot\rangle\cdots\langle\varphi_m,\cdot\rangle$, where $m\leqslant n$ and $\varphi_1,\ldots,\varphi_m\in\mathscr{S}_d$. Let $\mathscr{P}:=\bigcup_{n\geqslant 0}\mathscr{P}_n$ and $H_n:=\mathscr{P}_n\ominus\mathscr{P}_{n-1}$ be the space of nth degree Hermite polynomials. We denote by $\mathfrak{P}_n$ the orthogonal projection of L^2 onto H_n.

Let

$$\mathscr{D}:=[\varphi\ :\ \hat{\varphi}(\cdot)\in C_0(\mathbb{R}^d;\mathbb{R}^d)].$$

For any $\varphi_0\in\mathscr{D}$ there exists a sequence $(\varphi_n)_{n\geqslant 1}\subseteq\mathscr{S}_d$ whose Fourier transforms converge to $\hat{\varphi}_0$ as $n\to+\infty$ uniformly on any compact set. A direct calculation shows that then the sequence of random variables $\langle\varphi_n,\cdot\rangle$ converges in the L^2 sense to an element that we denote by $\langle\varphi_0,\cdot\rangle$.

We denote by $\mathscr{P}^{\mathrm{reg}}$ the space of all regular polynomials over $\mathbb{H}^m_\rho$, i.e.

$$\mathscr{P}^{\mathrm{reg}}:=\mathrm{span}[F:F(\cdot):=\langle\varphi_1,\cdot\rangle\cdots\langle\varphi_l,\cdot\rangle,$$

$$\text{for some positive integer } l\geqslant 1 \text{ and } \varphi_1,\ldots,\varphi_l\in\mathscr{D}].$$

Using Theorem 2.11 of Janson (1997) one can easily show that $\mathscr{P}^{\mathrm{reg}}$ is dense in L^p, $\forall p\in[1,+\infty)$.

2.2. Markovian dynamics

Let $\mathfrak{C}:=C([0,+\infty);\mathbb{H}^m_\rho)$. Let us also denote

$$V(t;\omega):=\mathbf{V}(t,\cdot;\omega),\qquad t\geqslant 0 \tag{2.5}$$

an $\mathbb{H}^m_\rho$-valued stochastic continuous trajectory process. With no loss of generality we may assume that $\Omega=\mathfrak{C}$ and $\mathbb{P}$ is the law of $V(\cdot)$ in $\mathfrak{C}$.

In this section we construct a C_0-continuous, Markovian L^2-semigroup $(P^t)_{t\geqslant 0}$ such that

$$\mathbb{E}[F(V(t+h))|\mathscr{V}_t]=P^hF(V(t)),\qquad \forall t,h\geqslant 0, \tag{2.6}$$

where F is a bounded, measurable function on $\mathbb{H}^m_\rho$ and $(\mathscr{V}_t)_{t\geqslant 0}$ is the natural filtration corresponding to $V(\cdot)$.

Let $W:[0,+\infty)\times\mathbb{R}^d\times\mathfrak{C}\to\mathbb{R}^d$ be a Gaussian random field over a certain probability space $\mathscr{T}_W:=(\mathfrak{C},\mathscr{B}(\mathfrak{C}),\mathbb{P}_W)$ whose covariance matrix equals

$$\begin{aligned}&\mathbb{E}_W[W(t,\mathbf{x})\otimes W(s,\mathbf{y})]\\&\quad=\int_{\mathbb{R}^d}\cos((\mathbf{x}-\mathbf{y})\cdot\mathbf{k})\mathrm{e}^{-r(\mathbf{k})|t-s|}[1-\mathrm{e}^{-r(\mathbf{k})(t\wedge s)}]\hat{\Gamma}(\mathbf{k})\,\mathrm{d}\mathbf{k}\end{aligned} \tag{2.7}$$

for all $(t,\mathbf{x}),(s,\mathbf{y})\in[0,+\infty)\times\mathbb{R}^d$. Here $\mathbb{E}_W$ is the expectation corresponding to $\mathbb{P}_W$. It is elementary to verify that the realizations of $W(\cdot,\cdot)$ are continuous in $(t,\mathbf{x})$ and C^∞-regular in $\mathbf{x}$ for any fixed t, $\mathbb{P}_W$-a.s. We define also $W(t):=W(t,\cdot)$, $t\geqslant 0$ an $\mathbb{H}^m_\rho$-valued continuous trajectory process.

Let also $S:[0,+\infty)\times\mathbb{R}^d\times\mathbb{H}^m_\rho\to\mathbb{R}^d$ be a Gaussian random field defined over $\mathcal{T}_2$ that is a continuous trajectory modification of

$$S(t,\mathbf{x}):=\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{k}}\mathrm{e}^{-r(\mathbf{k})t}\hat{S}_0(\mathrm{d}\mathbf{k}).$$

Such a field is spatially stationary and its covariance matrix equals

$$\int[S(t,\mathbf{x};\varphi)\otimes S(s,\mathbf{y};\varphi)]\mu(\mathrm{d}\varphi)$$
$$=\int_{\mathbb{R}^d}\cos((\mathbf{x}-\mathbf{y})\cdot\mathbf{k})\mathrm{e}^{-r(\mathbf{k})(t+s)}\hat{\Gamma}(\mathbf{k})\,\mathrm{d}\mathbf{k},\quad (t,\mathbf{x}),(s,\mathbf{y})\in[0,+\infty)\times\mathbb{R}^d.$$

Let $S(\cdot)$ denote the corresponding $\mathbb{H}^m_\rho$-valued stochastic process over $\mathcal{T}_2$.

The $\mathbb{H}^m_\rho$-valued stochastic process

$$V(t;\omega,\varphi)=S(t;\varphi)+W(t;\omega),\quad t\geqslant 0 \tag{2.8}$$

defined over $\mathcal{T}_W\otimes\mathcal{T}_2$ has the same law in $\mathfrak{C}$ as the process given by (2.5). Denote by $\mathbb{P}_\mu$, $\mathbb{E}_\mu$ the product measure of $\mathcal{T}_W\otimes\mathcal{T}_2$ and its respective expectation.

Suppose that

$$F(\cdot):=\langle\varphi_1,\cdot\rangle\cdots\langle\varphi_l,\cdot\rangle, \tag{2.9}$$

for some $l\geqslant 1$ and $\varphi_1,\dots,\varphi_l\in\mathcal{S}_d$. Let

$$P^tF(\varphi):=\mathbb{E}_W[\langle\varphi_1,V(t;\cdot,\varphi)\rangle\cdots\langle\varphi_l,V(t;\cdot,\varphi)\rangle],\quad\forall\varphi\in\mathbb{H}^m_\rho \tag{2.10}$$

for any $t\geqslant 0$. Suppose now that $\varphi_1,\dots,\varphi_l\in\mathcal{D}$. Let $(\psi_i^{(n)})_{n\geqslant 1}\subseteq\mathcal{S}_d$, $i=1,\dots,l$ be such that their Fourier transforms converge to the Fourier of the respective φ_i uniformly on compact sets. It can be shown, by a direct calculation, that for any fixed i the r.v.-s $\langle\psi_i^{(n)},V(t)\rangle$ converge as $n\to+\infty$ in any $L^p(\mathbb{P}_\mu)$ norm, $p\in[1,+\infty)$ to a certain r.v.-s that we denote $\langle\varphi_i,V(t)\rangle$. Using (2.10) we can extend therefore the definition of P^t to a linear operator on $\mathcal{P}^{\mathrm{reg}}$. The following result holds.

Proposition 2.1.

(1)

$$\int P^tFG\,\mathrm{d}\mu=\int FP^tG\,\mathrm{d}\mu,\quad\forall F,G\in\mathcal{P}^{\mathrm{reg}},\ t\geqslant 0. \tag{2.11}$$

(2) (2.6) *holds for any* $F\in\mathcal{P}^{\mathrm{reg}}$.

(3)

$$\|P^tF\|_{L^2}\leqslant\|F\|_{L^2},\quad\forall F\in\mathcal{P}^{\mathrm{reg}},\quad t\geqslant 0. \tag{2.12}$$

(4) $P^t(\mathcal{P}^{\mathrm{reg}})\subseteq\mathcal{P}^{\mathrm{reg}}$ *for all* $t\geqslant 0$.

Proof. Parts (1) and (2) of the proposition are the results of straightforward calculations using elementary properties of a Gaussian measure so we omit them. To show part (3) observe first that for any F of form (2.9) an application of Cauchy–Schwartz inequality yields

$$[P^tF(\varphi)]^2\leqslant\mathbb{E}_W[\langle\varphi_1,V(t;\cdot,\varphi)\rangle\cdots\langle\varphi_l,V(t;\cdot,\varphi)\rangle]^2=(P^tF^2)(\varphi),\quad\forall\varphi\in\mathbb{H}^m_\rho$$

and (2.12) follows for such elements. The same argument can in fact be used also for any $F=\sum_{i=1}^{q}\alpha_i F_i$, with F_i as in (2.9) and $\alpha_i\in\mathbb{R}$, $i=1,\ldots,d$, hence (2.12) follows for all $F\in\mathscr{P}^{\rm reg}$.

Part (4) can be shown calculating directly P^tF for $F\in\mathscr{P}^{\rm reg}$ using formula (2.10) and the rules of computing the expectation of a multiple product of normal variables. □

From the above proposition we can easily conclude the following

Corollary 2.2. *P^t can be extended to a Markov operator on L^2 for any $t\geqslant 0$ (i.e. it is positivity preserving and $P^t\mathbf{1}=\mathbf{1}$) satisfying (2.6), (2.11) and (2.12). In addition, $(P^t)_{t\geqslant 0}$ form a C_0-semigroup of self-adjoint operators on L^2.*

A standard computation shows that the correlation coefficient

$$\mathrm{Corr}(F(V(t+h)),G(V(t)))\leqslant \mathrm{e}^{-ah},\qquad \forall t\in\mathbb{R}^d, h\geqslant 0$$

and $F,G\in L^2$. Theorem 10.1, p. 181 of Rozanov (1969) implies that

$$\|P^tF\|_{L^2}\leqslant \mathrm{e}^{-at}\|F\|_{L^2},\qquad \forall t>0 \tag{2.13}$$

for any $F\in L^2_0:=[F\in L^2: \int F\,\mathrm{d}\mu=0]$. Using (2.13) and (2.11) we easily conclude that μ is ergodic, i.e. if $P^tF=F$ for some $t>0$, then $F\in\mathrm{span}(\mathbf{1})$.

Denote by $\mathscr{M}:D(\mathscr{M})\to L^2$, $\mathscr{E}_{\mathscr{M}}:D(\mathscr{E}_{\mathscr{M}})\times D(\mathscr{E}_{\mathscr{M}})\to\mathbb{R}$ the generator and the Dirichlet form corresponding to $(P^t)_{t\geqslant 0}$.

Proposition 2.3. *We have $\mathscr{P}^{\rm reg}\subseteq D(\mathscr{M})$ is a core of the generator $\mathscr{M}$. In addition, if $\varphi_1,\ldots,\varphi_l\in\mathscr{D}$ and $F(\cdot)=\langle\varphi_1,\cdot\rangle\cdots\langle\varphi_l,\cdot\rangle$ is given by (2.9) we have*

$$\mathscr{M}F=\sum_{p=1}^{l}F_p+2\sum_{1\leqslant p<q\leqslant l}F_{p,q} \tag{2.14}$$

where $F_p(\cdot):=\langle\varphi_1,\cdot\rangle\cdots\langle A\varphi_p,\cdot\rangle\cdots\langle\varphi_l,\cdot\rangle$ and the Fourier transform of $A\varphi_p$ is given by $-r(\mathbf{k})\hat\varphi_p(\mathbf{k})$, $\mathbf{k}\in\mathbb{R}^d$. Additionally,

$$F_{p,q}(\cdot):=\langle\varphi_1,\cdot\rangle\cdots\widehat{\langle\varphi_p,\cdot\rangle}\cdots\widehat{\langle\varphi_q,\cdot\rangle}\cdots\langle\varphi_l,\cdot\rangle\mathfrak{Q}(\varphi_p,\varphi_q), \tag{2.15}$$

where $\widehat{\langle\cdot,\cdot\rangle}$ means that the respective term should be omitted in the product and

$$\mathfrak{Q}(\varphi_p,\varphi_q)=\int_{\mathbb{R}^d}r(\mathbf{k})\hat\Gamma(\mathbf{k})\hat\varphi_p(\mathbf{k})\cdot\hat\varphi_q(\mathbf{k})\,\mathrm{d}\mathbf{k}.$$

Proof. The fact that $\mathscr{P}^{\rm reg}$ is a core of the generator can be seen from part (4) of Proposition 2.1 and Proposition 3.3 of Ethier and Kurtz (1986).

For any $\psi\in\mathscr{D}$ we define $S(t)\psi$ as an inverse Fourier transform of $\mathrm{e}^{-r(\mathbf{k})t}\hat\psi(\mathbf{k})$, $\mathbf{k}\in\mathbb{R}^d$. Since $\langle\varphi_i,S(t,\varphi)\rangle=\langle S(t)\varphi_i,\varphi\rangle$, μ-a.s. in φ we can write from (2.10)

$$P^tF(\varphi)-F(\varphi)=\langle S(t)\varphi_1,\varphi\rangle\cdots\langle S(t)\varphi_l,\varphi\rangle-\langle\varphi_1,\varphi\rangle\cdots\langle\varphi_l,\varphi\rangle$$
$$+2\sum_{1\leqslant p<q\leqslant l}F_{p,q}(t,\varphi)+\mathrm{o}(t). \tag{2.16}$$

Here the definition of $F_{p,q}(t,\cdot)$ differs from that of $F_{p,q}(\cdot)$ given in (2.15) only by replacing factor $\mathfrak{Q}$ by

$$\mathfrak{Q}(t,\varphi_p,\varphi_q)=\int_{\mathbb{R}^d}[1-\mathrm{e}^{-r(\mathbf{k})t}]\hat{\Gamma}(\mathbf{k})\hat{\varphi}_p(\mathbf{k})\cdot\hat{\varphi}_q(\mathbf{k})\,\mathrm{d}\mathbf{k}.$$

$\mathrm{o}(t)$ denotes a term such that $\mathrm{o}(t)/t\to 0$ as $t\to 0+$ in the L^2 sense. Dividing both sides of (2.16) by t and letting it tend to 0 we conclude (2.14). □

2.3. Finite dimensional approximation

In this section we construct a finite dimensional approximation of the field $\mathbf{V}(\cdot,\cdot)$ and present its basic properties. The results are presented mostly without proofs, which are contained in Komorowski (2001).

2.3.1. Approximation of a homogeneous Gaussian measure

For an arbitrary integer $N\geqslant 1$ let $\Lambda_N:=\{\mathbf{j}\in\mathbf{Z}^d:0<|\mathbf{j}|\leqslant N2^N\}$. Let

$$\Delta_{\mathbf{0}}^{(N)}:=\{\mathbf{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d:\ -2^{-N-1}\leqslant x_k<2^{-N-1}\}$$

and $\Delta_{\mathbf{j}}^{(N)}:=\mathbf{j}2^{-N}+\Delta_{\mathbf{0}}^{(N)}$, $\mathbf{j}\in\Lambda_N$. Let

$$X_{\mathbf{j}}^{(N)}(\varphi):=\mathrm{Re}\,\hat{S_0}(\Delta_{\mathbf{j}}^{(N)};\varphi),\qquad Y_{\mathbf{j}}^{(N)}(\varphi):=-\mathrm{Im}\,\hat{S_0}(\Delta_{\mathbf{j}}^{(N)};\varphi).$$

Thanks to (2.2) we have $X_{-\mathbf{j}}^{(N)}=X_{\mathbf{j}}^{(N)}$, $Y_{-\mathbf{j}}^{(N)}=-Y_{\mathbf{j}}^{(N)}$.

$X_{\mathbf{j}}^{(N)}$, $Y_{\mathbf{j}}^{(N)}$, $\mathbf{j}\in\Lambda_N^+$ are zero mean, independent Gaussian random vectors (after a suitable modification on a set of μ-zero measure) over probability space $\mathscr{T}_2$. Here Λ_N^+ is the subset of Λ_N consisting of those $\mathbf{j}=(j_1,\ldots,j_d)$ whose last non-vanishing component is positive. The covariance matrix of each vector $X_{\mathbf{j}}^{(N)}$, or $Y_{\mathbf{j}}^{(N)}$ equals

$$S_{\mathbf{j}}^{(N)}:=\int_{\Delta_{\mathbf{j}}^{(N)}}\hat{\Gamma}(\mathbf{k})\,\mathrm{d}\mathbf{k},\qquad \mathbf{j}\in\Lambda_N^+. \tag{2.17}$$

Let Υ_N denote the cardinality of Λ_N^+. Set $\pi_N:\mathbb{H}_\rho^m\to(\mathbb{R}^d)^{2\Upsilon_N}$ by

$$\pi_N(\varphi):=(X_{\mathbf{j}}^{(N)}(\varphi),\,Y_{\mathbf{j}}^{(N)}(\varphi))_{\mathbf{j}\in\Lambda_N^+},\qquad \varphi\in\mathbb{H}_\rho^m.$$

Let $j_N:(\mathbb{R}^d)^{2\Upsilon_N}\to\mathbb{H}_\rho^m$ be defined by

$$j_N(\mathbf{a},\mathbf{b})(\mathbf{x}):=\sum_{\mathbf{j}\in\Lambda_N}(a_{\mathbf{j}}\cos(\mathbf{k}_{\mathbf{j}}\cdot\mathbf{x})+b_{\mathbf{j}}\sin(\mathbf{k}_{\mathbf{j}}\cdot\mathbf{x})),\qquad \mathbf{x}\in\mathbb{R}^d,$$

with the convention $a_{-\mathbf{j}}=a_{\mathbf{j}}$, $b_{-\mathbf{j}}=-b_{\mathbf{j}}$. For the abbreviation sake we wrote $\mathbf{a}$ to denote the entire ensemble $a_{\mathbf{j}}$, $\mathbf{j}\in\Lambda_N^+$ and $\mathbf{b}$ for $b_{\mathbf{j}}$, $\mathbf{j}\in\Lambda_N^+$.

Let $\mu^{(N)}:=\mu\pi_N^{-1}$. It is a Gaussian measure on $(\mathbb{R}^d)^{2\Upsilon_N}$, which is the joint law of $(X_{\mathbf{j}}^{(N)},Y_{\mathbf{j}}^{(N)})_{\mathbf{j}\in\Lambda_N^+}$. Its characteristic function equals

$$S_0(\xi,\eta)=\prod_{\mathbf{j}\in\Lambda_N^+}\exp\left\{-\frac{1}{2}(S_{\mathbf{j}}^{(N)}\xi_{\mathbf{j}}\cdot\xi_{\mathbf{j}}+S_{\mathbf{j}}^{(N)}\eta_{\mathbf{j}}\cdot\eta_{\mathbf{j}})\right\}, \tag{2.18}$$

where $(\xi,\eta):=(\xi_{\mathbf{j}},\eta_{\mathbf{j}})_{\mathbf{j}\in\Lambda_N^+}\in(\mathbb{R}^d)^{2\Upsilon_N}$. We shall denote by $\mathscr{T}_2^{(N)}$ the probability triple $((\mathbb{R}^d)^{2\Upsilon_N},\mathscr{B}((\mathbb{R}^d)^{2\Upsilon_N}),\mu^{(N)})$.

Let $\tilde{\mu}^{(N)}$ be a Gaussian measure on $(\mathbb{H}^m_\rho, \mathscr{B}(\mathbb{H}^m_\rho))$ that corresponds to $\mu^{(N)}$ via embedding j_N, i.e.

$$\tilde{\mu}^{(N)} := \mu^{(N)} j_N^{-1}. \tag{2.19}$$

Checking that covariance matrices corresponding to the measures $\tilde{\mu}^{(N)}$ converge, as $N \to +\infty$, to that of μ we conclude that $\tilde{\mu}^{(N)} \Rightarrow \mu$ over $\mathbb{H}^m_\rho$, as $N \uparrow +\infty$.

For any $(\mathbf{a}, \mathbf{b}) := (a_\mathbf{j}, b_\mathbf{j})_{\mathbf{j} \in \Lambda_N^+}$ and $\mathbf{x} \in \mathbb{R}^d$ we define

$$\tau_\mathbf{x}^{(N)}(\mathbf{a}, \mathbf{b}) := (a_\mathbf{j} \cos(\mathbf{k_j} \cdot \mathbf{x}) + b_\mathbf{j} \sin(\mathbf{k_j} \cdot \mathbf{x}), -a_\mathbf{j} \sin(\mathbf{k_j} \cdot \mathbf{x}) + b_\mathbf{j} \cos(\mathbf{k_j} \cdot \mathbf{x}))_{\mathbf{j} \in \Lambda_N^+}.$$

Obviously $\mu^{(N)} \tau_\mathbf{x}^{(N)} = \mu^{(N)}$ and $j_N \circ \tau_\mathbf{x}^{(N)} = \tau_\mathbf{x} \circ j_N$, $\mathbf{x} \in \mathbb{R}^d$. We can introduce the gradient operator $\nabla^{(N)} = (D_{1,N}, \ldots, D_{d,N})$ as the $L^2(\mu_N)$-generator of the group of motions $U_N^\mathbf{x} F := F \circ \tau_\mathbf{x}^{(N)}$, $\mathbf{x} \in \mathbb{R}^d$. The abstract Laplacian is given by

$$\Delta^{(N)} F := D_{1,N}^2 F + \cdots + D_{d,N}^2 F, \quad \text{for any } F \in L^2(\mu^{(N)}) \cap C^\infty((\mathbb{R}^d)^{2\Upsilon_N}).$$

Let $n \geqslant 0$ be an integer. We denote by $\mathscr{P}_n^{(N)}$ the space of all polynomials in variables $(\mathbf{a}, \mathbf{b})$ of degree at most n, $\mathscr{P}^{(N)}$ the space of all polynomials and $H_n^{(N)} := \mathscr{P}_n^{(N)} \ominus \mathscr{P}_{n-1}^{(N)}$ the space of all Hermite polynomials of degree $n \geqslant 0$ corresponding to the measure $\mu^{(N)}$. Here $\mathscr{P}_{-1}^{(N)} := \{\mathbf{0}\}$. By $\mathfrak{P}_n^{(N)}$ we denote the $L^2(\mu^{(N)})$-orthogonal projection onto $H_n^{(N)}$.

Let $F \in \mathscr{P}^{(N)}$, a direct calculation shows that

$$\nabla^{(N)} F(\mathbf{a}, \mathbf{b}) = \sum_{\mathbf{j} \in \Lambda_N^+} \mathbf{k_j} \, (b_\mathbf{j} \cdot \nabla_{a_\mathbf{j}} - a_\mathbf{j} \cdot \nabla_{b_\mathbf{j}}) F(\mathbf{a}, \mathbf{b}), \tag{2.20}$$

and

$$\Delta^{(N)} F(\mathbf{a}, \mathbf{b}) = \sum_{\mathbf{j}, \mathbf{j}' \in \Lambda_N^+} \mathbf{k_j} \cdot \mathbf{k_{j'}} (b_\mathbf{j} \cdot \nabla_{a_\mathbf{j}} - a_\mathbf{j} \cdot \nabla_{b_\mathbf{j}})(b_{\mathbf{j}'} \cdot \nabla_{a_{\mathbf{j}'}} - a_{\mathbf{j}'} \cdot \nabla_{b_{\mathbf{j}'}}) F(\mathbf{a}, \mathbf{b}). \tag{2.21}$$

Here $\nabla_{a_\mathbf{j}}$, $\nabla_{b_\mathbf{j}}$ correspond to the "standard" gradient operators in $\mathbb{R}^d$ space with respect to the indicated variables.

2.3.2. Approximation of the Markovian dynamics

Let $\mathbf{w}_\mathbf{j}^{(N)}(\cdot)$, $\tilde{\mathbf{w}}_\mathbf{j}^{(N)}(\cdot)$, $\mathbf{j} \in \Lambda_N^+$ be independent standard d dimensional Brownian motions over $\mathscr{T}_W$ and

$$\varepsilon_\mathbf{j}^{(N)} := \sqrt{2 r(\mathbf{k_j}) S_\mathbf{j}^{(N)}}, \quad \forall \mathbf{j} \in \Lambda_N^+. \tag{2.22}$$

Let also $(\mathbf{a}, \mathbf{b}) := (a_\mathbf{j}, b_\mathbf{j})_{\mathbf{j} \in \Lambda_N^+}$ be given. We define

$$\Omega(t; \mathbf{a}, \mathbf{b}, \omega) := (a_\mathbf{j}(t; \mathbf{a}, \mathbf{b}, \omega), b_\mathbf{j}(t; \mathbf{a}, \mathbf{b}, \omega))_{\mathbf{j} \in \Lambda_N^+}, \quad t \geqslant 0$$

by

$$\mathrm{d}a_\mathbf{j}(t; \mathbf{a}, \mathbf{b}, \omega) = -r(\mathbf{k_j}) a_\mathbf{j}(t; \mathbf{a}, \mathbf{b}, \omega) \, \mathrm{d}t + \varepsilon_\mathbf{j}^{(N)} \, \mathrm{d}\mathbf{w}_\mathbf{j}^{(N)}(t; \omega),$$

$$\mathrm{d}b_\mathbf{j}(t; \mathbf{a}, \mathbf{b}, \omega) = -r(\mathbf{k_j}) b_\mathbf{j}(t; \mathbf{a}, \mathbf{b}, \omega) \, \mathrm{d}t + \varepsilon_\mathbf{j}^{(N)} \, \mathrm{d}\tilde{\mathbf{w}}_\mathbf{j}^{(N)}(t; \omega), \tag{2.23}$$

with $a_\mathbf{j}(0; \mathbf{a}, \mathbf{b}, \omega) = a_\mathbf{j}$, $b_\mathbf{j}(0; \mathbf{a}, \mathbf{b}, \omega) = b_\mathbf{j}$, $\mathbf{j} \in \Lambda_N^+$.

(2.23) can be explicitly solved and we can write that

$$\Omega(t;\mathbf{a},\mathbf{b},\omega) = \Theta(t;\mathbf{a},\mathbf{b}) + \Xi(t;\omega),$$

with

$$\Theta(t;\mathbf{a},\mathbf{b}) := (\mathrm{e}^{-r(\mathbf{k_j})t}a_{\mathbf{j}}, \mathrm{e}^{-r(\mathbf{k_j})t}b_{\mathbf{j}})_{\mathbf{j}\in\Lambda_N^+},$$

$$\Xi(t;\omega) := \left(\int_0^t \mathrm{e}^{-r(\mathbf{k_j})(t-s)}\varepsilon_{\mathbf{j}}^{(N)}\,\mathrm{d}\mathbf{w}_{\mathbf{j}}^{(N)}(s;\omega), \int_0^t \mathrm{e}^{-r(\mathbf{k_j})(t-s)}\varepsilon_{\mathbf{j}}^{(N)}\,\mathrm{d}\tilde{\mathbf{w}}_{\mathbf{j}}^{(N)}(s;\omega)\right).$$

Let

$$V_\varphi^{(N)}(t;\omega) := j_N \circ \Omega(t;\pi_N(\varphi),\omega), \qquad t \geqslant 0$$

be an Ornstein–Uhlenbeck process over $\mathscr{T}_W \otimes \mathscr{T}_2$ with values in $\mathbb{H}_\rho^m$ and $\mathbf{V}_\varphi^{(N)}$ the corresponding random field. We define also

$$S^{(N)}(t;\varphi) := j_N \circ \Theta(t;\pi_N(\varphi)),$$

and

$$W^{(N)}(t;\omega) := j_N \circ \Xi(t;\omega),$$

the respective $\mathbb{H}_\rho^m$-valued Gaussian processes over $\mathscr{T}_2$ and $\mathscr{T}_W$. To avoid introducing an additional notation we shall denote the corresponding random fields by the same symbols. A direct calculation shows that the covariance matrices of $W^{(N)}$ converge, as $N \to +\infty$ to the covariance matrix (2.7) so the laws of $W^{(N)}$ in $\mathfrak{C}$ weakly converge to the law of W. Similarly one can show that the laws of $S^{(N)}$ are convergent to the law of $S(\cdot)$.

The $\mathbb{H}_\rho^m$-valued process $V^{(N)}(\cdot) := S^{(N)}(\cdot) + W^{(N)}(\cdot)$ is stationary over probability space $\mathscr{T}_W \otimes \mathscr{T}_2$ and gives rise to a random, space-time stationary and spatially periodic, vector field $\mathbf{V}^{(N)}(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$.

Let P_N^t, $\mathscr{M}_N$, $\mathscr{E}_{\mathscr{M}_N}(\cdot,\cdot)$ be the respective $L^2(\mu^{(N)})$ semigroup, generator and Dirichlet form corresponding to the process $\Omega(\cdot)$ given by (2.23). The following fact is well known in the theory of Ornstein–Uhlenbeck processes.

Proposition 2.4. *$\mathscr{P}^{(N)}$ forms a core of the generator of $\mathscr{M}_N$ and*

$$\mathscr{M}_N F = \sum_{\mathbf{j}\in\Lambda_N^+} r(\mathbf{k_j})(\mathscr{M}_{a_{\mathbf{j}}} + \mathscr{M}_{b_{\mathbf{j}}})F, \tag{2.24}$$

with

$$\mathscr{M}_{a_{\mathbf{j}}}F(\mathbf{a},\mathbf{b}) := (S_{\mathbf{j}}^{(N)}\nabla_{a_{\mathbf{j}}},\nabla_{a_{\mathbf{j}}})F(\mathbf{a},\mathbf{b}) - a_{\mathbf{j}}\cdot\nabla_{a_{\mathbf{j}}}F(\mathbf{a},\mathbf{b}),$$

$$\mathscr{M}_{b_{\mathbf{j}}}F(\mathbf{a},\mathbf{b}) := (S_{\mathbf{j}}^{(N)}\nabla_{b_{\mathbf{j}}},\nabla_{b_{\mathbf{j}}})F(\mathbf{a},\mathbf{b}) - b_{\mathbf{j}}\cdot\nabla_{b_{\mathbf{j}}}F(\mathbf{a},\mathbf{b}), \quad \forall F\in\mathscr{P}^{(N)}. \tag{2.25}$$

Here, as before, $(\mathbf{a},\mathbf{b}) = (a_{\mathbf{j}}, b_{\mathbf{j}})_{\mathbf{j}\in\Lambda_N^+}$.

Proof. The fact that $\mathscr{P}^{(N)}$ is a core can be seen from the fact that it is a dense subset of $L^2(\mu^{(N)})$ which is invariant under the semigroup $(P_N^t)_{t\geqslant 0}$, see e.g. Proposition 3.3

of Ethier and Kurtz (1986). Formulas (2.4) and (2.25) follow from a direct calculation using Itô formula. □

$V^{(N)}(\cdot)$ is an approximation of $V(\cdot)$ in the following sense. Let $\Pi_N : L^2(\mu^{(N)}) \to L^2$, $J_N : \mathscr{P}^{\rm reg} \to \mathscr{P}^{(N)}$ be the linear maps given by $\Pi_N F(f) = F(\pi_N(f))$, $f \in \mathbb{H}^m_\rho$ and $J_N F(\mathbf{a}, \mathbf{b}) = F(j_N(\mathbf{a}, \mathbf{b}))$, $(\mathbf{a}, \mathbf{b}) \in (\mathbb{R}^d)^{2\Upsilon_N}$.

Proposition 2.5.

(i) *For any* $F \in \mathscr{P}^{\rm reg}$ *we have*

$$\lim_{N\uparrow+\infty} \Pi_N P_N^t J_N F = P^t F \ \textit{in any}\ L^p,\ p \in [1, +\infty). \tag{2.26}$$

(ii) *For any* $F \in \mathscr{P}^{\rm reg}$ *we have* $F \in D(\mathscr{E}_{\mathscr{M}})$ *and*

$$\lim_{N\to+\infty} \mathscr{E}_{\mathscr{M}_N}(J_N F, J_N F) = \mathscr{E}_{\mathscr{M}}(F, F). \tag{2.27}$$

(iii) *The Logarithmic Sobolev Inequality. For any* $N \geqslant 1$ *and* $F \in D(\mathscr{E}_{\mathscr{M}_N})$

$$a \int |F|^2 \log|F| \, \mathrm{d}\mu^{(N)} \leqslant \mathscr{E}_{\mathscr{M}_N}(F, F) + a\|F\|^2_{L^2(\mu^{(N)})} \log\|F\|_{L^2(\mu^{(N)})}. \tag{2.28}$$

(iv) *Similarly, for any* $F \in D(\mathscr{E}_{\mathscr{M}})$

$$a \int |F|^2 \log^+|F| \mathrm{d}\mu \leqslant \mathscr{E}_{\mathscr{M}}(F, F) + a\|F\|^2_{L^2} \log^+\|F\|_{L^2}. \tag{2.29}$$

Proof. The proof of the above proposition is standard. Part (i) has been shown in Komorowski (2001), see Proposition 1. To show part (ii) one can use formulas (2.14) and (2.24) to verify (2.27) for any $F(\cdot) = \langle \varphi_1, \cdot \rangle \cdots \langle \varphi_l, \cdot \rangle$, with $\varphi_1, \ldots, \varphi_l \in \mathscr{D}$. Part (iii) is a consequence of the classical logarithmic Sobolev inequality for finite dimensional Ornstein–Uhlenbeck processes, see e.g. Gross (1993), with the Sobolev constant independent of the mesh size of the periodic approximation. This fact, the result of part (ii) and the density argument, in consequence, yield part (iv) of the proposition. □

3. Lagrangian process

Suppose first that $\kappa > 0$. Let $\mathbf{x}(\cdot)$ be the stochastic process over $\mathscr{T}_W \otimes \mathscr{T}_1 \otimes \mathscr{T}_2$ given by (1.1). We introduce the process

$$\mathscr{Z}(t; \omega, \sigma, \varphi) := \tau_{\mathbf{x}(t;\omega,\sigma,\varphi)}(V(t; \omega, \varphi)), \quad t \geqslant 0 \tag{3.1}$$

over the probability space $\mathscr{T}_W \otimes \mathscr{T}_1 \otimes \mathscr{T}_2$, with the state space $\mathbb{H}^m_\rho$. It shall be called *the Lagrangian canonical process.*

Let

$$Q^t F(\varphi) := \mathbb{E}_W \mathbf{M} F(\mathscr{Z}(t; \cdot, \cdot, \varphi)), \quad F \in L^\infty,\ \varphi \in \mathbb{H}^m_\rho. \tag{3.2}$$

Here $\mathbf{M}$ is the expectation operator relative to the probability triple $\mathscr{T}_1$. It has been shown in Komorowski (2000), see Theorem 1, that

$$\mathbb{E}_\mu \mathbf{M}[F(\mathscr{Z}(t+h)) \,|\, \mathscr{V}_t \otimes \mathscr{Q}_t] = Q^h F(\mathscr{Z}(t)), \quad t, h \geqslant 0 \tag{3.3}$$

for any $F:\mathbb{H}^m_\rho\to\mathbb{R}$ bounded and measurable (recall that $\mathbb{E}_\mu$ is the expectation with respect to the product probability measure of $\mathscr{T}_W\otimes\mathscr{T}_2$). Here $(\mathscr{V}_t)_{t\geqslant 0}$, $(\mathscr{Q}_t)_{t\geqslant 0}$ are the natural filtrations corresponding to $V(\cdot)$ and $\mathbf{w}(\cdot)$ respectively.

Let $\mathscr{C}_{\mathscr{M}}:=D(\mathscr{M})\cap\mathscr{C}^2_b$. A direct calculation, see e.g. (Komorowski, 2002, Theorem 2, p. 424), shows that for any $F\in\mathscr{C}_{\mathscr{M}}$

$$\mathscr{L}F:=\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}Q^tF=(\kappa\Delta+\mathscr{M}+\mathscr{S}+\mathscr{A})F\quad\text{for any } F\in\mathscr{C}_{\mathscr{M}},\tag{3.4}$$

Here the derivative is taken in the L^2-sense,

$$\mathscr{S}F:=-\tfrac{1}{2}(\nabla\cdot\mathbb{V})F,\tag{3.5}$$

$$\mathscr{A}F:=\mathbb{V}\cdot\nabla F+\tfrac{1}{2}(\nabla\cdot\mathbb{V})F,\quad F\in\mathscr{C}_{\mathscr{M}}\tag{3.6}$$

and

$$\mathbb{V}(\varphi)=(\mathbb{V}_1(\varphi),\dots,\mathbb{V}_d(\varphi)):=(\varphi_1(\mathbf{0}),\dots,\varphi_d(\mathbf{0})).\tag{3.7}$$

The operators $\mathscr{S}$, $\mathscr{A}$ defined above satisfy the formal symmetry and anti-symmetry relations respectively, i.e.

$$(\mathscr{S}F,G)_{L^2}=(F,\mathscr{S}G)_{L^2},\quad(\mathscr{A}F,G)_{L^2}=-(F,\mathscr{A}G)_{L^2},\quad\forall F,G\in\mathscr{C}_{\mathscr{M}}.$$

One can write therefore the formal adjoint to $\mathscr{L}$ as

$$\mathscr{L}^*F=(\kappa\Delta+\mathscr{M}+\mathscr{S}-\mathscr{A})F,\quad F\in\mathscr{C}_{\mathscr{M}}.\tag{3.8}$$

Remark 3.1. Note that when the field $\mathbf{V}$ is compressible, i.e. $\nabla\cdot\mathbb{V}\not\equiv 0$, the measure μ *cannot* be invariant under $(\mathscr{Z}_t)_{t\geqslant 0}$. Indeed, from (3.8) we get $\mathscr{L}^*\mathbf{1}=-\nabla\cdot\mathbb{V}\not\equiv 0$, which in turn shows that the invariance of μ implies that the field must be incompressible.

The following result holds.

Theorem 3.2. *Under the assumptions of Theorem* 1.1 *there exists a constant* $C>0$ *depending only on* V_*, *see* (1.6), *such that for any* $a\geqslant C$ *there exists a Borel probability measure* ν_* *on* $\mathbb{H}^m_\rho$ *satisfying the following conditions.*

(i) ν_* *is invariant under* $(\mathscr{Z}_t)_{t\geqslant 0}$, *i.e.* $\int Q^tF\,\mathrm{d}\nu_*=\int F\,\mathrm{d}\nu_*$ *for all* $F\in C_b(\mathbb{H}^m_\rho)$ *and* $t\geqslant 0$. $(Q^t)_{t\geqslant 0}$ *can be therefore extended to a* C_0*-semigroup on* $L^2(\nu_*)$.
(ii) $\int|\mathbb{V}|\,\mathrm{d}\nu_*<+\infty$.
(iii) ν_* *is equivalent to* μ *and* $\Phi_*:=\mathrm{d}\nu_*/\mathrm{d}\mu$ *satisfies*

$$\int\Phi_*^2\log^+\Phi_*\,\mathrm{d}\mu<+\infty.\tag{3.9}$$

(iv) ν_* *is ergodic in the following sense: if* $F\in L^\infty$ *is such that* $Q^tF=F$ *for some* $t>0$ *then* $F\equiv const$, ν_**-a.s.*

The proof of this theorem is contained in Sections 4 and 5.1.

Remark 3.3. The proof of Theorem 1.1 can be concluded from Theorem 3.2. Indeed, let $\mathbb{P}$, P_μ be the laws of $V(\cdot)$ and $\mathscr{X}(\cdot)$, respectively in $\mathfrak{C}$. To simplify the notation we identify the probability space $\mathscr{T}_0$, appearing in Theorem 3.2, with $(\mathfrak{C},\mathscr{B}(\mathfrak{C}),\mathbb{P})$.

We can write then

$$P_\mu(A)=\int_{\mathbb{H}^m_\rho} P_\varphi(A)\,\mathrm{d}\mu(\varphi),\quad \text{and}$$

$$P_{\nu_*}(A)=\int_{\mathbb{H}^m_\rho} P_\varphi(A)\Phi_*(\varphi)\,\mathrm{d}\mu(\varphi),\ \ \forall A\in\mathscr{B}(\mathfrak{C}), \tag{3.10}$$

where $P_\varphi(\cdot):=P_\mu(\cdot\,|\,V(0)=\varphi)$. Obviously, from part (iii) of Theorem 3.2 we conclude that P_{ν_*} is equivalent to P_μ. Let $\mathfrak{F}:=\mathrm{d}P_{\nu_*}/\mathrm{d}P_\mu$ and let $\mathscr{X}:\mathfrak{C}\times\Sigma\to\mathfrak{C}$ be given by

$$(\mathscr{X}(\omega,\sigma))(t):=\mathbf{V}(t,\mathbf{x}(t;\omega,\sigma)+\cdot;\omega),\quad t\geqslant 0,(\omega,\sigma)\in\mathfrak{C}\times\Sigma,$$

where $\mathbf{V}(\cdot,\cdot)$ and $\mathbf{x}(\cdot)$ are as in (1.1). Set $\mathbb{P}_*(\mathrm{d}\omega,\mathrm{d}\sigma):=\mathfrak{F}\circ\mathscr{X}(\omega,\sigma)\mathbb{P}(\mathrm{d}\omega)\otimes\mathbb{W}(\mathrm{d}\sigma)$ a probability measure on $(\mathfrak{C}\times\Sigma,\mathscr{V}\otimes\mathscr{W})$. It can be easily concluded from Theorem 3.2 that $\mathbb{P}_*$ is a regular invariant measure in the sense of the definition given in Section 1, which concludes the proof of Theorem 1.1.

Remark 3.4. A direct consequence of parts (iii) and (iv) of Theorem 3.2 is the following.

Corollary 3.5. *ν_* is a unique invariant probability measure for $(\mathscr{X}_t)_{t\geqslant 0}$ that is absolutely continuous w.r.t. μ.*

Remark 3.6. Theorem 3.2 implies also the existence of the Stokes drift, when $\kappa>0$. Namely, the following result holds.

Corollary 3.7 (The vanishing of the Stokes drift). *Suppose that $\kappa>0$ and the conditions* V(1)–(3) *hold. Then,*

$$\lim_{t\uparrow+\infty}\frac{\mathbf{x}(t;\omega,\sigma)}{t}=\mathbf{0} \tag{3.11}$$

for $\mathbb{P}\otimes\mathbb{W}$-a.s. (ω,σ).

Proof. Indeed, note that

$$\int\int_{\Omega\times\Sigma}|\mathbf{V}(0,\mathbf{0};\omega)|\mathbb{P}_*(\mathrm{d}\omega,\mathrm{d}\sigma)=\int_{\mathbb{H}^m_\rho}|\mathbb{V}|\,\mathrm{d}\nu_*=<+\infty.$$

Hence by virtue of the Individual Ergodic Theorem we conclude that the limit of the expression on the left hand side of (3.11) exists $\mathbb{P}\otimes\mathbb{W}$-a.s. and equals a certain deterministic constant, say $\mathbf{v}$.

Let us define $\tilde{\mathbf{V}}(t,\mathbf{x}):=-\mathbf{V}(t,-\mathbf{x})$, $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d$. Note that the laws of $\tilde{\mathbf{V}}$ and $\mathbf{V}$ in $C(\mathbb{R}\times\mathbb{R}^d;\mathbb{R}^d)$ coincide. However $\mathbf{y}(t):=-\mathbf{x}(t)$, $t\geqslant 0$ satisfies (1.1) with the drift

$\mathbf{V}$ replaced by $\tilde{\mathbf{V}}$ and Brownian motion $\mathbf{w}(\cdot)$ replaced by $-\mathbf{w}(\cdot)$. This, in turn, shows that the laws of $-\mathbf{x}(\cdot)$ and $\mathbf{x}(\cdot)$ in $C([0,+\infty);\mathbb{R}^d)$ coincide, so we must have $\mathbf{v}=-\mathbf{v}$, which yields (3.11). □

In the case when $\kappa=0$ (1.1) becomes an ordinary differential equation (1.2) with a random right hand side, (3.1) defines then the canonical process $(\mathscr{Z}(t))_{t\geqslant 0}$ over $\mathscr{T}_W\otimes\mathscr{T}_2$. For any $t\geqslant 0$ we can also define an operator Q^t via an appropriate modification of (3.2). In this case for any $F\in\mathscr{C}_{\mathscr{M}}$ we obtain

$$\mathscr{L}F := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} Q^t F = (\mathscr{M}+\mathscr{S}+\mathscr{A})F, \tag{3.12}$$

where the derivative is taken in the L^2-sense and $\mathscr{S}$, $\mathscr{A}$ are defined by (3.5), (3.6). In this case we have the following.

Theorem 3.8. *Under the assumptions of Theorem* 1.2, *there exists a constant $C>0$ depending only on V_*, such that if $a\geqslant C$ then there exists a Borel probability measure ν_* on $\mathbb{H}^m_\rho$ satisfying* (i), (ii) *of Theorem* 3.2 *and*

(iii) *ν_* is absolutely continuous with respect to μ, with $\Phi_*=\mathrm{d}\nu_*/\mathrm{d}\mu$ satisfying* (3.9).

(iv) *If, in addition, we assume condition* (A) *we have the following weaker version of ergodicity than the one stated in part* (iv) *of the previous theorem. Namely, any function $F\in L^\infty(\nu_*)$ satisfying $Q^tF=F$ (understood as the equality of elements from $L^\infty(\nu_*)$) for all $t>0$ must be equivalent to a constant (in $L^\infty(\nu_*)$). Moreover, ν_* is a unique, absolutely continuous, invariant measure among those possessing densities satisfying* (3.9).

Remark 3.9. The proofs of Theorems 1.2 and 1.4 can be concluded from Theorem 3.8 in the same fashion as Theorem 1.1 has been obtained from Theorem 3.2, see Remark 3.3.

Remark 3.10. The above result can be interpreted in terms of the existence of an invariant measure for the dynamics that comes from a solution of a certain stochastic partial differential equation (S.P.D.E.). Namely, in the case when $r(\mathbf{k})\equiv a$, for some constant $a>0$, it has been shown in Fannjiang et al. (2002) that the process $\mathscr{Z}(\cdot)$ can be constructed as a solution of a S.P.D.E.

$$\mathrm{d}\mathscr{Z}(t) = (-a\mathscr{Z}(t)+\mathscr{Z}(t,\mathbf{0})\cdot\nabla\mathscr{Z}(t))\,\mathrm{d}t + C\,\mathrm{d}B(t), \tag{3.13}$$

with $B(\cdot)$ a $\mathbb{L}^2$-cylindrical Wiener process, $C:\mathbb{L}^2\to\mathbb{H}^m_\rho$ a Hilbert–Schmidt operator that is the unique extension of $C:\mathscr{S}_d\to\mathbb{H}^m_\rho$, given by $\widehat{C\varphi}(\mathbf{k})=\sqrt{2a\hat{\Gamma}(\mathbf{k})}\hat{\varphi}(\mathbf{k})$, $\varphi\in\mathscr{S}_d$. We assume also that μ is the law of $\mathscr{Z}(0)$. It is known, see Theorems 2 and 3 of Fannjiang et al. (2002), p. 179 and p. 185, that (3.13) possesses a unique strong solution. Theorem 3.8 guarantees the existence of a stationary solution to (3.13) whose law in $\mathfrak{C}$ is absolutely continuous w.r.t. the law of $\mathscr{Z}(\cdot)$.

4. The construction of an invariant measure

4.1. Finite dimensional approximation of the Lagrangian process

Let $q \geqslant 0$, $\varphi \in \mathbb{H}^m_\rho$ and $\mathbf{x}^{(N)}_{q,\varphi}(\cdot)$ be the solution of

$$\mathrm{d}\mathbf{x}^{(N)}_{q,\varphi}(t;\omega,\sigma) = q\mathbf{V}^{(N)}_{\varphi}(t,\mathbf{x}^{(N)}_{q,\varphi}(t;\omega,\sigma);\omega)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}\mathbf{w}(t;\sigma), \qquad t \geqslant 0,$$
$$\mathbf{x}^{(N)}_{q,\varphi}(0;\omega,\sigma) = \mathbf{0}. \tag{4.1}$$

Set

$$\mathscr{Z}^{(N)}_{q,\mathbf{a},\mathbf{b}}(t;\omega,\sigma) := \tau^{(N)}_{\mathbf{x}^{(N)}_{q,j_N(\mathbf{a},\mathbf{b})}(t;\omega,\sigma)} \circ \Omega(t;\mathbf{a},\mathbf{b},\omega),$$
$$(\omega,\sigma) \in \mathfrak{C} \times \Sigma, \qquad t \geqslant 0 \tag{4.2}$$

for any $(\mathbf{a},\mathbf{b}) = (a_\mathbf{j}, b_\mathbf{j})_{\mathbf{j}\in\Lambda_N^+} \in (\mathbb{R}^d)^{2\Upsilon_N}$. It is an $(\mathbb{R}^d)^{2\Upsilon_N}$-valued process defined over $\mathscr{T}_W \otimes \mathscr{T}_1$.

Proposition 4.1. (i) *For any $(\mathbf{a},\mathbf{b}) \in (\mathbb{R}^d)^{2\Upsilon_N}$ the process $\mathscr{Z}^{(N)}_{q,\mathbf{a},\mathbf{b}}(\cdot) = (\tilde{a}_\mathbf{j}(\cdot;\mathbf{a},\mathbf{b}), \tilde{b}_\mathbf{j}(\cdot;\mathbf{a},\mathbf{b}))_{\mathbf{j}\in\Lambda_N^+}$ is an $(\mathbb{R}^d)^{2\Upsilon_N}$-valued diffusion described by the stochastic differential equation*

$$\mathrm{d}\tilde{a}_\mathbf{j}(t;\mathbf{a},\mathbf{b}) = \left[-r(\mathbf{k_j})\tilde{a}_\mathbf{j}(t;\mathbf{a},\mathbf{b}) + 2q\left(\sum_{\mathbf{j}'\in\Lambda_N^+} \tilde{a}_{\mathbf{j}'}(t;\mathbf{a},\mathbf{b})\right)\cdot \mathbf{k_j}\tilde{b}_\mathbf{j}(t;\mathbf{a},\mathbf{b}) - \kappa|\mathbf{k_j}|^2\tilde{a}_\mathbf{j}(t;\mathbf{a},\mathbf{b})\right]\mathrm{d}t$$
$$+\ \varepsilon^{(N)}_\mathbf{j}\,\mathrm{d}\tilde{\mathbf{w}}_\mathbf{j}(t) + \sqrt{2\kappa}\,\tilde{b}_\mathbf{j}(t;\mathbf{a},\mathbf{b})\,\mathbf{k_j}\cdot \mathrm{d}\tilde{\mathbf{w}}(t)$$
$$\mathrm{d}\tilde{b}_\mathbf{j}(t;\mathbf{a},\mathbf{b}) = \left[-r(\mathbf{k_j})\tilde{b}_\mathbf{j}(t;\mathbf{a},\mathbf{b}) - 2q\left(\sum_{\mathbf{j}'\in\Lambda_N^+} \tilde{a}_{\mathbf{j}'}(t;\mathbf{a},\mathbf{b})\right)\cdot \mathbf{k_j}\,\tilde{a}_\mathbf{j}(t;\mathbf{a},\mathbf{b}) - \kappa|\mathbf{k_j}|^2\tilde{b}_\mathbf{j}(t;\mathbf{a},\mathbf{b})\right]\mathrm{d}t$$
$$+\ \varepsilon^{(N)}_\mathbf{j}\,\mathrm{d}\tilde{\mathbf{w}}'_\mathbf{j}(t) - \sqrt{2\kappa}\tilde{a}_\mathbf{j}(t;\mathbf{a},\mathbf{b})\,\mathbf{k_j}\cdot\,\mathrm{d}\tilde{\mathbf{w}}(t)$$

$$\tilde{a}_\mathbf{j}(0;\mathbf{a},\mathbf{b}) = a_\mathbf{j}, \qquad \tilde{b}_\mathbf{j}(0;\mathbf{a},\mathbf{b}) = b_\mathbf{j}, \qquad \forall \mathbf{j}\in\Lambda_N^+. \tag{4.3}$$

Here $(\tilde{\mathbf{w}}_\mathbf{j}(\cdot), \tilde{\mathbf{w}}'_\mathbf{j}(\cdot))_{\mathbf{j}\in\Lambda_N^+}$, $\tilde{\mathbf{w}}(\cdot)$ are mutually independent standard d-dimensional Brownian motions (recall also that $\varepsilon^{(N)}_\mathbf{j}$ is given by (2.22)).

(ii) *The generator of the diffusion described by (4.3) is given by*

$$\mathscr{L}_{q,N}F = [\kappa\Delta^{(N)} + \mathscr{M}_N + q(\mathscr{S}_N + \mathscr{A}_N)]F, \qquad F \in C_0^\infty((\mathbb{R}^d)^{2\Upsilon_N}), \tag{4.4}$$

where

$$\mathscr{S}_N F := -\tfrac{1}{2}(\nabla^{(N)}\cdot\mathbb{V}^{(N)})F,$$
$$\mathscr{A}_N F := \mathbb{V}^{(N)}\cdot\nabla^{(N)}F + \tfrac{1}{2}(\nabla^{(N)}\cdot\mathbb{V}^{(N)})F, \qquad F \in C_0^\infty((\mathbb{R}^d)^{2\Upsilon_N}) \tag{4.5}$$

are finite dimensional analogs of $\mathscr{S}$, $\mathscr{A}$ *defined in* (3.5) *and* (3.6). *Here*

$$\mathbb{V}^{(N)}(\mathbf{a},\mathbf{b}) := (\mathbb{V}_1^{(N)}(\mathbf{a},\mathbf{b}),\ldots,\mathbb{V}_d^{(N)}(\mathbf{a},\mathbf{b})) := 2\sum_{\mathbf{j}\in\Lambda_N^+} a_{\mathbf{j}}. \tag{4.6}$$

(iii) *Set*

$$Q_{q,N}^t F(\mathbf{a},\mathbf{b}) := \mathbb{E}_W \mathbf{M} F(\mathscr{Z}_{q,\mathbf{a},\mathbf{b}}^{(N)}(t)), \tag{4.7}$$

it is the transition of probability semigroup for $\mathscr{Z}_{q,\cdot,\cdot}^{(N)}(\cdot)$, *i.e.*

$$\mathbb{E}_W \mathbf{M}[F(\mathscr{Z}_{q,\mathbf{a},\mathbf{b}}^{(N)}(t+h))|\mathscr{W}_t^{(N)}\otimes\mathscr{W}_t] = Q_{q,N}^h F(\mathscr{Z}_{q,\mathbf{a},\mathbf{b}}(t)), \tag{4.8}$$

$\forall t,h\geqslant 0$, $F\in C_b((\mathbb{R}^d)^{2\Upsilon_N})$, $(\mathbf{a},\mathbf{b})\in(\mathbb{R}^d)^{2\Upsilon_N}$. *Here* $(\mathscr{W}_t^{(N)})_{t\geqslant 0}$, $(\mathscr{W}_t)_{t\geqslant 0}$ *are the natural filtrations corresponding to* $(\mathbf{w}_{\mathbf{j}}(\cdot),\mathbf{w}'_{\mathbf{j}}(\cdot))_{\mathbf{j}\in\Lambda_N^+}$, $\mathbf{w}(\cdot)$, *respectively.*

The proof of the proposition can be obtained via a standard application of Itô stochastic calculus so we omit it.

In what follows we shall denote $Q_N^t := Q_{1,N}^t$. Let $\mathbf{x}^{(N)}(\cdot)$ be the solution of (1.1) with the drift replaced by $\mathbf{V}^{(N)}$. Recall that $m > d/2+1$ (then, $\mathbb{H}_\varrho^m \subseteq C^1(\mathbb{R}^d,\mathbb{R}^d)$). To state our next result let us define $J_N : C_b(\mathbb{H}_\rho^m) \to C_b((\mathbb{R}^d)^{2\Upsilon_N})$, $J_N F(\mathbf{a},\mathbf{b}) := F(j_N(\mathbf{a},\mathbf{b}))$ for any $F\in C_b(\mathbb{H}_\rho^m)$, $(\mathbf{a},\mathbf{b})\in(\mathbb{R}^d)^{2\Upsilon_N}$. We show that $(Q_N^t)_{t\geqslant 0}$ approximates $(Q^t)_{t\geqslant 0}$, as $N\uparrow+\infty$.

Proposition 4.2. *For any* $F\in C_b(\mathbb{H}_\rho^m)$ *we have*

$$\lim_{N\uparrow+\infty} \|\Pi_N Q_N^t J_N F - Q^t F\|_{L^2} = 0. \tag{4.9}$$

Proof. Thanks to Skorokhod's representation theorem, see e.g. Theorem 2.7, p. 9 of Ikeda and Watanabe (1981), there exist $\mathbb{H}_\rho^m$-valued processes $\tilde{S}^{(N)}(\cdot)$, $N\geqslant 1$ and $\tilde{S}(\cdot)$ given over a certain probability space $\tilde{\mathscr{T}}_1 := (\tilde{\Omega}_1,\tilde{\mathscr{V}}_1,\tilde{\mathbb{P}}_1)$ and $\mathbb{H}_\rho^m$-valued processes $\tilde{W}^{(N)}(\cdot)$, $N\geqslant 1$ and $\tilde{W}(\cdot)$ given over another probability space $\tilde{\mathscr{T}}_2 := (\tilde{\Omega}_2,\tilde{\mathscr{V}}_2,\tilde{\mathbb{P}}_2)$, for which

(1)

$$\lim_{N\uparrow+\infty}\sup_{t\in[0,T]} \|\tilde{S}^{(N)}(t)-\tilde{S}(t)\|_{\mathbb{H}_\rho^m} = 0, \quad \tilde{\mathbb{P}}_1\text{-a.s.}, \tag{4.10}$$

$$\lim_{N\uparrow+\infty}\sup_{t\in[0,T]} \|\tilde{W}^{(N)}(t)-\tilde{W}(t)\|_{\mathbb{H}_\rho^m} = 0, \quad \tilde{\mathbb{P}}_2\text{-a.s.} \tag{4.11}$$

(2) The laws of $\tilde{S}^{(N)}(\cdot)$, $S^{(N)}(\cdot)$ and $\tilde{W}^{(N)}(\cdot)$, $W^{(N)}(\cdot)$ in $\mathfrak{C}$ coincide for each $N\geqslant 1$. The same also holds for the laws of $\tilde{S}(\cdot)$, $S(\cdot)$ and $\tilde{W}(\cdot)$, $W(\cdot)$.

We denote by $\tilde{\mathbb{E}}_1$, $\tilde{\mathbb{E}}_2$ the expectation operators corresponding to $\tilde{\mathbb{P}}_1$ and $\tilde{\mathbb{P}}_2$, respectively.

Since $\mathbb{H}_\varrho^m$ is continuously embedded into $C^1(\mathbb{R}^d,\mathbb{R}^d)$ we have almost sure convergence of the respective random fields $\tilde{\mathbf{V}}^{(N)}(\cdot,\cdot) := \tilde{S}^{(N)}(\cdot,\cdot)+\tilde{W}^{(N)}(\cdot,\cdot)$ defined over

$\tilde{\mathscr{T}}_1 \otimes \tilde{\mathscr{T}}_2$ to $\tilde{\mathbf{V}}(\cdot,\cdot) := \tilde{S}(\cdot,\cdot) + \tilde{W}(\cdot,\cdot)$ in $C^1([0,T] \times B, \mathbb{R}^d)$ for any $T > 0$ and closed ball $B \subseteq \mathbb{R}^d$. This fact, in turn, implies that

$$\lim_{N\uparrow+\infty} \sup_{t\in[0,T]} |\tilde{\mathbf{x}}^{(N)}(t;\tilde{\omega}_1,\tilde{\omega}_2,\sigma) - \tilde{\mathbf{x}}(t;\tilde{\omega}_1,\tilde{\omega}_2,\sigma)| = 0,$$

for $\tilde{\mathbb{P}}_1 \otimes \tilde{\mathbb{P}}_2 \otimes \mathbb{W}$-a.s. $(\tilde{\omega}_1,\tilde{\omega}_2,\sigma) \in \tilde{\Omega}_1 \times \tilde{\Omega}_2 \times \Sigma$. Here $\tilde{\mathbf{x}}^{(N)}(\cdot)$, $\tilde{\mathbf{x}}(\cdot)$ denote the solutions of (1.1) with the drift replaced by $\tilde{\mathbf{V}}^{(N)}(\cdot,\cdot)$ and $\tilde{\mathbf{V}}(\cdot,\cdot)$ correspondingly. In consequence,

$$\lim_{N\uparrow+\infty} \sup_{t\in[0,T]} \|\tau_{\tilde{\mathbf{x}}^{(N)}(t)} \tilde{V}^{(N)}(t) - \tau_{\tilde{\mathbf{x}}(t)} \tilde{V}(t)\|_{\mathbb{H}^m_\rho} = 0, \quad \tilde{\mathbb{P}}_1 \otimes \tilde{\mathbb{P}}_2 \otimes \mathbb{W}. \tag{4.12}$$

In light of condition (2) spelled above, the laws of $(\tau_{\tilde{\mathbf{x}}^{(N)}(t)} \tilde{V}^{(N)}(t), \tilde{V}^{(N)}(0))$ and $(\tau_{\mathbf{x}^{(N)}(t)} V^{(N)}(t), V^{(N)}(0))$ in $\mathbb{H}^m_\rho \times \mathbb{H}^m_\rho$ coincide for each $N \geqslant 1$ and $t \geqslant 0$. Thus, for any $F \in C_b(\mathbb{H}^m_\rho)$, the law of r.v. $[\Pi_N Q^t_N J_N F]^2(V^{(N)}(0))$, cf. (2.1), considered over $\mathscr{T}_2$ coincides with that of

$$[\tilde{\mathbb{E}}_2 \mathbf{M} F(\tau_{\tilde{\mathbf{x}}^{(N)}(t;\tilde{\omega}_1,\cdot,\cdot)} \tilde{V}^{(N)}(t;\tilde{\omega}_1,\cdot,\cdot))]^2$$

considered over $\tilde{\mathscr{T}}_1$. Hence

$$\lim_{N\uparrow+\infty} \int_{\mathbb{H}^m_\varrho} [\Pi_N Q^t_N J_N F]^2 \, d\mu = \lim_{N\uparrow+\infty} \tilde{\mathbb{E}}_1 [\tilde{\mathbb{E}}_2 \mathbf{M} F(\tau_{\tilde{\mathbf{x}}^{(N)}(t)} \tilde{V}^{(N)}(t))]^2$$

$$= \tilde{\mathbb{E}}_1 [\tilde{\mathbb{E}}_2 \mathbf{M} F(\tau_{\tilde{\mathbf{x}}(t)} \tilde{V}(t))]^2 = \int_{\mathbb{H}^m_\rho} (Q^t F)^2 \, d\mu. \tag{4.13}$$

The last equality follows from the fact that the laws of r.v.-s $(Q^t F)^2 \ (V(0))$ and

$$[\tilde{\mathbb{E}}_2 \mathbf{M} F(\tau_{\tilde{\mathbf{x}}(t;\tilde{\omega}_1,\cdot)} \tilde{V}(t;\tilde{\omega}_1,\cdot))]^2,$$

considered over $\mathscr{T}_2$ and $\tilde{\mathscr{T}}_1$, respectively, coincide.

For any $F, G \in C_b(\mathbb{H}^m_\rho)$, we also have

$$\begin{aligned} \lim_{N\uparrow+\infty} \int_{\mathbb{H}^m_\varrho} \Pi_N Q^t_N J_N F \Pi_N J_N G \, d\mu &= \lim_{N\uparrow+\infty} \tilde{\mathbb{E}}_1 \tilde{\mathbb{E}}_2 \mathbf{M}[F(\tau_{\tilde{\mathbf{x}}^{(N)}(t)} \tilde{V}^{(N)}(t)) G(\tilde{V}^{(N)}(0))] \\ &\overset{(4.12)}{=} \tilde{\mathbb{E}}_1 \tilde{\mathbb{E}}_2 \mathbf{M}[F(\tau_{\tilde{\mathbf{x}}(t)} \tilde{V}(t)) G(\tilde{V}(0))] \\ &= \int_{\mathbb{H}^m_\rho} Q^t F G \, d\mu. \end{aligned} \tag{4.14}$$

The conclusion of the proposition follows from (2.26), (4.13) and (4.14). □

Proposition 4.3. *Suppose that the field* $\mathbf{V}$ *satisfies the assumptions of Theorem* 1.1, $(Q^t_{q,N})_{t\geqslant 0}$ *is given by* (4.7). *Then, there is* $C > 0$ *independent of* N *such that for all* $q < Ca$ *there exists a Borel measure* $\nu^{(N)}_q$ *on* $(\mathbb{R}^d)^{2\Upsilon_N}$ *for which*

(1)

$$\int_{(\mathbb{R}^d)^{2\Upsilon_N}} Q^t_{q,N} F \, d\nu^{(N)}_q = \int_{(\mathbb{R}^d)^{2\Upsilon_N}} F \, d\nu^{(N)}_q, \quad \forall F \in L^\infty(\mu^{(N)}), \quad t \geqslant 0,$$

i.e. $\nu^{(N)}_q$ *is invariant under* $\mathscr{L}^{(N)}_{q,\cdot,\cdot}(\cdot)$.

(2) *$v_q^{(N)}$ is absolutely continuous w.r.t. $\mu^{(N)}$. Let $\Phi_q^{(N)} := \mathrm{d}v_q^{(N)}/\mathrm{d}\mu^{(N)}$. We have*

$$\sup_{N\geqslant 1}\|\Phi_q^{(N)}\|_{L^2(\mu^{(N)})} < +\infty. \tag{4.15}$$

Proof. First, notice that without any loss of generality we may assume that all matrices $S_{\mathbf{j}}^{(N)}$, $\mathbf{j}\in\Lambda_N^+$ are non-degenerate. Otherwise we would reduce the phase space $(\mathbb{R}^d)^{2\Upsilon_N}$ to $\mathfrak{R} := \oplus_{\mathbf{j}\in\Lambda_N^+}(\mathscr{R}_{\mathbf{j}}\oplus\mathscr{R}_{\mathbf{j}})$, where $\mathscr{R}_{\mathbf{j}}$ is the range of $S_{\mathbf{j}}^{(N)}$ for a given $\mathbf{j}\in\Lambda_N^+$ and consider an isometric image of $\mathscr{L}_{q,\cdot,\cdot}^{(N)}(\cdot)$ in that space. As a result of this simplification we can assume that the diffusion given by (4.3) is non-degenerate.

Suppose first that $\kappa>0$. By $\Phi_{\kappa,q}^{(N)}$ we shall denote the invariant density corresponding to the given value of κ. Note that for $q=0$ we obviously have $\Phi_{\kappa,0}^{(N)}=\mathbf{1}$. Let us suppose that

$$\Phi_{\kappa,q}^{(N)} = \mathbf{1} + \sum_{k=1}^{+\infty} q^k F_k^{(N)} \in L^2(\mu^{(N)}). \tag{4.16}$$

We are seeking $\Phi_{\kappa,q}^{(N)}\in C^2((\mathbb{R}^d)^{2\Upsilon_N})\cap L^2(\mu^{(N)})$ that is nonnegative, satisfies $\int \Phi_{\kappa,q}^{(N)} d\mu^{(N)} = 1$ and

$$[\mathscr{L}_{0,N} + q(\mathscr{S}_N - \mathscr{A}_N)]\Phi_{\kappa,q}^{(N)} = 0. \tag{4.17}$$

Equivalently, we can rewrite (4.17) in the form

$$\mathscr{L}_{0,N}F_k^{(N)} + (\mathscr{S}_N - \mathscr{A}_N)F_{k-1}^{(N)} = 0, \quad \forall k \geqslant 1. \tag{4.18}$$

Here $F_0^{(N)} = \mathbf{1}$. Suppose that we have already found $F_i^{(N)}\in\mathscr{P}_i^{(N)}$, $i=0,\dots,k-1$ and such that $\int F_i^{(N)}\,\mathrm{d}\mu^{(N)}=0$, $i=1,\dots,k-1$ for some $k\geqslant 1$. Thus, in particular $F_{k-1}^{(N)}$ is C^∞-smooth, $(\mathscr{S}_N-\mathscr{A}_N)F_{k-1}^{(N)}\in\mathscr{P}_k^{(N)}$ and

$$\int(\mathscr{S}_N-\mathscr{A}_N)F_{k-1}^{(N)}\,\mathrm{d}\mu^{(N)}$$
$$= -\int \mathbb{V}^{(N)}\cdot\nabla^{(N)}F_{k-1}^{(N)}\,\mathrm{d}\mu^{(N)} - \int(\nabla^{(N)}\cdot\mathbb{V}^{(N)})F_{k-1}^{(N)}\mathrm{d}\mu^{(N)} = 0,$$

which in turn implies that $G_k^{(N)} := (\mathscr{S}_N-\mathscr{A}_N)F_{k-1}^{(N)}\in\mathscr{P}_k^{(N)}$ belongs to the range of the operator $\mathscr{L}_{0,N}$ and

$$F_k^{(N)} := \mathscr{L}_{0,N}^{-1}G_k^{(N)}\in\mathscr{P}_k^{(N)}, \tag{4.19}$$

with $\int F_k^{(N)}\,\mathrm{d}\mu^{(N)}=0$.

Let $f_{k,n} := \mathfrak{P}_n^{(N)}F_k^{(N)}$. Note that $f_{k,0}=0$ for $k\geqslant 1$ and $f_{k,n}=0$ for $n\geqslant k+1$. Rewriting (4.18), using orthogonal projections onto the Hermite polynomial spaces, we obtain

$$\begin{aligned}&\mathscr{L}_{0,N}f_{k,n} + \mathfrak{P}_n^{(N)}\mathscr{S}_N f_{k-1,n+1} + \mathfrak{P}_n^{(N)}\mathscr{S}_N f_{k-1,n-1}\\ &\quad -\mathfrak{P}_n^{(N)}\mathscr{A}_N f_{k-1,n+1} - \mathfrak{P}_n^{(N)}\mathscr{A}_N f_{k-1,n-1} = 0, \quad \forall n\geqslant 1.\end{aligned} \tag{4.20}$$

Multiplying both sides of (4.20) by $-f_{k,n}$ and summing up over n we obtain the following estimate:

$$\sum_{n\geqslant 1}(-\mathscr{L}_{0,N}f_{k,n},f_{k,n})_{L^2(\mu^{(N)})}$$

$$\leqslant \sum_{n\geqslant 1}(|(\mathscr{S}_N f_{k-1,n-1},f_{k,n})_{L^2(\mu^{(N)})}|+|(\mathscr{A}_N f_{k-1,n-1},f_{k,n})_{L^2(\mu^{(N)})}|$$

$$+|(\mathscr{S}_N f_{k-1,n+1},f_{k,n})_{L^2(\mu^{(N)})}|+|(\mathscr{A}_N f_{k-1,n+1},f_{k,n})_{L^2(\mu^{(N)})}|). \tag{4.21}$$

The hypercontractivity property of L^p norms over Gaussian measures, see the proof of Theorem 5.10 in Janson (1997) (in particular the first formula after (5.4) there), implies that there exists a constant $C>0$ independent of N,n,k,κ such that (both here and in the ensuing estimates we denote any such generic constant by C)

$$|(\mathscr{S}_N f_{k-1,n-1},f_{k,n})_{L^2(\mu^{(N)})}|\leqslant C\sqrt{n}\|f_{k-1,n-1}\|_{L^2(\mu^{(N)})}\|f_{k,n}\|_{L^2(\mu^{(N)})}$$

$$\leqslant \frac{1}{2}(C^2n\|f_{k-1,n-1}\|^2_{L^2(\mu^{(N)})}+\|f_{k,n}\|^2_{L^2(\mu^{(N)})}). \tag{4.22}$$

Here we used an elementary inequality

$$ab\leqslant 1/2(a^2+b^2),\qquad a,b\in\mathbb{R}. \tag{4.23}$$

Analogously,

$$|(\mathscr{A}_N f_{k-1,n-1},f_{k,n})_{L^2(\mu^{(N)})}|=|(f_{k-1,n-1},\mathscr{A}_N f_{k,n})_{L^2(\mu^{(N)})}|$$

$$\leqslant \frac{1}{2}\,|(f_{k-1,n-1},(\nabla^{(N)}\cdot\mathbb{V}^{(N)})f_{k,n})_{L^2(\mu^{(N)})}|+|(f_{k-1,n-1},\mathbb{V}^{(N)}\cdot\nabla^{(N)}f_{k,n})_{L^2(\mu^{(N)})}|$$

$$\leqslant C\sqrt{n}(\|f_{k,n}\|_{L^2(\mu^{(N)})}+\|\nabla^{(N)}f_{k,n}\|_{L^2(\mu^{(N)})})\|f_{k-1,n-1}\|_{L^2(\mu^{(N)})}.$$

Using again inequality (4.23) we conclude that

$$|(\mathscr{A}_N f_{k-1,n-1},f_{k,n})_{L^2(\mu^{(N)})}|$$

$$\leqslant \frac{1}{2}\left[C^2n\left(1+\frac{1}{\kappa}\right)\|f_{k-1,n-1}\|^2_{L^2(\mu^{(N)})}+\|f_{k,n}\|^2_{L^2(\mu^{(N)})}+\kappa\|\nabla^{(N)}f_{k,n}\|^2_{L^2(\mu^{(N)})}\right]. \tag{4.24}$$

Analogous estimates can be derived for $|(\mathscr{S}_N f_{k-1,n+1},f_{k,n})_{L^2(\mu^{(N)})}|$ and $|(\mathscr{A}_N f_{k-1,n+1},f_{k,n})_{L^2(\mu^{(N)})}|$.

From (4.22), (4.24) we obtain that the left hand side of (4.21) can be estimated from above by

$$2C^2\left(1+\frac{1}{2\kappa}\right)|||F^{(N)}_{k-1}|||^2+\kappa\|\nabla^{(N)}F^{(N)}_k\|^2_{L^2_d(\mu^{(N)})}+2\|F^{(N)}_k\|^2_{L^2(\mu^{(N)})}. \tag{4.25}$$

Here

$$|||F|||^2:=\sum_{n\geqslant 0}(n+1)\|\mathfrak{P}^{(N)}_nF\|^2_{L^2(\mu^{(N)})},\qquad F\in\mathscr{P}^{(N)}.$$

On the other hand we can estimate the left hand side of (4.21) from below by

$$a\sum_{n\geqslant 1} n\|f_{k,n}\|^2_{L^2(\mu^{(N)})}+\kappa\|\nabla^{(N)}F_k^{(N)}\|^2_{L^2_d(\mu^{(N)})}. \tag{4.26}$$

This estimate is obtained using the fact that on each space $H_n^{(N)}$ the spectral gap of the operator $\mathscr{M}_N$ is greater than or equal to na, see Proposition 5.1 below.

Using both (4.25) and (4.26) we conclude that

$$|||F_k^{(N)}|||^2 \leqslant \frac{8C^2}{a}\left(1+\frac{1}{2\kappa}\right)|||F_{k-1}^{(N)}|||^2. \tag{4.27}$$

Here we used the fact that $an-2\geqslant a(n+1)/4$ for $a\geqslant 2$. (4.27) leads to an estimate

$$|||F_k^{(N)}|||^2 \leqslant \left[\frac{8C^2}{a}\left(1+\frac{1}{2\kappa}\right)\right]^k, \quad \forall k\geqslant 0.$$

Hence, $\Phi_{\kappa,q}^{(N)}$ is well defined by means of (4.16) for a sufficiently large $a>0$ and satisfies (4.17). The fact that $\Phi_{\kappa,q}^{(N)}\geqslant 0$ can be shown by proving that the space of solutions to Eq. (4.17) is of linear dimension one, which is a standard fact for a non-degenerate finite dimensional diffusion.

Now we remove the assumption on positivity of the molecular diffusivity by proving an estimate

$$\|\Phi_{\kappa,q}^{(N)}\|_{L^2(\mu^{(N)})}\leqslant C, \tag{4.28}$$

with the constant C that may depend on q but is independent of κ and N.

Let $\tilde{\Phi}_\kappa := \Phi_{\kappa,q}^{(N)}-\mathbf{1}$. It satisfies the equation

$$\mathscr{L}_{0,N}\tilde{\Phi}_\kappa + q\mathscr{S}_N\tilde{\Phi}_\kappa - q\mathscr{A}_N\tilde{\Phi}_\kappa = q\nabla^{(N)}\cdot\mathbb{V}_N. \tag{4.29}$$

Projecting both sides of (4.29) onto $H_n^{(N)}$ and letting $\tilde{\Phi}_{\kappa,n} := \mathfrak{P}_n^{(N)}\tilde{\Phi}_\kappa$, we obtain the equations

$$\mathscr{L}_{0,N}\tilde{\Phi}_{\kappa,1}+\mathfrak{P}_1^{(N)}\mathscr{S}_N\tilde{\Phi}_{\kappa,2}-\mathfrak{P}_1^{(N)}\mathscr{A}_N\tilde{\Phi}_{\kappa,2}=q\nabla^{(N)}\cdot\mathbb{V}_N, \tag{4.30}$$

$$\begin{aligned}\mathscr{L}_{0,N}\tilde{\Phi}_{\kappa,n}&+\mathfrak{P}_n^{(N)}\mathscr{S}_N\tilde{\Phi}_{\kappa,n+1}+\mathfrak{P}_n^{(N)}\mathscr{S}_N\tilde{\Phi}_{\kappa,n-1}\\ &-\mathfrak{P}_n^{(N)}\mathscr{A}_N\tilde{\Phi}_{\kappa,n+1}-\mathfrak{P}_n^{(N)}\mathscr{A}_N\tilde{\Phi}_{\kappa,n-1}=0, \quad \forall n\geqslant 2.\end{aligned} \tag{4.31}$$

We take the $L^2(\mu^{(N)})$-scalar products of both sides of (4.30) and (4.31) against $\tilde{\Phi}_{\kappa,1}$ and $\tilde{\Phi}_{\kappa,n}$ respectively. Then,

$$\begin{aligned}a\|\tilde{\Phi}_{\kappa,1}\|^2_{L^2(\mu^{(N)})}&\leqslant q|(\nabla^{(N)}\cdot\mathbb{V}_N,\tilde{\Phi}_{\kappa,1})_{L^2(\mu^{(N)})}|\\ &\quad+(\mathscr{S}_N\tilde{\Phi}_{\kappa,2},\tilde{\Phi}_{\kappa,1})_{L^2(\mu^{(N)})}-(\mathscr{A}_N\tilde{\Phi}_{\kappa,2},\tilde{\Phi}_{\kappa,1})_{L^2(\mu^{(N)})}\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}&an\|\tilde{\Phi}_{\kappa,n}\|^2_{L^2(\mu^{(N)})}\\ &\quad\leqslant(\mathscr{S}_N\tilde{\Phi}_{\kappa,n+1},\tilde{\Phi}_{\kappa,n})_{L^2(\mu^{(N)})}+(\mathscr{S}_N\tilde{\Phi}_{\kappa,n-1},\tilde{\Phi}_{\kappa,n})_{L^2(\mu^{(N)})}\\ &\qquad-(\mathscr{A}_N\tilde{\Phi}_{\kappa,n+1},\tilde{\Phi}_{\kappa,n})_{L^2(\mu^{(N)})}-(\mathscr{A}_N\tilde{\Phi}_{\kappa,n-1},\tilde{\Phi}_{\kappa,n})_{L^2(\mu^{(N)})}, \quad \forall n\geqslant 2.\end{aligned} \tag{4.33}$$

Performing the summation on both sides of (4.32), (4.33) over all $n \geqslant 1$ we conclude the following estimate

$$a\sum_{n\geqslant 1} n\|\tilde{\Phi}_{\kappa,n}\|^2_{L^2(\mu^{(N)})}$$
$$\leqslant q|(\nabla^{(N)}\cdot\mathbb{V}_N,\tilde{\Phi}_{\kappa,1})_{L^2(\mu^{(N)})}| + 2\sum_{n\geqslant 1}|(\mathscr{S}_N\tilde{\Phi}_{\kappa,n+1},\tilde{\Phi}_{\kappa,n})_{L^2(\mu^{(N)})}|. \tag{4.34}$$

Here we have used the fact that

$$(\mathscr{A}_N\tilde{\Phi}_{\kappa,n+1},\tilde{\Phi}_{\kappa,n})_{L^2(\mu^{(N)})} = -(\mathscr{A}_N\tilde{\Phi}_{\kappa,n},\tilde{\Phi}_{\kappa,n+1})_{L^2(\mu^{(N)})}.$$

Applying (4.22) we conclude that

$$a\sum_{n\geqslant 1} n\|\tilde{\Phi}_{\kappa,n}\|^2_{L^2(\mu^{(N)})}$$
$$\leqslant q\|\nabla^{(N)}\cdot\mathbb{V}_N\|_{L^2(\mu^{(N)})}\|\tilde{\Phi}_{\kappa,1}\|_{L^2(\mu^{(N)})} + 4C\sum_{n\geqslant 1}\sqrt{n}\|\tilde{\Phi}_{\kappa,n}\|^2_{L^2(\mu^{(N)})}. \tag{4.35}$$

Supposing that a is chosen sufficiently large (but independent of N) we obtain (4.28) and the conclusion of the lemma follows. □

4.2. The construction of the invariant measure via finite dimensional approximations

Let $q=1$ and choose $C>0$ from part (iv) of Proposition 4.1 so that (4.15) holds for all $N\geqslant 1$. Let $\nu^{(N)} := \nu_1^{(N)}$ and

$$\Phi^{(N)} := \Phi_1^{(N)} = \frac{\mathrm{d}\nu^{(N)}}{\mathrm{d}\mu^{(N)}}.$$

Let also $\tilde{\nu}^{(N)} := \nu^{(N)} j_N^{-1}$ and

$$\Psi^{(N)} := \frac{\mathrm{d}\tilde{\nu}^{(N)}}{\mathrm{d}\tilde{\mu}^{(N)}}, \tag{4.36}$$

where $\tilde{\mu}^{(N)}$ is given by (2.19).

Obviously, in light of (4.28),

$$\sup_{N\geqslant 1}\|\Psi^{(N)}\|_{L^2(\tilde{\mu}^{(N)})} \leqslant C. \tag{4.37}$$

The sequence $(\tilde{\nu}^{(N)})_{N\geqslant 1}$ is tight. Indeed, let $\varepsilon>0$ be arbitrary and $K\subseteq\mathbb{H}^m_\rho$ be a compact set such that $\tilde{\mu}^{(N)}[K^c]\leqslant\varepsilon$, $\forall N\geqslant 1$. Using Cauchy–Schwartz inequality and (4.15) we conclude that

$$\tilde{\nu}^{(N)}[K^c] \leqslant \|\Psi^{(N)}\|_{L^2(\tilde{\mu}^{(N)})}\sqrt{\tilde{\mu}^{(N)}[K^c]} \leqslant C\sqrt{\varepsilon}, \quad \forall N\geqslant 1$$

and tightness follows from an application of Prokhorov's theorem.

Suppose that ν_* is a weak limiting point of $(\tilde{\nu}^{(N)})_{N\geqslant 1}$. We show the following.

Proposition 4.4.

(1) $\nu_* \ll \mu$.

(2) *Let* $\Phi_* := \mathrm{d}\nu_*/\mathrm{d}\mu$. *We have* $\Phi_* \in L^2 \log^+ L$, *i.e. the Orlicz space consisting of all functions* F *that satisfy* $\int F^2 \log^+ F \,\mathrm{d}\mu < +\infty$.
(3) ν_* *is invariant for* $\mathscr{L}(\cdot)$.

Proof. *Part* (1). Suppose that $\varepsilon > 0$ and $A \in \mathscr{B}(\mathbb{H}_\rho^m)$ is such that $\mu[A] < \delta := \varepsilon^2/(8C)$. Let $F \in C_b(\mathbb{H}_\rho^m)$ be such that $0 \leqslant F \leqslant 1$ and

$$\int |F - \mathbf{1}_A| \,\mathrm{d}\nu < \varepsilon/4. \tag{4.38}$$

We have

$$\begin{aligned} \nu_*[A] &\leqslant \int F \,\mathrm{d}\nu_* + \varepsilon/4 = \lim_{N\uparrow+\infty} \int F \,\mathrm{d}\tilde{\nu}^{(N)} + \varepsilon/4 \\ &\leqslant \limsup_{N\uparrow+\infty} \left(\int F(\Psi^{(N)})^2 \,\mathrm{d}\tilde{\mu}^{(N)} \right)^{1/2} \left(\int F \mathrm{d}\tilde{\mu}^{(N)} \right)^{1/2} + \varepsilon/4. \end{aligned} \tag{4.39}$$

In light of (4.28) and (4.38) the expression on the utmost right hand side of (4.39) can be estimated by

$$C \left(\int F \,\mathrm{d}\mu \right)^{1/2} + \varepsilon/4 < \varepsilon$$

for a given choice of F and absolute continuity of ν_* w.r.t. μ follows.

Part (2). First, we prove that $\Phi_* \in L^2$. Suppose that $F \in C_b(\mathbb{H}_\rho^m)$ and $\varepsilon > 0$. Then for sufficiently large N we have

$$\int F\Phi_* \,\mathrm{d}\mu \leqslant \int F\Psi^{(N)} \,\mathrm{d}\tilde{\mu}^{(N)} + \varepsilon. \tag{4.40}$$

The right hand side of (4.40) is, by virtue of (4.37) less than or equal to

$$C \left(\int F^2 \,\mathrm{d}\tilde{\mu}^{(N)} \right)^{1/2} + \varepsilon, \quad \forall N \geqslant 1.$$

Letting first $N \uparrow +\infty$ and then subsequently $\varepsilon \downarrow 0$ we conclude that

$$\int F\Phi_* \,\mathrm{d}\mu \leqslant C\|F\|_{L^2}, \quad \forall F \in C_b(\mathbb{H}_\rho^m)$$

hence $\Phi_* \in L^2$.

Note also that in light of (4.28) sequence $(\Pi_N \Phi^{(N)})_{N\geqslant 1}$ is weakly compact in L^2. Let us choose a subsequence $(\Pi_{N'}\Phi^{(N')})$ that corresponds to a subsequence $(\tilde{\nu}^{(N')})$ that weakly converges to $\mathrm{d}\nu_* = \Phi_* \,\mathrm{d}\mu$. We show that $w - \lim_{N'\to+\infty} \Pi_{N'}\Phi^{(N')} = \Phi_*$. With some abuse of the notation we shall omit writing the prime by the subsequence.

In fact, thanks to Propositions 4.2 and 4.3 applied with $t = 0$ we have

$$\begin{aligned} \lim_{N\uparrow+\infty} \int F\Pi_N\Phi^{(N)} \,\mathrm{d}\mu &= \lim_{N\uparrow+\infty} \int \Pi_N J_N F \Pi_N \Phi^{(N)} \,\mathrm{d}\mu \\ &= \lim_{N\uparrow+\infty} \int F \,\mathrm{d}\tilde{\nu}^{(N)} = \int F \,\mathrm{d}\nu_* \\ &= \int F\Phi_* \,\mathrm{d}\mu, \quad \forall F \in C_b(\mathbb{H}_\rho^m). \end{aligned}$$

Hence $(\Pi_N \Phi^{(N)})_{N\geqslant 1}$ converges L^2-weakly to Φ_*. There exists therefore a sequence $(Y_N)_{N\geqslant 1}$ of convex combinations of $(\Pi_N \Phi^{(N)})_{N\geqslant 1}$ that converges L^2-strongly to Φ_*. We can assume, with no loss of generality, that it is also pointwise convergent.

On the other hand, we can conclude from the argument contained in Section 4.1 that $\Phi^{(N)} \in D(\mathscr{E}_{\mathscr{M}_N})$ and

$$S = \sup_{N\geqslant 1} \mathscr{E}_{\mathscr{M}_N}(\Phi^{(N)}, \Phi^{(N)}) < +\infty.$$

Hence, by virtue of part (iii) of Proposition 2.5, we conclude that there exists a constant $C > 0$ such that

$$\int [\Pi_N \Phi^{(N)}]^2 [\log^+(\Pi_N \Phi^{(N)}) + 1]\, \mathrm{d}\mu \leqslant C < +\infty. \tag{4.41}$$

The set $\mathscr{C}$ of elements of L^2 satisfying (4.41) is convex. Thus $(Y_N)_{N\geqslant 1} \subset \mathscr{C}$ and from (4.41) we obtain by virtue of the Fatou lemma that (3.9) holds.

Part (3). Let $F \in C_b(\mathbb{H}^m_\rho)$. Using Proposition 4.2 and the weak convergence of $(\Pi_N \Phi^{(N)})_{N\geqslant 1}$ we conclude that

$$\int Q^t F \,\mathrm{d}\nu_* = \int Q^t F \Phi_* \,\mathrm{d}\mu = \lim_{N\uparrow+\infty} \int \Pi_N Q_N^t J_N F \Pi_N \Phi^{(N)} \,\mathrm{d}\mu. \tag{4.42}$$

Using the definition of $\mu^{(N)}$ we can rewrite the utmost right hand side of (4.42) as being equal to

$$\lim_{N\uparrow+\infty} \int Q_N^t J_N F \Phi^{(N)} \,\mathrm{d}\mu^{(N)} = \lim_{N\uparrow+\infty} \int J_N F \Phi^{(N)} \,\mathrm{d}\mu^{(N)}. \tag{4.43}$$

The last equality follows from the invariance of $\Phi^{(N)}$ under $(Q_N^t)_{t\geqslant 0}$. The right hand side of (4.43) can be further rewritten as being equal to

$$\lim_{N\uparrow+\infty} \int F \,\mathrm{d}\tilde{\nu}^{(N)} = \int F \,\mathrm{d}\nu_*. \qquad \square$$

Remark 4.5. Suppose that $\varepsilon > 0$ is arbitrary. Let us fix a certain $V_0 > 0$. Assume further that μ, the law of $\mathbf{V}(0,\cdot)$, is such that $V_*(\mu) \leqslant V_0$, where $V_*(\mu)$ is defined in (1.6). It is clear from (4.34) that, there exists $a_0(\varepsilon, V_0)$ such that

$$\|\Phi_* - \mathbf{1}\|_{L^2} < \varepsilon, \tag{4.44}$$

provided that $a \geqslant a_0$.

Recall the γ-distance between two Borel probability laws $\mathscr{L}_1$ and $\mathscr{L}_2$ defined on $\mathbb{R}^d$ and possessing the first moments, see p. 330 of Dudley (1989),

$$\gamma(\mathscr{L}_1, \mathscr{L}_2) = \sup\left\{ \left| \int G(\mathbf{x})\, \mathscr{L}_1(\mathrm{d}\mathbf{x}) - \int G(\mathbf{x}) \mathscr{L}_2(\mathrm{d}\mathbf{x}) \right| : \mathrm{Lip}(G) \leqslant 1, \quad G(\mathbf{0}) = 0 \right\}.$$

Here $\mathrm{Lip}(\cdot)$ is the Lipschitz constant of a given function and we assume that both laws possess the first absolute moments. Convergence of laws in γ-metric implies the weak convergence.

Let $\mathscr{L}_1$ be the law of $\mathbf{V}(0,\mathbf{0})$ over the probability space corresponding to probability $\mathbb{P}$ and $\mathscr{L}_2$ be the law $\mathbf{V}(t,\mathbf{x}(t))$ over the probability space corresponding to $\mathbb{P}_*$

(according to Theorem 1.1 it is independent of t). Note that (4.44) implies that,

$$\gamma(\mathscr{L}_1,\mathscr{L}_2)=\sup\left\{\left|\int G(\mathbb{V})(\Phi_*-\mathbf{1})\,\mathrm{d}\mu\right| : G:\mathbb{R}^d\to\mathbb{R},\right.$$
$$\left.\mathrm{Lip}(G)\leqslant 1,\quad G(\mathbf{0})=0\right\}\leqslant\|\mathbb{V}\|_{L^2_d}\|\Phi_*-\mathbf{1}\|_{L^2}<\varepsilon V_0$$

for $a>a_0$. Thus the γ-distance between the law of the Eulerian velocity and that of the stationary Lagrangian velocity asymptotically vanishes as the spectral gap increases to infinity, provided the second absolute moments of the field together with its gradient remain bounded. This result would be of even greater importance if we had an accompanying result proving some sort of statistical stability of the Lagrangian process. More specifically, assume that μQ^t denotes the law of $\mathscr{X}(t)$, see (3.1). It can be quite easily shown, at least in the case when $\kappa>0$, that $\mu Q^t\ll\mu$, see the argument contained in Section 5.1 below. Denote $\mathbf{1}Q^t:=\mathrm{d}\mu Q^t/\mathrm{d}\mu$. Suppose we could prove that

$$\lim_{t\to+\infty}\|\mathbf{1}Q^t-\Phi_*\|_{L^2}=0. \tag{4.45}$$

Then, denoting by $\mathscr{L}(t)$ the law of $\mathbf{V}(t,\mathbf{x}(t))$ over the probability space corresponding to $\mathbb{P}\otimes\mathbb{W}$ we would also have $\gamma(\mathscr{L}(t),\mathscr{L}_2)\to 0$, as $t\to+\infty$. We could claim therefore that the laws of the Lagrangian and Eulerian velocities get closer in γ-distance with the spectral gap increasing to infinity. To show (4.45) one would need to prove for example the spectral gap estimate for the Lagrangian dynamics, which is currently beyond our reach.

5. Ergodicity of the invariant measure

5.1. The case $\kappa>0$

We prove that $\Phi_*>0$, μ-a.s. Indeed, let $A_*:=\operatorname{supp}\Phi_*$. We have $\mu(A_*)>0$ and $(Q^t\mathbf{1}_{A_*^c},\Phi_*)_{L^2}=0$ for all $t\geqslant 0$. In consequence,

$$\mathbb{E}_\mu\int_{\mathbb{R}^d}\mathbf{1}_{A_*^c}(\tau_{\mathbf{y}}(V(t)))p(0,\mathbf{0},t,\mathbf{y};V)\mathbf{1}_{A_*}(V(0))\,\mathrm{d}\mathbf{y}=0$$

which in turn implies

$$0=\mathbb{E}_\mu\mathbf{1}_{A_*^c}(\tau_{\mathbf{y}}(V(t)))\mathbf{1}_{A_*}(V(0))=(U^{\mathbf{y}}P^t\mathbf{1}_{A_*^c},\mathbf{1}_{A_*})_{L^2} \tag{5.1}$$

for m-a.e. $\mathbf{y}\in\mathbb{R}^d$. Here $p(\cdot,\cdot,\cdot,\cdot;V)$ denotes the transition of probability density corresponding to diffusion given by (4.1). C_0-continuity of the group $U^{\cdot}$ implies that (5.1) holds for all $\mathbf{y}\in\mathbb{R}^d$ and in particular also for $\mathbf{y}=\mathbf{0}$. Therefore, $P^t\mathbf{1}_{A_*^c}\leqslant\mathbf{1}_{A_*^c}$. Self-adjointness of P^t implies that also $P^t\mathbf{1}_{A_*}\leqslant\mathbf{1}_{A_*}$. Thus, $\mathbf{1}_{A_*}$ is invariant under P^t and thanks to ergodicity of that semigroup A_* must be of full μ measure.

Let us assume that

$$\mathbf{1}_A(\varphi)=Q^t\mathbf{1}_A(\varphi),\qquad \nu_*\text{-a.s.} \tag{5.2}$$

Since the invariant measure is equivalent with μ we conclude that

$$\mathbf{1}_A(\varphi)=Q^t\mathbf{1}_A(\varphi)=\mathbb{E}_W\int_{\mathbb{R}^d}\mathbf{1}_A(\tau_{\mathbf{y}}(V(t;\cdot,\varphi)))p(0,\mathbf{0},t,\mathbf{y};V(\cdot,\varphi))\,\mathrm{d}\mathbf{y},\qquad\mu\text{-a.s.} \tag{5.3}$$

Then multiplying both sides of (5.3) by $\mathbf{1}_{A^c}(\varphi)$ and integrating over φ with respect to μ we get (since $p(0,\mathbf{0},t,\mathbf{y};V)>0$, $\mathbb{P}_\mu$ a.s.)

$$\mathbf{1}_A(\tau_{\mathbf{y}}(V(t)))\mathbf{1}_{A^c}(V(0))=0$$

for m a.e. $\mathbf{y}$ and $\mathbb{P}_\mu$ a.s. in V. Repeating, from this point on, the argument used to show that A_* is of full measure we infer that the set A is μ, thus also, ν_*-trivial.

5.2. The case $\kappa=0$

We shall assume throughout this section that condition (A) holds, see the statement of Theorem 4.1. Under this assumption we shall prove that Φ_* is a unique invariant density for $(Q^t)_{t\geqslant 0}$ belonging to $L^2\log^+L$. This implies ergodicity of ν_*. Indeed, the existence of $A\in\mathscr{B}(\mathbb{H}^m_\rho)$ that satisfies

$$Q^t\mathbf{1}_A=\mathbf{1}_A,\ \text{in } L^\infty(\nu_*),\quad \forall t\geqslant 0\quad\text{and}\quad 0<\nu_*(A)<1$$

would imply that $\Phi_{1,*}:=\nu_*^{-1}(A)\Phi_*\mathbf{1}_A$ is another such invariant density, thus leading to a contradiction.

Suppose therefore that $\Phi_{1,*}\in L^2\log^+L$ is non-negative, satisfies $\int\Phi_{1,*}\,\mathrm{d}\mu=1$ and

$$\int\Phi_{1,*}Q^tF\,\mathrm{d}\mu=\int\Phi_{1,*}F\,\mathrm{d}\mu,\quad \forall t\geqslant 0,\ F\in L^\infty. \tag{5.4}$$

Then $\Psi:=\Phi_*-\Phi_{1,*}\in L^2\log^+L$ satisfies $\int\Psi\,\mathrm{d}\mu=0$ and

$$\int\Psi Q^tF\,\mathrm{d}\mu=\int\Psi F\,\mathrm{d}\mu,\quad \forall t\geqslant 0,\ F\in L^\infty. \tag{5.5}$$

Recall, see Section 2.1, that $\mathscr{P}$, H_n denote the spaces of all polynomials and Hermite polynomials of degree n in L^2 correspondingly. The following proposition holds.

Proposition 5.1. (i) *For any $F\in H_n$ we have $F\in D(\mathscr{M})$ and*

$$an\|F\|^2_{L^2}\leqslant|(\mathscr{M}F,F)_{L^2}|,\quad \forall n\geqslant 1. \tag{5.6}$$

(ii)

$$\|\nabla F\|_{L^2_d}\leqslant 4Kn^{4/3}\|F\|_{L^2},\quad \forall n\geqslant 1, F\in H_n. \tag{5.7}$$

The constant $K>0$ comes from condition (A).

(iii) *$Q^tF\in\bigcap_{p>1}L^p$ for any $F\in\mathscr{P}$.*

(iv) *Suppose that $G\in L^2$ and $F\in\mathscr{P}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(Q^tF,G)_{L^2}=(\mathscr{L}F,G)_{L^2}, \tag{5.8}$$

where $\mathscr{L}F$ is given by (3.12). *Note that in light of the results contained in parts* (i), (ii) *we have $F\in D(\mathscr{M})\cap W^{1,2}$.*

The proof of this proposition is a bit technical. Not to distract our attention from the principal objective of this section we postpone briefly its presentation.

An immediate corollary of the above proposition and (5.5) is the following.

Corollary 5.2.

$$(\Psi, \mathscr{L}F)_{L^2} = 0, \quad \forall F \in \mathscr{P}. \tag{5.9}$$

Let $\Psi_n := \mathfrak{P}_n \Psi$ and let $l_n := \|\Psi_n\|_{L^2}$. We show that

$$l_n = 0, \quad \forall n \geqslant 1. \tag{5.10}$$

From (5.9) we obtain

$$\begin{aligned}(\Psi_n, \mathscr{M}\Psi_n)_{L^2} &+ (\mathscr{S}\Psi_{n+1}, \Psi_n)_{L^2} + (\mathscr{S}\Psi_{n-1}, \Psi_n)_{L^2}\\ &+ (\Psi_{n+1}, \mathscr{A}\Psi_n)_{L^2} - (\mathscr{A}\Psi_{n-1}, \Psi_n)_{L^2} = 0, \quad \forall n \geqslant 1,\end{aligned} \tag{5.11}$$

with $\mathscr{S}$, $\mathscr{A}$ given by (3.5), (3.6) correspondingly.

Let us fix $N \geqslant 1$. Summing up both sides of equations (5.11) for $1 \leqslant n \leqslant N$ we obtain

$$\sum_{n=1}^{N} (\Psi_n, \mathscr{M}\Psi_n)_{L^2} + 2\sum_{n=1}^{N-1} (\Psi_n, \mathscr{S}\Psi_{n+1})_{L^2} + (\mathbb{V} \cdot \nabla\Psi_N, \Psi_{N+1})_{L^2} = 0.$$

However,

$$(\mathbb{V} \cdot \nabla\Psi_N, \Psi_{N+1})_{L^2} = \int \nabla\Psi_N \cdot \mathfrak{P}_n(\mathbb{V}\Psi)\, \mathrm{d}\mu - (\mathbb{V} \cdot \nabla\Psi_N, \Psi_{N-1})_{L^2}. \tag{5.12}$$

We claim that $\mathbb{V}\Psi \in L^2_d$. This can be seen as follows. Let $k \in \{1, \ldots, d\}$. Suppose that $A > 0$ is sufficiently small so that $\int \mathrm{e}^{A\mathbb{V}_k^2}\, \mathrm{d}\mu < +\infty$. By virtue of Young's inequality

$$\begin{aligned}\int \mathbb{V}_k^2 \Psi^2\, \mathrm{d}\mu &\leqslant \int \left[\left(1 + \frac{\Psi^2}{A}\right) \log\left(1 + \frac{\Psi^2}{A}\right) - \frac{\Psi^2}{A}\right] \mathrm{d}\mu\\ &+ \int \left[\mathrm{e}^{A\mathbb{V}_k^2} - 1 - A\mathbb{V}_k^2\right] \mathrm{d}\mu < +\infty.\end{aligned}$$

Letting $c_N := \|\mathfrak{P}_N(\mathbb{V}\Psi)\|_{L^2}$, we can write using parts (i) and (ii) of Proposition 5.1

$$a\sum_{n=1}^{N} n l_n^2 \leqslant C_1 \left(\sum_{n=1}^{N-1} \sqrt{n}\, l_n l_{n+1} + N^{4/3} c_N l_N + N^{11/6} l_{N-1} l_N\right)$$

for some constant $C_1 > 0$ independent of $N \geqslant 1$, $K > 0$ and $a > 0$. Hence, for a sufficiently large $a > 0$, we have

$$\sum_{n=1}^{N} n l_n^2 \leqslant C_2[N^{4/3} c_N l_N + \tfrac{1}{2} N^{11/6}(l_{N-1}^2 + l_N^2)]. \tag{5.13}$$

We claim first that in fact (5.13) implies that $\sum_{n=1}^{+\infty} n l_n^2 < +\infty$. Indeed, denote the left hand side of (5.13) by S_N and assume that $S_N \uparrow +\infty$. From (5.13) we conclude then

$$\frac{1}{2}(l_{N-1}^2 + l_N^2) + \frac{l_N c_N}{N^{1/2}} \geqslant \frac{1}{C_2} \times \frac{S_N}{N^{11/6}}, \tag{5.14}$$

hence

$$\frac{N}{2}(l_{N-1}^2+l_N^2)+N^{1/2}l_Nc_N \geqslant \frac{1}{C_2}\times\frac{S_N}{N^{5/6}}.$$

We conclude therefore that there exists $C_3>0$ such that

$$N(l_{N-1}^2+l_N^2)+c_N^2 \geqslant C_3\frac{S_N}{N^{5/6}}. \tag{5.15}$$

Since for sufficiently large N we have $S_N\geqslant 1$, (5.15) implies that (remember that $\sum c_N^2<+\infty$) in fact

$$S_N\geqslant C_4N^{1/6} \tag{5.16}$$

for some $C_4>0$. Using once more (5.15), this time together with estimate (5.16), we get $S_N\geqslant C_5N^{1/3}$ for some $C_5>0$. Iterating this procedure 5 times we conclude that $S_N\geqslant C_6N^{5/6}$ for some $C_6>0$. Thus from (5.14) we get

$$\frac{1}{2}(l_{N-1}^2+l_N^2)+\frac{l_Nc_N}{N^{1/2}}\geqslant\frac{C_6}{N}, \tag{5.17}$$

hence we conclude that $\sum_{N\geqslant 1} l_N^2=+\infty$, which leads to a contradiction caused by our assumption that $\lim_{N\to+\infty}S_N=+\infty$.

We have proved therefore that

$$\sum_{N\geqslant 0}(c_N^2+Nl_{N-1}l_N)\leqslant\sum_{N\geqslant 0}(c_N^2+N(l_{N-1}^2+Nl_N^2))<+\infty. \tag{5.18}$$

On the other hand, (5.18) implies in particular that there exists a subsequence $(N_k)_{k\geqslant 1}$, for which

$$\lim_{k\uparrow+\infty}[N_kc_{N_k}^2+N_k^2(l_{N_k-1}^2+l_{N_k}^2)]=0. \tag{5.19}$$

From (5.13) we get however that

$$\sum_{n=1}^{N} nl_n^2\leqslant C_7[Nc_N^2+N^{11/6}(l_{N-1}^2+l_N^2)]$$

for some constant $C_7>0$ and all $N\geqslant 1$. Thus, by virtue of (5.19), we have $\lim_{k\uparrow+\infty}S_{N_k}=0$, which in turn implies (5.10).

5.3. *Proof of Proposition 5.1*

5.3.1. *Part (i)*

To prove (5.6) it suffices to show that

$$na\|F\|^2_{L^2(\mu^{(N)})}\leqslant\mathscr{E}_{\mathscr{M}_N}(F,F) \tag{5.20}$$

for any $F\in H_n^{(N)}$ and all $n,N\geqslant 1$.

With no loss of generality we shall assume for the sake of transparency of the ensuing calculation that the matrices $S_{\mathbf{j}}^{(N)}$ given by (2.17) are non-singular for $\mathbf{j} \in \Lambda_N^+$. Let $e_{1,\mathbf{j}}, \dots, e_{d,\mathbf{j}}$ be the eigenvectors of $S_{\mathbf{j}}^{(N)}$ and $\lambda_{1,\mathbf{j}} \geqslant \cdots \geqslant \lambda_{d,\mathbf{j}} > 0$ be the corresponding to them eigenvalues. For any $\mathbf{j} \in \Lambda_N^+$, $n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$, $a = \sum_{p=1}^d a_p e_{p,\mathbf{j}}$ define

$$h_{\mathbf{j},n}(a) := \bigotimes_{p=1}^{d} h_{n_p}(\lambda_{p,\mathbf{j}}^{-1/2} a_p), \tag{5.21}$$

where $h_n(\cdot)$, $n \geqslant 0$ the standard orthonormal system of Hermite polynomials on $L^2(\mathbb{R}, \nu)$, with ν is the standard d-dimensional Gaussian measure. For any $\mathbf{n} = (n_{\mathbf{j}} \in \mathbb{Z}_+^d;\ \mathbf{j} \in \Lambda_N^+)$, $\mathbf{a} = (a_{\mathbf{j}};\ \mathbf{j} \in \Lambda_N^+)$ we set

$$h_{\mathbf{n}}(\mathbf{a}) := \bigotimes_{\mathbf{j} \in \Lambda_N^+} h_{\mathbf{j},n_{\mathbf{j}}}(a_{\mathbf{j}}). \tag{5.22}$$

The set of all $h_{\mathbf{n}}(\mathbf{a}) \otimes h_{\mathbf{m}}(\mathbf{b})$, with $|\mathbf{n}| + |\mathbf{m}| = n$ forms an orthonormal basis of $H_n^{(N)}$, $n \geqslant 0$. In addition, $h_{\mathbf{n}}(\cdot) \otimes h_{\mathbf{m}}(\cdot)$ are the eigenvectors of the generator of the Ornstein–Uhlenbeck process given by (2.24) corresponding to the eigenvalue $\sum_{\mathbf{j} \in \Lambda_N^+} r(\mathbf{k}_{\mathbf{j}})(|n_{\mathbf{j}}| + |m_{\mathbf{j}}|)$.

Let

$$F(\mathbf{a}, \mathbf{b}) := \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha(\mathbf{n}, \mathbf{m})\, h_{\mathbf{n}}(\mathbf{a}) \otimes h_{\mathbf{m}}(\mathbf{b}), \tag{5.23}$$

where $\alpha(\mathbf{n}, \mathbf{m})$ are certain coefficients. Obviously

$$\|F\|^2_{L^2(\mu^{(N)})} = \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha^2(\mathbf{n}, \mathbf{m}).$$

On the other hand, we obtain that

$$\mathscr{E}_{\mathscr{M}_N}(F, F) = \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha^2(\mathbf{n}, \mathbf{m}) \sum_{\mathbf{j} \in \Lambda_N^+} r(\mathbf{k}_{\mathbf{j}})(|n_{\mathbf{j}}| + |m_{\mathbf{j}}|). \tag{5.24}$$

Using (V3) we conclude (5.20).

5.3.2. Part (ii)

In order to prove (5.7) it suffices to show that

$$\|\nabla^{(N)} F\|_{L_d^2(\mu^{(N)})} \leqslant 4Kn^{4/3} \|F\|_{L^2(\mu^{(N)})} \tag{5.25}$$

for any $F \in H_n^{(N)}$ and all $n, N \geqslant 1$.

For any $\mathbf{n} = (n_{\mathbf{j}};\ \mathbf{j} \in \Lambda_N^+)$ define $\mathbf{n}(p, \mathbf{j}, +) := (\tilde{n}_{\mathbf{j}'};\ \mathbf{j}' \in \Lambda_N^+)$, with $\tilde{n}_{\mathbf{j}'} = n_{\mathbf{j}'}$, $\mathbf{j}' \neq \mathbf{j}$, $\tilde{n}_{q,\mathbf{j}} = n_{q,\mathbf{j}}$, for $q \neq p$ and $\tilde{n}_{p,\mathbf{j}} = n_{p,\mathbf{j}} + 1$. Similarly, $\mathbf{n}(p, \mathbf{j}, -) := (\tilde{n}_{\mathbf{j}'};\ \mathbf{j}' \in \Lambda_N^+)$, with $\tilde{n}_{\mathbf{j}'} = n_{\mathbf{j}'}$, $\mathbf{j}' \neq \mathbf{j}$, $\tilde{n}_{q,\mathbf{j}} = n_{q,\mathbf{j}}$, for $q \neq p$ and

$$\tilde{n}_{p,\mathbf{j}} = (n_{p,\mathbf{j}} - 1)_+ := \begin{cases} n_{p,\mathbf{j}} - 1 & \text{if } n_{p,\mathbf{j}} \geqslant 1, \\ 0 & \text{if } n_{p,\mathbf{j}} = 0. \end{cases}$$

A similar notation is introduced for the multi-index $\mathbf{m}$.

Applying (2.20) to F given by (5.23) we can write, with the help of the above notation,

$$\nabla^{(N)}F(\mathbf{a},\mathbf{b})=\sum_{p=1}^{d}\sum_{\mathbf{j}\in\Lambda_N^+}\sum_{|\mathbf{n}|+|\mathbf{m}|=n}\mathbf{k}_{\mathbf{j}}\alpha(\mathbf{n},\mathbf{m})[\varepsilon(p,\mathbf{j},\mathbf{n},\mathbf{m})h_{\mathbf{n}(p,\mathbf{j},-)}(\mathbf{a})\otimes h_{\mathbf{m}(p,\mathbf{j},+)}(\mathbf{b})$$

$$-\gamma(p,\mathbf{j},\mathbf{n},\mathbf{m})\,h_{\mathbf{n}(p,\mathbf{j},+)}(\mathbf{a})\otimes h_{\mathbf{m}(p,\mathbf{j},-)}(\mathbf{b})],\tag{5.26}$$

with $\gamma(p,\mathbf{j},\mathbf{n},\mathbf{m}):=\sqrt{(n_{p,\mathbf{j}}+1)m_{p,\mathbf{j}}}$, $\varepsilon(p,\mathbf{j},\mathbf{n},\mathbf{m}):=\sqrt{n_{p,\mathbf{j}}(m_{p,\mathbf{j}}+1)}$. To get (5.26) we use the following elementary formulas for Hermite polynomials

$$h_n'(a)=\sqrt{n}\,h_{n-1}(a)\quad\text{ and }\quad ah_n(a)=\sqrt{n+1}\,h_{n+1}(a)-\sqrt{n}\,h_{n-1}(a).$$

A direct calculation shows that

$$\|\nabla^{(N)}F\|^2_{L^2_d(\mu_N)}=\sum_{\mathbf{j},\mathbf{j}'\in\Lambda_N^+}\sum_{p,p'=1}^{d}\sum_{\substack{|\mathbf{n}|+|\mathbf{m}|=n\\|\mathbf{n}'|+|\mathbf{m}'|=n}}\mathbf{k}_{\mathbf{j}}\cdot\mathbf{k}_{\mathbf{j}'}\,\alpha(\mathbf{n},\mathbf{m})\alpha(\mathbf{n}',\mathbf{m}')$$

$$\times[\gamma\gamma'\delta_{\mathbf{n},+,+}\delta_{\mathbf{m},-,-}+\varepsilon\varepsilon'\delta_{\mathbf{n},-,-}\delta_{\mathbf{m},+,+}$$

$$-\gamma\varepsilon'\delta_{\mathbf{n},+,-}\delta_{\mathbf{m},-,+}-\varepsilon\gamma'\delta_{\mathbf{n},-,+}\delta_{\mathbf{m},+,-}].\tag{5.27}$$

Here

$$\delta_{\mathbf{n},s_1,s_2}:=\delta(\mathbf{n}(p,\mathbf{j},s_1),\mathbf{n}'(p',\mathbf{j}',s_2)),$$

$$\delta_{\mathbf{m},s_1,s_2}:=\delta(\mathbf{m}(p,\mathbf{j},s_1),\mathbf{m}'(p',\mathbf{j}',s_2)),$$

for any $s_1,s_2\in\{-,+\}$, γ, γ' are the abbreviations for $\gamma(p,\mathbf{j},\mathbf{n},\mathbf{m})$, $\gamma(p',\mathbf{j}',\mathbf{n}',\mathbf{m}')$ and a similar convention concerns also ε and ε'. The expression corresponding to each of the four terms appearing in parentheses on the right hand side of (5.27) can be dealt with separately.

Let us consider the first term. Using an elementary inequality $ab\leqslant a^2+b^2$ it can be estimated by

$$\sum_{\mathbf{j},\mathbf{j}'\in\Lambda_N^+}\sum_{p,p'=1}^{d}\sum_{\substack{|\mathbf{n}|+|\mathbf{m}|=n\\|\mathbf{n}'|+|\mathbf{m}'|=n}}(|\mathbf{k}_{\mathbf{j}}|^2m_{p,\mathbf{j}}\alpha^2(\mathbf{n},\mathbf{m})+|\mathbf{k}_{\mathbf{j}'}'|^2m'_{p',\mathbf{j}'}\alpha^2(\mathbf{n}',\mathbf{m}'))$$

$$\times[(n_{p,\mathbf{j}}+1)(n'_{p',\mathbf{j}'}+1)]^{1/2}\delta_{\mathbf{n},+,+}\delta_{\mathbf{m},-,-}.\tag{5.28}$$

We split the summation in the expression above into 2 sums corresponding to $\alpha^2(\mathbf{n},\mathbf{m})$ and $\alpha^2(\mathbf{n}',\mathbf{m}')$. We deal with them in the same fashion so we only estimate the first one. For any multi-index $\mathbf{n}=(n_{\mathbf{j}};\mathbf{j}\in\Lambda_N^+)$ we let $J(\mathbf{n}):=[\mathbf{j}\in\Lambda_N:|n_{\mathbf{j}}|>0]$.

$$K^2\sum_{p,\mathbf{j}\in J(\mathbf{m})}\sum_{|\mathbf{n}|+|\mathbf{m}|=n}(n_{p,\mathbf{j}}+1)^{1/2}m_{p,\mathbf{j}}\alpha^2(\mathbf{n},\mathbf{m})$$

$$\times\sum_{p',\mathbf{j}'\in J(\mathbf{n}(p,\mathbf{j},+))}\sum_{|\mathbf{n}'|+|\mathbf{m}'|=n}(n'_{p',\mathbf{j}'}+1)^{1/2}\delta_{\mathbf{n},+,+}\delta_{\mathbf{m},-,-}.\tag{5.29}$$

Here we used the fact that $|\mathbf{k}_\mathbf{j}| \leqslant K$. Because of the presence of the Kronecker symbols the summation over $\mathbf{n}', \mathbf{m}'$, for a given $p, \mathbf{j}, \mathbf{m}, \mathbf{n}, \mathbf{j}', p'$ reduces only to those terms for which

$$\mathbf{n}'(p', \mathbf{j}', +) = \mathbf{n}(p, \mathbf{j}, +), \tag{5.30}$$

$$\mathbf{m}'(p', \mathbf{j}', -) = \mathbf{m}(p, \mathbf{j}, -). \tag{5.31}$$

The solution $\tilde{\mathbf{n}}(p, p', \mathbf{j}, \mathbf{j}') = (\tilde{n}_\mathbf{i}(p, p', \mathbf{j}, \mathbf{j}');\ \mathbf{i} \in \Lambda_N^+)$ of (5.30) is given by

$$\tilde{n}_{q,\mathbf{i}}(p, p', \mathbf{j}, \mathbf{j}') = \begin{cases} n_{q,\mathbf{i}} & \text{if } (q, \mathbf{i}) \notin \{(p, \mathbf{j}), (p', \mathbf{j}')\}, \\ n_{p,\mathbf{j}} + 1 - \delta((p, \mathbf{j}), (p', \mathbf{j}')) & \text{if } (q, \mathbf{i}) = (p, \mathbf{j}), \\ n_{p',\mathbf{j}'} - 1 + \delta((p, \mathbf{j}), (p', \mathbf{j}')) & \text{if } (q, \mathbf{i}) = (p', \mathbf{j}'). \end{cases}$$

Note that $|\tilde{\mathbf{n}}(p, p', \mathbf{j}, \mathbf{j}')| = |\mathbf{n}|$.

Eq. (5.31) has only one solution $\tilde{\mathbf{m}}(p, p', \mathbf{j}, \mathbf{j}') = (\tilde{m}_\mathbf{i}(p, p', \mathbf{j}, \mathbf{j}');\ \mathbf{i} \in \Lambda_N^+)$ satisfying $|\tilde{\mathbf{m}}(p, p', \mathbf{j}, \mathbf{j}')| = |\mathbf{m}|$ whose form depends on the size of $m_{p',\mathbf{j}'}$ and whether, or not $(p, \mathbf{j})$ and $(p', \mathbf{j}')$ coincide.

First, suppose that $m_{p',\mathbf{j}'} \geqslant 1$ or $m_{p',\mathbf{j}'} \geqslant 1$ and $(p, \mathbf{j}) \neq (p', \mathbf{j}')$ then, taking into account that $|\tilde{\mathbf{m}}(p, p', \mathbf{j}, \mathbf{j}')| = |\mathbf{m}|$, the solution to (5.31) is given by

$$\tilde{m}_{q,\mathbf{i}}(p, p', \mathbf{j}, \mathbf{j}') = \begin{cases} m_{q,\mathbf{i}} & \text{if } (q, \mathbf{i}) \notin \{(p, \mathbf{j}), (p', \mathbf{j}')\}, \\ m_{p,\mathbf{j}} - 1 & \text{if } (q, \mathbf{i}) = (p, \mathbf{j}), \\ m_{p',\mathbf{j}'} + 1 & \text{if } (q, \mathbf{i}) = (p', \mathbf{j}'). \end{cases} \tag{5.32}$$

If however $m_{p',\mathbf{j}'} = m_{p,\mathbf{j}} = 0$, or $(p, \mathbf{j}) = (p', \mathbf{j}')$ we have $\tilde{\mathbf{m}}(p, p', \mathbf{j}, \mathbf{j}') = \mathbf{m}$.

The expression appearing (5.29) can be estimated by

$$\begin{aligned} & K^2 \sum_{p,\mathbf{j} \in J(\mathbf{m})} \sum_{|\mathbf{n}|+|\mathbf{m}|=n} (n_{p,\mathbf{j}} + 1)^{1/2} m_{p,\mathbf{j}} \alpha^2(\mathbf{n}, \mathbf{m}) \sum_{p',\mathbf{j}' \in J(\mathbf{n}(p,\mathbf{j},+))} (\tilde{n}_{p',\mathbf{j}'}(p, p', \mathbf{j}, \mathbf{j}') + 1)^{1/2} \\ & = K^2 \sum_{p,\mathbf{j} \in J(\mathbf{m})} \sum_{|\mathbf{n}|+|\mathbf{m}|=n} (n_{p,\mathbf{j}} + 1)^{1/2} m_{p,\mathbf{j}} \alpha^2(\mathbf{n}, \mathbf{m}) \\ & \quad \times \sum_{p',\mathbf{j}' \in J(\mathbf{n}(p,\mathbf{j},+))} (n_{p',\mathbf{j}'}(p, p', \mathbf{j}, \mathbf{j}') + \delta((p, \mathbf{j}), (p', \mathbf{j}')))^{1/2}. \end{aligned} \tag{5.33}$$

The summation over $p', \mathbf{j}' \in J(\mathbf{n}(p, \mathbf{j}, +))$ can be split into the summation over those $p', \mathbf{j}'$-s for which $n_{p',\mathbf{j}'}(p, p', \mathbf{j}, \mathbf{j}') \geqslant n^{1/3}$ (since $\sum |n_\mathbf{j}| \leqslant n$, there are at most $n^{2/3}$ such terms) and those $p', \mathbf{j}'$-s for which the opposite holds. We can estimate therefore the right hand side of (5.33) by

$$2K^2 n^{3/2}(n^{2/3} \times \sqrt{n} + n \times \sqrt{n^{1/3}}) \|F\|^2_{L^2(\mu_N)} = 4K^2 n^{8/3} \|F\|^2_{L^2(\mu_N)}.$$

The remaining terms appearing in (5.27) can be dealt with similarly so the right hand side of this equation can be estimated by $16K^2 n^{8/3} \|F\|^2_{L^2(\mu_N)}$ and inequality (5.25) follows.

5.3.3. Part (iii)

Thanks to (5.6) we have $\mathscr{P} \subseteq D(\mathscr{M})$. In addition, by virtue of assumption (A) we also have $\mathscr{P} \subseteq \bigcap_{m=1}^{+\infty} \bigcap_{p\geqslant 1} W^{p,m}$ and

(1)

$$(t,\mathbf{x}) \mapsto F(\tau_{\mathbf{x}} V(t)), \quad (t,\mathbf{x}) \mapsto \nabla F(\tau_{\mathbf{x}} V(t)), \quad (t,\mathbf{x}) \mapsto \mathscr{M}F(\tau_{\mathbf{x}} V(t))$$

are continuous $\mathbb{P}$-a.s.

(2) For any $q > 0$ the random variables

$$\mathscr{F}_q(\varphi) := \sup_{\mathbf{x}\in\mathbb{R}^d} \frac{|F(\tau_{\mathbf{x}}\varphi)|}{(1+|\mathbf{x}|)^q}, \quad \mathscr{G}_q(\varphi) := \sup_{\mathbf{x}\in\mathbb{R}^d} \frac{|\nabla F(\tau_{\mathbf{x}}\varphi)|}{(1+|\mathbf{x}|)^q} \tag{5.34}$$

belong to $\bigcap_{p\geqslant 1} L^p$.

Using (3.2) and (5.34) we can write

$$|Q^T F(\varphi)| \leqslant \mathbb{E}_W[\mathscr{F}_q(V(T;\cdot,\varphi))(1+|\mathbf{x}(T;\cdot,\varphi)|)^q], \quad \mu\text{-a.s.} \tag{5.35}$$

For each n we define

$$Y_n(t,\mathbf{x};\omega) := \frac{\mathbf{V}(t,\mathbf{x};\omega,\varphi)}{(|\mathbf{x}|+n)^{1/2}}, \quad (t,\mathbf{x})\in[0,T]\times\mathbb{R}^d$$

a Gaussian random field defined over $\mathscr{T}_W \otimes \mathscr{T}_2$. Let

$$K_n(\lambda) := \left[(\omega,\varphi):\ \sup_{(t,\mathbf{x})\in[0,T]\times\mathbb{R}^d} |Y_n(t,\mathbf{x})| \leqslant \lambda\right], \quad n \geqslant 1.$$

By virtue of Theorem 5.2, p. 120 of Adler (1990), there exist λ_0, C_1, C_2 independent of n such that

$$\mathbb{P}_\mu(K_n^c(\lambda_0)) \leqslant C_1 \exp\{-C_2 n\}, \quad \forall n \geqslant 1. \tag{5.36}$$

From (1.2) we obtain that $X_T(\omega,\varphi) := \sup_{0\leqslant t\leqslant T} |\mathbf{x}(t;\omega,\varphi)|$ satisfies

$$X_T(\omega,\varphi) \leqslant C_3(1+n^{1/2}) \quad \text{for } (\omega,\varphi)\in K_n(\lambda_0), \quad n \geqslant 1 \tag{5.37}$$

for some deterministic constant $C_3 > 0$ depending only on λ_0, T. Thus,

$$\mathbb{P}_\mu(X_T \geqslant C_3 n^{1/2}) \leqslant C_1 \exp\{-C_2 n\}, \quad \forall n \geqslant 1. \tag{5.38}$$

In particular, (5.38) implies that $X_T \in \bigcap_{p\geqslant 1} L^p(\mathbb{P}_\mu)$ and the conclusion of part (iii) follows from this, condition (2) and (5.35).

5.3.4. Part (iv)

For any $F\in\mathscr{P}$ and $G\in L^2$ we have

$$(Q^t F, G)_{L^2} = \mathbb{E}_\mu[G(V(0))F(\tau_{\mathbf{x}(t)}V(t))]. \tag{5.39}$$

Note that

$$F(\tau_{\mathbf{x}(t)}V(t)) = F(V(t)) + \int_0^t \nabla F(\tau_{\mathbf{x}(s)}V(t))\cdot\mathbf{V}(s,\mathbf{x}(s))\,\mathrm{d}s \tag{5.40}$$

and

$$F(V(t)) - F(V(0)) - \int_0^t \mathscr{M}F(V(s))\,\mathrm{d}s, \quad t \geqslant 0 \tag{5.41}$$

is a $(\mathscr{V}_t)_{t\geqslant 0}$ martingale. Using both (5.40), (5.41) we obtain

$$\frac{1}{t}[(Q^tF,G)_{L^2}-(F,G)_{L^2}]$$

$$=\frac{1}{t}\left\{\int_0^t (P^s\mathscr{M}F,G)_{L^2}\,\mathrm{d}s\right.$$

$$\left.+\int_0^t \mathbb{E}_\mu[\nabla F(\tau_{\mathbf{x}(s)}V(t))\cdot\mathbf{V}(s,\mathbf{x}(s))G(V(0))]\,\mathrm{d}s\right\}. \tag{5.42}$$

Formula (5.8) is obtained after taking the limit with $t\downarrow 0$ in (5.42). Passage to the limit under the integrals can be justified with the help of the estimates obtained in the previous section.

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