Probab. Theory Relat. Fields (2001)
Digital Object Identifier (DOI) 10.1007/s004400100142

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On homogenization of time-dependent random flows

Received: 25 February 2000 / Revised version: 11 December 2000 /
Published online: 14 June 2001 –

Abstract. We study a diffusion with a random, time dependent drift. We prove the invariance principle when the spectral measure of the drift satisfies a certain integrability condition. This result generalizes the results of [13, 7].

1. Introduction

We consider the long time behavior of a diffusive particle in a random flow described by the Itô stochastic differential equation

$$\begin{cases} d\mathbf{x}(t) = \mathbf{V}(t, \mathbf{x}(t); \omega)dt + \sqrt{2}d\mathbf{w}(t), \\ \mathbf{x}(0) = \mathbf{0}, \end{cases} \tag{1.1}$$

where $\mathbf{V} : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a zero mean, stationary, ergodic random vector field over a certain probability space $(\Omega, \mathscr{V}, P)$. We assume further that the realizations of $\mathbf{V}$ are almost surely of divergence free in the sense that, for any smooth compactly supported function $\varphi(\mathbf{x})$ on $\mathbb{R}^d$, and any $t \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \mathbf{V}(t, \mathbf{x}; \omega) \cdot \nabla_x \varphi(\mathbf{x})\, d\mathbf{x} = 0 \qquad P - a.s.,$$

with P the underlying probability measure. In order to guarantee the existence of the solution of (1.1) we will also assume that $\mathbf{V}(t, \mathbf{x}; \omega)$ is $(P-a.s.)$ locally Lipschitz in $\mathbf{x}$. We are interested in proving an invariance principle for $\mathbf{x}(t)$, i.e. the convergence in distribution of the process $\varepsilon\mathbf{x}(\varepsilon^{-2}t)$ to a Brownian motion with a certain co-variance matrix, sometimes referred to as the *effective diffusivity*, $\mathbf{D} \geq 2\mathbf{I}$. This problem has been widely studied under various conditions on the random flow. Typically the flow is assumed to be the divergence of a stationary random anti-symmetric matrix valued field $\mathbf{H}(t, \mathbf{x}; \omega) = \{H_{p,q}(t, \mathbf{x}; \omega)\}$ – the so-called *stream matrix*:

$$\mathbf{V}(t, \mathbf{x}; \omega) = \nabla_{\mathbf{x}} \cdot \mathbf{H}(t, \mathbf{x}; \omega)\ .$$

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Mathematics Subject classification (2000): Primary 60F17, 35B27; Secondary 60G44

Key words or phrases: Invariance principle – Homogenization – Martingale

This stream matrix was assumed bounded ([13]), or with some finite p-moments for $p > d + 2$ ([7]). The novelty of the present paper is that our assumption carries only on the spatial energy spectrum of the flow. More precisely, suppose that $\mathbf{V}$ has the second absolute moment and consider its co-variance:

$$C_{p,q}(t, \mathbf{x}) = \int V_p(t, \mathbf{x}, \omega) V_q(0, \mathbf{0}, \omega)\, dP(\omega), \qquad p, q = 1, \cdots, d.$$

The *spatial energy spectrum* $e_{\mathbf{V}}(d\mathbf{k})$ of $\mathbf{V}$ is a measure defined by

$$\mathrm{trace}[C(0, \mathbf{x})] = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(\mathbf{x}\cdot\mathbf{k})}\, e_{\mathbf{V}}(d\mathbf{k})$$

We assume that

$$\int_{\mathbb{R}^d} \frac{e_{\mathbf{V}}(d\mathbf{k})}{|\mathbf{k}|^2} < +\infty.$$

This condition implies the existence of a stationary stream matrix with the right properties. Observe that we do not put any restrictions on the time energy spectrum. As a consequence our result includes the static flow ($\mathbf{V}(t, \mathbf{x}, \omega) = \mathbf{V}(\mathbf{x}, \omega)$) considered earlier in section 7 of [8].

However we should point out that our argument does not take any advantage from eventual mixing properties of the flow in the time direction. It remains an open problem how to improve this condition taking into account the time-mixing of the flow (cf. [2, 5, 6] for results in this direction, assuming a Markovian evolution of the flow).

We will outline now the main ideas of the proof. The goal is to show that the rescaled process

$$\varepsilon\mathbf{x}(\varepsilon^{-2}t) = \mathbf{x}_\varepsilon(t) = \sqrt{2}\varepsilon\mathbf{w}\left(\frac{t}{\varepsilon^2}\right) + \varepsilon \int_0^{t/\varepsilon^2} \mathbf{V}(s, \mathbf{x}(s); \omega)\, ds, \tag{1.2}$$

converges in law to a Brownian motion with a certain positive variance matrix. The general strategy (cf. [11] for example) suggests to find an approximation of the second term on the right hand side of (1.2) by a sequence of martingales, and then apply a central limit theorem for martingales. It turns out that the Brownian motion coming out from this term is orthogonal to $\mathbf{w}(t)$.

In order to find such martingale approximation, one exploits the translation invariance of the system. Since $\mathbf{V}(s, \mathbf{x}; \omega)$ is a stationary process on $\mathbb{R}^{d+1}$, it can be written as $\tilde{\mathbf{V}}(\tau_{t,\mathbf{x}}\omega)$ for a corresponding vector valued function $\tilde{\mathbf{V}}$ on Ω and a group of measure preserving transformation $\{\tau_{t,\mathbf{x}}\}$. Then one can look at the *environment as seen form the particle*: this is a Markov process on the probability space Ω defined by

$$\eta(t) = \tau_{t,\mathbf{x}(t)}\omega \tag{1.3}$$

The generator $\mathscr{L}$ of this Markov process can be explicitly computed (cf. (3.2)), and it turns out that $P(d\omega)$ is invariant and ergodic. The term of (1.2) that we want to study can be written as a functional of the process $\eta(s)$:

$$\varepsilon \int_0^{t/\varepsilon^2} \mathbf{V}(s, \mathbf{x}(s); \omega)\, ds = \varepsilon \int_0^{t/\varepsilon^2} \tilde{\mathbf{V}}(\eta(s))\, ds \, . \tag{1.4}$$

At this point one should look for a sequence of functions $u_n(\omega)$ in the domain of the generator $\mathscr{L}$ such that $\mathscr{L}u_n$ approximate $\tilde{V}_p$ $(p = 1, \ldots, d)$, in the sense that the variance of

$$\varepsilon \int_0^{t/\varepsilon^2} \left(\tilde{V}_p - \mathscr{L}u_n\right)(\eta(s))\, ds \tag{1.5}$$

will be small for large n and small ε. This will permit to approximate (1.4) with the martingales

$$\varepsilon \left[u_n(\eta(t\varepsilon^{-2})) - u_n(\eta(0))\right] - \varepsilon \int_0^{t/\varepsilon^2} \mathscr{L}u_n(\eta(s))\, ds \tag{1.6}$$

In order to find such sequence of functions one can look at the solution of the resolvent equation

$$\lambda u_\lambda - \mathscr{L}u_\lambda = -\tilde{V}_p \tag{1.7}$$

and choosing $\lambda = \varepsilon^2$ one tries to make the approximation at the same time as the limit in $\varepsilon \to 0$ (cf. [10]). Because of the lack of spectral gap of generator $\mathscr{L}$ (as an operator on $L^2(\Omega, P)$), the L^2–norm of u_λ can explode as $\lambda \to 0$. The above strategy will work if one has some control on this L^2–norm, in particular if

$$\lambda \int u_\lambda(\omega)^2 \; P(d\omega) \underset{\lambda \to 0}{\longrightarrow} 0 \tag{1.8}$$

the one can easily see that the boundary terms arising in the corresponding martingale approximation (1.6) will vanish in the limit. Furthermore (1.8) also implies the convergence of the quadratic variation of these martingales. So the (1.8) is the central point of the proof and it is the content of Lemma 5.1.

Our proof of (1.8) is inspired by [14]. The difference here is that, because of the time dependence of the random field, an extra term appears in the generator $\mathscr{L}$, due to the shift in the time direction. To deal with this term (the first in the equation 5.2), we need to prove a sublinear growth of the L^1–norm of the *correctors* (instead of the usual L^2 sublinear growth, cf. [15], sufficient in the static case, cf. [14]). This key estimate is proven in section 4, proposition 4.2.

The paper is organized as follows. In the next section we give the precise setup and we state the main result. In section 3 we construct the *environment process* $\eta(t)$ and we study the main properties of it. In particular, we need to prove certain regularity properties of the functions contained in the domain of the generator. Section 4 contains the proof of the L^1 sublinear growth of the correctors. Section 5 contains the proof of (1.8) and the proof of the main theorem. Tightness is shown following the argument of [17]. In the appendix we prove the global existence of the process.

2. Setup and the statement of the main result

Let P be a certain Borel probability measure on a Polish space (Ω, d). Let $\tau_{t,\mathbf{x}}$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ be a *stochastically continuous, measurable* group of *measure preserving* transformations acting *ergodically* on Ω, see [13] pp. 204-206 for definitions. The stochastic continuity of the group $\tau_{t,\mathbf{x}}$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ implies that $U^{t,\mathbf{x}}\tilde{f}(\omega) := \tilde{f}(\tau_{t,\mathbf{x}}(\omega))$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is a strongly continuous group of unitary maps in L^2. Here $L^p := L^p(\Omega, \mathscr{B}(\Omega), P)$, $1 \le p \le +\infty$. The group possesses $d + 1$ generators defined as

$$D_q \tilde{f} := \frac{d}{dh} U^{h\mathbf{e}_q} \tilde{f}\,|_{h=0} \qquad q = 0, 1, \cdots, d$$

with $\mathbf{e}_p$, $p = 0, 1, \cdots, d$ the canonical basis of $\mathbb{R} \times \mathbb{R}^d$. Using these generators, given a positive integer m, we can define for any $1 \le p < +\infty$ the Sobolev spaces $W^{p,m}$ consisting of those $\tilde{f} \in L^p$ that belong to the domains of $D_1^{m_1} \cdots D_d^{m_d}$, where $m_1 + \cdots + m_d \le m$. On $W^{p,m}$ we consider the norm $\|\tilde{f}\|_{p,m}^p := \sum_{m_1+\cdots+m_d \le m} \|D_1^{m_1} \cdots D_d^{m_d} \tilde{f}\|_{L^p}^p$. This definition can be extended in an obvious way to include also the case when $p = +\infty$. By $C_b^{k,m}$ we denote the space consisting of those elements $\tilde{f} \in L^\infty$, for which $\tilde{f}(\tau_{t,\mathbf{x}}(\omega))$ are k times differentiable in t, m times differentiable in $\mathbf{x}$, P a.s., and whose all relevant derivatives are bounded. We adopt the usual convention that vanishing of either k or m means just continuity in the appropriate variable. Let $\mathscr{C} := \bigcap_{k,m \ge 0} C_b^{k,m}$. Another consequence of the strong continuity of $U^{t,\mathbf{x}}$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is the fact that there exists an orthogonal projection valued measure $E(d\tau, d\mathbf{k})$ defined on $(\mathbb{R} \times \mathbb{R}^d, \mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R}^d))$ such that for any $\tilde{f}, \tilde{g} \in L^2$

$$(U^{t,\mathbf{x}}\tilde{f}, \tilde{g})_{L^2} = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(t\tau + \mathbf{x}\cdot\mathbf{k})} (E(d\tau, d\mathbf{k})\tilde{f}, \tilde{g})_{L^2} \qquad \forall (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d. \quad (2.1)$$

For any $\tilde{f} \in L^2$ we can define its *spatial spectral measure* as the random measure given by $\hat{f}(A) := E(\mathbb{R} \times A)\tilde{f}$, $A \in \mathscr{B}(\mathbb{R}^d)$. Its structure measure $e_f(A) := \|\hat{f}(A)\|_{L^2}^2$ is sometimes called the *spatial energy spectrum* of $\tilde{f}$.

We denote by L_d^2 the space of all square integrable d-dimensional random vectors $\tilde{F} = (\tilde{F}_1, \cdots, \tilde{F}_d)$ equipped with the usual Hilbert space norm $\|\tilde{F}\|_{L_d^2}^2 := (\tilde{F}, \tilde{F})_{L_d^2}$ with $(\tilde{F}, \tilde{G}) := \sum_{p=1}^{d} (\tilde{F}_p, \tilde{G}_p)_{L^2}$. $\hat{F}(A) := (\hat{F}_1(A), \cdots, \hat{F}_d(A))$ is the *spatial spectral measure* of any $\tilde{F} \in L_d^2$. The numerical measure $e_F(A) :=$ trace $\int \hat{F}(A) \otimes \hat{F}(A) dP$, $A \in \mathscr{B}(\mathbb{R}^d)$ is referred to as the *spatial energy spectrum* of $\tilde{F}$.

The *abstract gradient* operator $\nabla : W^{2,1} \to L_d^2$ is defined by

$$\nabla \tilde{f} := (D_1 \tilde{f}, \cdots, D_d \tilde{f}) \in L_d^2,$$

and the *abstract Laplacian* as $\triangle \tilde{f} := \sum_{q=1}^d D_q^2 \tilde{f}$, defined for sufficiently smooth $\tilde{f}$ e.g. $\tilde{f} \in W^{p,2}$. Suppose now that

V 1) $\tilde{\mathbf{V}} \in L^2_d$ and it has *mean zero* i.e. $\int \tilde{\mathbf{V}} dP = \mathbf{0}$
V 2) $\tilde{\mathbf{V}}$ is of *divergence free*, i.e. for any $\tilde{\phi} \in W^{2,1}$ we have $\int \tilde{\mathbf{V}} \nabla \tilde{\phi} dP = 0$.
V 3) $e_{\mathbf{V}}$, the spatial energy spectrum of $\tilde{\mathbf{V}}$, satisfies $\int\limits_{\mathbb{R}^d} \frac{e_{\mathbf{V}}(d\mathbf{k})}{|\mathbf{k}|^2} < +\infty$.

Conditions V 3) allows to define an anti-symmetric matrix $\tilde{\mathbf{H}} = [\tilde{H}_{p,q}]$ - the so-called *stream matrix* - by

$$\tilde{H}_{p,q} := -\mathrm{i} \int\limits_{\mathbb{R}^d} \frac{k_p \hat{V}_q(d\mathbf{k}) - k_q \hat{V}_p(d\mathbf{k})}{|\mathbf{k}|^2}. \tag{2.2}$$

Here $\hat{\mathbf{V}}(d\mathbf{k}) = (\hat{V}_1(d\mathbf{k}), \cdots, \hat{V}_d(d\mathbf{k}))$ is the spatial spectral measure corresponding to $\tilde{\mathbf{V}}$. By condition V 3) it is easy to see that $\tilde{H}_{p,q} \in L^2$. Then, using conditions V 1) and V 2), it is not difficult to prove that $\tilde{\mathbf{V}} = \nabla \cdot \tilde{\mathbf{H}}$.

We define the random velocity as

$$\mathbf{V}(t, \mathbf{x}; \omega) := \tilde{\mathbf{V}}(\tau_{t,\mathbf{x}}(\omega)). \tag{2.3}$$

Remark. In what follows tilde sign will be used to distinguish between a random element $\tilde{f}$ defined on Ω and the corresponding stationary object $f(t, \mathbf{x}; \omega) := \tilde{f}(\tau_{t,\mathbf{x}}(\omega))$. With this notation we have

$$\mathbf{V}(t, \mathbf{x}; \omega) = \nabla_{\mathbf{x}} \cdot \mathbf{H}(t, \mathbf{x}; \omega)$$

We assume that $\mathbf{V}(t, \mathbf{x}; \omega)$ satisfies the following regularity condition.

V 4) For any compact set $K \subseteq \mathbb{R}^d$ and $T > 0$ there exists a random constant $C_\omega(T, K) < +\infty$ such that

$$\sup_{0 \le t \le T} |\mathbf{V}(t, \mathbf{x}; \omega) - \mathbf{V}(t, \mathbf{y}; \omega)| \le C_\omega(T, K)|\mathbf{x} - \mathbf{y}|, \qquad \mathbf{x},\ \mathbf{y} \in K,\ P\ a.s.$$

and $\sup\limits_{0 \le t \le T} |\mathbf{V}(t, \mathbf{0}; \omega)| < +\infty,\ P\ a.s.$.

By Itô's existence and uniqueness theorem for solutions of stochastic differential equations it is well known that under the above condition the Itô equation

$$d\mathbf{x}_{s,\mathbf{x}}(t; \omega, \sigma) = \mathbf{V}(t, \mathbf{x}_{s,\mathbf{x}}(t; \omega, \sigma); \omega)\, dt + \sqrt{2} d\mathbf{w}(t; \sigma), \tag{2.4}$$
$$\mathbf{x}_{s,\mathbf{x}}(s; \omega, \sigma) = \mathbf{x}$$

possesses a unique local solution, i.e. determined up to a possible explosion time, for P a.s. ω. Here $\mathbf{w}(t; \sigma)$, $t \ge 0$ is a standard Brownian Motion given on a certain probability space $(\Sigma, \mathscr{B}, Q)$. We denote by $\mathbf{M}$, the mathematical expectation corresponding to measure Q and by $\mathscr{W}_t$, $t \ge 0$ the filtration of σ-algebras generated by $\mathbf{w}_s$, $s \le t$. In the special case when both $s = 0$ and $\mathbf{x} = \mathbf{0}$ we omit writing the subscript by the trajectory.

If we assume in addition that $\|\tilde{\mathbf{V}}\|_{L^\infty} < +\infty$ then, due to incompressibility of the drift, it can be shown, see [16] Theorem 2 p. 500, that for any $f \in L^p$, $p \ge 1$

the process $\tilde{f}(\tau_{t,\mathbf{x}(t)}(\omega))$, $t \geq 0$ is stationary with respect to $P \otimes Q$. This fact can be used to show that the trajectories of (2.4) do not explode P a.s. for any velocity field $\mathbf{V}(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ satisfying V 1) - 4). Indeed the following proposition holds.

Proposition 2.1. *Suppose that* $\tilde{\mathbf{V}}$ *satisfies conditions V 1), 2) and 4). Then the following hold.*

a) There exists a sequence of $\tilde{\mathbf{V}}^{(n)} \in \mathscr{C}$, $n \geq 1$ *such that each* $\tilde{\mathbf{V}}^{(n)}$ *satisfies V 1), 2) and for any compact* $K \subseteq \mathbb{R}^d$ *and* $T > 0$ *we have*

$$\lim_{n\uparrow+\infty} \int_0^T \sup_{\mathbf{x}\in K} |\mathbf{V}^{(n)}(t,\mathbf{x};\omega) - \mathbf{V}(t,\mathbf{x};\omega)|\, dt = 0,$$

P a.s. in ω.

b) Suppose $\mathbf{x}^{(n)}_{s,\mathbf{x}}(t;\omega)$, $t \geq s$ *is the sequence of the solutions of*

$$d\mathbf{x}^{(n)}_{s,\mathbf{x}}(t;\omega) = \mathbf{V}^{(n)}(t,\mathbf{x}^{(n)}_{s,\mathbf{x}}(t;\omega);\omega)\, dt + \sqrt{2}d\mathbf{w}(t), \tag{2.5}$$

with $\mathbf{x}^{(n)}_{s,\mathbf{x}}(s) = \mathbf{x}$. *Then, for any* $T > s$, $\mathbf{x} \in \mathbb{R}^d$

$$\lim_{m,n\uparrow+\infty} \sup_{s\leq t\leq T} |\mathbf{x}^{(n)}_{s,\mathbf{x}}(t) - \mathbf{x}^{(m)}_{s,\mathbf{x}}(t)| = 0 \tag{2.6}$$

for P a.s. ω.

c) Denote by $\mathbf{x}_{s,\mathbf{x}}(t;\omega)$, $t \geq s$, $\mathbf{x} \in \mathbb{R}^d$ *the respective limits of the solution of (2.5). For P a.s. ω they are unique global solutions of (2.4). Furthermore, they are non-degenerate, i.e. for any Borel measurable set* $A \in \mathbb{R}^d$, *for which* $m_d(A) > 0$ *we have* $P^\omega(s,\mathbf{x},t,A) > 0$, *$P$ a.s. Here m_d denotes the d-dimensional Lebesgue measure and* $P^\omega(s,\mathbf{x},t,\cdot)$ *is the distribution of* $\mathbf{x}_{s,\mathbf{x}}(t,\omega)$ *in* $\mathbb{R}^d$.

Because of the technical character of this proposition we present its proof in the appendix.

The main theorem we set out to prove in this paper can be formulated as follows.

Theorem 2.2. *We suppose that a velocity field* $\mathbf{V}(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ *is given by (2.3), satisfies the assumptions V 1) - 4). Then, the laws of the trajectories* $\mathbf{x}_\varepsilon(t) := \varepsilon\mathbf{x}(t/\varepsilon^2)$, $t \geq 0$ *converge, in probability with respect to ω, as* $\varepsilon \downarrow 0$, *to the law of a Brownian motion with a non-trivial co-variance matrix* $\mathbf{D} \geq 2\mathbf{I}$.

The co-variance matrix $\mathbf{D} \geq 2\mathbf{I}$ can be also expressed with variational formulas, exactly as in [13].

3. The environment process

For any vector field $\tilde{\mathbf{V}}$ satisfying the assumptions of Theorem 2.2 we define the Markovian process over the probability space $(\Sigma, \mathscr{B}, Q)$ with the state space $(\Omega, \mathscr{B}(\Omega))$ by

$$\eta(t) := \tau_{t,\mathbf{x}(t)}(\omega). \tag{3.1}$$

We denote by $P^t_{\mathbf{V}}\tilde{f} := \mathbf{M}\tilde{f}(\tau_{t,\mathbf{x}(t)}(\omega))$, $t \geq 0$, $\tilde{f} \in L^2$ its L^2 semigroup and by $\mathscr{L}_{\mathbf{V}}$ its generator. The measure P is invariant i.e. $\int P^t\tilde{f}\,dP = \int \tilde{f}\,dP$, for any $\tilde{f} \in L^2$. In fact, it is ergodic, i.e. any $\tilde{f} \in L^2$ satisfying $\int P^t\tilde{f}\,\tilde{g}\,dP = \int \tilde{f}\,\tilde{g}\,dP$, for all $t > 0$ and $\tilde{g} \in L^2$, must be P-a.s. constant. To show this assertion we suppose that $C \in \mathscr{B}(\Omega)$ is such that

$$\int \mid P^t\mathbf{1}_C - \mathbf{1}_C \mid dP = \int \left| \int_{\mathbb{R}^d} \mathbf{1}_C(\tau_{t,\mathbf{y}}(\omega))P^\omega(t,dy) - \mathbf{1}_C(\omega) \right| P(d\omega) = 0.$$

We have then $\int_{\mathbb{R}^d} \mathbf{1}_C(\tau_{t,\mathbf{y}}(\omega))P^\omega(t,dy) \equiv 0$ or 1, P a.s. depending on whether $\omega \notin C$ or $\omega \in C$. Since $P^\omega(t,\cdot)$ charges any set of positive Lebesgue measure we conclude that $P(\tau_{t,\mathbf{y}}(C)\triangle C) = 0$, for all $(t,\mathbf{y}) \in \mathbb{R}\times\mathbb{R}^d$ and in consequence $P(C) = 0$ or 1 by virtue of ergodicity of the group τ.

An elementary calculation shows that for any $\tilde{f} \in \mathscr{C}$ we have

$$\mathscr{L}_{\mathbf{V}}\tilde{f} = D_0\tilde{f} + \triangle\tilde{f} + \tilde{\mathbf{V}}\cdot\nabla\tilde{f} \tag{3.2}$$

In fact the following proposition holds, cf. Proposition 3.1 of [13].

Proposition 3.1. *Assume that* $\tilde{\mathbf{V}} \in L^\infty$. *Then* $\mathscr{C}$ *is a core of* $\mathscr{L}_{\mathbf{V}}$. *The domain of the generator* $\mathscr{D}(\mathscr{L}_{\mathbf{V}}) \subseteq W^{2,1} \cap \mathscr{D}(D_0)$ *and*

$$-(\mathscr{L}_{\mathbf{V}}\tilde{f},\tilde{f})_{L^2} = \|\nabla\tilde{f}\|^2_{L^2_d}, \quad \textit{for all } \tilde{f} \in \mathscr{D}(\mathscr{L}_{\mathbf{V}}). \tag{3.3}$$

For any $\lambda > 0$ we denote by R_λ the resolvent of the generator, that is the operator determined by $I = (\lambda - \mathscr{L}_{\mathbf{V}})R_\lambda = R_\lambda(\lambda - \mathscr{L}_{\mathbf{V}})$. The operator norm of the resolvent in L^2 satisfies

$$\|R_\lambda\| \leq 1/\lambda. \tag{3.4}$$

Proposition 3.2. *Assume that* $\tilde{\mathbf{V}}$ *satisfies V1) - 4). Then the domain of the generator* $D(\mathscr{L}_{\mathbf{V}}) \subseteq W^{2,1}$.

Proof. By virtue of Proposition 3.1 the conclusion of Proposition 3.2 holds for $\tilde{\mathbf{V}} \in L^\infty$. To show the proposition it suffices to prove that for any $\tilde{g} \in \mathscr{C}$ we have $\tilde{f}_\lambda := R_\lambda\tilde{g} \in W^{2,1}$ and $\|\tilde{f}_\lambda\|_{L^2} \leq \frac{1}{\lambda}\|\tilde{g}\|_{L^2}$, $\|\nabla\tilde{f}_\lambda\|_{L^2_d} \leq \frac{1}{\sqrt{\lambda}}\|\tilde{g}\|_{L^2}$. The first inequality follows from (3.4). The second can be argued by approximation as follows. Let $n \geq 1$ be an integer. Suppose that $\mathbf{x}^{(n)}(t)$, $t \geq 0$ are the solutions of (2.5) satisfying $\mathbf{x}^{(n)}(0) = \mathbf{0}$ with $\tilde{\mathbf{V}}^{(n)}$ as in the statement of Proposition 2.1. We define then $\tilde{f}_\lambda^{(n)} := R_\lambda^{(n)}\tilde{g}$, where $R_\lambda^{(n)}$ is the resolvent of $\mathscr{L}_{\mathbf{V}^{(n)}}$, i.e. the unique solutions of

$$\lambda \tilde{f}_\lambda^{(n)} - \mathscr{L}_{\mathbf{V}^{(n)}} \tilde{f}_\lambda^{(n)} = \tilde{g}. \tag{3.5}$$

It is well known that

$$\tilde{f}_\lambda^{(n)}(\omega) = \int_0^{+\infty} e^{-\lambda s} \mathbf{M}\tilde{g}(\tau_{s,\mathbf{x}^{(n)}(s)}(\omega))\, ds.$$

Using a similar expression for $\tilde{f}_\lambda = R_\lambda \tilde{g}$ and (2.6) we infer, by Proposition 2.1 and via the Lebesgue Dominated Convergence Theorem - applicable here since $\tilde{g}$ is bounded – that

$$\lim_{n\uparrow+\infty} \|\tilde{f}_\lambda^{(n)} - \tilde{f}_\lambda\|_{L^2} = 0. \tag{3.6}$$

Multiplying both sides of (3.5) by $\tilde{f}_\lambda^{(n)}$ and integrating over $P(d\omega)$ we obtain that $\|\tilde{f}_\lambda^{(n)}\|_{L^2} \le \|\tilde{g}\|_{L^2}/\lambda$ and $\|\nabla \tilde{f}_\lambda^{(n)}\|^2_{L^2_d} \le \|\tilde{g}\|_{L^2}\|\tilde{f}_\lambda^{(n)}\|_{L^2}$. We conclude therefore that $\tilde{f}_\lambda \in W^{2,1}$, $\nabla \tilde{f}_\lambda^{(n)} \rightharpoonup \nabla \tilde{f}_\lambda$, weakly in L^2_d, as $n \uparrow +\infty$. Since $\liminf_n \|\nabla \tilde{f}_\lambda^{(n)}\|_{L^2_d} \ge \|\nabla \tilde{f}_\lambda\|_{L^2_d}$, we have

$$\lambda\|\tilde{f}_\lambda\|^2_{L^2} + \|\nabla \tilde{f}_\lambda\|^2_{L^2_d} \le (\tilde{g}, \tilde{f}_\lambda)_{L^2} = \lambda\|\tilde{f}_\lambda\|^2_{L^2} + ((-\mathscr{L}_{\mathbf{V}}) f_\lambda, f_\lambda)_{L^2} \tag{3.7}$$

Notice also that (3.7) remains valid for an arbitrary $\tilde{g} \in L^2$ with no assumption on its boundedness. □

It follows immediately from 3.7 that for any $\tilde{g} \in L^2$, for $\tilde{f}_\lambda = R_\lambda \tilde{g}$

$$\|\nabla \tilde{f}_\lambda\|^2_{L^2_d} \le ((-\mathscr{L}_{\mathbf{V}})\tilde{f}_\lambda, \tilde{f}_\lambda)_{L^2} \tag{3.8}$$

In fact, we will prove, cf. Proposition 3.4 below, that equality holds in (3.8). This will be a consequence of the following proposition that establishes the regularity of $\tilde{f}_\lambda$. Recall that $f_\lambda(t, \mathbf{x}; \omega) = \tilde{f}_\lambda(\tau_{t,\mathbf{x}}(\omega))$.

Proposition 3.3. *For P a.s. ω we have $f_\lambda(\cdot, \cdot; \omega) \in H^{1,2}_{loc}(\mathbb{R} \times \mathbb{R}^d)$ – the Sobolev space consisting of functions that are locally square integrable together with one generalized derivative in t and two in $\mathbf{x}$. Furthermore*

$$\lambda f_\lambda(t, \mathbf{x}) - (\partial_t + L_{t,\mathbf{x}}) f_\lambda(t, \mathbf{x}) = g(t, \mathbf{x}), \tag{3.9}$$

with $L_{t,\mathbf{x}} = \triangle_{\mathbf{x}} + \mathbf{V}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}}$ and

$$f_\lambda(t, \mathbf{x}) = \mathscr{G}(f_\lambda(0, \cdot), \nabla \tilde{f}_\lambda, g - \lambda f_\lambda)(t, \mathbf{x}), \quad t < 0, \ \mathbf{x} \in \mathbb{R}^d, \tag{3.10}$$

where

$$\mathscr{G}(f,\tilde F,g)(t,\mathbf{x}) := \int_{\mathbb{R}^d} G(-t,\mathbf{x}-\mathbf{y}) f(\mathbf{y})d\mathbf{y} + \int_t^0 \int_{\mathbb{R}^d} G(s-t,\mathbf{x}-\mathbf{y})\left[\mathbf{V}(s,\mathbf{y})\cdot F(s,\mathbf{y}) + g(s,\mathbf{y})\right] ds\, d\mathbf{y} \tag{3.11}$$

for any $f \in L^2_{loc}(\mathbb{R}^d)$, $g \in L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$, $\tilde F \in L^2_d$. *Here*

$$G(t,\mathbf{x}) := (2\pi t)^{d/2}\exp\{-|\mathbf{x}|^2/(2t)\}, \qquad t>0, \qquad \mathbf{x}\in\mathbb{R}^d.$$

Proof. Suppose first that $\tilde{\mathbf{V}} \in L^\infty$. We choose $\tilde f_\lambda \in \mathscr{C}$. Then

$$\tilde g := (\lambda - \mathscr{L}_{\mathbf{V}})\,\tilde f_\lambda = [\lambda - (\triangle + \tilde{\mathbf{V}}\cdot\nabla + D_0)]\tilde f_\lambda \in L^\infty.$$

A standard argument shows that $f_\lambda(t,\mathbf{x};\omega)$ is a solution of (3.9), for P a.s. ω.

Since $(\partial_t + \triangle_{\mathbf{x}})f_\lambda = -\mathbf{V}\cdot\nabla_{\mathbf{x}} f_\lambda + \lambda f_\lambda - g$ and f_λ is bounded we conclude, with the help of the argument contained in Section 4.1 of [12], that f_λ is given by (3.10). Now let $\tilde g \in L^2$ be arbitrary and $\tilde f_\lambda = R_\lambda \tilde g$. By virtue of Proposition 3.1 we can find an L^2 approximation of $\tilde f_\lambda$ by elements $\tilde u_n \in \mathscr{C}$, $n \geq 1$ such that $\lim_{n\uparrow+\infty}(\lambda - \mathscr{L}_{\mathbf{V}})\tilde u_n = \tilde g$, in the L^2 sense and $\nabla \tilde u_n \rightharpoonup \nabla \tilde f_\lambda$ weakly in L^2_d. This approximation allows to obtain the representation of f_λ in terms of (3.10). A standard a'priori estimates, see e.g. [12] Theorem 9.1 p. 342, show also that $f_\lambda \in H^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^d)$ and (3.9) holds.

For a general $\tilde{\mathbf{V}} \in L^2$ we can use the approximation by $\tilde{\mathbf{V}}^{(n)}$ introduced in Proposition 2.1. The corresponding $\tilde f^{(n)}_\lambda := R^{(n)}_\lambda \tilde g$ satisfy both (3.9) and (3.10). Since $\tilde f^{(n)}_\lambda \to \tilde f_\lambda$ and $\nabla \tilde f^{(n)}_\lambda \rightharpoonup \nabla \tilde f_\lambda$ as $n\uparrow+\infty$ strongly in L^2 and weakly in L^2_d we conclude that $f_\lambda(\cdot,\cdot)$, $\partial_{x_p} f_\lambda(\cdot,\cdot) \in L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$, $p=1,\cdots,d$ and, in consequence, f_λ satisfies (3.10) for P a.s. ω.

Let

$$\phi^{(\delta)}(t,\mathbf{x}) := \frac{1}{\delta^{d+1}}\phi(\frac{t}{\delta},\frac{\mathbf{x}}{\delta}), \tag{3.12}$$

where $\phi(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$ be a nonnegative, smooth, compactly supported function satisfying $\phi(-t,-\mathbf{x}) = \phi(t,\mathbf{x})$ and $\int\int\phi(t,\mathbf{x})dtd\mathbf{x} = 1$. Then $f^{(\delta)}_\lambda(t,\mathbf{x}) := f_\lambda * \phi^{(\delta)}(t,\mathbf{x})$ satisfies

$$(\partial_t + \triangle_{\mathbf{x}})f^{(\delta)}_\lambda = (-\mathbf{V}\cdot\nabla_{\mathbf{x}} f_\lambda + \lambda f_\lambda - g) * \phi^{(\delta)}.$$

Notice that $\mathbf{V}(\cdot,\cdot;\omega) \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^d)$, for P a.s. ω, thanks to the assumption V 4). Using, again, Theorem 9.1 from [12] we conclude that $\sup_{\delta>0}\|f^{(\delta)}_\lambda(\cdot,\cdot;\omega)\|^{(K)}_{1,2} < +\infty$, P a.s. in ω for any cylinder $K := [a,b]\times[|\mathbf{x}|\leq R] \subseteq \mathbb{R}\times\mathbb{R}^d$, $a<b$, $R>0$. Here $\|\cdot\|^{(K)}_{1,2}$ denotes the norm in $H^{1,2}(K)$. The conclusion of the proposition follows upon the passage to the limit with $\delta\downarrow 0$. □

Proposition 3.4. *For any $f \in D(\mathscr{L}_{\mathbf{V}})$ (3.3) holds.*

We postpone the proof of this proposition till Section 5 after we demonstrate the proof of Lemma 5.1, since the arguments used to obtain both these results are quite similar.

4. The corrector field

We set $\tilde{\chi}_{p,\lambda} := -R_\lambda \tilde{V}_p$. Since $|(\tilde{V}_p, \tilde{f})_{L^2}| \le \|\widetilde{\mathbf{H}}\|_{L^2_{d\times d}} \|\nabla \tilde{f}\|_{L^2_d}$, we have the bounds

$$\lambda \|\tilde{\chi}_{p,\lambda}\|^2_{L^2} + \|\nabla \tilde{\chi}_{p,\lambda}\|^2_{L^2_d} \le \lambda \|\tilde{\chi}_{p,\lambda}\|^2_{L^2_d} + (\tilde{\chi}_{p,\lambda}, (-\mathscr{L}_{\mathbf{V}})\tilde{\chi}_{p,\lambda})_{L^2} = |(\tilde{V}_p, \tilde{\chi}_{p,\lambda})_{L^2}|$$
$$\le \|\widetilde{\mathbf{H}}\|_{L^2_{d\times d}} \|\nabla \tilde{\chi}_{p,\lambda}\|_{L^2_d}$$

we deduce that both $\lambda \|\tilde{\chi}_{p,\lambda}\|^2_{L^2} \le C$ and $\|\nabla \tilde{\chi}_{p,\lambda}\|^2_{L^2_d} \le C$ with constant $C > 0$ independent of $\lambda > 0$. In consequence

$$\lim_{\lambda \downarrow 0} \lambda \, \|\tilde{\chi}_{p,\lambda}\|_{L^2} = 0. \tag{4.1}$$

We can also find $\tilde{\mathbf{E}}_p = (\tilde{E}_{p,1}, \cdots, \tilde{E}_{p,d})$ - an L^2_d-weak limiting point of $\{\nabla \tilde{\chi}_{p,\lambda}\}_\lambda$ – obtained by taking a subsequence in $\lambda \downarrow 0$. We construct a non-stationary field

$$\theta_p(\mathbf{x}; \omega) := -\mathrm{i} \sum_{q=1}^{d} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} - 1}{|\mathbf{k}|^2} k_q \hat{E}_{p,q}(d\mathbf{k}) \tag{4.2}$$

whose spatial gradient equals $\nabla_{\mathbf{x}} \theta_p(\mathbf{x}) = \mathbf{E}_p(0, \mathbf{x})$, where $\mathbf{E}_p(t, \mathbf{x}; \omega) := \tilde{\mathbf{E}}_p(\tau_{t,\mathbf{x}}(\omega))$. Let $\chi_p(t, \mathbf{x})$, $t < 0$, $\mathbf{x} \in \mathbb{R}^d$ be defined by

$$\chi_p(t, \mathbf{x}) := \mathscr{G}(\theta_p, \tilde{\mathbf{E}}_p, V_p)(t, \mathbf{x}). \tag{4.3}$$

where $\mathscr{G}$ is defined by (3.11). The following proposition holds.

Proposition 4.1. *We have*

$$\nabla_{\mathbf{x}} \chi_p(t, \mathbf{x}; \omega) = \mathbf{E}_p(t, \mathbf{x}; \omega), \tag{4.4}$$

P a.s. in $\omega \in \Omega$,

$$\chi_p(t, \mathbf{x}) = \int_{\mathbb{R}^d} G(-t, \mathbf{x} - \mathbf{y}) \theta_p(\mathbf{y}) d\mathbf{y}$$
$$+ \int_t^0 \int_{\mathbb{R}^d} G(s - t, \mathbf{x} - \mathbf{y}) \left[\mathbf{V}(s, \mathbf{y}) \cdot \nabla_{\mathbf{x}} \chi_p(s, \mathbf{y}) + V_p(s, \mathbf{y})\right] ds \, d\mathbf{y}. \tag{4.5}$$

In addition $\chi_p(\cdot, \cdot; \omega) \in H^{1,2}_{loc}((-\infty, 0) \times \mathbb{R}^d)$,

$$\partial_t \chi_p(t, \mathbf{x}) + L_{t,\mathbf{x}} \chi_p(t, \mathbf{x}) = -V_p(t, \mathbf{x}), \;\; t < 0, \; \mathbf{x} \in \mathbb{R}^d \tag{4.6}$$

for any $p = 1, \cdots, d$ with $\chi_p(0, \mathbf{x}) = \theta_p(\mathbf{x})$.

Proof. The proof is quite routine so we only highlight its main points. First, from Proposition 3.3 we conclude that

$$\nabla_{\mathbf{x}}\chi_{p,\lambda}(t,\mathbf{x}) = \int_{\mathbb{R}^d} G(-t,\mathbf{x}-\mathbf{y})\mathbf{E}_{p,\lambda}(0,\mathbf{y})d\mathbf{y} + \int_t^0\int_{\mathbb{R}^d} \nabla_{\mathbf{x}} G(s-t,\mathbf{x}-\mathbf{y}) \times \left[\mathbf{V}(s,\mathbf{y})\cdot\mathbf{E}_{p,\lambda}(s,\mathbf{y}) + V_p(s,\mathbf{y}) - \lambda\chi_{p,\lambda}(s,\mathbf{y})\right] ds\, d\mathbf{y}, \quad (4.7)$$

with $\tilde{\mathbf{E}}_{p,\lambda} := \nabla\tilde{\chi}_{p,\lambda}$. Passing to the limit for a sub-sequence corresponding to $\lambda \downarrow 0$ we conclude (4.4) from (4.7), using (4.3) and (4.1). (4.5) now follows from (4.4) and $\chi_p(0,\mathbf{x}) = \theta_p(\mathbf{x})$. To obtain the second part of the proposition we consider $\chi_p^{(\delta)} := \chi_p * \phi^{(\delta)}$ with $\phi^{(\delta)}$ given by (3.11). From (4.5) we conclude that $\chi_p^{(\delta)} \in H^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^d)$ and $\partial_t\chi_p^{(\delta)} + \Delta_{\mathbf{x}}\chi_p^{(\delta)} = -[\mathbf{V}\cdot\nabla_{\mathbf{x}}\chi_p + V_p] * \phi^{(\delta)}$, in the classical sense. The remark on local boundedness of $\mathbf{V}(\cdot,\cdot;\omega)$ together with the classical a'priori estimates, cf. the corresponding part of the proof of Proposition 3.3, imply that $\chi_p(\cdot,\cdot;\omega) \in H^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^d)$ and (4.6) follows. □

For an arbitrary $a > 0$ we define $\chi_p^{(a)}(t,\mathbf{x}) := a^{-1}\chi_p(a^2t, a\mathbf{x})$, $\theta_p^{(a)}(\mathbf{x}) = \theta_p(a\mathbf{x})/a$ and $\mathbf{H}^{(a)}(t,\mathbf{x}) := \mathbf{H}(a^2t,a\mathbf{x})$, $\mathbf{H}_p^{(a)}(t,\mathbf{x}) := \mathbf{H}_p(a^2t,a\mathbf{x})$, $\mathbf{E}_p^{(a)}(t,\mathbf{x}) := \mathbf{E}_p(a^2t,a\mathbf{x})$. Here $\tilde{\mathbf{H}}_p := (\tilde{H}_{p,1},\cdots,\tilde{H}_{p,d})$. From (4.3) we get that for $t < 0$, $\mathbf{x}\in\mathbb{R}^d$

$$\chi_p^{(a)}(t,\mathbf{x}) = \int_{\mathbb{R}^d} G(-t,\mathbf{x}-\mathbf{y})\theta_p^{(a)}(\mathbf{y})d\mathbf{y} + \int_t^0\int_{\mathbb{R}^d} \nabla_{\mathbf{y}} G(s-t,\mathbf{x}-\mathbf{y})\cdot\left[\mathbf{H}_p^{(a)}(s,\mathbf{y}) + \mathbf{H}^{(a)}(s,\mathbf{y})\mathbf{E}_p^{(a)}(s,\mathbf{y})\right] ds\, d\mathbf{y}, \quad (4.8)$$

The following result shows sub-linear growth of the corrector.

Proposition 4.2. *For any positive* $\phi \in C_0^\infty((-\infty,0)\times\mathbb{R}^d)$ *we have*

$$\lim_{a\uparrow+\infty}\int_{-\infty}^0\int_{\mathbb{R}^d} \phi(t,\mathbf{x})\|\chi_p^{(a)}(t,\mathbf{x})\|_{L^1}\, dt\, d\mathbf{x} = 0. \quad (4.9)$$

Proof. Using (4.8) we get that the expression under the limit in (4.9) can be estimated by

$$\frac{1}{a}\int_{R^d} u(\mathbf{y})\mathbf{E}|\theta_p(a\mathbf{y})|d\mathbf{y} + \int_{-\infty}^0\int_{R^d} |\phi(t,\mathbf{x})|v_a(t,\mathbf{x})\, dt\, d\mathbf{x} \quad (4.10)$$

where

$$u(\mathbf{y}) := \int_{-\infty}^0\int_{R^d} G(-t,\mathbf{x}-\mathbf{y})|\phi(t,\mathbf{x})|\, dt\, d\mathbf{x}$$

and

$$v_a(t,\mathbf{x}) := \mathbf{E}\left|\int_t^0\int_{R^d}\nabla_{\mathbf{y}}G(s-t,\mathbf{x}-\mathbf{y})\cdot\left[\mathbf{H}_p^{(a)}(s,\mathbf{y})+\mathbf{H}^{(a)}(s,\mathbf{y})\mathbf{E}_p^{(a)}(s,\mathbf{y})\right]ds\,d\mathbf{y}\right|.$$

Due to the fact that $\nabla_{\mathbf{y}}G(\cdot-t,\mathbf{x}-\cdot)\in L^1([t,0]\times R^d)$ we can use the Mean Ergodic Theorem to conclude that

$$\lim_{a\uparrow+\infty}v_a(t,\mathbf{x})=\left|\left(\int_t^0\int_{R^d}\nabla_{\mathbf{y}}G(s-t,\mathbf{x}-\mathbf{y})\,ds\,d\mathbf{y}\right)\cdot\mathbf{E}\left(\tilde{\mathbf{H}}_p+\tilde{\mathbf{H}}\tilde{\mathbf{E}}_p\right)\right|=0$$

for any $(t,\mathbf{x})\in(-\infty,0)\times R^d$. On the other hand

$$|v_a(t,\mathbf{x})|\le\mathbf{E}\left(\left|\mathbf{H}_p+\mathbf{H}\mathbf{E}_p\right|\right)\int_t^0\int_{R^d}\left|\nabla_{\mathbf{y}}G(s-t,\mathbf{x}-\mathbf{y})\right|ds\,d\mathbf{y}\le C\sqrt{-t}$$

for some constant $C>0$. Thanks to the Lebesgue convergence theorem we conclude that the second term of (4.10) vanishes with $a\uparrow+\infty$. After routine calculations it can be shown that the first term of (4.10) vanishes, as $a\uparrow+\infty$. This is due to the fact that $\nabla_{\mathbf{y}}\theta_p(\mathbf{y})=\tilde{\mathbf{E}}_p(\tau_{0,\mathbf{y}}(\omega))$ is stationary and square integrable, and $\theta_p(\mathbf{0})=0$. See Proposition 3 of [4] for details on this point. □

5. The proof of Theorem 2.2.

We shall follow a version of the argument contained in [14]. The key element is the following.

Lemma 5.1. *For an arbitrary* $p\in\{1,\cdots,d\}$

$$\lim_{\lambda\downarrow0}\lambda\|\tilde{\chi}_{p,\lambda}\|_{L^2}^2=0. \tag{5.1}$$

Proof. For arbitrary $R,a>0$ we set $h_{a,R}(t,\mathbf{x};\omega):=f_a(\chi_p(t,\mathbf{x};\omega))\varphi_a(\mathbf{x})\psi_a(t)\,j_R(\omega)$, where $f_a(r):=-a\vee(r\wedge a)$, $\varphi_a(\mathbf{x}):=a^{-d}\varphi(\mathbf{x}/a)$, $\psi_a(t):=a^{-2}\psi(t/a^2)$, $j_R(\omega):=\mathbf{1}_{[|\tilde{\mathbf{H}}|\le R]}(\omega)$. Here $\varphi\ge0$, $\psi\ge0$ are compactly supported, smooth probability densities on $\mathbb{R}^d$ and $(-\infty,0)$ correspondingly. Equation (4.6) yields

$$\int_{-\infty}^0\int_{\mathbb{R}^d}\int\partial_t[F_a(\chi_p(t,\mathbf{x}))]\varphi_a(\mathbf{x})\psi_a(t)\,j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$-\int_{-\infty}^0\int_{\mathbb{R}^d}\int\nabla_{\mathbf{x}}\chi_p(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}[f_a(\chi_p(t,\mathbf{x}))\varphi_a(\mathbf{x})]\psi_a(t)\,j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$-\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int \left(\mathbf{H}(t,\mathbf{x})\nabla_{\mathbf{x}}\chi_p(t,\mathbf{x})\right)\cdot\nabla_{\mathbf{x}}\left[f_a(\chi_p(t,\mathbf{x}))\varphi_a(\mathbf{x})\right]\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$=-\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int \mathbf{H}_p(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}\left[f_a(\chi_p(t,\mathbf{x}))\varphi_a(\mathbf{x})\right]\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega), \tag{5.2}$$

with F_a - the primitive of f_a - satisfying $F_a(0)=0$. Obviously

$$|f_a(r)|\le a\wedge|r|,\quad F_a(r)\le a|r|\,,\ |f_a'(r)|\le 1\wedge\left|\frac{r}{a}\right|\,,\ \text{ for all } r\in\mathbb{R}. \tag{5.3}$$

Consider now the first term on the left hand side of (5.2). After integration by parts in t variable we find that its absolute value equals

$$\frac{1}{a^4}\left|\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int F_a(\chi_p(t,\mathbf{x}))\varphi_a(\mathbf{x})\psi'\left(\frac{t}{a^2}\right)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)\right|$$

$$\stackrel{(5.3)}{\le}\int_{-\infty}^{0}\int_{\mathbb{R}^d}\|\chi_p^{(a)}(t,\mathbf{x})\|_{L^1}\varphi(\mathbf{x})|\psi'(t)|\,dt\,d\mathbf{x}. \tag{5.4}$$

where $\chi_p^{(a)}(t,\mathbf{x})=a^{-1}\chi_p(a^2t,a\mathbf{x})$. By virtue of Proposition 4.2 the right hand side of (5.4) vanishes with $a\uparrow+\infty$. The second term on the left hand side of (5.2) equals

$$\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int |\mathbf{E}_p(t,\mathbf{x})|^2\varphi_a(\mathbf{x})\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$+\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int |\mathbf{E}_p(t,\mathbf{x})|^2[f_a'(\chi_p(t,\mathbf{x}))-1]\varphi_a(\mathbf{x})\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$+\frac{1}{a^{d+1}}\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int \mathbf{E}_p(t,\mathbf{x})\cdot(\nabla_{\mathbf{x}}\varphi)\left(\frac{\mathbf{x}}{a}\right)f_a(\chi_p(t,\mathbf{x}))\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega). \tag{5.5}$$

Allowing $a\uparrow+\infty$ and, then subsequently letting $R\uparrow+\infty$ we obtain that, in the limit, the first term of (5.5) becomes $\|\tilde{\mathbf{E}}_p\|^2_{L^2_d}$. Since $|f_a'(r)-1|=1_{[|r|\ge a]}$, the second term of (5.5) can be estimated by

$$\frac{1}{a}\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int |\mathbf{E}_p(t,\mathbf{x})|^2 1_{[|\chi_p(t,\mathbf{x})|\ge a]}\varphi_a(\mathbf{x})\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$\le K^2 \int_{-\infty}^{0} \int_{\mathbb{R}^d} \int 1_{[|\mathbf{E}_p(t,\mathbf{x})| \le K]} 1_{[|\chi_p(t,\mathbf{x})| \ge a]} \varphi_a(\mathbf{x}) \psi_a(t)\, dt\, d\mathbf{x}\, dP$$

$$+ \int 1_{[|\tilde{\mathbf{E}}_p| > K]} |\tilde{\mathbf{E}}_p|^2 dP$$

$$\le K^2 \int_{-\infty}^{0} \int_{\mathbb{R}^d} \int \frac{1}{a} |\chi_p(t,\mathbf{x})| \varphi_a(\mathbf{x}) \psi_a(t)\, dt\, d\mathbf{x}\, dP + \int 1_{[|\tilde{\mathbf{E}}_p| > K]} |\tilde{\mathbf{E}}_p|^2 dP$$

$$= K^2 \int_{-\infty}^{0} \int_{\mathbb{R}^d} \int |\chi_p^{(a)}(t,\mathbf{x})| \varphi(\mathbf{x}) \psi(t)\, dt\, d\mathbf{x}\, dP + \int 1_{[|\tilde{\mathbf{E}}_p| > K]} |\tilde{\mathbf{E}}_p|^2 dP \quad (5.6)$$

for an arbitrary $K > 0$. Passing to the limit with $a \uparrow +\infty$ we obtain from Proposition 4.2 that the left member of (5.6) is estimated by $\int 1_{[|\tilde{\mathbf{E}}_p| > K]} |\tilde{\mathbf{E}}_p|^2 dP$ for an arbitrary $K > 0$. Letting then $K \to \infty$ we obtain that the second member of (5.5) vanishes, as $a \uparrow +\infty$. The last remaining term of (5.5) can be estimated by

$$\frac{K}{a^{d+1}} \int_{-\infty}^{0} \int_{\mathbb{R}^d} \int f_a(\chi_p(t,\mathbf{x})) |\nabla_{\mathbf{x}} \varphi\left(\frac{\mathbf{x}}{a}\right)| \psi_a(t)\, dt\, d\mathbf{x}\, dP$$

$$+ \frac{1}{a^d} \int_{-\infty}^{0} \int_{\mathbb{R}^d} \int 1_{[|\mathbf{E}_p(t,x)| > K]} |\mathbf{E}_p(t,x)| |\nabla_{\mathbf{x}} \varphi\left(\frac{\mathbf{x}}{a}\right)| \psi_a(t)\, dt\, d\mathbf{x}\, dP$$

$$\le K \int_{-\infty}^{0} \int_{\mathbb{R}^d} \|\chi_p^{(a)}(t,\mathbf{x})\|_{L^1} |\nabla_{\mathbf{x}} \varphi(\mathbf{x})| \psi(t)\, dt\, d\mathbf{x}$$

$$+ \int 1_{[|\tilde{\mathbf{E}}_p| > K]} |\tilde{\mathbf{E}}_p| dP \int_{\mathbb{R}^d} |\nabla_{\mathbf{x}} \varphi(\mathbf{x})|\, d\mathbf{x} \quad (5.7)$$

Arguing as in (5.6) we conclude that this expression vanishes in the limit as $a \uparrow +\infty$ and $K \to \infty$. Summarizing, we conclude that, in the limit, first in $a \uparrow +\infty$, then in $R \uparrow +\infty$, the second term on the left hand side of (5.2) becomes $\|\tilde{E}_p\|^2_{L^2_d}$. Similar argument applied to the right member of (5.2) shows that in the aforementioned limit the term becomes $-\left(\tilde{\mathbf{H}}_p, \tilde{\mathbf{E}}_p\right)_{L^2_d}$. In fact this can be written as

$$-\mathbf{E}\left(\tilde{\mathbf{H}}_p \cdot \tilde{\mathbf{E}}_p\right) \quad (5.8)$$

$$- \int_{-\infty}^{0} \int_{\mathbb{R}^d} \int \mathbf{H}_p(t,\mathbf{x}) \cdot \mathbf{E}_p(t,\mathbf{x}) [f_a'(\chi_p(t,\mathbf{x})) - 1] \varphi_a(\mathbf{x}) \psi_a(t) j_R(\omega)\, dt\, d\mathbf{x}\, P(d\omega)$$

$$(5.9)$$

$$-\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int \mathbf{H}_p(t,\mathbf{x})\cdot\nabla\varphi_a(\mathbf{x})f_a(\chi_p(t,\mathbf{x}))\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega) \tag{5.10}$$

and by the same arguments used in (5.5), one can show that the expressions in line (5.9) and line (5.10) vanish as $a\to\infty$.

Finally we consider the third member of the left hand side of (5.2). Due to anti-symmetry of $\mathbf{H}$ it is equal to

$$-\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int \big(\mathbf{H}(t,\mathbf{x})\nabla_{\mathbf{x}}\chi_p(t,\mathbf{x})\big)\cdot\nabla_{\mathbf{x}}\varphi_a(\mathbf{x})f_a(\chi_p(t,\mathbf{x}))\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

and its absolute value can be bounded by

$$RK\int_{-\infty}^{0}\int_{\mathbb{R}^d}\int |\nabla\varphi_a(\mathbf{x})|f_a(\chi_p(t,\mathbf{x}))\psi_a(t)\,dt\,d\mathbf{x}\,P(d\omega) \tag{5.11}$$

$$+R\left(\int 1_{[|\tilde{\mathbf{E}}_p|>K]}|\tilde{\mathbf{E}}_p|dP\right)\,a\int_{\mathbb{R}^d}|\nabla\varphi_a(\mathbf{x})|\,d\mathbf{x} \tag{5.12}$$

for an arbitrary $K>0$. The term (5.11) goes to zero as $a\uparrow+\infty$ exactly like (5.7), while (5.11) is independent of a and goes to 0 as $K\to\infty$.

Summarizing we have shown that

$$\|\tilde{E}_p\|^2_{L^2_d} = -(\tilde{\mathbf{H}}_p,\tilde{E}_p)_{L^2_d}. \tag{5.13}$$

On the other hand multiplying both sides of (3.5) where $\tilde{g}=-\tilde{V}_p$ by $\tilde{\chi}^{(n)}_{p,\lambda}:=-R^{(n)}_\lambda\tilde{V}_p$, integrating over $P(d\omega)$ and then passing to the respective subsequences, first in $n\uparrow+\infty$ and then in $\lambda\downarrow 0$ we conclude that

$$\limsup_{\lambda\downarrow 0}\lambda\|\tilde{\chi}_{p,\lambda}\|^2_{L^2}+\|\tilde{E}_p\|^2_{L^2_d}\le-\int\tilde{\mathbf{H}}_p\cdot\tilde{E}_p\,dP \tag{5.14}$$

and (5.1) follows from (5.13). □

Proof of Proposition 3.4. It suffices to prove the proposition for $\tilde{f}_\lambda=R_\lambda\tilde{g}$, where $\tilde{g}\in L^2$. By Proposition 3.1 $\tilde{f}_\lambda\in W^{2,1}$ and, by Proposition 3.3, $f_\lambda(\cdot,\cdot;\omega)\in H^{1,2}_{loc}(\mathbb{R}\times\mathbb{R}^d)$ and satisfies equation (3.9), ω-a.s. Multiplying both sides of the equation by a test function $h_{a,R}(t,\mathbf{x};\omega):=f_a(f_\lambda(t,\mathbf{x};\omega))\varphi_a(\mathbf{x})\psi_a(t)j_R(\omega)$, with φ_a,ψ_a,j_R,f_a the same as in the proof of Lemma 5.1, and subsequently integrating over $(-\infty,0)\times R^d$ and Ω we obtain

$$\int_{-\infty}^{0}\int_{R^d}\int \partial_t[F_a(f_\lambda(t,\mathbf{x}))]\varphi_a(\mathbf{x})\psi_a(t)j_R(\omega)\,dt\,d\mathbf{x}\,P(d\omega)$$

$$- \int_{-\infty}^{0} \int_{R^d} \int \nabla_{\mathbf{x}} f_\lambda(t,\mathbf{x}) \cdot \nabla_{\mathbf{x}}[f_a(f_\lambda(t,\mathbf{x}))\varphi_a(\mathbf{x})]\psi_a(t) j_R(\omega)\, dt\, d\mathbf{x}\, P(d\omega)$$

$$- \int_{-\infty}^{0} \int_{R^d} \int (\mathbf{H}(t,\mathbf{x})\nabla_{\mathbf{x}} f_\lambda(t,\mathbf{x})) \cdot \nabla_{\mathbf{x}} [f_a(f_\lambda(t,\mathbf{x}))\varphi_a(\mathbf{x})]\, \psi_a(t) j_R(\omega)\, dt\, d\mathbf{x}\, P(d\omega)$$

$$= \int_{-\infty}^{0} \int_{R^d} \int \mathscr{L}_{\mathbf{V}} \tilde{f}_\lambda(\tau_{t,\mathbf{x}}\omega)\, f_a(f_\lambda(t,\mathbf{x}))\varphi_a(\mathbf{x})\psi_a(t) j_R(\omega)\, dt\, d\mathbf{x}\, P(d\omega). \quad (5.15)$$

Repeating exactly the same argument as in the proof of Lemma 5.1 will lead us to (3.3). □

We also have the following.

Corollary 5.2. $\lim_{\lambda\downarrow 0} \|\nabla\tilde{\chi}_{p,\lambda} - \tilde{\mathbf{E}}_p\|_{L^2_d} = 0.$

Proof. It follows, by a standard argument, from (5.1). By (5.13) and since $\tilde{\mathbf{E}}_p$ is a weak limit of $\nabla_{\mathbf{x}}\tilde{\chi}_{p,\lambda}$,

$$\begin{aligned} -(\tilde{\mathbf{H}}_p, \tilde{\mathbf{E}}_p)_{L^2_d} &= \|\tilde{\mathbf{E}}_p\|^2_{L^2_d} \le \lim_{\lambda\to 0} \|\nabla_{\mathbf{x}}\tilde{\chi}_{p,\lambda}\|^2_{L^2_d} \\ &= \lim_{\lambda\to 0} -(\tilde{\mathbf{H}}_p, \nabla_{\mathbf{x}}\tilde{\chi}_{p,\lambda})_{L^2_d} = -(\tilde{\mathbf{H}}_p, \tilde{\mathbf{E}}_p)_{L^2_d} \end{aligned}$$

This implies $\lim_{\lambda\downarrow 0} \|\nabla\tilde{\chi}_{p,\lambda}\|^2_{L^2_d} = \|\tilde{\mathbf{E}}_p\|^2_{L^2_d}$, which in turn yields that $\nabla\tilde{\chi}_{p,\lambda} \to \tilde{\mathbf{E}}_p$ strongly in L^2_d, as $\lambda \downarrow 0$. □

The proof of Theorem 2.2 follows in two steps.

Convergence of finite dimensional distributions. By the definition (3.1) of the environment process and proposition 3.3, Ito's formula can be applied to $\tilde{\chi}_{p,\lambda}$ and we obtain

$$\int_0^t \mathscr{L}_{\mathbf{V}}\tilde{\chi}_{p,\lambda}(\eta(s))\, ds = \tilde{\chi}_{p,\lambda}(\eta(t)) - \tilde{\chi}_{p,\lambda}(\eta(0)) - \sqrt{2}\int_0^t \nabla\tilde{\chi}_{p,\lambda}(\eta(s)) \cdot d\mathbf{w}(s)$$

According to (2.4) and choosing $\lambda = \varepsilon^2$, we have

$$\begin{aligned} \mathbf{e}_p \cdot \mathbf{x}_\varepsilon(t) &= \sqrt{2}\varepsilon w_p\left(\frac{t}{\varepsilon^2}\right) + \varepsilon \int_0^{t/\varepsilon^2} \tilde{V}_p(\eta(s))ds \\ &= \varepsilon M_p\left(\frac{t}{\varepsilon^2}\right) + \sqrt{2}\varepsilon \int_0^{t/\varepsilon^2} \left[\nabla\tilde{\chi}_{p,\varepsilon^2}(\eta(s)) - \tilde{\mathbf{E}}_p(\eta(s))\right] \cdot d\mathbf{w}(s) \quad (5.16) \\ &+ \varepsilon\tilde{\chi}_{p,\varepsilon^2}(\eta(0)) - \varepsilon\tilde{\chi}_{p,\varepsilon^2}\left(\eta\left(\frac{t}{\varepsilon^2}\right)\right) + \varepsilon^3 \int_0^{t/\varepsilon^2} \tilde{\chi}_{p,\varepsilon^2}(\eta(s))ds, \quad (5.17) \end{aligned}$$

where

$$M_p(t) := \sqrt{2}\int_0^t \left[\mathbf{e}_p + \tilde{\mathbf{E}}_p(\eta(s))\right] \cdot d\mathbf{w}(s).$$

A standard calculation shows that

$$\mathbf{ME}\left|\sqrt{2}\varepsilon \int_0^{t/\varepsilon^2} \left[\nabla\tilde{\chi}_{p,\varepsilon^2}(\eta(s)) - \tilde{\mathbf{E}}_p(\eta(s))\right] \cdot d\mathbf{w}(s)\right|^2 = 2t\|\nabla\tilde{\chi}_{p,\varepsilon^2} - \tilde{\mathbf{E}}_p\|^2_{L^2_d}.$$

Then the convergence of $\nabla\tilde{\chi}_{p,\varepsilon^2}$ given by Corollary 5.2 implies that the mean of the square of the second term in line (5.16) tends to 0 as $\varepsilon \to 0$. By Lemma 5.1 and stationarity, the same is true for the expression in line (5.17).

$(\varepsilon M_1(t/\varepsilon^2), \cdots, \varepsilon M_d(t/\varepsilon^2))$, $t \geq 0$ is a d-dimensional continuous trajectory martingale with joint quadratic variation of its p-th and q-th components equal to

$$2\varepsilon^2 \int_0^{t/\varepsilon^2} \left[\mathbf{e}_p + \tilde{\mathbf{E}}_p(\eta(s))\right] \cdot \left[\mathbf{e}_q + \tilde{\mathbf{E}}_q(\eta(s))\right] ds.$$

By the ergodic theorem it converges, as $\varepsilon \downarrow 0$, to

$$2t \int [\mathbf{e}_p + \tilde{\mathbf{E}}_p] \cdot [\mathbf{e}_q + \tilde{\mathbf{E}}_q]\, dP = 2t\left(\delta_{p,q} + \int \tilde{\mathbf{E}}_p \cdot \tilde{\mathbf{E}}_q\, dP\right)$$

(since $\int \tilde{\mathbf{E}}_p \; dP = 0$). The martingale central limit theorem applied to a d-dimensional martingale $(M_1(t), \cdots, M_d(t))$, $t \geq 0$ (cf. e.g. [3], Theorem 7.1.4 pp. 339-340) allows to conclude the convergence of the finite dimensional distributions of $\mathbf{x}_\varepsilon(t)$, $t \geq 0$ to a Brownian motion with the co-variance matrix given by $2[\delta_{p,q} + (\tilde{\mathbf{E}}_p, \tilde{\mathbf{E}}_q)_{L^2_d}]$. □

Tightness. As in [13], we use the argument of [17] (see also [18]). By $\mathscr{H}_1$ and $\mathscr{H}_{-1}$ we denote respectively the completion of $\mathscr{C}$ with respect to the norm $\|f\|_1 := (u, -\Delta u)^{1/2}_{L^2}$, and $\|f\|_{-1} := \sup_{\|\phi\|_1=1} |(f, \phi)_{L^2}|$, $f \in \mathscr{C}$. Notice that V 3) implies that $V_p \in \mathscr{H}_{-1}$, $p = 1, \cdots, d$. Since $\mathscr{C}$ is L^2-dense and invariant under the group $U^{0,\mathbf{x}}$, $\mathbf{x} \in \mathbb{R}^d$, it is a core of Δ. Then for any $\sigma > 0$ there exists $u_p \in \mathscr{C}$ such that $\|\Delta u_p - V_p\|_{L^2} < \sigma$. We are then in the position to apply Theorem 2.2 of [17] that leads to the estimate

$$\mathbf{ME}\left\{\sup_{0\leq t\leq T}\left|\varepsilon \int_0^{t/\varepsilon^2} V_p(\eta(s))\, ds\right|^2\right\} \leq 14T\|V_p\|^2_{-1} \tag{5.18}$$

This implies the compactness of $\mathbf{x}_\varepsilon(t)$, $t \geq 0$ in $D([0, T]; \mathbb{R}^d)$. □

Appendix: the proof of Proposition 2.1.

Let $\tilde{\mathbf{V}} = \nabla \cdot \tilde{\mathbf{H}}$ with $\tilde{\mathbf{H}}$ given by (2.2). Set

$$\tilde{\mathbf{H}}_0^{(n)}(\omega) := \int\limits_{\mathbb{R}^d} \phi^{(n)}(s, \mathbf{y})\tilde{\mathbf{H}}(\tau_{-s,-\mathbf{y}}(\omega))\, ds\, d\mathbf{y}$$

with $\phi^{(n)} := n^{-d-1}\phi(n\,t, n\,\mathbf{x})$, $n \geq 1$, where $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_+$, is nonnegative, compactly supported and smooth. Each $\tilde{\mathbf{V}}_0^{(n)} := \nabla \cdot \tilde{\mathbf{H}}_0^{(n)}$ has C^∞ smooth trajectories and satisfies part a) of the proposition except for the fact that it needs not be bounded. However one can find $\tilde{\mathbf{H}}^{(n)}$ with components in $\mathscr{C}$ such that $\|\tilde{\mathbf{H}}^{(n)} - \tilde{\mathbf{H}}_0^{(n)}\|_{2,m} < 1/n$, for some sufficiently large m, say $m > d + 2$.

Then $\tilde{\mathbf{V}}^{(n)} := \nabla \cdot \tilde{\mathbf{H}}^{(n)}$ satisfies part a) of the proposition. Moreover all fields $\mathbf{V}^{(n)}(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$, $n \geq 1$ satisfy condition V 4) uniformly in $n \geq 1$.

Let us denote by $\mathbf{x}_{s,\mathbf{x}}^{(n)}(t; \omega)$, $t \geq s$ the sequence of solutions of (2.5). Then

$$\mathbf{ME}(\sup_{s \leq t \leq T} |\mathbf{x}_{s,\mathbf{x}}^{(n)}(t)|)^2 \leq 2\left[T^2\|\tilde{\mathbf{V}}^{(n)}\|_{L_d^2} + 2\mathbf{M}(\sup_{s \leq t \leq T} |\mathbf{w}(t) - \mathbf{w}(s)|)^2\right] \quad (5.19)$$

Let $\tau_K^{(n)}$ be the exit time of $\mathbf{x}_{n,\omega}^{s,\mathbf{x}}(t)$, $t \geq 0$ from a given compact set K. Then, with $\tau_K^{(m,n)} := \tau_K^{(m)} \wedge \tau_K^{(n)}$, we have

$$\begin{aligned}
&|\mathbf{x}_{s,\mathbf{x}}^{(n)}(t \wedge \tau_K^{(m,n)}) - \mathbf{x}_{s,\mathbf{x}}^{(m)}(t \wedge \tau_K^{(m,n)})| \\
&\leq \int_s^{t\wedge\tau_K^{(m,n)}} |\mathbf{V}^{(n)}(u, \mathbf{x}_{s,\mathbf{x}}^{(n)}(u)) - \mathbf{V}^{(m)}(u, \mathbf{x}_{s,\mathbf{x}}^{(m)}(u))|\, du \\
&\leq \int\limits_s^T \sup_{\mathbf{x}\in K} |\mathbf{V}^{(n)}(u, \mathbf{x}) - \mathbf{V}^{(m)}(u, \mathbf{x})|\, du \\
&\quad + C_\omega(K, T) \int_s^{t\wedge\tau_K^{(m,n)}} |\mathbf{x}_{s,\mathbf{x}}^{(n)}(u) - \mathbf{x}_{s,\mathbf{x}}^{(m)}(u)|\, du
\end{aligned}$$

that in turn implies that the sequence of stopping times $\tau_K^{(n)}$, $n \geq 1$ is $P \otimes Q$ a.s. convergent to a certain τ_K. In addition the trajectories $\mathbf{x}^{(n)}(t \wedge \tau_K)$, $t \geq s$ are uniformly convergent to $\mathbf{x}_{s,\mathbf{x}}(t \wedge \tau_K)$, $t \in [s, T]$ - a solution up to time τ_K, of (2.4). Additionally, in view of (5.19) we deduce that

$$\mathbf{ME}(\sup_{s \leq t \leq \tau_K} |\mathbf{x}_{s,\mathbf{x}}(t)|)^2 \leq 2\left[T^2\|\mathbf{V}\|_{L_d^2} + 2\mathbf{M}(\sup_{s \leq t \leq T} |\mathbf{w}(t) - \mathbf{w}(s)|)^2\right]. \quad (5.20)$$

Note that the right hand side of (5.20) is independent of K. Allowing K to be as large as we wish, we conclude that $\mathbf{x}_{s,\mathbf{x}}(t)$, $T \geq t \geq s$ is well defined and does not explode $P \otimes Q$ a.s. for any $T > s$.

We show now non-degeneracy of the constructed diffusion. Let A be a Borel measurable set with $m_d(A) > 0$. We denote by ϱ the exit time of $\mathbf{x}_{s,\mathbf{x}}(t; \omega)$, $t \geq s$

out of the ball B centered at $\mathbf{0}$ with radius sufficiently large to have $m_d(B \cap A) > 0$ and $\mathbf{U}(t, \mathbf{x}) = \varphi(\mathbf{x})\mathbf{V}(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$, where φ is a nonnegative, smooth function equal to 1 in B and 0 outside $2B$. We have $\mathbf{x}_{s,\mathbf{x}}(t \wedge \varrho) = \mathbf{y}(t \wedge \varrho)$, $t \geq 0$, $P \otimes Q$ a.s. with $\mathbf{y}(t)$, $t \geq 0$ the solution of Itô stochastic differential equation (2.4) where $\mathbf{V}$ is replaced by $\mathbf{U}$. We conclude therefore that $Q(\mathbf{x}_{s,\mathbf{x}}(t) \in A) \geq Q(\mathbf{y}(t) \in A, t \leq \varrho) > 0$. The last inequality follows from the Girsanov Theorem, see e.g. Theorem 3.5.1 of [9]; notice that $Q(\mathbf{w}(t) \in A, t \leq \varrho) > 0$. □

Acknowledgements. Research of T.K. was partially supported by a grant (Nr 2 PO3A 017 17) from the State Committee for Scientific Research of Poland.

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