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On the sector condition and homogenization of diffusions with a Gaussian drift

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Abstract

In this paper we present the functional central limit theorem for a class of Markov processes, whose L^2-generator satisfies the so-called *graded sector condition*. We apply the result to obtain homogenization theorems for certain classes of diffusions with a random Gaussian drift. Additionally, we present a result concerning the regularity of the effective diffusivity tensor with respect to the parameters related to the statistics of the drift. The abstract central limit theorem, see Theorem 2.2, is obtained by applying the technique used in Sethuraman et al. (Comm. Pure Appl. Math. 53 (2000) 972) to the case of infinite particle systems.

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1. Introduction

In the present paper we wish to show applications of various versions of the sector condition to homogenization of diffusions with a stationary Gaussian drift. We

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formulate and present a certain abstract result, see Theorem 2.2 of Section 2, that asserts the functional central limit theorem (FCLT) for a class of additive functionals of a Markov stochastic process $(\eta_t)_{t\geqslant 0}$ that possesses a stationary ergodic measure μ and whose $L^2(\mu)$ generator L satisfies *the graded sector condition*, see condition (H4) of Section 2. This condition has been introduced and applied to infinite particle systems by Sethuraman et al. [19], see also [11].

It is well known, see [9], that FCLT holds in the case when L is self-adjoint. The generalization of this result to the so-called *quasi-reversible case*, i.e. processes whose generator satisfies the *strong sector condition* $(|(Lf,g)_{L^2}|\leqslant C|(Lf,f)_{L^2}|^{1/2}|(Lg,g)_{L^2}|^{1/2}$, for some $C>0$ and $f,g\in D(L))$, has been done in [15,20]. In Theorem 2.2 we show that the FCLT holds even in the case when the generator does not satisfy the sector condition globally but there exists a family of mutually orthogonal subspaces H_n of L^2 such that L maps $H_n\cap D(L)$ into $H_{n-1}\oplus H_n\oplus H_{n+1}$, cf. condition (H3), and the sector condition holds on each H_n with the sector constant possibly increasing at the rate no faster than n^b for some $b\in[0,1)$, cf. condition (H4). The version of this theorem has been proven for infinite particle systems in [19], cf. also [11,14], and the present paper essentially follows the proof shown there. Our main objective is to show applications of this result in the context of diffusions with a Gaussian incompressible drift. In this case we suppose that the particle trajectory is described by Itô's stochastic differential equation

$$d\mathbf{x}(t)=\mathbf{V}(t,\mathbf{x}(t))\,dt+\sqrt{2\kappa}\,d\mathbf{w}(t),\quad \mathbf{x}(0)=\mathbf{0}, \tag{1.1}$$

where $\mathbf{V}=(V_1,\dots,V_d):\mathbb{R}\times\mathbb{R}^d\times\Omega\to\mathbb{R}^d$ is a d-dimensional stationary, Gaussian random field with incompressible $(\nabla_{\mathbf{x}}\cdot\mathbf{V}(t,\mathbf{x})\equiv 0)$ realizations given over a probability space $\mathscr{T}_1:=(\Omega,\mathscr{B}(\Omega),\mu)$, $\mathbf{w}(\cdot)$ is a d-dimensional standard Brownian motion, independent of $\mathbf{V}$, given on another probability space $\mathscr{T}_0:=(\Sigma,\mathscr{W},Q)$. The parameter $\kappa\geqslant 0$ is sometimes referred to as *the molecular diffusivity*. When $\kappa=0$ Eq. (1.1) becomes an ordinary differential equation.

We are interested in proving the FCLT (or homogenization) for the family of continuous trajectory processes $\mathbf{x}_\epsilon(t):=\epsilon\mathbf{x}(\frac{t}{\epsilon^2})$, $t\geqslant 0$, as $\epsilon\downarrow 0$. Gaussianity of $\mathbf{V}$ implies that there is a natural way of selecting H_n as the space of the n-degree Hermite polynomials built over the Gaussian Hilbert space generated by the field, cf. Section 4 below for the steady case (time-independent field). Using the abstract results of Section 2, we show (cf. Theorem 4.2) that the homogenization holds when $\mathbf{V}$ is a divergence of an anti-symmetric Gaussian stationary tensor. This result is quite well known and follows also from [4,13]. With the described technique we are able to show, see Theorem 5.1 of Section 5, that the covariance tensor of the limiting Wiener measure, the so-called *effective diffusivity*, depends smoothly on the drift. In proving regularity of the effective diffusivity we follow closely the method developed in [12] for proving regularity of the self-diffusion coefficient for a tagged particle in a simple exclusion process.

Finally, in Section 6.4.3 we apply Theorem 2.2 to prove FCLT for a class of time dependent Gaussian, Markovian drifts (the so-called *Ornstein–Uhlenbeck* drifts), see

Theorem 6.3. In particular, we prove homogenization for $\mathbf{V}(\cdot,\cdot)$ with the power-law energy spectrum, i.e. a field whose covariance matrix $\mathbf{R}(\cdot,\cdot)$ can be written as

$$\mathbf{R}(t,\mathbf{x}) = \int_{\mathbb{R}^d} e^{i\mathbf{x}\cdot\mathbf{k}} \exp\{-|\mathbf{k}|^{2\beta}t\} \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}} \left(\mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right) d\mathbf{k}, \tag{1.2}$$

with $a:[0,+\infty)\to[0,+\infty)$ a certain compactly supported cut-off function, $\beta \geqslant 0$ and $\alpha<1$—the latter condition is needed to ensure the square integrability of the field. This class of fields plays an important role in statistical hydrodynamics because in $d=3$ for $(\alpha,\beta)=(4/3,1/3)$ the energy spectrum corresponding to (1.2) satisfies Kolmogorov–Obukhov self-similarity hypothesis for the velocity field of a turbulent flow.

In case when $\kappa>0$, one can show that the generator of the corresponding Lagrangian canonical process, see Section 6.3 for the definition, satisfies the strong sector condition provided that $\alpha<0$, or $\alpha\in(0,1)$ and $\alpha+\beta<1$, see part (i) of Lemma 6.2. This fact in turn implies the FCLT. We point out here that the FCLT has been established in this case, via a different technique, by Fannjiang and Komorowski [5,6]. Using the method presented here we show that the homogenization takes place also in the case when $\kappa=0$ (random motions), provided that $\alpha+\beta<1$ and $\beta\in[0,1]$. Then, the generator of the Lagrangian process satisfies the graded sector condition with $b=3/4$.

2. Preliminaries and the formulation of the abstract result

Suppose that (Ω,d) is a Polish metric space and μ is a probability measure on $\mathscr{B}(\Omega)$—the σ-algebra of Borel subsets. Let $(P^t)_{t\geqslant 0}$ be a strongly continuous semigroup of Markov operators on $L^2 := L^2(\Omega,\mathscr{B}(\Omega),\mu)$, i.e. $P^t f\geqslant 0$ for $f\geqslant 0$, $P^t\mathbf{1}=\mathbf{1}$. We assume that

(M) μ is invariant, i.e.

$$\int P^t f\, d\mu = \int f\, d\mu, \quad \forall f\in L^2, t\geqslant 0, \tag{2.1}$$

and *ergodic*, i.e. any f such that $P^t f = f$ for all $t\geqslant 0$ must satisfy $f\in\mathrm{span}\{\mathbf{1}\}$.

We suppose that $(\eta_t)_{t\geqslant 0}$ is a stationary, Ω-valued Markov process, defined over a probability space $\mathscr{T} := (\mathscr{A},\mathscr{V},\mathbb{P}_\mu)$, with $(P^t)_{t\geqslant 0}$ its transition of probability semigroup, i.e.

$$\mathbb{E}_\mu[f(\eta_{t+h})|\mathscr{Z}_t] = P^h f(\eta_t)$$

for any $f\in L^2$, $t,h\geqslant 0$ and μ is the law of η_0. Here $(\mathscr{Z}_t)_{t\geqslant 0}$ is the natural filtration corresponding to the process, $\mathbb{E}_\mu$, $\mathbb{E}_\mu[\cdot|\mathscr{Z}_t]$ stand for the respective expectation and conditional expectation relative to $\mathbb{P}_\mu$.

We denote by $L : \mathscr{D}(L) \to L^2$ the generator of the semigroup and by

$$\mathscr{E}_L(f, g) := (-Lf, g)_{L^2}, \quad (f, g) \in \mathscr{D}(L) \times L^2 \tag{2.2}$$

a bilinear form corresponding to L. We denote L^* its adjoint in L^2. Below, we list the assumptions made about L.

(H1) There exists a common core $\mathscr{C}$ for L and L^*. Let $S : \mathscr{C} \to L^2$ be given by $S = -(L + L^*)/2$. We suppose that it is essentially self-adjoint.

We denote the self-adjoint closure of S by the same symbol. It is clearly a non-negative definite operator satisfying $(Sf, f)_{L^2} = -(Lf, f)_{L^2}$, $f \in \mathscr{C}$.

(H2) There exists an orthogonal decomposition $L^2 = \oplus_{n=0}^{\infty} H_n$, where H_n are closed subspaces of L^2 with $H_0 := \mathrm{span}\{\mathbf{1}\}$, such that $D_\infty := (\oplus_{n=0}^{\infty} D_n) \cap \mathscr{D}(L)$, with $D_n := \mathscr{D}(L) \cap H_n$, forms a *core* of L.

We denote by Π_n the orthogonal projection onto H_n.

(H3)

$$L_n := L_{|D_n} : D_n \to H_{n-1} \oplus H_n \oplus H_{n+1} \quad \text{for } n \geqslant 1 \tag{2.3}$$

and

$$\Pi_n(\mathscr{C}) \subseteq \mathscr{D}(S) \quad \text{and} \quad S(\Pi_n(\mathscr{C})) \subseteq H_n \text{ for all } n \geqslant 1 \tag{2.4}$$

(note that this condition implies $S\Pi_n f = \Pi_n S f$ for all $f \in \mathscr{C}, n \geqslant 1$).

(H4) There exist $C > 0$, $b \in [0, 1)$ such that for any $n \geqslant 1$

$$|(Lf, g)_{L^2}| \leqslant Cn^b \mathscr{E}_L(f, f)^{1/2} \mathscr{E}_L(g, g)^{1/2}, \quad \forall f \in D_n, g \in \mathscr{D}(L). \tag{2.5}$$

From (H3) and (H4) we deduce that for any $f \in D_\infty$ and arbitrary $n \geqslant 1$

$$|(\Pi_n Lf, g)_{L^2}| \leqslant Cn^b\, \mathscr{E}_L(\Pi_{n-1}^{n+1} f, \Pi_{n-1}^{n+1} f)^{1/2} \mathscr{E}_L(g, g)^{1/2}, \forall g \in \mathscr{D}(L), \tag{2.6}$$

with $\Pi_k^n = \sum_{j=k}^{n} \Pi_j$ and b, C as in condition (H4). Using (2.4) and the fact that $\mathscr{C}$ is the core of S we get.

Proposition 2.1.

$$|(\Pi_n Lf, g)_{L^2}| \leqslant Cn^b\, (S\Pi_{n-1}^{n+1} f, \Pi_{n-1}^{n+1} f)^{1/2} (S\Pi_n g, \Pi_n g)_{L^2}^{1/2}, \tag{2.7}$$

for all $f \in \mathscr{C}$, $g \in \mathscr{D}(S)$, $n \geqslant 1$.

We claim that the null space of S is spanned on $\mathbf{1}$. Indeed, suppose that $f \in \mathscr{D}(S)$ and $Sf = 0$. Let $f = \sum_{n=0}^{\infty} f_n$, with $f_n \in H_n$. From (2.4) we conclude that $f_n \in \mathscr{D}(S)$ and $Sf_n = 0$. From Proposition 2.1 we deduce therefore that $f \in \mathscr{D}(L^*)$ and $L^* f = 0$, hence f is orthogonal to the closure of the range of L. Since μ is ergodic (condition M)) this fact implies that $f \in \mathrm{span}\{\mathbf{1}\}$.

Let $L^2_0 := \{f \in L^2 : \int f\, d\mu = 0\}$. We define a pre-Hilbert space $\mathscr{H}^0_1 := \mathscr{D}(S^{1/2}) \cap L^2_0$, with the norm given by $||f||_1^2 := ||S^{1/2}f||^2_{L^2}$. Note that $\mathscr{D}(L) \subseteq \mathscr{H}^0_1$ and $||u||_1^2 = -(Lu, u)_{L^2}$. Let $\mathscr{H}_1$ be the completion of $\mathscr{H}^0_1$ under the norm $||\cdot||_1$.

Let $\mathscr{H}^0_{-1}$ be the space consisting of all $f \in L^2_0$ satisfying

$$||f||^2_{-1} := \sup\,[2(f,g)_{L^2} - ||g||_1^2] < +\infty. \tag{2.8}$$

The supremum here is taken over all $g \in L^2_0$, with the convention that $||g||_1 = +\infty$, when the respective norm is undefined. One can easily check that $||\cdot||_{-1}$ defines a norm on $\mathscr{H}^0_{-1}$ that is pre-Hilbert. The completion of this space under $||\cdot||_{-1}$ shall be denoted by $\mathscr{H}_{-1}$. Let $f \in \mathscr{H}^0_{-1}$ and $F_f(\cdot) = (f, \cdot)_{L^2}$. One can easily show that

$$\sup_{||g||_1 = 1} |F_f(g)| = ||f||_{-1},$$

so the map $f \mapsto F_f$ extends to a unitary isomorphism of $\mathscr{H}_{-1}$ with the dual to $\mathscr{H}_1$ under $(\cdot,\cdot)_{L^2}$ pairing.

Thanks to (2.4) we have

$$||f||_1^2 = \sum_{n \geqslant 1} ||\Pi_n f||_1^2 \quad \text{and} \quad ||f||^2_{-1} = \sum_{n \geqslant 1} ||\Pi_n f||^2_{-1}. \tag{2.9}$$

For an arbitrary $k \geqslant 0$ we introduce the following two norms:

$$|||f|||^2_{-1,k} := \sum_{n \geqslant 1} n^{2k} ||\Pi_n f||^2_{-1} \quad \text{and} \quad |||f|||^2_{1,k} := \sum_{n \geqslant 1} n^{2k} ||\Pi_n f||^2_1 \tag{2.10}$$

and denote by $\mathscr{H}_{1,k}$, $\mathscr{H}_{-1,k}$ the completions of D_∞ under the respective norms.

From (2.8) we immediately conclude that for any $u \in \mathscr{D}(L) \cap L^2_0$ such that $Lu \in \mathscr{H}_{-1}$

$$||Lu||_{-1} \geqslant ||u||_1. \tag{2.11}$$

Furthermore, see [19, Theorem 2.2], for any $f \in \mathscr{H}^0_{-1}$ we have

$$\mathbb{E}_\mu \left| \sup_{0 \leqslant t \leqslant T} \int_0^t f(\eta_s)\, ds \right|^2 \leqslant 8T ||f||^2_{-1}. \tag{2.12}$$

Now we are ready to formulate the abstract invariance principle.

Theorem 2.2. *Suppose that $(\eta_t)_{t \geqslant 0}$ is a Markov process, with the corresponding L^2-semigroup satisfying* (H1)–(H4). *Assume further that $f \in \mathscr{H}^0_{-1}$ satisfies $|||f|||_{-1,k} < +\infty$, for some $k \geqslant b$ (with b as in condition* (H4)). *Then, as $\epsilon \downarrow 0$, the family of processes*

$$Y_\epsilon(t) := \epsilon \int_0^{t/\epsilon^2} f(\eta_s)\, ds, \quad t \geqslant 0$$

satisfies the invariance principle in probability with respect to the initial configuration, i.e. the laws Q^ω *of* $Y_\epsilon(\cdot)$ *over* $C[0,+\infty)$, *given the initial configuration* $\eta_0=\omega$, *converge weakly, in* μ*-probability with respect to* ω, *to the law of a Brownian motion with a non-trivial, deterministic covariance matrix.*

3. The proof of the invariance principle

3.1. Convergence of finite-dimensional distributions

Assume first that we can prove that there exists a sequence of elements $u_n \in D(L)$, $n\geqslant 1$ such that

$$\lim_{n\uparrow+\infty} ||Lu_n - f||_{-1} = 0. \tag{3.1}$$

Notice that $(u_n)_{n\geqslant 1}$ is convergent in $\mathscr{H}_1$. Indeed, from (2.11) we have

$$||Lu_n - Lu_m||_{-1} \geqslant ||u_n - u_m||_1, \quad \forall\, n,\, m\geqslant 1$$

and our claim follows. Taking (3.1) for granted we proceed with the remainder of the proof.

We can write then that

$$Y_\epsilon(t) = \epsilon M_n\left(\frac{t}{\epsilon^2}\right) + R_{n,\epsilon}(t)$$

with

$$M_n(t) := -u_n(\eta(t)) + u_n(\eta(0)) + \int_0^t Lu_n(\eta_s)\, ds \tag{3.2}$$

a $(\mathscr{F}_t)$-martingale and the remainder term

$$R_{n,\epsilon}(t) := -\epsilon u_n(\eta(0)) + \epsilon u_n\left(\eta\left(\frac{t}{\epsilon^2}\right)\right) + \epsilon \int_0^{t/\epsilon^2} [f(\eta_s) - Lu_n(\eta_s)]\, ds.$$

Using (2.12) we obtain that

$$\mathbb{E}_\mu\left(\sup_{0\leqslant s\leqslant t} |R_{n,\epsilon}(s)|^2\right) \leqslant 3\left[\epsilon^2||u_n||_{L^2}^2 + \epsilon^2\mathbb{E}_\mu\left(\sup_{0\leqslant s\leqslant t/\epsilon^2} |u_n(\eta(s))|^2\right) + 8t||f - Lu_n||_{-1}^2\right].$$

Note that from (3.2) it follows that

$$\mathbb{E}_\mu\left(\sup_{0\leqslant s\leqslant 1}|u_n(\eta(s))|^2\right)\leqslant 3\left[||u_n||_{L^2}^2+||Lu_n||_{L^2}^2+\mathbb{E}_\mu\left(\sup_{0\leqslant s\leqslant 1}|M_n(s)|^2\right)\right]. \quad (3.3)$$

Using Doob's inequality we conclude that the left-hand side of (3.3) is less than or equal to $C(||u_n||_{L^2}^2+||Lu_n||_{L^2}^2+||u_n||_1^2)<+\infty$ for some constant $C>0$. By virtue of Birkhoff's ergodic theorem we conclude therefore from (3.3) that

$$\lim_{\epsilon\downarrow 0}\epsilon^2\,\mathbb{E}_\mu\left(\sup_{0\leqslant s\leqslant t/\epsilon^2}|u_n(\eta(s))|^2\right)=0. \quad (3.4)$$

Hence

$$\limsup_{\epsilon\downarrow 0}\ \mathbb{E}_\mu\left(\sup_{0\leqslant s\leqslant t}|R_{n,\epsilon}(s)|^2\right)\leqslant 24t||f-Lu_n||_{-1}^2,$$

thus, the remainder term vanishes, as $n\uparrow+\infty$.

For an arbitrary $\sigma>0$ we can choose n_0 such that $||u_n-u_{n_0}||_1<\sigma$ for all $n\geqslant n_0$. From Doob's inequality we conclude that

$$\mathbb{E}_\mu\left(\sup_{0\leqslant t\leqslant T}\left|\epsilon M_n\left(\frac{t}{\epsilon^2}\right)-\epsilon M_{n_0}\left(\frac{t}{\epsilon^2}\right)\right|^2\right)\leqslant C\mathbb{E}_\mu\left|\epsilon M_n\left(\frac{T}{\epsilon^2}\right)-\epsilon M_{n_0}\left(\frac{T}{\epsilon^2}\right)\right|^2$$
$$=C||u_n-u_{n_0}||_1^2T<C\sigma^2T. \quad (3.5)$$

Since $\sigma>0$ has been chosen arbitrarily it suffices only to show that the finite-dimensional distributions of $Z_\epsilon(t)\coloneqq\epsilon M_{n_0}(\frac{t}{\epsilon^2})$, $t\geqslant 0$ converge weakly to a normal law. This however follows from the standard Central Limit Theorem for martingales with stationary, ergodic increments, see e.g. Billingsley [2, Theorem 23.1, p. 206]. We conclude also that the limiting variance of $(Z_\epsilon(t_1),\dots,Z_\epsilon(t_N))$, as $\epsilon\downarrow 0$, is given by $2||u_{n_0}||_1^2[t_i\wedge t_j]_{i,j=1,\dots,N}$, thus the limiting variance of $(Y_\epsilon(t_1),\dots,Y_\epsilon(t_N))$ is given by $2||u_*||_1^2[t_i\wedge t_j]_{i,j=1,\dots,N}$, where u_* is the $\mathscr{H}_1$-limit of u_n, as $n\uparrow+\infty$.

3.2. Proof of (3.1)

We construct the required approximation for any element $f\in\mathscr{H}_{-1}^0$ with $|||f|||_{-1,k}<+\infty$ for some $k\geqslant b$. Let $\lambda>0$ and u_λ be the solution of the resolvent equation

$$\lambda u_\lambda-Lu_\lambda=-f. \quad (3.6)$$

The following proposition holds.

Proposition 3.1. *Assume that*

$$\sup_{\lambda>0} \lambda\|u_\lambda\|_{-1} < +\infty, \tag{3.7}$$

then, there exists a sequence $u_n \in \mathscr{D}(L)$, $n \geqslant 1$ *satisfying* (3.1). *Furthermore,*

$$\lim_{\lambda\downarrow 0} \lambda\|u_\lambda\|_{L^2}^2 = 0, \quad \lim_{\lambda\downarrow 0} \|u_\lambda - u_*\|_1 = 0 \tag{3.8}$$

for some $u_* \in \mathscr{H}_1$.

Proof. The proof follows closely the argument used in the proof of Theorem 2.4 of [19], we present it here for completeness sake. First, let us multiply both sides of (3.6) by u_λ and perform integration. We get

$$\lambda\|u_\lambda\|_{L^2}^2 + \|u_\lambda\|_1^2 = -(f, u_\lambda)_{L^2} \leqslant \|f\|_{-1}\|u_\lambda\|_1. \tag{3.9}$$

We can therefore conclude that

$$\sup_{\lambda>0} \|u_\lambda\|_1 < +\infty \quad \text{and} \quad \sup_{\lambda>0} \lambda\|u_\lambda\|_{L^2}^2 < +\infty. \tag{3.10}$$

In conclusion, there is a subsequence $\lambda_n \downarrow 0$ such that $v_n := u_{\lambda_n}$ has a $\mathscr{H}_1$-weak limit w. Thanks to (3.10) we infer that $\lambda u_\lambda \to 0$ in L^2, as $\lambda \downarrow 0$, and consequently from (3.7) we infer that $\lambda u_\lambda \rightharpoonup 0$ in $\mathscr{H}_{-1}$, as $\lambda \downarrow 0$. This, in turn, implies that $Lu_\lambda \rightharpoonup f$, as $\lambda \downarrow 0$, $\mathscr{H}_{-1}$-weakly. There exists therefore $(u_n)_{n\geqslant 1}$, a sequence of convex combinations of $v_1, \ldots, v_n$, such that $u_n \to w$ and $Lu_n \to f$, as $n \uparrow +\infty$, in $\mathscr{H}_1$ and $\mathscr{H}_{-1}$ correspondingly. Additionally, we have established that

$$\|w\|_1^2 = -(w, f)_{L^2}. \tag{3.11}$$

Eq. (3.9) implies that

$$\limsup_{\lambda\downarrow 0} \lambda\|u_\lambda\|_{L^2}^2 + \|w\|_1^2 \leqslant \limsup_{\lambda\downarrow 0} (\lambda\|u_\lambda\|_{L^2}^2 + \|u_\lambda\|_1^2) = -(f, w)_{L^2}. \tag{3.12}$$

Eq. (3.8) follows then from (3.11) and(3.12). Note that we have also shown

$$\|w\|_1^2 = \lim_{n\uparrow+\infty} \|u_{\lambda_n}\|_1^2,$$

which proves that $\lim_{n\uparrow+\infty}\|u_{\lambda_n} - w\|_1 = 0$. The proof that w is a unique $\mathscr{H}_1$-limiting point for u_λ, as $\lambda \downarrow 0$, consists in showing that for any two $\mathscr{H}_1$-limiting points w_1, w_2, both satisfying (3.11), the midpoint $w := 1/2(w_1 + w_2)$ must also satisfy (3.11), hence

$$\|1/2(w_1 + w_2)\|_1^2 = 1/2(\|w_1\|_1^2 + \|w_2\|^2)$$

and, in consequence $w_1 = w_2$. □

By virtue of the above proposition (3.1) holds, provided that we can show (3.7). To that purpose we choose $n_2 > n_1 \geqslant 1$ positive integers and for a given $k \geqslant 0$ define a bounded linear operator $T : L^2 \to L^2$ by

$$Tu \coloneqq \sum_{n \geqslant 1} t(n) \Pi_n u$$

with $t(n) \coloneqq n_1^k \vee (n^k \wedge n_2^k)$. Obviously $T(D_\infty) = D_\infty$. The following proposition holds.

Proposition 3.2. *$T(\mathscr{H}_1) = \mathscr{H}_1$ and for any $u_n \to u$ in $\mathscr{H}_1$ we have $Tu_n \to Tu$ in $\mathscr{H}_1$, as $n \to +\infty$. Furthermore, there exists a constant $C > 0$ independent of n_1, n_2, k such that for any $u \in D_\infty \cap L_0^2$*

$$|([T, L]u, Tu)_{L^2}| \leqslant \frac{Ck}{n_1^{1-b}} ||Tu||_1^2. \tag{3.13}$$

Here the commutator $[T, L]u \coloneqq TLu - LTu$, $u \in D_\infty \cap L_0^2$.

Proof. Suppose that $u \in \mathscr{H}_1$. Set

$$g \coloneqq \sum_{n \geqslant 1} t(n)^{-1} \Pi_n u.$$

Obviously, $g \in \mathscr{H}_1$ and $Tg = u$, so $T(\mathscr{H}_1) = \mathscr{H}_1$. The $\mathscr{H}_1$-continuity of T is straightforward. Recalling condition (H3) we obtain, after a simple calculation, that for any $u \in D_\infty \cap L_0^2$

$$[T, L]u = \sum_{n \geqslant 1} [(t(n+1) - t(n)) L_{n,n+1} u_n + (t(n-1) - t(n)) L_{n,n-1} u_n],$$

where $L_{n,k} \coloneqq \Pi_k L \Pi_n$ and $u_n \coloneqq \Pi_n u$. Hence, from condition (H4) we get

$$\begin{aligned} &|([T, L]u, Tu)_{L^2}| \\ &\quad = \Bigg| \sum_{n \geqslant 1} [t(n+1)(t(n+1) - t(n))(L_{n,n+1} u_n, u_{n+1})_{L^2} \\ &\qquad + t(n-1)(t(n-1) - t(n))(L_{n,n-1} u_n, u_{n-1})_{L^2}] \Bigg| \\ &\quad \leqslant C \sum_{n \geqslant 1} \left[\frac{n^b |t(n+1) - t(n)|}{t(n)} ||Tu_n||_1 ||Tu_{n+1}||_1 \right. \\ &\qquad \left. + \frac{n^b |t(n-1) - t(n)|}{t(n-1)} ||Tu_n||_1 ||Tu_{n-1}||_1 \right]. \end{aligned}$$

Notice that for any $k<n_1\leqslant n\leqslant n_2$ we have

$$\begin{aligned}\frac{n^b|t(n\pm 1)-t(n)|}{t(n)} &\leqslant \left| n^b\left[\left(1\pm\frac{1}{n}\right)^k-1\right]\right| \\ &\leqslant \sum_{j=1}^{k}\binom{k}{j} n^{-j+b}\leqslant n_1^b\sum_{j=1}^{k}\frac{1}{j!}\left(\frac{k}{n_1}\right)^j \\ &\leqslant \frac{k}{n_1^{1-b}}\sum_{j=1}^{k}\frac{1}{j!}\leqslant\frac{ek}{n_1^{1-b}}.\end{aligned}$$

Eq. (3.13) follows then from the above bound, an elementary inequality

$$||Tu_n||_1||Tu_{n-1}||_1\leqslant\tfrac{1}{2}(||Tu_n||_1^2+||Tu_{n-1}||_1^2)$$

and the first identity of (2.9). □

Thanks to assumption (H2) the conclusion of the proposition can be extended easily to the entire $\mathscr{D}(L)\cap L_0^2$.

Corollary 3.3.

$$|(TLu,Tu)_{L^2}+||Tu||_1^2|\leqslant\frac{Ck}{n_1^{1-b}}||Tu||_1^2,\quad \textit{for all } u\in\mathscr{D}(L)\cap L_0^2. \tag{3.14}$$

Next we show that

Lemma 3.4. *Let u_λ be the solution of* (3.6) *and $f\in\mathscr{H}^0_{-1}$ satisfies $|||f|||_{-1,k}<+\infty$ for some $k\geqslant 0$. Then,*

$$\sup_{\lambda>0}|||u_\lambda|||_{1,k}\leqslant C(b,k)|||f|||_{-1,k}. \tag{3.15}$$

The constant appearing on the right-hand side of (3.15) *depends only on b,k.*

Proof. We apply the operator T to both sides of (3.6), multiply them by Tu_λ and perform integration. As a result we get (note that $u_\lambda\in L_0^2$)

$$\lambda||Tu_\lambda||_{L^2}^2+||Tu_\lambda||_1^2=-(Tf,Tu_\lambda)_{L^2}+(TLu_\lambda,Tu_\lambda)_{L^2}+||Tu_\lambda||_1^2. \tag{3.16}$$

The right-hand side of (3.16) can be estimated with the help of Corollary 3.3 by

$$||Tf||_{-1}||Tu_\lambda||_1+\frac{Ck}{n_1^{1-b}}||Tu_\lambda||_1^2\leqslant\frac{1}{2}(||Tf||_{-1}^2+||Tu_\lambda||_1^2)+\frac{Ck}{n_1^{1-b}}||Tu_\lambda||_1^2.$$

Now, for the given k we can choose n_1 so large that $Ckn_1^{b-1}<1/4$. In consequence, we deduce from this and (3.16) that

$$\sum_{1\leqslant n\leqslant n_2} n^{2k}||u_{\lambda,n}||_1^2 \leqslant ||Tu_\lambda||_1^2 \leqslant 2||Tf||_{-1}^2. \tag{3.17}$$

Finally, note that the right-hand side does not depend on n_2 so we can send it to infinity obtaining, in consequence (3.15). $\square$

We finish the proof of (3.1) by showing that (3.7) holds, cf. Proposition 3.1. Using Proposition 2.1 we can write, with the notation $u_{\lambda,n} := \Pi_n u_\lambda$, that for any $g\in\mathscr{H}_1^0$

$$\begin{aligned} |(Lu_\lambda, g)_{L^2}| &\leqslant \sum_{n\geqslant 1} |(\Pi_n Lu_\lambda, g)_{L^2}| \\ &\overset{(2.6)}{\leqslant} C\sum_{n\geqslant 1} n^b(||u_{\lambda,n-1}||_1^2 + ||u_{\lambda,n}||_1^2 + ||u_{\lambda,n+1}||_1^2)^{1/2}||g_n||_1 \\ &\leqslant C_1||g||_1\, |||u_\lambda|||_{1,b}, \end{aligned} \tag{3.18}$$

with $C_1 := (2^{2b}+2)^{1/2}C$ (recall that $u_{\lambda,0}=0$). Here $g_n := \Pi_n g$ and the constant $C_1>0$ appearing in the rightmost member does not depend on $\lambda>0$. Thus,

$$\sup_{\lambda>0} ||Lu_\lambda||_{-1} \leqslant C_1 < +\infty,$$

which in turn implies (3.7).

Remark. Note that in view of Proposition 3.1 and the argument of Lemma 3.4 we have in fact shown that for any $f\in\bigcap_{k\geqslant 1}\mathscr{H}_{-1,k}$ we have $u_\lambda\in\bigcap_{k\geqslant 1}\mathscr{H}_{1,k}$ and

$$\lim_{\lambda\downarrow 0} |||u_\lambda - w|||_{1,k} = 0, \quad \forall\, k\geqslant 0.$$

3.3. Tightness

We use the argument of [19,21]. Notice that according to the proof of Proposition 3.1 we have shown that there exists $(u_n)_{n\geqslant 1}\subseteq\mathscr{D}(L)$ such that $(Lu_n)_{n\geqslant 1}\subseteq\mathscr{H}_{-1}^0$ and

$$\lim_{n\uparrow+\infty} ||Lu_n - f||_{-1} = 0 \quad \text{and} \quad \lim_{n\uparrow+\infty} ||u_n - w||_1 = 0.$$

Let us choose an arbitrary $\sigma>0$. By (2.12) and (3.1):

$$\begin{aligned} &\mathbb{E}_\mu\left\{\epsilon^2 \sup_{0\leqslant t\leqslant T}\left|\int_0^{t/\epsilon^2} [f(\eta(s)) - Lu_n(\eta(s))]\,ds\right|^2\right\} \\ &\quad\leqslant 8T||f - Lu_n||_{-1}^2 < 8T\sigma^2 \end{aligned} \tag{3.19}$$

for sufficiently large n. But

$$\epsilon \int_0^{t/\epsilon^2} Lu_n(\eta(s))\,ds = -\epsilon N_{u_n}(t/\epsilon^2) + \epsilon u_n(\eta(t/\epsilon^2)) - \epsilon u_n(\eta(0)),$$

where

$$N_{u_n}(t) := u_n(\eta(t)) - u_n(\eta(0)) - \int_0^t Lu_n(\eta(s))\,ds, \quad t \geqslant 0$$

is a cadlag martingale with respect to the standard filtration $(\mathscr{Z}_t)_{t\geqslant 0}$.

By virtue of Theorem 23.1, p. 206 of [2] the family $(\epsilon N_{u_n}([t/\epsilon^2]))_{t\geqslant 0}$, $\epsilon \in (0,1)$, is tight in $D[0,T]$ for any $T>0$, $p=1,\dots,d$. Let $X_k := \sup_{k-1\leqslant t\leqslant k}|N_{u_n}(t) - N_{u_n}(k-1)|^2$. The sequence $(X_k)_{k\geqslant 1}$ is stationary with $\mathbb{E}_\mu X_1 = \|u_n\|_1^2 < +\infty$. By virtue of Birkhoff's ergodic theorem we can easily conclude that $1/N \max\{X_1,\dots,X_N\} \to 0$ as $N \to +\infty$ both a.s. and in the L^1 sense, thus,

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_\mu \left(\sup_{0\leqslant t\leqslant T} \epsilon^2 |N_{u_n}(t/\epsilon^2) - N_{u_n}([t/\epsilon^2]|^2 \right) = 0. \tag{3.20}$$

We conclude therefore that the family of processes $(\epsilon N_{u_n}(t/\epsilon^2))_{t\geqslant 0}$, $\epsilon \in (0,1)$ is tight in $D[0,T]$ for any $T>0$, $p=1,\dots,d$. This together with (3.19) yield tightness of the laws of $(\epsilon \int_0^{t/\epsilon^2} f(\eta(s))\,ds)_{t\geqslant 0}$, $\epsilon \in (0,1)$ in $D[0,T]$. The continuity of trajectories implies tightness in $C[0,+\infty)$.

4. Application to convection–diffusion with a steady Gaussian drift

Let us denote by $\mathscr{A}(d)$ the space of all real valued, anti-symmetric $d\times d$ matrices. Let Ω be the Frechet space $C(\mathbb{R}^d;\mathscr{A}(d))$ of all anti-symmetric matrix valued continuous functions and assume that $\tau_{\mathbf{x}}(\omega)(\cdot) = \omega(\mathbf{x}+\cdot)$, $\mathbf{x}\in\mathbb{R}^d$, $\omega\in\Omega$. We suppose that μ is a homogeneous, Gaussian measure on $\mathscr{B}(\Omega)$, i.e.

(HG) $\mu\tau_{\mathbf{x}} = \mu$, $\mathbf{x}\in\mathbb{R}^d$ and for any $m\geqslant 1$, $\varphi_1,\dots,\varphi_m \in C_0^\infty(\mathbb{R}^d)$ the random variables

$$\omega \mapsto \langle \omega_{i,j}, \varphi_k \rangle := \int_{\mathbb{R}^d} \omega_{i,j}(\mathbf{x})\varphi_k(\mathbf{x})\,d\mathbf{x},$$
$$i,j = 1,\dots,d,\; k=1,\dots,m$$

are jointly Gaussian.

Recall that $\mathscr{T}_1 := (\Omega, \mathscr{B}(\Omega), \mu)$ and $L^p := L^p(\mathscr{T}_1)$, $p\in[1,+\infty]$. We suppose further that the measure is ergodic under the spatial translations, i.e. any $g\in L^\infty$ satisfying $g\circ\tau_{\mathbf{x}} = g$, $\forall\,\mathbf{x}\in\mathbb{R}^d$ must be constant μ-a.s. For any $g\in L^p$ we adopt the notation $D_k g := \frac{d}{dh} g(\tau_{h\mathbf{e}_k}\omega)|_{h=0}$, where $\mathbf{e}_k$, $k=1,\dots,d$ is the kth vector of the canonical basis in $\mathbb{R}^d$. The derivatives are understood in the L^p sense. For $p\in[1,+\infty)$ we let $\mathscr{W}^{p,m}$ be the Banach space consisting of those elements $g \in L^p$, for which

$$||g||_{p,m}^p := \sum_{i_1+\cdots+i_d \leqslant m} ||D_1^{i_1} \cdots D_d^{i_d} g||_{L^p}^p < +\infty.$$

We can extend the definition to cover the case $p = +\infty$ in the usual way.

Let $\mathbf{H}(\omega) := \omega(\mathbf{0})$. We suppose additionally that

(R) the entries of $\mathbf{H}$ belong to $\mathscr{W}^{2,4}$. Note that this assumption implies in particular that the field $\mathbf{x} \mapsto \mathbf{H}(\mathbf{x};\omega) := \mathbf{H}(\tau_{\mathbf{x}}\omega)$ is four times continuously differentiable for μ-a.s. ω.

Let $\mathscr{T}_0 := (\Sigma, \mathscr{B}, Q)$ be a certain probability space. For any $\mathbf{x} \in \mathbb{R}^d$ we consider the process $(\mathbf{x}^{\mathbf{x}}(t;\sigma,\omega))_{t \geqslant 0}$ over the product probability space $\mathscr{T}_0 \otimes \mathscr{T}_1$ defined as a solution, for a given ω, of the following Itô stochastic differential equation:

$$\begin{cases} d\mathbf{x}^{\mathbf{x}}(t;\sigma,\omega) = \mathbf{V}(\mathbf{x}^{\mathbf{x}}(t;\sigma,\omega);\omega)\,dt + \sqrt{2}\,d\mathbf{w}(t;\sigma), \\ \mathbf{x}^{\mathbf{x}}(\sigma,\omega) = \mathbf{x}, \end{cases} \tag{4.1}$$

where $\mathbf{w}(\cdot) = (w_1(\cdot), \ldots, w_d(\cdot))$ is a d-dimensional standard Brownian motion over $\mathscr{T}_0$.

The drift $\mathbf{V}(\cdot;\omega)$ is given by

(S) $\mathbf{V}(\mathbf{x};\omega) := \mathbf{V}(\tau_{\mathbf{x}}\omega)$, $\mathbf{x} \in \mathbb{R}^d$, where $\mathbf{V} = (V_1, \ldots, V_d) := \nabla \cdot \mathbf{H}$, i.e.

$$V_p = \sum_{q=1}^d D_q H_{p,q}, \quad p = 1, \ldots, d.$$

Note that $\mathbf{V}(\omega) = \mathbf{V}(\mathbf{0};\omega)$.

Under the above assumptions (4.1) possesses a unique global solution for μ-a.s. ω, see [10]. We define

$$\eta_t(\sigma,\omega) := \tau_{\mathbf{x}^{\mathbf{0}}(t;\sigma,\omega)}\omega, \quad (\sigma,\omega) \in \Sigma \times \Omega,\ t \geqslant 0$$

the canonical Lagrangian process over $\mathscr{T}_0 \otimes \mathscr{T}_1$. It is Markovian, i.e. $\mathbf{E}^Q[f(\eta_{t+h})|\mathscr{Z}_t] = P^h f(\eta_t)$, for any $f \in L^2$, where $P^t : L^2 \to L^2$, $t \geqslant 0$ is a semigroup of Markovian operators given by

$$P^t f(\omega) := \int_{\mathbb{R}^d} p^\omega(t,\mathbf{0},\mathbf{y}) f(\tau_{\mathbf{y}}\omega)\,d\mathbf{y},$$

with $p^\omega(t,\mathbf{x},\mathbf{y}) > 0$, $(t,\mathbf{x},\mathbf{y}) \in [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ the transition of probability densities for the diffusion given by (4.1) corresponding to a fixed ω. Here $(\mathscr{Z}_t)_{t \geqslant 0}$ is the natural filtration corresponding to $(\eta_t)_{t \geqslant 0}$. Measure μ is *invariant* under the semigroup, i.e. (2.1) holds. Positivity of transition of probability densities implies, via a standard argument, that μ is *ergodic*. Hence, condition (M) from Section 2 is fulfilled. A direct computation also shows that the generator L of the semigroup is given by

$$Lg = \Delta g + \mathbf{V}\cdot\nabla g, \quad \text{for } g\in\mathscr{W}^{2+\varrho,2},\ \varrho>0,$$

with $\nabla g := (D_1 g, \dots, D_d g)$, $\Delta := D_1^2 + \cdots + D_d^2$.

Lemma 4.1. *$\mathscr{C} := \bigcap_{\varrho>0}\mathscr{W}^{2+\varrho,2}$ is a core of L.*

Proof. Obviously $\mathscr{C}$ is dense in L^2. We show that for any fixed $\varrho>0$ we have $P^t(\mathscr{W}^{2+\varrho,2})\subseteq\mathscr{W}^{2+\varrho',2}$ for any $\varrho'\in(0,\varrho)$ and $t\geqslant 0$, which would then imply $P^t(\mathscr{C})\subseteq\mathscr{C}$. The conclusion of the proposition follows then from e.g. [3, Proposition 3.3].

Let us choose an arbitrary $f\in\mathscr{W}^{2+\varrho,2}$. It is elementary to verify that, for any $\omega\in\Omega$,

$$\begin{aligned} P^t f(\tau_{\mathbf{h}}\omega) &= \int f(\tau_{\mathbf{x}^{\mathbf{0}}(t;\sigma,\tau_{\mathbf{h}}\omega)}(\tau_{\mathbf{h}}\omega))Q(d\sigma) \\ &= \int f(\tau_{\mathbf{x}^{\mathbf{h}}(t;\sigma,\omega)}\omega)Q(d\sigma), \quad \text{for any } \mathbf{h}\in\mathbb{R}^d. \end{aligned} \tag{4.2}$$

Differentiating both sides of (4.2) with respect to $\mathbf{h}$ at $\mathbf{0}$ in the $L^{2+\varrho'}$-sense, with $\varrho'\in(0,\varrho)$, we obtain

$$\nabla P^t f(\omega) = \int \mathbf{J}(t,\sigma,\omega)\nabla f(\eta_t(\sigma,\omega))Q(d\sigma), \tag{4.3}$$

where the matrix $\mathbf{J}(t,\sigma,\omega) = [J_{i,j}(t,\sigma,\omega)]$ is defined as $J_{i,j}(t,\sigma,\omega) := \partial_{h_i}\mathbf{x}_j^{\mathbf{h}}(t;\sigma,\omega)|_{\mathbf{h}=\mathbf{0}}$. It satisfies the equation

$$\mathbf{J}(t,\sigma,\omega) = \mathbf{I} + \int_0^t \mathbf{J}(s,\sigma,\omega)\nabla\mathbf{V}(\eta_s)\,ds. \tag{4.4}$$

To justify the equality in (4.5) it suffices to prove that

$$\int\int |\mathbf{J}(t,\sigma,\omega)|^r\, Q(d\sigma)\mu(d\omega) < +\infty \quad \text{for all } r>1. \tag{4.5}$$

In order to show (4.5) we consider the Gaussian field

$$Y_n(\mathbf{x}) := \left(\frac{\mathbf{V}(\mathbf{x})}{|\mathbf{x}|+n}, \frac{\nabla_{\mathbf{x}}\mathbf{V}(\mathbf{x})}{|\mathbf{x}|+n}, \frac{\nabla_{\mathbf{x}}^2\mathbf{V}(\mathbf{x})}{|\mathbf{x}|+n}\right), \quad \mathbf{x}\in\mathbb{R}^d.$$

Set

$$K_n(\lambda) := \left[\omega : \sup_{\mathbf{x}\in\mathbb{R}^d} |Y_n(\mathbf{x})| \leqslant \lambda\right], \quad n \geqslant 1.$$

By virtue of Theorem 5.2, [1, p. 120], there exist λ_0, C_1, C_2 independent of n such that

$$\mu(K_n^c(\lambda_0)) \leqslant C_1 \exp\{-C_2 n^2\}, \quad \forall n \geqslant 1. \tag{4.6}$$

Let us define $L_n := \{\sup_{0\leqslant t\leqslant T}|\mathbf{w}(t)| \leqslant n\}$. By a standard estimate $Q(L_n^c) \leqslant C_3 \exp\{-C_4 n^2\}$. From (4.1) we obtain that $X_T := \sup_{0\leqslant t\leqslant T}|\mathbf{x}^{\mathbf{0}}(t)|$ satisfies

$$X_T(\sigma,\omega) \leqslant C_5\, n \quad \text{for } (\sigma,\omega) \in L_n \times K_n(\lambda_0),\ n \geqslant 1 \tag{4.7}$$

for some deterministic constant $C_5 = C(\lambda_0, T)$. Hence,

$$\sup_{t\in[0,T]} |\nabla \mathbf{V}(\eta_t)| \leqslant \lambda_0(X_T + n) \leqslant C_6 n \quad \text{for } (\sigma,\omega)\in L_n \times K_n(\lambda_0),\ n \geqslant 1 \tag{4.8}$$

for some deterministic constant $C_6 > 0$.

From (4.8) we obtain that

$$|\mathbf{J}(t,\sigma,\omega)| \leqslant e^{C_6 n t} \quad \text{for } t \in [0,T] \text{ and } (\sigma,\omega) \in L_n \times K_n(\lambda_0),\ n \geqslant 1, \tag{4.9}$$

hence $Q\otimes\mu[(\sigma,\omega) : |\mathbf{J}(t,\sigma,\omega)| > e^{C_6 n t}] < e^{-C_7 n^2}$ for some constant $C_7 > 0$ and all $n \geqslant 1$ and (4.5) follows.

Similar considerations, using the second derivative of $\mathbf{V}$, lead to the conclusion that $P^t f \in \mathscr{W}^{2+\varrho',2}$ for any $\varrho' \in (0,\varrho)$. $\square$

A straightforward calculation shows that the L^2-adjoint to P^t is equal to

$$(P^t)^* f = \int_{\mathbb{R}^d} p^{\omega}(t,\mathbf{y},\mathbf{0}) f(\tau_{\mathbf{y}}\omega)\, d\mathbf{y}.$$

The family $((P^t)^*)_{t\geqslant 0}$ forms a C_0-semigroup on L^2 whose generator equals L^*, see [16, Corollary 10.6, p. 41]. A direct calculation shows that $\mathscr{C} \subseteq D(L^*)$ and

$$L^* g = \Delta g - \mathbf{V}\cdot\nabla g, \quad g \in \mathscr{C}.$$

The argument contained in the proof of Lemma 4.1 shows that $\mathscr{C}$ is a core of L^*. Hence $Sg = \Delta g$, $g \in \mathscr{C}$. The closure of S is the self-adjoint generator of C_0-semigroup

$$Q^t f = \int_{\mathbb{R}^d} q(t,\mathbf{y}) f(\tau_{\mathbf{y}}\omega)\, d\mathbf{y},$$

where $q(t,\mathbf{y}) := (2\pi t)^{-d/2} \exp\{-\frac{|\mathbf{y}|^2}{2t}\}$. Hence condition (H1) holds.

We can define the respective spaces $\mathscr{H}_1$ and $\mathscr{H}_{-1}$. Denote by $\mathscr{P}_n$ the closure in L^2 of the linear space consisting of the elements $p(X_1,\dots,X_m)$, where p is an nth degree polynomial in m variables and each X_k is a random variable of the form $\langle H_{i,j}(\cdot),\varphi\rangle$, $i,j=1,\dots,d$, $\varphi\in C_0^\infty(\mathbb{R}^d)$. We let H_0 be the space of constants and $H_n := \mathscr{P}_n \ominus \mathscr{P}_{n-1}$. The elements of H_n are sometimes called Hermite polynomials of degree n. It is well known, see e.g. [8, Theorem 2.6], that $L^2=\oplus_{n=0}^\infty H_n$. Since $V_1,\dots,V_d$, the components of $\mathbf{V}$, belong to H_1 condition (S) implies that $V_p\in\mathscr{H}_{-1}$, $p=1,\dots,d$. In fact, one has $V_p\in\mathscr{H}_{-1,k}$ for all $k\geqslant 0$, $p=1,\dots,d$.

Denote by Π_n the L^2-orthogonal projection onto H_n. It is elementary to observe that $\Pi_n(\mathscr{W}^{2,2})\subseteq\mathscr{W}^{2,2}$, see [8, Theorem 5; 14, p. 63]. In fact, due to the equivalence of all L^p norms on H_n, see e.g. [8], we can write that $D_n^0 := \Pi_n(\mathscr{W}^{2+\varrho,2})\subseteq\mathscr{W}^{2+\varrho,2}$, hence condition (H2) of Section 3 is satisfied. Additionally, thanks to the fact that H_n remains invariant under the action of the group of spatial shifts $\tau_{\mathbf{x}}$, $\mathbf{x}\in\mathbb{R}^d$, we conclude that also (H3) holds.

For any $f\in D_n^0$ and $g\in\mathscr{W}^{2+\varrho,2}$ we have, thanks to Lemma 4.1,

$$-(Lf,g)_{L^2}=\int\nabla f\nabla g\,d\mu-\int(\mathbf{V}\cdot\nabla f)g\,d\mu=(Sf,g)_{L^2}+\int\mathbf{H}\nabla f\cdot\nabla g\,d\mu. \tag{4.10}$$

To estimate the second term on the utmost right-hand side of (4.10) we use Hölder inequality, which implies that it is less than or equal to

$$\begin{aligned}||\mathbf{H}\nabla f||_{L^2_d}||\nabla g||_{L^2_d}&\leqslant||\mathbf{H}||_{L^{2n}_{d\times d}}||\nabla f||_{L^{2n/(n-1)}_d}||\nabla g||_{L^2_d}\\&\leqslant\left(\frac{n+1}{n-1}\right)^{n/2}||\mathbf{H}||_{L^{2n}_{d\times d}}||\nabla f||_{L^2_d}||\nabla g||_{L^2_d},\end{aligned}$$

by virtue of the hyper-contractivity estimate of L^p norms on Gaussian spaces, see the proof of Theorem 5.10 in [8] (in particular the first formula after (5.4) there). Here $||\cdot||_{L^p_d}$, $||\cdot||_{L^p_{d\times d}}$ denote the respective L^p norms of random vectors and matrices. Notice that by Stirling's formula $||\mathbf{H}||_{L^{2n}_{d\times d}}\sim\sqrt{n}$, thus

$$|(Lf,g)_{L^2}|\leqslant C\sqrt{n}|(Sf,f)_{L^2}|^{1/2}|(Sg,g)_{L^2}|^{1/2} \tag{4.11}$$

and (H4) of Section 3 follows.

For any $\lambda>0$ and $p=1,\dots,d$ we denote by $u_\lambda^{(p)}$ the solutions of

$$\lambda u_\lambda^{(p)}-Lu_\lambda^{(p)}=-V_p. \tag{4.12}$$

According to the results of Section 3 there exists $u_*^{(p)}\in\mathscr{H}_1$, $p=1,\dots,d$ satisfying

$$\lim_{\lambda\downarrow 0}||u_\lambda^{(p)}-u_*^{(p)}||_1=0.$$

In fact the following result holds.

Theorem 4.2. *Under assumption* (HG), (R), (S) *the laws of trajectories* $(\epsilon\mathbf{x}(\frac{t}{\epsilon^2}))_{t\geqslant 0}$ *in* $C([0,+\infty);\mathbb{R}^d)$ *satisfy the invariance principle, as* $\epsilon\downarrow 0$. *The limiting Wiener measure has the covariance matrix given by* $\tilde{\mathbf{D}} = [\tilde{D}_{p,q}]$, *where*

$$\tilde{D}_{p,q} = 2[\delta_{p,q} + (u_*^{(p)}, u_*^{(q)})_1], \quad p,q = 1, \dots, d. \tag{4.13}$$

Proof. Note that

$$\epsilon\mathbf{x}\left(\frac{t}{\epsilon^2}\right) = \epsilon\int_0^{t/\epsilon^2} \mathbf{V}(\eta_s)\,ds + \sqrt{2}\,\epsilon\mathbf{w}\left(\frac{t}{\epsilon^2}\right). \tag{4.14}$$

By virtue of Theorem 2.2 the first term on the right-hand side of (4.14) converges weakly to a Brownian motion with covariance matrix given by $\mathbf{D} = [D_{p,q}]$, where

$$D_{p,q} = 2(u_*^{(p)}, u_*^{(q)})_1. \tag{4.15}$$

To prove the invariance principle for the sum of the terms appearing on the right-hand side of (4.14) we observe that for each $p = 1, \dots, d$ one can choose a sequence $(u_n^{(p)})_{n\geqslant 1} \subseteq \mathscr{W}^{2+\varrho,2}$ corresponding to V_p, as in (3.1) with $f = V_p$. Denoting by $M_n^{(p)}$ the respective martingale, see (3.2), we get, with the help of Itô–Krylov formula, that the joint quadratic variation equals

$$\langle M_n^{(p)}, w_q \rangle_t = \sqrt{2}\int_0^t D_q u_n^{(p)}(\eta_s)\,ds.$$

Hence

$$\epsilon^2 \langle M_n^{(p)}, w_q \rangle_{t/\epsilon^2} \to \sqrt{2}\int D_q u_n^{(p)}\,d\mu = 0$$

as $\epsilon\downarrow 0$, both a.s. and in the mean, by virtue of the ergodic theorem. Thus, we can conclude that $\mathbf{x}_\epsilon(\cdot)$ satisfies the invariance principle, with the limiting Brownian motion having the covariance matrix given by $\tilde{\mathbf{D}}$. □

5. Regularity of the effective diffusivity tensor with respect to the drift

Let $\mathscr{G}_{\text{reg}}$ be the space consisting of all random matrices $\mathbf{K} : \Omega \to \mathscr{A}(d)$ whose entries belong to $\mathscr{P}_1 \cap \mathscr{W}^{2,4}$, with the topology on $\mathscr{G}_{\text{reg}}$ induced by the $\mathscr{W}^{2,4}$-norm. Let $\mathbf{H} : \Theta \to \mathscr{G}_{\text{reg}}$ be a C^∞ smooth map acting on an open domain $\Theta \subseteq R^N$, with a certain integer $N \geqslant 1$.

Denote by $\tilde{\mathbf{D}}(\theta)$ the appropriate limiting covariance matrix corresponding to the drift $\mathbf{V}(\theta) = \nabla \cdot \mathbf{H}(\theta)$, cf. (4.13). We show the following result.

Theorem 5.1. $\tilde{\mathbf{D}}(\theta)$ *depends* C^∞ *smoothly on the parameter* $\theta \in \Theta$.

Proof. With no loss of generality, we assume that $\Theta \subseteq \mathbb{R}^1$. We denote by $(u_\lambda^{(1)}(\theta), \dots, u_\lambda^{(d)}(\theta))$ the respective solutions of (4.12) corresponding to $L(\theta) = \Delta + \mathbf{V}(\theta) \cdot \nabla$. From the remark made after the proof of Lemma 3.4 we conclude that $u_\lambda^{(p)}(\theta) \in \bigcap_{k \geqslant 1} \mathscr{H}_{1,k}$.

The results of Section 3 imply that there exist $u_*^{(p)}(\theta)$, for which

$$\lim_{\lambda \downarrow 0} |||u_\lambda^{(p)}(\theta) - u_*^{(p)}(\theta)|||_{1,k} = 0$$

and $\tilde{\mathbf{D}}(\theta) = 2\mathbf{I} + 2[(u_*^{(p)}(\theta), u_*^{(q)}(\theta))_1]$. We prove first that $\theta \mapsto u_\lambda^{(p)}(\theta)$ is C^∞ smooth function mapping Θ into $\mathscr{H}_{1,k}$ for any $k \geqslant 1$. In order to do so it suffices only to show that the solutions of

$$\lambda v_\lambda^{(p)}(\theta) - L(\theta) v_\lambda^{(p)}(\theta) = f_\lambda(\theta), \tag{5.1}$$

$\mathscr{H}_{1,k}$-converge to a certain element $v_*^{(p)}(\theta)$, when $\lambda \downarrow 0$. Here

$$f_\lambda(\theta) \coloneqq -V_p'(\theta) + \mathbf{V}'(\theta) \cdot \nabla u_\lambda^{(p)}(\theta).$$

In fact, a standard argument using approximate difference quotients, shows that $\theta \mapsto u_*^{(p)}(\theta)$ is differentiable and

$$v_*^{(p)}(\theta) = \frac{d}{d\theta} u_*^{(p)}(\theta) \tag{5.2}$$

holds in the $\mathscr{H}_{1,k}$-sense.

Note that $f_\lambda(\theta) \in \bigcap_{k \geqslant 1} \mathscr{H}_{-1,k}$, due to the fact that $u_\lambda^{(p)} \in \bigcap_{k \geqslant 1} \mathscr{H}_{1,k}$. On the other hand, for any $g \in H_n$ and $n \geqslant 2$

$$\begin{aligned} |(f_\lambda(\theta), g)_{L^2}| &= |(\mathbf{V}'(\theta) \cdot \nabla u_\lambda^{(p)}(\theta), g)_{L^2}| = |(\mathbf{H}'(\theta) \cdot \nabla u_\lambda^{(p)}(\theta), \nabla g)_{L_d^2}| \\ &\leqslant C\sqrt{n}\, ||u_{\lambda,n}||_1\, ||g||_1. \end{aligned}$$

Thus $|||f_\lambda(\theta)|||_{-1,k} \leqslant C|||u_\lambda|||_{1,k+1}$, with constant C independent of n and λ. In conclusion, we can find a unique solution $v_\lambda^{(p)} \in \bigcap_{k \geqslant 1} \mathscr{H}_{1,k}$ of (5.1) for any $\lambda > 0$. Since

$$\lim_{\lambda \downarrow 0} |||f_\lambda(\theta) - f(\theta)|||_{-1,k} = 0, \quad \text{for all } k \geqslant 1,$$

where $f(\theta) := -V_p'(\theta) + \mathbf{V}'(\theta) \cdot \nabla u_*^{(p)}(\theta) \in \bigcap_{k \geqslant 1} \mathscr{H}_{-1,k}$, we also can find a certain $v_*^{(p)}(\theta) \in \bigcap_{k \geqslant 1} \mathscr{H}_{1,k}$, such that

$$\lim_{\lambda \downarrow 0} |||v_\lambda^{(p)}(\theta) - v_*^{(p)}(\theta)|||_{1,k} = 0, \quad \text{for all } k \geqslant 1.$$

Recursion allows us to extend the above argument to the derivatives of any order. □

6. Homogenization of diffusions with Ornstein–Uhlenbeck drifts

In this section we admit temporal dynamics of the drift that is assumed to be both Markovian and Gaussian.

6.1. Homogeneous Gaussian measures on Hilbert spaces

To give an appropriate setting for that situation we suppose that Ω is the Hilbert space of d-dimensional incompressible vector fields that is the completion of $C_{0,\text{div}}^\infty := \{\omega \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) : \nabla_\mathbf{x} \cdot \omega = 0\}$ with respect to the norm

$$||\omega||_\Omega^2 := \int_{\mathbb{R}^d} (|\omega(\mathbf{x})|^2 + |\nabla_\mathbf{x}\omega(\mathbf{x})|^2 + \cdots + |\nabla_\mathbf{x}^m \omega(\mathbf{x})|^2) \vartheta_\rho(\mathbf{x})\, d\mathbf{x}$$

for any positive integer m and the weight function $\vartheta_\rho(\mathbf{x}) := (1 + |\mathbf{x}|^2)^{-\rho}$, where $\rho > d/2$. We shall also assume that $m > d/2 + 1$ so any $\omega \in \Omega$ is of C^1 class of regularity. On Ω we have a group of transformations $\tau_\mathbf{x} : \Omega \to \Omega$, given by $\tau_\mathbf{x}\omega(\cdot) := \omega(\cdot + \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$.

The measure μ, cf. Section 4, is supposed to be homogeneous, Gaussian of zero mean given by the covariance

$$\int_\Omega \langle \varphi_1, \omega \rangle \langle \varphi_2, \omega \rangle \mu(d\omega) = \int_{\mathbb{R}^d} \frac{\mathscr{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \hat{\varphi}_1(\mathbf{k}) \cdot \hat{\varphi}_2(\mathbf{k})\, d\mathbf{k}.$$

Here $\varphi_1, \varphi_2 \in C_{0,\text{div}}^\infty$ and $\hat{\varphi}$ denotes the Fourier transform of a given function φ. We assume that $\mathscr{E} : [0, +\infty) \to [0, +\infty)$ satisfies

(E) there exists $K, C > 0$ and $\alpha < 1$ such that

$$\text{supp } \mathscr{E}(\cdot) \subseteq [0, K] \quad \text{and} \quad \mathscr{E}(k) \leqslant \frac{C}{k^{2\alpha - 1}}, \ \forall k \in (0, K].$$

Remark. We note that condition (E) can be somewhat weakened by replacing the assumption of compact support of $\mathscr{E}(\cdot)$ by a hypothesis guaranteeing a sufficient decay rate of the energy spectrum, e.g. the condition that $\sup_{k > 0}(1 + k)^n \mathscr{E}(k) < +$

∞ for all $n \geqslant 1$ certainly suffices for the validity of Theorem 6.3 presented below. However, due to the fact that we wish to focus our exposition on the role of the sector condition and not on the computational side of verifying it, we do not present the calculations in the most general case

$\mathscr{P}_n$ is the L^2 closure of the linear space $\mathscr{P}_n^{\text{reg}}$ spanned by the monomials $\langle \varphi_1, \cdot \rangle \cdots \langle \varphi_m, \cdot \rangle$, where $m \leqslant n$ and $\varphi_1, \ldots, \varphi_m \in \mathscr{S}_{\text{div}}(\mathbb{R}^d, \mathbb{R}^d)$—the Schwartz space of divergence-less fields—satisfy $\mathbf{0} \notin \operatorname{supp} \hat{\varphi}_k$, $k = 1, \ldots, m$. Let $\mathscr{P}^{\text{reg}} := \bigcup_{n \geqslant 0} \mathscr{P}_n^{\text{reg}}$ and $H_n := \mathscr{P}_n \ominus \mathscr{P}_{n-1}$ be the space of nth degree Hermite polynomials and Π_n be, as before, the L^2 projection onto H_n. Note that $\Pi_n(\mathscr{W}^{p,m}) \subseteq \mathscr{W}^{p,m} \cap \mathscr{P}_n$ for any $p \in (1, +\infty)$ and $m \geqslant 1$. Here $\mathscr{W}^{p,m}$ are the Sobolev spaces that can be introduced in complete analogy to Section 4.

6.2. Markovian dynamics

Let

$$V_\omega(t) = S(t)\omega + \int_0^t S(t-s) B \, dW(s). \tag{6.1}$$

Here $(W(t))_{t \geqslant 0}$ is a cylindrical Wiener process on $L^2_{\text{div}}(\mathbb{R}^d, \mathbb{R}^d)$ over the probability space $\mathscr{T}_2 := (\Omega, \mathscr{Z}, \mathbb{P})$ and $B : L^2_{\text{div}}(\mathbb{R}^d, \mathbb{R}^d) \to \Omega$ is the continuous extension of

$$\widehat{B\psi}(\mathbf{k}) = \sqrt{2\mathscr{E}(|\mathbf{k}|)}\, |\mathbf{k}|^{(1+2\beta-d)/2} \hat{\psi}(\mathbf{k}), \quad \psi \in \mathscr{C}^\infty_{0,\text{div}}, \tag{6.2}$$

where $\beta \geqslant 0$. It can be shown, see part (1) of Proposition 2 of [7], that B is a Hilbert–Schmidt operator.

$S(t)$, on the other hand, is given by

$$\widehat{S(t)\psi}(\mathbf{k}) := e^{-|\mathbf{k}|^{2\beta} t} \hat{\psi}(\mathbf{k}), \quad \psi \in \mathscr{C}^\infty_{0,\text{div}}. \tag{6.3}$$

It can be shown, see part (2) of Proposition 2 of [7], that $(S(t))_{t \geqslant 0}$ extends to a C_0-semigroup of operators on Ω, provided that β is an integer. For a non-integral β the above is still true provided that $d/2 < \rho < d/2 + \beta$. $\mathscr{C}^\infty_{0,\text{div}}$ is a core of the generator $-A$ of the semigroup and

$$\widehat{A\psi}(\mathbf{k}) = |\mathbf{k}|^{2\beta} \hat{\psi}(\mathbf{k}), \quad \psi \in \mathscr{C}^\infty_{0,\text{div}}.$$

Let $P^t f(\omega) := \mathbb{E} f(V_\omega(t))$ for any $f \in L^2$, where $\mathbb{E}$ is the expectation corresponding to measure $\mathbb{P}$ on the probability space $\mathscr{T}_2$. The measure μ is invariant under $(P^t)_{t \geqslant 0}$ in the sense of (2.1). This relation can be shown for elements from $\mathscr{P}_n$ with the help of (6.1) and then extended by the density argument to the entire L^2.

By $(V_\mu(t))_{t \geqslant 0}$ we denote the process $(V_f(t))_{t \geqslant 0}$ over $\mathscr{T}_1 \otimes \mathscr{T}_2$ with a randomized initial condition, distributed according to μ and independent of the cylindrical

Wiener process $W(\cdot)$. $V_\mu(\cdot)$, as a time stationary process, can always be extended to a process $t \mapsto V_\mu(t)$ defined for all real t. Let $\mathbf{V}(t,\mathbf{x}) := V_\mu(t)(\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. It is an $\mathbb{R}^d$-valued, continuous trajectory Gaussian random vector field, with the covariance matrix given by

$$\mathbf{R}(t,\mathbf{x}) = \int_{\mathbb{R}^d} \cos(\mathbf{x}\cdot\mathbf{k}) e^{-|\mathbf{k}|^{2\beta} t} \frac{\mathscr{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \Gamma(\mathbf{k})\, d\mathbf{k},$$

where $\Gamma(\mathbf{k}) = \mathbf{I} - \mathbf{k} \otimes \mathbf{k}|\mathbf{k}|^{-2}$. A standard computation shows that the correlation coefficient

$$\mathrm{Corr}(f_{\varphi_1}(V_\mu(t+h)), f_{\varphi_2}(V_\mu(t))) \leqslant e^{-l^{2\beta} h},$$

with $f_\varphi(\cdot) := \langle \varphi, \ \cdot \ \rangle$, $\mathrm{supp}(\hat{\varphi}_i) \cap B(\mathbf{0}, l) = \phi$, $i = 1,2$ where $\varphi_1, \varphi_2 \in C^\infty_{0,\mathrm{div}}$. Theorem 10.1 [18, p. 181] implies that

$$||P^t f||_{L^2} \leqslant e^{-l^{2\beta} t} ||f||_{L^2} \tag{6.4}$$

for any $f(\cdot) = \langle \varphi_1, \cdot \rangle \cdots \langle \varphi_N, \cdot \rangle$, with $\mathrm{supp}(\hat{\varphi}_i) \cap B(\mathbf{0}, l) = \phi$, $i = 1, \ldots, N$.

A simple calculation shows that

$$\int P^t f g \, d\mu = \int f P^t g \, d\mu, \quad \text{for all } t \geqslant 0, \ f, g \in L^2. \tag{6.5}$$

As a consequence of (6.5) we conclude that $(P^t)_{t \geqslant 0}$ is a C_0-continuous semigroup of self-adjoint Markovian contractions on L^2. Using (6.4) and (6.5) we easily conclude that μ is ergodic, i.e. if $P^t f = f$ for some $t > 0$ then $f \in \mathrm{span}(\mathbf{1})$.

Denote by $M : \mathscr{D}(M) \to L^2$, $\mathscr{E}_M : \mathscr{D}(\mathscr{E}_M) \times \mathscr{D}(\mathscr{E}_M) \to \mathbb{R}$ its generator and the Dirichlet form respectively. Since $F(V_f(t)) \in \mathscr{P}_n$, $\mathbb{P}$-a.s. for any $F \in \mathscr{P}_n$ we conclude that $P^t(H_n) \subseteq H_n$, $t \geqslant 0$ and, in consequence, $M(\mathscr{D}(M) \cap H_n) \subseteq H_n$. In addition $\mathscr{P}^{\mathrm{reg}}$ is a core of M, see the proof of part (ii) of Lemma 6.4.

6.3. *Diffusions with Ornstein–Uhlenbeck drift*

Let $\mathbf{x}(\cdot)$ be a stochastic process over $\mathscr{T}_0 \otimes \mathscr{T}_1 \otimes \mathscr{T}_2$ given by

$$\begin{cases} d\mathbf{x}(t;\omega,\sigma) = \mathbf{V}(t, \mathbf{x}(t;\omega,\sigma);\omega)\, dt + \sqrt{2\kappa}\, d\mathbf{w}(t;\sigma), t \geqslant 0, \\ \mathbf{x}(0;\omega,\sigma) = \mathbf{0}. \end{cases} \tag{6.6}$$

The Lagrangian canonical process, considered over the probability space $\mathscr{T}_0 \otimes \mathscr{T}_2$, is given by $\eta_\omega(t;\sigma) := \tau_{\mathbf{x}(t;\omega,\sigma)}(V_\omega(t))$, $t \geqslant 0$. Let $\kappa > 0$. We set

$$Q^t f(\omega) = \mathbf{E}^Q \mathbb{E} f(\eta_\omega(t)), \tag{6.7}$$

with $\mathbf{E}^Q$ denoting the expectation corresponding to probability measure Q. Then,

$$\mathbf{E}^Q\mathbb{E}[f(\eta_\omega(t+h))|\mathscr{V}_t] = Q^h f(\eta_\omega(t)), \quad t,h \geqslant 0, \tag{6.8}$$

with $(\mathscr{V}_t)_{t\geqslant 0}$ the natural filtration corresponding to the Lagrangian process and $\mathbf{E}^Q\mathbb{E}[\cdot|\cdot]$ the conditional expectation operator associated with the probability space $\mathscr{T}_0 \otimes \mathscr{T}_2$. When, on the other hand, $\kappa = 0$ we modify accordingly formulas (6.7) and (6.8) by omitting $\mathbf{E}^Q$.

Measure μ is stationary and ergodic for $(Q^t)_{t\geqslant 0}$ in the sense of condition (M) (of course one should substitute Q^t in place of P^t there). For $\kappa > 0$ this can be seen directly from Theorem 1, [10, p. 424]. In case $\kappa = 0$ the stationarity of μ is a straightforward consequence of the invariance of μ for positive κ. After letting $\kappa \downarrow 0$ one concludes easily that the invariance persists also in the case of vanishing molecular diffusivity. Ergodicity can be seen from the fact that μ is ergodic under $(P^t)_{t\geqslant 0}$, cf. Section 6.2, and formula (6.9) below.

Using the argument contained in the proof of Lemma 4.1 we conclude.

Lemma 6.1. *$\mathscr{C} := \mathscr{D}(M) \cap \bigcap_{\varrho>0} \mathscr{W}^{2+\varrho,2}$ is a core of $L : \mathscr{D}(L) \to L^2$, the generator of $(Q^t)_{t\geqslant 0}$. In addition,*

$$Lf = \kappa\Delta f + Mf + \mathbf{V}\cdot\nabla f, \quad \textit{for any } f \in \mathscr{C}, \tag{6.9}$$

with $\mathbf{V}(\omega) = (V_1(\omega), \ldots, V_d(\omega)) := (\omega_1(\mathbf{0}), \ldots, \omega_d(\mathbf{0}))$. Here $\omega = (\omega_1, \ldots, \omega_d) \in \Omega$.

We also have $\mathscr{C} \subseteq \mathscr{D}(L^*)$ is a core of L^*, with

$$L^* f = \kappa\Delta f + Mf - \mathbf{V}\cdot\nabla f, \quad f \in \mathscr{C}.$$

$S = -1/2(L + L^*)$ defined on $\mathscr{C}$ is essentially self-adjoint and

$$Sf = \kappa\Delta f + Mf, \quad f \in \mathscr{C}.$$

By virtue of the above lemma we conclude that (H1) of Section 3 is satisfied. Let $\mathscr{C}_n := \mathscr{C} \cap H_n$. Thanks to (6.9) one can easily prove that $\mathscr{C}_n$ is a core of L_n. Moreover, $\Pi_n(\mathscr{C}) \subseteq \mathscr{C}$ and $S\Pi_n f = \Pi_n Sf$ on $\mathscr{C}$. Therefore, condition (H3) holds. Additionally, $\mathscr{C}_\infty := \bigoplus_{n\geqslant 0} \mathscr{C}_n$ is a core of L and therefore (H2) is satisfied.

Note that $\alpha + \beta < 1$ implies $V_p \in \mathscr{H}_{-1}$. Indeed, let $f \in \mathscr{H}_1 \cap L^2$ of the form $f_\varphi(\cdot) = \langle \varphi, \cdot \rangle$ for any $\varphi \in C^\infty_{0,\mathrm{div}}$. We have $f_\varphi \in \mathscr{C}_1$ and

$$(V_p, f_\varphi)_{L^2} = \int_{\mathbb{R}^d} \frac{\mathscr{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \hat{\varphi}_p(\mathbf{k})\, d\mathbf{k}. \tag{6.10}$$

Hence, by the Cauchy inequality

$$|(V_p,f_\varphi)_{L^2}|\leqslant \left\{\int_{\mathbb{R}^d}\frac{\mathscr{E}(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\beta+d-1}}\right\}^{1/2}\left\{\int_{\mathbb{R}^d}|\mathbf{k}|^{2\beta}\mathscr{E}(|\mathbf{k}|)|\hat{\varphi}_p(\mathbf{k})|^2\frac{d\mathbf{k}}{|\mathbf{k}|^{d-1}}\right\}^{1/2}$$
$$\leqslant C\mathscr{E}_M(f_\varphi,f_\varphi)^{1/2}\leqslant C\|f_\varphi\|_1.$$

Note that

$$\int_{\mathbb{R}^d}\frac{\mathscr{E}(|\mathbf{k}|)\,d\mathbf{k}}{|\mathbf{k}|^{2\beta+d-1}}<+\infty$$

thanks to the fact that $\alpha+\beta<1$.

Since $V_p\in H_1$ (Hermite polynomial of first degree) we have, for an arbitrary $f\in\mathscr{H}_1$,

$$|(V_p,f)_{L^2}|=|(V_p,\Pi_1 f)_{L^2}|\leqslant C\|\Pi_1 f\|_1\leqslant C\|f\|_1$$

and therefore $V_p\in\mathscr{H}_{-1}$.

The key observation is contained in the following lemma, which guarantees that condition (H4) holds for $\kappa\geqslant 0$.

Lemma 6.2. *Suppose that* $\alpha+\beta<1$. *Then, there exists a constant* $C>0$ *independent of* n, $\kappa\geqslant 0$, *such that*

(i)
$$|(\mathbf{V}\cdot\nabla f,g)_{L^2}|\leqslant C[\mathscr{E}_M^{1/2}(f,f)\|\nabla g\|_{L^2_d}+\mathscr{E}_M^{1/2}(g,g)\|\nabla f\|_{L^2_d}] \tag{6.11}$$

for all $n\geqslant 1$, $f\in\mathscr{C}_n$, $g\in\mathscr{W}^{2,1}\cap\mathscr{D}(\mathscr{E}_M)$.

(ii) *if, in addition,* $\beta\in[0,1]$ *we have*

$$\|\nabla f\|_{L^2_d}\leqslant Cn^{3/4}\mathscr{E}_M^{1/2}(f,f),\quad \forall n\geqslant 1, f\in\mathscr{C}_n. \tag{6.12}$$

Remark 1. It is easy to verify these inequalities for *linear* functions, i.e. functions in H_1. For example let $f_\varphi(\omega)=\langle\varphi,\omega\rangle$ for $\varphi\in C^\infty_{0,\mathrm{div}}$, then $f_\varphi\in H_1$ and

$$\|\nabla f_\varphi\|^2_{L^2_d}=\sum_{p=1}^d\int_{\mathbb{R}^d}k_p^2|\hat{\varphi}_p(\mathbf{k})|^2\frac{\mathscr{E}(\mathbf{k})}{|\mathbf{k}|^{d-1}}\,d\mathbf{k}$$
$$\leqslant K^{2(1-\beta)}\int_{\mathbb{R}^d}|\mathbf{k}|^{2\beta}|\hat{\varphi}(\mathbf{k})|^2\frac{\mathscr{E}(\mathbf{k})}{|\mathbf{k}|^{d-1}}\,d\mathbf{k},$$

which proves (6.12) for $n=1$. A similar elementary argument shows (6.11) for functions in H_1. In order to prove (6.11) and (6.12) for general functions we need an approximation procedure by periodization, and this will be shown in the next section.

Remark 2. Obviously, in case when $\kappa > 0$ (6.11) implies the *strong* sector condition for the generator of the Lagrangian process (6.9). If, on the other hand, $\kappa = 0$ (6.12) implies the *graded sector condition* (H4) of Section 2.

In light of the results presented in Section 3 we have $u_\lambda^{(p)}$, the solutions of (4.12), are $\mathscr{H}_1$-convergent, as $\lambda \downarrow 0$, to certain $u_*^{(p)}$, $p = 1, \ldots, d$. Repeating now the argument of Section 4 we conclude the following result.

Theorem 6.3. *Suppose that the spectrum of the field* $\mathbf{V}$ *satisfies condition* (E), $\kappa > 0$ *and* $\beta \geqslant 0$ *is such that* $\alpha + \beta < 1$. *Then, the laws of* $(\epsilon \mathbf{x}(\frac{t}{\epsilon^2}))_{t \geqslant 0}$ *satisfy the invariance principle, as* $\epsilon \downarrow 0$. *The covariance matrix of the limiting Wiener measure is given by* $\mathbf{D} = 2[\kappa \delta_{p,q} + (u_*^{(p)}, u_*^{(q)})_1]$. *When* $\kappa = 0$ *the invariance principle still holds, provided that* $\beta \in [0, 1]$.

6.4. The proof of Lemma 6.2

The proof will be carried out in several steps.

6.4.1. Periodization of the field in the spatial variable

For an arbitrary integer $N \geqslant 1$ let $\Lambda_N := \{\mathbf{j} \in \mathbf{Z}^d : N^{-1} 2^N < |\mathbf{j}| \leqslant N 2^N\}$ and Λ_N^+ be the subset of Λ_N consisting of those $\mathbf{j} = (j_1, \ldots, j_d)$ whose last non-vanishing component is positive. By Υ_N we denote the cardinality of Λ_N^+.

Suppose that $0 \leqslant \phi_{\mathbf{0}}^{(N)} \leqslant 1$ is a C^∞ smooth function such that

$$\operatorname{supp}(\phi_{\mathbf{0}}^{(N)}) \subseteq \Delta_{\mathbf{0}}^{(N)} := \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : -2^{-N-1} \leqslant x_k < 2^{-N-1}\}$$

and $\phi_{\mathbf{0}}^{(N)}(\mathbf{x}) \equiv 1$, when $-2^{-N-1}(1 - 2^{-N}) \leqslant x_k \leqslant 2^{-N-1}(1 - 2^{-N})$, $k = 1, \ldots, d$. Let $\phi_{\mathbf{j}}^{(N)}(\mathbf{x}) := \phi_{\mathbf{0}}^{(N)}(\mathbf{x} - \mathbf{k}_{\mathbf{j}})$, where $\mathbf{k}_{\mathbf{j}} := \mathbf{j} 2^{-N}$ for $\mathbf{j} = (j_1, \ldots, j_d) \in \Lambda_N$.

Since $\Omega \subseteq \mathscr{S}'_{\mathrm{div}}(\mathbb{R}^d; \mathbb{R}^d)$—the space of divergence free Schwartz distribution—we can define for any $\omega \in \Omega$ its Fourier transform $\hat{\omega} = (\hat{\omega}_1, \ldots, \hat{\omega}_d)$, via the relation $\hat{\omega}_k(\psi) := \omega_k(\hat{\psi})$ for any $\psi \in \mathscr{S}(\mathbb{R}^d)$ (see [17, p. 5]). Let

$$X_{\mathbf{j}}^{(N)}(\omega) := \operatorname{Re} \Gamma(\mathbf{k}_{\mathbf{j}}) \hat{\omega}(\phi_{\mathbf{j}}^{(N)}), \quad Y_{\mathbf{j}}^{(N)}(\omega) := -\operatorname{Im} \Gamma(\mathbf{k}_{\mathbf{j}}) \hat{\omega}(\phi_{\mathbf{j}}^{(N)}), \quad \mathbf{j} \in \Lambda_N^+.$$

$X_{\mathbf{j}}^{(N)}$, $Y_{\mathbf{j}}^{(N)}$, $\mathbf{j} \in \Lambda_N^+$ are independent, centered, real Gaussian vectors over the probability space $\mathscr{T}_1$. The covariance matrix of each of them equals

$$S_{\mathbf{j}}^{(N)} := \Gamma(\mathbf{k}_{\mathbf{j}}) \left(\int_{R^d} [\phi_{\mathbf{j}}^{(N)}(\mathbf{k})]^2 \frac{\mathscr{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \Gamma(\mathbf{k}) \, d\mathbf{k} \right) \Gamma(\mathbf{k}_{\mathbf{j}}).$$

A direct calculation shows that

$$\lim_{N\uparrow+\infty}\sup_{\mathbf{j}\in\Lambda_N^+,}|2^{Nd}\,S_{\mathbf{j}}^{(N)}-\sigma_{\mathbf{j}}^2\,\Gamma(\mathbf{k_j})|=0, \tag{6.13}$$

$$\operatorname{tr} S_{\mathbf{j}}^{(N)}\leqslant 2(d-1)\sigma_{\mathbf{j}}^2\,2^{-Nd},\quad \forall \mathbf{j}\in\Lambda_N^+, \tag{6.14}$$

where $\sigma_{\mathbf{j}}^2 := \mathscr{E}(|\mathbf{k_j}|)|\mathbf{k_j}|^{1-d}$.

Set $\pi_N : \Omega \to \overset{2\Upsilon_N}{(\mathbb{R}^d)}$ by $\pi_N(\omega) := (X_{\mathbf{j}}^{(N)}(\omega), Y_{\mathbf{j}}^{(N)}(\omega); \mathbf{j}\in\Lambda_N^+)$ and $j_N : \overset{2\Upsilon_N}{(\mathbb{R}^d)} \to \Omega$ by

$$j_N(\mathbf{a},\mathbf{b})(\mathbf{x}) := \sum_{\mathbf{j}\in\Lambda_N}\Gamma(\mathbf{k_j})[a_{\mathbf{j}}\cos(\mathbf{k_j}\cdot\mathbf{x})+b_{\mathbf{j}}\sin(\mathbf{k_j}\cdot\mathbf{x})],\quad \mathbf{x}\in\mathbb{R}^d,$$

with the convention $a_{-\mathbf{j}} = a_{\mathbf{j}}$, $b_{-\mathbf{j}} = -b_{\mathbf{j}}$. For the abbreviation sake we wrote $\mathbf{a}$ to denote the entire ensemble $a_{\mathbf{j}}$, $\mathbf{j}\in\Lambda_N^+$ and $\mathbf{b}$ for $b_{\mathbf{j}}, \mathbf{j}\in\Lambda_N^+$. Note that, obviously, $\pi_N j_N$ is the co-ordinatewise projection onto $\prod_{\mathbf{j}\in\Lambda_N^+} K_{\mathbf{j}}\times K_{\mathbf{j}}$, where $K_{\mathbf{j}} := [a\in\mathbb{R}^d : a\perp\mathbf{k_j}]$. Let $e_{d,\mathbf{j}} := \mathbf{k_j}/|\mathbf{k_j}|$, $e_{1,\mathbf{j}},\dots,e_{d-1,\mathbf{j}}$ be the eigenvectors of $S_{\mathbf{j}}^{(N)}$ contained in $K_{\mathbf{j}}$ and $\lambda_{1,\mathbf{j}}\geqslant\cdots\geqslant\lambda_{d-1,\mathbf{j}}\geqslant 0$ be the corresponding to them eigenvalues. In view of (6.13) we have

$$\lim_{N\uparrow+\infty}\sup_{\mathbf{j}\in\Lambda_N^+}\max_{p=1,\dots,d-1}|2^{Nd}\lambda_{p,\mathbf{j}}-\sigma_{\mathbf{j}}^2|=0.$$

Denote by $\nu_N := \mu\pi_N^{-1}$ the joint law of $X_{\mathbf{j}}^{(N)}$, $Y_{\mathbf{j}}^{(N)}$, $\mathbf{j}\in\Lambda_N^+$ on $\overset{2\Upsilon_N}{(\mathbb{R}^d)}$. Its characteristic function equals

$$\Phi(\xi_{\mathbf{j}},\eta_{\mathbf{j}};\mathbf{j}\in\Lambda_N^+)=\prod_{\mathbf{j}\in\Lambda_N^+}\exp\left\{-\frac{1}{2}(S_{\mathbf{j}}^{(N)}\xi_{\mathbf{j}}\cdot\xi_{\mathbf{j}}+S_{\mathbf{j}}^{(N)}\eta_{\mathbf{j}}\cdot\eta_{\mathbf{j}})\right\}. \tag{6.15}$$

π_N induces an isometric embedding $\wp_N : L^2(\nu_N)\to L^2(\mu)$, given by $\wp_N f(\omega) := f(\pi_N(\omega))$, while j_N induces a linear operator $J_N : \mathscr{P}^{\mathrm{reg}}\to L^2(\nu_N)$ by $J_N f(\mathbf{a},\mathbf{b}) := f(j_N(\mathbf{a},\mathbf{b}))$.

For any ensemble $(\mathbf{a},\mathbf{b}) := \{a_{\mathbf{j}}, b_{\mathbf{j}}; \mathbf{j}\in\Lambda_N^+\}$ and $\mathbf{x}\in\mathbb{R}^d$ we define

$$\begin{aligned}\tau_{\mathbf{x}}(\mathbf{a},\mathbf{b}) := \{&a_{\mathbf{j}}\cos(\mathbf{k_j}\cdot\mathbf{x})+b_{\mathbf{j}}\sin(\mathbf{k_j}\cdot\mathbf{x}),\\ &-a_{\mathbf{j}}\sin(\mathbf{k_j}\cdot\mathbf{x})+b_{\mathbf{j}}\cos(\mathbf{k_j}\cdot\mathbf{x});\mathbf{j}\in\Lambda_N^+\}.\end{aligned}$$

For any f that is a polynomial in the variables $\mathbf{a}, \mathbf{b}$ we set

$$\nabla^{(N)} f(\mathbf{a},\mathbf{b}) \coloneqq \nabla_{\mathbf{x}|\mathbf{x}=\mathbf{0}} f(\tau_{\mathbf{x}}(\mathbf{a},\mathbf{b})) = \sum_{\mathbf{j}\in\Lambda_N^+} \mathbf{k_j}\,(b_{\mathbf{j}}\cdot\nabla_{a_{\mathbf{j}}} - a_{\mathbf{j}}\cdot\nabla_{b_{\mathbf{j}}}) f(\mathbf{a},\mathbf{b}). \tag{6.16}$$

The images of Wiener process $B\mathbf{W}(\cdot)$ under π_N are finite-dimensional Brownian motions given by

$$\pi_N B\mathbf{W}(t) = (|\mathbf{k_j}|^\beta \sqrt{2S_{\mathbf{j}}^{(N)}}\mathbf{w}_{\mathbf{j}}^{(N)}(t), |\mathbf{k_j}|^\beta \sqrt{2S_{\mathbf{j}}^{(N)}}\tilde{\mathbf{w}}_{\mathbf{j}}^{(N)}(t); \mathbf{j}\in\Lambda_N^+), \quad t\geqslant 0, \tag{6.17}$$

where $\mathbf{w}_{\mathbf{j}}^{(N)}(\cdot)$, $\tilde{\mathbf{w}}_{\mathbf{j}}^{(N)}(\cdot)$, $\mathbf{j}\in\Lambda_N^+$ are independent standard d-dimensional Brownian motions and $\mathbf{w}_{-\mathbf{j}}^{(N)}(\cdot) = \mathbf{w}_{-\mathbf{j}}^{(N)}(\cdot)$, $\tilde{\mathbf{w}}_{-\mathbf{j}}^{(N)}(\cdot) = -\tilde{\mathbf{w}}_{-\mathbf{j}}^{(N)}(\cdot)$.

Let us consider the d-dimensional Ornstein–Uhlenbeck processes $a_{\mathbf{j}}(\cdot;\mathbf{a},\mathbf{b})$, $b_{\mathbf{j}}(\cdot;\mathbf{a},\mathbf{b})$, given for $\mathbf{j}\in\Lambda_N^+$ by

$$\begin{cases} da_{\mathbf{j}}(t;\mathbf{a},\mathbf{b}) = -|\mathbf{k_j}|^{2\beta} a_{\mathbf{j}}(t;\mathbf{a},\mathbf{b})\,dt + |\mathbf{k_j}|^\beta \sqrt{2S_{\mathbf{j}}^{(N)}}\,d\mathbf{w}_{\mathbf{j}}^{(N)}(t), \\ a_{\mathbf{j}}(0;\mathbf{a},\mathbf{b}) = a_{\mathbf{j}}, \end{cases} \tag{6.18}$$

$$\begin{cases} db_{\mathbf{j}}(t;\mathbf{a},\mathbf{b}) = -|\mathbf{k_j}|^{2\beta} b_{\mathbf{j}}(t;\mathbf{a},\mathbf{b})dt + |\mathbf{k_j}|^\beta \sqrt{2S_{\mathbf{j}}^{(N)}}\,d\tilde{\mathbf{w}}_{\mathbf{j}}^{(N)}(t), \\ b_{\mathbf{j}}(0;\mathbf{a},\mathbf{b}) = b_{\mathbf{j}}, \end{cases} \tag{6.19}$$

and $a_{-\mathbf{j}} = a_{\mathbf{j}}$, $b_{-\mathbf{j}} \coloneqq -b_{\mathbf{j}}$, $\mathbf{j}\in\Lambda_N^+$.

We denote by $(P_N^t)_{t\geqslant 0}$, M_N, $\mathscr{E}_{M_N}(\cdot,\cdot)$ the $L^2(\nu_N)$-semigroup, generator and Dirichlet form corresponding to the $(\mathbb{R}^d)^{2\Upsilon_N}$-valued Markovian family of processes $\mathbf{V}^{(N)}(\cdot;\mathbf{a},\mathbf{b})$ given by (6.18), and (6.19). Let $\mathbf{V}^{(N)}(\cdot;\nu_N)$ be the stationary solution of those equations and

$$\mathbf{V}^{(N)}(t,\mathbf{x};\mathbf{a},\mathbf{b}) \coloneqq j_N(\mathbf{V}^{(N)}(t;\mathbf{a},\mathbf{b}))(\mathbf{x}), \quad \mathbf{V}^{(N)}(t,\mathbf{x};\nu_N) \coloneqq j_N(\mathbf{V}^{(N)}(t;\nu_N))(\mathbf{x}).$$

For any polynomial f in variables $\mathbf{a}$, $\mathbf{b}$ we have $f\in\mathscr{D}(\mathscr{E}_{M_N})$ and

$$\mathscr{E}_{M_N}(f,f) = \sum_{\mathbf{j}\in\Lambda_N^+} |\mathbf{k_j}|^{2\beta}\mathscr{E}_{\mathbf{j}}(f,f), \tag{6.20}$$

with

$$\mathscr{E}_{\mathbf{j}}(f,f) \coloneqq \frac{1}{2}\underbrace{\int\cdots\int}_{(\mathbb{R}^d)^{2\Upsilon_N}} (S_{\mathbf{j}}^{(N)}\nabla_{a_{\mathbf{j}}}f\cdot\nabla_{a_{\mathbf{j}}}f + S_{\mathbf{j}}^{(N)}\nabla_{b_{\mathbf{j}}}f\cdot\nabla_{b_{\mathbf{j}}}f)\,d\nu_N.$$

We also have $\wp_N(\mathscr{D}(\mathscr{E}_{M_N}))\in\mathscr{D}(\mathscr{E}_M)$ and $\mathscr{E}_M(\wp_N f,\wp_N f)=\mathscr{E}_{M_N}(f,f)$, $f\in\mathscr{D}(\mathscr{E}_{M_N})$. Let $H_n^{(N)}$, $\Pi_n^{(N)}$ be the space of all nth degree Hermite polynomials over $(\mathbb{R}^d)^{2\Upsilon_N}$ corresponding to measure ν_N and the respective orthogonal projection. For any $\mathbf{j}\in\Lambda_N^+$, $n=(n_1,\dots,n_{d-1})\in\mathbb{Z}_+^{d-1}$, $a=\sum_{p=1}^d a_p\, e_{p,\mathbf{j}}$ define

$$h_{\mathbf{j},n}(a):=\bigotimes_{p=1}^{d-1} h_{n_p}(\lambda_{p,\mathbf{j}}^{-1/2}a_p), \tag{6.21}$$

where $h_n(\cdot)$, $n\geqslant 0$ the standard orthonormal system of Hermite polynomials on $L^2(\mathbb{R},\nu)$, with ν is the standard d-dimensional Gaussian measure. For any $\mathbf{n}=(n_{\mathbf{j}}\in\mathbb{Z}_+^{d-1};\ \mathbf{j}\in\Lambda_N^+)$, $\mathbf{a}=(a_{\mathbf{j}};\ \mathbf{j}\in\Lambda_N^+)$ we set

$$h_{\mathbf{n}}(\mathbf{a}):=\bigotimes_{\mathbf{j}\in\Lambda_N^+} h_{\mathbf{j},n_{\mathbf{j}}}(a_{\mathbf{j}}). \tag{6.22}$$

The set of all $h_{\mathbf{n}}(\mathbf{a})\otimes h_{\mathbf{m}}(\mathbf{b})$, with $|\mathbf{n}|+|\mathbf{m}|=n$ forms an orthonormal basis of $H_n^{(N)}$, $n\geqslant 0$.

Let

$$\mathbf{V}^{(N)}:=2\sum_{p=1}^{d-1}\sum_{\mathbf{j}\in\Lambda_N^+} a_{p,\mathbf{j}}e_{p,\mathbf{j}}, \tag{6.23}$$

with $a_{\mathbf{j}}=\sum_{p=1}^d a_{p,\mathbf{j}}e_{p,\mathbf{j}}$. Using elementary properties of Hermite polynomials, stating that $ah_n(a)=\sqrt{n+1}h_{n+1}(a)+\sqrt{n}h_{n-1}(a)$, see e.g. (3.14) of [8] (after taking into account the normalizaton factor), we get

$$\mathbf{V}^{(N)}\,h_{\mathbf{n}}(\mathbf{a})$$
$$=2\sum_{p=1}^{d-1}\sum_{\mathbf{j}\in\Lambda_N^+}[\sqrt{\lambda_{p,\mathbf{j}}(n_{p,\mathbf{j}}+1)}h_{\mathbf{n}(p,\mathbf{j},+)}(\mathbf{a})+\sqrt{\lambda_{p,\mathbf{j}}n_{p,\mathbf{j}}}h_{\mathbf{n}(p,\mathbf{j},-)}(\mathbf{a})]e_{p,\mathbf{j}}. \tag{6.24}$$

Here for any $\mathbf{n}=(n_{\mathbf{j}};\mathbf{j}\in\Lambda_N^+)$ we have $\mathbf{n}(p,\mathbf{j},+):=(m_{\mathbf{j}'};\ \mathbf{j}'\in\Lambda_N^+)$, with $m_{\mathbf{j}'}=n_{\mathbf{j}'}$, $\mathbf{j}'\neq\mathbf{j}$, $m_{q,\mathbf{j}}=n_{q,\mathbf{j}}$, for $q\neq p$ and $m_{p,\mathbf{j}}=n_{p,\mathbf{j}}+1$. Similarly, $\mathbf{n}(p,\mathbf{j},-):=(m_{\mathbf{j}'};\mathbf{j}'\in\Lambda_N^+)$, with $m_{\mathbf{j}'}=n_{\mathbf{j}'}$, $\mathbf{j}'\neq\mathbf{j}$, $m_{q,\mathbf{j}}=n_{q,\mathbf{j}}$, for $q\neq p$ and

$$m_{p,\mathbf{j}}=(n_{p,\mathbf{j}}-1)_+:=\begin{cases} n_{p,\mathbf{j}}-1 & \text{if } n_{p,\mathbf{j}}\geqslant 1,\\ 0 & \text{if } n_{p,\mathbf{j}}=0.\end{cases}$$

6.4.2. *The approximation result*

We show that $\mathbf{V}^{(N)}(\cdot;\nu_N)$ is an approximation of $\mathbf{V}_\mu(\cdot)$ in an appropriate sense.

Lemma 6.4. (i) *For any* $f \in \mathscr{P}^{\rm reg}$ *we have*

$$\lim_{N\uparrow+\infty} \wp_N P_N^t J_N f = P^t f \text{ in } L^2. \tag{6.25}$$

(ii) *We have* $\mathscr{P}^{\rm reg} \subseteq \mathscr{D}(\mathscr{E}_M)$. *In addition,* $J_N(\mathscr{P}^{\rm reg}) \in \mathscr{D}(\mathscr{E}_{M_N})$ *and*

$$\lim_{N\uparrow+\infty} \mathscr{E}_{M_N}(J_N f, J_N f) = \mathscr{E}_M(f,f), \quad f \in \mathscr{P}^{\rm reg}. \tag{6.26}$$

Proof. (i) It suffices to verify that (6.25) holds for polynomials of the form

$$f(\cdot) = \langle \varphi_1, \cdot \rangle \cdots \langle \varphi_m, \cdot \rangle, \tag{6.27}$$

where $\varphi_1, \ldots, \varphi_m \in C^\infty_{0,{\rm div}}$. Then,

$$\begin{aligned} &\wp_N P_N^t J_N f(\omega) \\ &\quad = \mathbf{E}[\langle \varphi_1, j_N(\mathbf{V}^{(N)}(t;\pi_n(\omega)))\rangle \cdots \langle \varphi_m, j_N(\mathbf{V}^{(N)}(t;\pi_n(\omega)))\rangle] \end{aligned} \tag{6.28}$$

and the right-hand side of (6.28) can be expressed as a finite sum of certain products made of the expressions of the form

$$\sum_{\mathbf{j}\in\Lambda_N^+} \int_{\mathbb{R}^d} [X_{\mathbf{j}}^{(n)} \cos(\mathbf{k_j}\cdot\mathbf{x}) + Y_{\mathbf{j}}^{(n)} \sin(\mathbf{k_j}\cdot\mathbf{x})]\varphi_k(\mathbf{x})\, d\mathbf{x}$$

and

$$\sum_{\mathbf{j}\in\Lambda_N^+} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 - e^{-2|\mathbf{k_j}|^{2\beta} t}) S_{\mathbf{j}}^{(n)} \cos(\mathbf{k_j}\cdot(\mathbf{x}-\mathbf{x}'))\varphi_k(\mathbf{x})\cdot\varphi_l(\mathbf{x}')\, d\mathbf{x}\, d\mathbf{x}'.$$

Taking into account the definitions of $X_{\mathbf{j}}^{(n)}$, $Y_{\mathbf{j}}^{(n)}$ we conclude that the right-hand side of (6.28) tends to

$$\mathbf{E}[\langle \varphi_1, \mathbf{V}_\omega(t)\rangle \cdots \langle \varphi_m, \mathbf{V}_\omega(t)\rangle] = P^t f(\omega),$$

in the L^2 sense, as $n\uparrow+\infty$.

(ii) Notice that the argument from the proof of part (i) also shows that $P_N^t J_N(\mathscr{P}^{\rm reg}) \subseteq J_N(\mathscr{P}^{\rm reg})$, $\forall t \geqslant 0$, so $J_N(\mathscr{P}^{\rm reg})$ is a core of M_N and

$$\begin{aligned} M_N f(\mathbf{a},\mathbf{b}) = &-\sum_{k=1}^m \langle \varphi_1, j_N(\mathbf{a},\mathbf{b})\rangle \cdots \langle A\varphi_k, j_N(\mathbf{a},\mathbf{b})\rangle \cdots \langle \varphi_N, j_N(\mathbf{a},\mathbf{b})\rangle \\ &+ 2\sum_{k<l} \langle \varphi_1, j_N(\mathbf{a},\mathbf{b})\rangle \cdots \widehat{\langle \varphi_k, \cdot\rangle} \cdots \widehat{\langle \varphi_l, \cdot\rangle} \cdots \langle \varphi_N, j_N(\mathbf{a},\mathbf{b})\rangle \\ &\times \sum_{j\in\Lambda_N} |\mathbf{k_j}|^{2\beta} S_{\mathbf{j}}^{(N)} \hat{\varphi}_k(\mathbf{k_j})\cdot\hat{\varphi}_l(\mathbf{k_j}), \end{aligned}$$

where $\widehat{\langle\cdot,\cdot\rangle}$ means that the respective term should be omitted in the product.

$$\wp_N J_N f(\mathbf{a},\mathbf{b}) = \langle \varphi_1, \pi_N j_N(\mathbf{a},\mathbf{b})\rangle \cdots \langle \varphi_m, \pi_N j_N(\mathbf{a},\mathbf{b})\rangle$$

for $f(\cdot) = \langle \varphi_1,\cdot\rangle\cdots\langle\varphi_N,\cdot\rangle \in \mathscr{P}^{\mathrm{reg}}$. Likewise $P^t(\mathscr{P}^{\mathrm{reg}})\subseteq\mathscr{P}^{\mathrm{reg}}$ so $\mathscr{P}^{\mathrm{reg}}$ is a core of M and

$$\begin{aligned} Mf(\omega) = &-\sum_{k=1}^{m}\langle\varphi_1,\omega\rangle\cdots\langle A\varphi_k,\omega\rangle\cdots\langle\varphi_m,\omega\rangle \\ &+2\sum_{k<l}\langle\varphi_1,\omega\rangle\cdots\widehat{\langle\varphi_k,\cdot\rangle}\cdots\widehat{\langle\varphi_l,\cdot\rangle}\cdots\langle\varphi_N,\omega\rangle \\ &\times\int_{\widehat{\mathbb{R}}^d}|\mathbf{k}|^{2\beta}\Gamma(\mathbf{k})\hat{\varphi}_k(\mathbf{k})\cdot\hat{\varphi}_l(\mathbf{k})\frac{\varepsilon(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\,d\mathbf{k}. \end{aligned}$$

Thus,

$$\lim_{N\uparrow+\infty}(||\wp_N J_N f - f||_{L^2} + ||\wp_N M_N J_N f - Mf||_{L^2}) = 0, \quad f\in\mathscr{P}^{\mathrm{reg}}$$

and (6.26) follows. □

6.4.3. Proof of (6.11)

Suppose that $f\in\mathscr{C}_n$ and $g\in\mathscr{C}_\infty$. Note that

$$\begin{aligned} |(\mathbf{V}\cdot\nabla f, g)_{L^2}| &\leqslant \sum_{k=n-1}^{n+1}|(\mathbf{V}\cdot\nabla f,\Pi_k g)_{L^2}| \\ &= |(\nabla f,\Pi_n(\mathbf{V}\Pi_{n+1}g))_{L^2_d}| + |(\mathbf{V}\cdot\nabla f,\Pi_n g)_{L^2}| \\ &\quad + |(\nabla f,\Pi_n(\mathbf{V}\Pi_{n-1}g))_{L^2_d}|. \end{aligned} \tag{6.29}$$

Note that since $f\in H_n$ we have $\mathbf{V}\cdot\nabla f\in H_{n-1}\oplus H_{n+1}$, thus the middle term appearing in (6.29) vanishes. The estimates for each of the two remaining terms are quite similar, so we only conduct them for the first one and show that it is less than or equal to

$$C\mathscr{E}_M^{1/2}(g,g)||\nabla f||_{L^2_d}. \tag{6.30}$$

Let $\mathbf{V}^{(N)}$ be given by (6.23). We prove that for any polynomial $g\in H_{n+1}^{(N)}$

$$||\Pi_n^{(N)}(\mathbf{V}^{(N)}g)||_{L^2_d(\nu_N)}\leqslant C\mathscr{E}^{1/2}_{M_N}(g,g), \tag{6.31}$$

with the constant $C>0$ independent of g, n and N. Eq. (6.31) implies (6.30), thanks to Lemma 6.4 and the fact that $\mathscr{P}^{\mathrm{reg}}$ is a core of $\mathscr{E}_M$.

Let $\mathbf{n}=(n_{\mathbf{j}};\ \mathbf{j}\in\Lambda_N^+)$ and $\mathbf{m}=(m_{\mathbf{j}};\ \mathbf{j}\in\Lambda_N^+)$, with $n_{\mathbf{j}}=(n_{\mathbf{j},1},\dots,n_{\mathbf{j},d-1})$, $m_{\mathbf{j}}=(m_{\mathbf{j},1},\dots,m_{\mathbf{j},d-1})\in\mathbb{Z}_+^{d-1}$, and

$$g(\mathbf{a},\mathbf{b}):=\sum_{|\mathbf{n}|+|\mathbf{m}|=n+1}\alpha(\mathbf{n},\mathbf{m})\,h_{\mathbf{n}}(\mathbf{a})\otimes h_{\mathbf{m}}(\mathbf{b})\in H_{n+1}^{(N)}, \tag{6.32}$$

cf. (6.21) and (6.22). Since $h_{\mathbf{n}}(\cdot)\otimes h_{\mathbf{m}}(\cdot)$ are the eigenvectors of the generator of the Ornstein–Uhlenbeck process given by (6.18), (6.19) corresponding to the eigenvalue $\sum_{\mathbf{j}\in\Lambda_N^+}|\mathbf{k}_{\mathbf{j}}|^{2\beta}(|n_{\mathbf{j}}|+|m_{\mathbf{j}}|)$ we obtain that

$$\mathscr{E}_{M_N}(g,g)=\sum_{|\mathbf{n}|+|\mathbf{m}|=n+1}\alpha^2(\mathbf{n},\mathbf{m})\sum_{\mathbf{j}\in\Lambda_N^+}|\mathbf{k}_{\mathbf{j}}|^{2\beta}(|n_{\mathbf{j}}|+|m_{\mathbf{j}}|). \tag{6.33}$$

Using (6.24) we get

$$\Pi_n(\mathbf{V}^{(N)}g)=2\sum_{p=1}^{d-1}\sum_{\mathbf{j}\in\Lambda_N^+}\sum_{|\mathbf{n}|+|\mathbf{m}|=n+1}\alpha(\mathbf{n},\mathbf{m})\sqrt{\lambda_{p,\mathbf{j}}\,n_{p,\mathbf{j}}}\,h_{\mathbf{n}(p,\mathbf{j},-)}(\mathbf{a})\otimes h_{\mathbf{m}}(\mathbf{b})\,e_{p,\mathbf{j}}.$$

Hence,

$$\begin{aligned}&\|\Pi_n(\mathbf{V}^{(N)}\,g)\|_{L_d^2}^2\\&=4\sum_{p,p'=1}^{d-1}\sum_{\mathbf{j},\mathbf{j}'\in\Lambda_N^+}\sum_{\substack{|\mathbf{n}|+|\mathbf{m}|=n+1\\|\mathbf{n}'|+|\mathbf{m}'|=n+1}}\alpha(\mathbf{n},\mathbf{m})\,\alpha(\mathbf{n}',\mathbf{m}')\\&\quad\times\sqrt{\lambda_{p,\mathbf{j}}\,n_{p,\mathbf{j}}}\sqrt{\lambda_{p',\mathbf{j}'}n_{p',\mathbf{j}'}}e_{p,\mathbf{j}}\cdot e_{p',\mathbf{j}'}\\&\quad\times\delta(\mathbf{n}(p,\mathbf{j},-),\mathbf{n}'(p',\mathbf{j}',-))\,\delta(\mathbf{m},\mathbf{m}'),\end{aligned} \tag{6.34}$$

with $\delta(\cdot,\cdot)$ the Kronecker symbol. The right-hand side of (6.34) is therefore less than or equal to

$$\begin{aligned}&4\sum_{p,p'=1}^{d-1}\sum_{\mathbf{j},\mathbf{j}'\in\Lambda_N^+}\sum_{\substack{|\mathbf{n}|+|\mathbf{m}|=n+1\\|\mathbf{n}'|+|\mathbf{m}'|=n+1}}\left[\left(\frac{|\mathbf{k}_{\mathbf{j}}|}{|\mathbf{k}_{\mathbf{j}'}|}\right)^{2\beta}\alpha^2(\mathbf{n},\mathbf{m})\lambda_{p,\mathbf{j}}\,n_{p,\mathbf{j}}\right.\\&\left.+\left(\frac{|\mathbf{k}_{\mathbf{j}'}|}{|\mathbf{k}_{\mathbf{j}}|}\right)^{2\beta}\alpha^2(\mathbf{n}',\mathbf{m}')\,\lambda_{p',\mathbf{j}'}n_{p',\mathbf{j}'}\right]\delta(\mathbf{n}(p,\mathbf{j},-),\mathbf{n}'(p',\mathbf{j}',-))\delta(\mathbf{m},\mathbf{m}').\end{aligned} \tag{6.35}$$

The expression corresponding to the first term in parentheses can be estimated by

$$\begin{aligned}&\sum_{|\mathbf{n}|+|\mathbf{m}|=n+1}\alpha^2(\mathbf{n},\mathbf{m})\sum_{\mathbf{j}\in\Lambda_N^+}|\mathbf{k}_{\mathbf{j}}|^{2\beta}\,|n_{\mathbf{j}}|\\&\times\sum_{p,p'=1}^{d-1}\sum_{|\mathbf{n}'|+|\mathbf{m}'|=n+1}\sum_{\mathbf{j}'\in\Lambda_N^+}\delta(\mathbf{n}(p,\mathbf{j},-),\mathbf{n}'(p',\mathbf{j}',-))\,\frac{\mathrm{tr}S_{\mathbf{j}'}^{(N)}}{|\mathbf{k}_{\mathbf{j}'}|^{2\beta}}\\&\leqslant C\,\mathscr{E}_{M_N}(g,g),\end{aligned}$$

by virtue of (6.33). Here C is a constant, independent of N and satisfying

$$\sum_{p,p'=1}^{d-1} \sum_{|\mathbf{n}'|+|\mathbf{m}'|=n+1} \sum_{\mathbf{j}' \in \Lambda_N^+} \delta(\mathbf{n}(p,\mathbf{j},-),\mathbf{n}'(p',\mathbf{j}',-)) \frac{\operatorname{tr} S_{\mathbf{j}'}^{(N)}}{|\mathbf{k}_{\mathbf{j}'}|^{2\beta}} \leqslant 2 \sum_{\mathbf{j}' \in \Lambda_N^+} \frac{\operatorname{tr} S_{\mathbf{j}'}^{(N)}}{|\mathbf{k}_{\mathbf{j}'}|^{2\beta}} \leqslant C, \quad \forall N \geqslant 1.$$

The same considerations lead to an estimate of the second term in (6.35) by $C\mathscr{E}_{M_N}(g,g)$ with the constant C independent of N and (6.30) follows.

Using the above method one can show that the third term on the right-hand side of (6.29) can be estimated by

$$C\mathscr{E}_M(f,f)^{1/2}||\nabla g||_{L_d^2}.$$

and (6.11) follows.

6.4.4. Proof of (6.12)

A straightforward calculation shows that

$$\lim_{N\uparrow+\infty} ||\wp_N \nabla^{(N)} J_N f - \nabla f||_{L_d^2} = 0$$

for any $f \in \mathscr{P}^{\text{reg}}$. Therefore, in order to prove (6.12) it suffices to show that

$$||\nabla^{(N)} f||_{L_d^2(\nu_N)} \leqslant C n^{3/4} \mathscr{E}_{M_N}^{1/2}(f,f) \tag{6.36}$$

for any $f \in H_n^{(N)}$, with C independent of f, N and n.

Applying (6.16) to

$$f(\mathbf{a},\mathbf{b}) := \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha(\mathbf{n},\mathbf{m}) h_{\mathbf{n}}(\mathbf{a}) \otimes h_{\mathbf{m}}(\mathbf{b}) \in H_n^{(N)}, \tag{6.37}$$

using elementary properties of orthonormal Hermite polynomials, namely $h_n'(a) = \sqrt{n} h_{n-1}(a)$ and the (3.14) of [8] we conclude the following formula:

$$\nabla^{(N)} f(\mathbf{a},\mathbf{b}) = \sum_{p=1}^{d-1} \sum_{\mathbf{j} \in \Lambda_N^+} \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \mathbf{k}_{\mathbf{j}} \alpha(\mathbf{n},\mathbf{m}) \times [\gamma(p,\mathbf{j},\mathbf{n},\mathbf{m}) h_{\mathbf{n}(p,\mathbf{j},+)}(\mathbf{a}) \otimes h_{\mathbf{m}(p,\mathbf{j},-)}(\mathbf{b}) - \varepsilon(p,\mathbf{j},\mathbf{n},\mathbf{m}) h_{\mathbf{n}(p,\mathbf{j},-)}(\mathbf{a}) \otimes h_{\mathbf{m}(p,\mathbf{j},+)}(\mathbf{b})], \tag{6.38}$$

with $\gamma(p,\mathbf{j},\mathbf{n},\mathbf{m}) := \sqrt{(n_{p,\mathbf{j}}+1)\,m_{p,\mathbf{j}}}$ and $\varepsilon(p,\mathbf{j},\mathbf{n},\mathbf{m}) := \sqrt{n_{p,\mathbf{j}}(m_{p,\mathbf{j}}+1)}$. A direct calculation shows that

$$||\nabla^{(N)} f||^2_{L^2_d(\nu_N)} = \sum_{p,p'=1}^{d-1} \sum_{\mathbf{j},\mathbf{j}' \in \Lambda_N^+} \sum_{\substack{|\mathbf{n}|+|\mathbf{m}|=n \\ |\mathbf{n}'|+|\mathbf{m}'|=n}} \mathbf{k}_{\mathbf{j}} \cdot \mathbf{k}_{\mathbf{j}'} \alpha(\mathbf{n},\mathbf{m})\alpha(\mathbf{n}',\mathbf{m}')$$
$$\times [\gamma\gamma'\delta_{\mathbf{n},+,+}\delta_{\mathbf{m},-,-} + \varepsilon\varepsilon'\delta_{\mathbf{n},-,-}\delta_{\mathbf{m},+,+}$$
$$- \gamma\varepsilon'\delta_{\mathbf{n},+,-}\delta_{\mathbf{m},-,+} - \varepsilon\gamma'\delta_{\mathbf{n},-,+}\delta_{\mathbf{m},+,-}]. \tag{6.39}$$

Here

$$\delta_{\mathbf{n},s_1,s_2} := \delta(\mathbf{n}(p,\mathbf{j},s_1), \mathbf{n}'(p',\mathbf{j}',s_2)),$$

$$\delta_{\mathbf{m},s_1,s_2} := \delta(\mathbf{m}(p,\mathbf{j},s_1), \mathbf{m}'(p',\mathbf{j}',s_2)),$$

for any $s_1, s_2 \in \{-,+\}$, γ, γ' are the abbreviations for $\gamma(p,\mathbf{j},\mathbf{n},\mathbf{m})$, $\gamma(p',\mathbf{j}',\mathbf{n}',\mathbf{m}')$ and a similar convention is used also for ε and ε'. The expressions corresponding to each of the four terms appearing in parentheses on the right-hand side of (6.39) can be dealt with separately. Since the estimates for each term are quite similar we deal with the first term only. Using an elementary inequality we can estimate that term by

$$\sum_{p,p'=1}^{d-1} \sum_{\mathbf{j},\mathbf{j}' \in \Lambda_N^+} \sum_{\substack{|\mathbf{n}|+|\mathbf{m}|=n \\ |\mathbf{n}'|+|\mathbf{m}'|=n}} [\alpha^2(\mathbf{n},\mathbf{m})|\mathbf{k}_{\mathbf{j}}|^2|m_{\mathbf{j}}| + \alpha^2(\mathbf{n}',\mathbf{m}')|\mathbf{k}_{\mathbf{j}'}|^2|m'_{\mathbf{j}'}|]\delta_{\mathbf{n},+,+}\delta_{\mathbf{m},-,-}[(n_{p,\mathbf{j}}+1)(n'_{p',\mathbf{j}}+1)]^{1/2}.$$

We can estimate the first term in the above sum by

$$(n+1)^{1/2} \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha^2(\mathbf{n},\mathbf{m}) \sum_{\mathbf{j}\in\Lambda_N^+} |m_{\mathbf{j}}|\,|\mathbf{k}_{\mathbf{j}}|^2 \sum_{p,p'=1}^{d-1} \sum_{\mathbf{j}'\in\Lambda_N^+} \sum_{|\mathbf{n}'|+|\mathbf{m}'|=n} \delta_{\mathbf{n},+,+}\delta_{\mathbf{m},-,-}(n'_{p',\mathbf{j}}+1)^{1/2}$$
$$\leqslant 4\,(n+1)^{3/2} \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha^2(\mathbf{n},\mathbf{m}) \sum_{\mathbf{j}\in\Lambda_N^+} |m_{\mathbf{j}}|\,|\mathbf{k}_{\mathbf{j}}|^2. \tag{6.40}$$

Recalling that $\beta \in [0,1]$ and denoting by $K>0$ any real number satisfying $\operatorname{supp}\mathscr{E}(\cdot) \subseteq [0,K]$ we can estimate the right-hand side of (6.40) by

$$4(n+1)^{3/2}K^{2(1-\beta)} \sum_{|\mathbf{n}|+|\mathbf{m}|=n} \alpha^2(\mathbf{n},\mathbf{m}) \sum_{\mathbf{j}\in\Lambda_N^+} |m_{\mathbf{j}}|\,|\mathbf{k}_{\mathbf{j}}|^{2\beta}$$
$$\leqslant 4(n+1)^{3/2}K^{2(1-\beta)}\mathscr{E}_{M_N}(f,f) \tag{6.41}$$

and (6.36) follows. The last inequality in (6.41) follows from (6.33).

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