A NOTE ON THE CENTRAL LIMIT THEOREM FOR TWO-FOLD STOCHASTIC RANDOM WALKS IN A RANDOM ENVIRONMENT.

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Abstract. We consider a class of two-fold stochastic random walks in a random environment. The transition probability is given by an ergodic random field on $\mathbb{Z}^d$ with two-fold stochastic realizations. The central limit theorem for this class of random walks has been claimed by Kozlov under certain strong mixing conditions (cf. [4], Theorem 3, p. 121). However the statement and the argument used in [4] are not correct, and we provide a counterexample in dimension two (cf. example 2.3 below). We give a sufficient condition for the walk to satisfy the central limit theorem (see condition (H) below). Then we give some spectral and mixing conditions that imply condition (H).

1. Introduction

We consider a two-fold stochastic random walk in a random environment (R.W.R.E.) with zero local drift. The random walk starting from $\mathbf{x} \in \mathbb{Z}^d$ in the given environment $\omega \in \Omega$ is a Markov chain $(X_{n,\omega})_{n\geq 0}$ with the state space $\mathbb{Z}^d$, over a probability space $\mathcal{T}_1 := (\Sigma, \mathcal{W}, Q)$, that satisfies

$$
\begin{aligned}
(1.1)\qquad & Q[X^{\mathbf{x}}_{n+1,\omega} = X^{\mathbf{x}}_{n,\omega} + \mathbf{z} | X^{\mathbf{x}}_{0,\omega}, \ldots, X^{\mathbf{x}}_{n,\omega}] \overset{Q\text{ a.s.}}{=} p(X^{\mathbf{x}}_{n,\omega}, X^{\mathbf{x}}_{n,\omega} + \mathbf{z}; \omega), \quad n \geq 0, \quad \mathbf{z} \in \Lambda, \\
& Q[X^{\mathbf{x}}_{0,\omega} = \mathbf{x}] = 1.
\end{aligned}
$$

Here (Ω, d) is a Polish metric space, $\mathcal{B}(\Omega)$ the σ-algebra of Borel sets and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{B}(\Omega))$. Let $\mathcal{T}_0 := (\Omega, \mathcal{B}(\Omega), \mathbb{P})$. We assume further that the transition of probabilities $p : \mathbb{Z}^d \times \Lambda \times \Omega \to [0, +\infty)$ is a stationary random field on $\mathbb{Z}^d$, i.e. it is given by

$$
p(\mathbf{x}, \mathbf{x} + \mathbf{z}; \omega) := p_{\mathbf{z}}(T_{\mathbf{x}}\omega) \qquad \forall (\mathbf{x}, \mathbf{z}, \omega) \in \mathbb{Z}^d \times \Lambda \times \Omega,
$$

where $\{T_{\mathbf{x}} : \Omega \to \Omega, \mathbf{x} \in \mathbb{Z}^d\}$ is a group of transformations preserving measure $\mathbb{P}$, i.e. $T_{\mathbf{x}} T_{\mathbf{y}} = T_{\mathbf{x}+\mathbf{y}}$, $T_{\mathbf{x}}(A) \in \mathcal{B}(\Omega)$ and $\mathbb{P}[T_{\mathbf{x}}(A)] = \mathbb{P}[A]$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ and $A \in \mathcal{B}(\Omega)$. Here $\Lambda \subset \mathbb{Z}^d$ is of finite cardinality and such that $\mathrm{span}(\Lambda) = \mathbb{Z}^d$.

The random variables $\{p_{\mathbf{z}} : \Omega \to [0, +\infty),\ \mathbf{z} \in \Lambda\}$ satisfy

(S) (*Normalization*)

$$
\sum_{\mathbf{z} \in \Lambda} p_{\mathbf{z}}(\omega) = 1, \quad \mathbb{P} - a.s.
$$

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By the designation that the walk is *two-fold stochastic*, cf. [4] p. 119, we mean that it satisfies the following conservation law:

$$\sum_{\mathbf{x}\in\Lambda} p(\mathbf{x},\mathbf{y};\omega) \equiv 1, \quad \mathbb{P}\ a.s., \quad \forall\, \mathbf{y}\in\mathbb{Z}^d,$$

or equivalently that

(2S)

$$\sum_{\mathbf{z}\in\Lambda} p_{\mathbf{z}}(T_{-\mathbf{z}}\omega) = 1, \qquad \mathbb{P}\ a.s.$$

The random vector

$$\mathbf{v}(\omega) = (v^{(1)}(\omega),\cdots,v^{(d)}(\omega)) := \sum_{\mathbf{z}\in\Lambda} \mathbf{z}p_{\mathbf{z}}(\omega) \tag{1.2}$$

is called *the local drift* of the walk. We suppose that the local drift is of zero mean, i.e.

(ND)

$$\langle\mathbf{v}\rangle = \mathbf{0},$$

where $\langle\cdot\rangle$ denotes the mathematical expectation corresponding to $\mathbb{P}$.

The random walk in a random environment is a stochastic process $(X_n)_{n\ge 0}$ over the product probability space $\mathcal{T}_0\otimes\mathcal{T}_1 := (\Omega\times\Sigma, \mathcal{B}(\Omega)\otimes\mathcal{W}, \mathbb{P}\otimes Q)$ defined by $X_n(\omega,\sigma) := X^{\mathbf{0}}_{n,\omega}(\sigma)$ for any $(\omega,\sigma)\in\Omega\times\Sigma$. With no loss of generality we assume that our random walk starts at the origin.

We are interested in proving the central limit theorem (C.L.T.) for $X_n/\sqrt{n}$ as $n\to+\infty$. In the following section we formulate sufficient conditions for such a theorem, see Theorem 2.2 and in particular the condition (H) below.

The question of obtaining the C.L.T. for two-fold stochastic R.W.R.E. has been considered in [4]. Yet, in our view, the proofs of the main results concerning such walks formulated in Theorems 3 and 4 on pp. 89, 91 and Theorem 3 on p. 121 of that paper are not complete. *In particular, the statement of Theorem 3 on p. 121 is not correct in dimensions one and two.* The crucial estimate of the field E contained in the fourth formula after (2.11) on p. 90 also seems to be incorrect. Because of the significance of the results announced in Kozlov's paper, the fact that they are very frequently cited elsewhere in the literature and since we could not find any other source dealing with the issue of two-fold stochastic random walks we felt compelled to compose this brief note in order to fill the existing gap.

The main ideas of homogenization of diffusions in a random environment were laid out by Kozlov ([3]) and by Papanicolaou and Varadhan ([9]). These ideas were generalized in the context of general central limit theorems for reversible Markov processes by Kipnis and Varadhan ([2]), and in [8, 11] for non-reversible processes satisfying a strong sector condition. In general the two-fold stochastic R.W.R.E. does not satisfy a strong sector condition, and the problem corresponds, in the continuous setting, to a diffusion in a divergence-free random field with unbounded stream matrix (cf. [5]).

Let us briefly outline the principal ideas of the homogenization approach, actually borrowed from [4], and explain the nature of the missing part of the argument contained there. In what follows we shall denote by $\mathbf{M}$ the expectation with respect to Q. Fix $\omega \in \Omega$. Following [4] we introduce the *environment chain*

$$\underline{\omega}_n^{\omega} := T_{X_{n,\omega}}(\omega), \quad n \geq 1, \tag{1.3}$$

defined over $\mathcal{T}_1$ with Ω its state space. The transition operator P of this chain can be explicitly computed, see (3.1) below, and it turns out that $\mathbb{P}$ is its invariant measure.

We call $L = P - I$ the *generator* of the chain. The process $\underline{\omega}_n(\omega, \sigma) := \underline{\omega}_n^{\omega}(\sigma)$, considered over $\mathcal{T}_0 \otimes \mathcal{T}_1$, is therefore stationary. As a simple consequence of the Markov property for $(X_{n,\omega})_{n\geq 1}$, with $\omega \in \Omega$ fixed, we conclude that

$$M_{n,\omega} := X_{n,\omega} - \sum_{k=0}^{n-1} \mathbf{v}(\underline{\omega}_k^{\omega}) \tag{1.4}$$

is an $\mathbb{R}^d$-valued, zero mean martingale with respect to the natural filtration corresponding to $(X_{n,\omega})_{n\geq 0}$.

Next, we solve the resolvent equation, in the L^2 space over the invariant measure $\mathbb{P}$ (see Section 2 for defintion),

$$\lambda E_\lambda^{(p)} - L E_\lambda^{(p)} = v^{(p)} \tag{1.5}$$

for any $p = 1, \cdots, d$, $\lambda > 0$. Let $M_\lambda := (M_\lambda^{(1)}, \cdots, M_\lambda^{(d)})$, where

$$M_{\lambda,n}^{(p)} := E_\lambda^{(p)}(\underline{\omega}_n) - \sum_{k=0}^{n-1} L E_\lambda^{(p)}(\underline{\omega}_k).$$

It is a square integrable $\mathbb{R}^d$-valued martingale corresponding to the natural filtration $(\mathcal{F}_n)$ of $(\underline{\omega}_n)_{n\geq 0}$. Letting $\mathcal{M}_{\lambda,n} := M_{n,\omega} + M_{\lambda,n}$, $n \geq 1$, we conclude that it is a square integrable $\mathbb{R}^d$-valued martingale with respect to $(\mathcal{F}_n)$. A straightforward elementary calculation shows that

$$\left\langle \mathbf{M} | \mathcal{M}_{\lambda,n} - \mathcal{M}_{\lambda',n}|^2 \right\rangle = n \sum_{p=1}^{d} \int \left\{ \sum_{\mathbf{z}\in\Lambda} [D_{\mathbf{z}} E_\lambda^{(p)}(\omega) - D_{\mathbf{z}} E_{\lambda'}^{(p)}(\omega)] p_{\mathbf{z}}(\omega) \right\}^2 \mathbb{P}(d\omega) \tag{1.6}$$

for any $\lambda, \lambda' > 0$. Defining $E_\lambda := (E_\lambda^{(1)}, \cdots, E_\lambda^{(d)})$ we conclude from (1.4) that

$$\frac{X_n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \mathcal{M}_{\lambda,n} + \frac{1}{\sqrt{n}} E_\lambda(\underline{\omega}_0) - \frac{1}{\sqrt{n}} E_\lambda(\underline{\omega}_n) + \frac{\lambda}{\sqrt{n}} \sum_{k=0}^{n-1} E_\lambda(\underline{\omega}_k). \tag{1.7}$$

Choosing $\lambda = n^{-1}$ we can prove, with the help of a standard C.L.T. for ergodic martingales (see Theorem 6.11 of [12]), that the laws of $\frac{1}{\sqrt{n}}\mathcal{M}_{1/n,n}$ converge to the law of a centered normal random vector, provided that

$$\lim_{\lambda\to 0} D_{\mathbf{z}} E_\lambda^{(p)} = e_{\mathbf{z}}^{(p)}, \quad \forall\, \mathbf{z} \in \Lambda,\, p = 1, \cdots, d, \tag{1.8}$$

L^2-strongly. This fact, in turn, is implied by the condition

$$\lim_{\lambda\to 0} \lambda \|E_\lambda^{(p)}\|_{L^2}^2 = 0, \tag{1.9}$$

see (1.2.9) in [6]. (1.9) also implies that all the non-martingale terms appearing on the right hand side of (1.7) must vanish, in the mean square sense, thanks to the stationarity of $(\underline{\omega}_n)_{n\geq 0}$. It appears that the proof of (1.9) is missing in [4] and the present article is devoted to its demonstration. In order to show (1.9) one needs assumption (H), see the following section. In proving that (H) implies (1.9) we adopt the argument laid out in [5] by Oelschläger.

In addition we provide a number of easy to check conditions, expressed in terms of the statistical properties of random variables $p_{\mathbf{z}}$, $\mathbf{z} \in \Lambda$ (see Section 4 below) that suffice for the condition (H) to hold. We give an example, see Example 2.3 below, of a two–dimensional *shear layer* R.W.R.E. that does not satisfy (H) and for which the C.L.T. fails.

2. Preliminaries and the formulation of the main result

By $B(\Omega)$ we denote the space of all bounded Borel measurable random variables. Let L^2 denote the space of all square integrable random variables over the probability space $\mathcal{T}_0$ equipped with the usual scalar product

$$(F, G)_{L^2} := \int FGd\mathbb{P}, \quad \forall\, F, G \in L^2,$$

and the norm $\|F\|_{L^2}^2 = (F, F)_{L^2}$. We also define $U_{\mathbf{x}}F(\omega) := F(T_{\mathbf{x}}\omega)$ for all $\mathbf{x} \in \mathbb{Z}^d$, $F \in L^2$. For any random variable $F : \Omega \to \mathbb{R}$ and $\mathbf{z} \in \mathbb{Z}^d$ we define *the abstract gradient in the direction of* $\mathbf{z}$ as

$$D_{\mathbf{z}}F(\omega) := F(T_{\mathbf{z}}\omega) - F(\omega).$$

Likewise, for any function $f : \mathbb{Z}^d \to \mathbb{R}$ and $\mathbf{z} \in \mathbb{Z}^d$ we define *the lattice gradient in the direction of* $\mathbf{z}$ as

$$\partial_{\mathbf{z}} f(\mathbf{x}) := f(\mathbf{x} + \mathbf{z}) - f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{Z}^d.$$

(E) (*Ergodicity of the environment*) Any $F \in B(\Omega)$ such that $U_{\mathbf{x}}F = F$ for all $\mathbf{x} \in \mathbb{Z}^d$ must satisfy $F \equiv \text{const}$ $\mathbb{P}$ a.s.

Recall we assumed that Λ generates $\mathbb{Z}^d$. We suppose further that $p_{\mathbf{z}}$, $\mathbf{z} \in \Lambda$, satisfy the following ellipticity condition.

(Ell) (*Ellipticity*) There exists a constant $\kappa > 0$ such that

$$\inf_{\mathbf{z}\in\Lambda} p_{\mathbf{z}}(\omega) \ge \kappa, \quad \mathbb{P}\ a.s.$$

From (E) and (Ell) it follows that $\mathbb{P}$ is an ergodic measure for the environment chain.

Our crucial assumption about the local drift is the following:

(H) there exists a constant $C > 0$ such that

$$|(v^{(p)}, F)_{L^2}| \le C \sum_{\mathbf{z}\in\Lambda} \|D_{\mathbf{z}} F\|_{L^2}, \quad \forall\, F \in L^2,\, p = 1, \cdots, d.$$

Recall that $\mathbf{v} = (v^{(1)}, \cdots, v^{(d)})$.

Note that it follows from (H) that there exist $H^{(p)}_{\mathbf{z}} \in L^2$, $\mathbf{z} \in \Lambda$, such that

$$v^{(p)} = \sum_{\mathbf{z}\in\Lambda} D^*_{\mathbf{z}} H^{(p)}_{\mathbf{z}}, \quad \forall\, p = 1, \cdots, d. \tag{2.1}$$

Remark 2.1. Condition (H) guarantees that the components $v^{(p)}$ of the local drift belong to the range of $(-L^s)^{1/2}$, where L^s is the symmetric part of the generator $L = P - I$ of the environment process. It is well known, cf. [2], that in the symmetric case, i.e. when $p_{\mathbf{z}}(\omega) = p_{\mathbf{z}}(T_{-\mathbf{z}}\omega)$ for all $\mathbf{z} \in \Lambda$ and $\mathbb{P}$ a.s. ω, (H) suffices for the C.L.T. to hold. Theorem 2.2 below says that this is also the case for the environment process corresponding to R.W.R.E. that is non-reversible.

The following theorem is the main result of this article.

Theorem 2.2. *Under the assumptions* (E), (S), (Ell), (2S), (ND) *and* (H) *the sequence of the laws corresponding to the random vectors $Z_n := X_n/\sqrt{n}$, $n \ge 1$ converges weakly, as $n \to +\infty$ to the law of a Gaussian random vector with zero mean and a non-trivial covariance matrix.*

Example 2.3. We provide here an example of a two-fold stochastic random walk in a random environment that does not satisfy condition (H) and that is characterized by a superdiffusive behavior. Observe that in Theorem 3, p. 121, of [4] it is not assumed that the drift satisfies condition (H).

Let $\mathcal{T}_0$ be the probability triple as in Section 1, equipped with a one–dimensional group of motions T_y, $y \in \mathbb{R}$, for which, measure $\mathbb{P}$ is both invariant and ergodic. We let $d = 2$, $\kappa \in (0, 1/4)$, $\Lambda = \{\pm e_1, \pm e_2\}$. Let also $p_{\pm e_2}(\omega) \equiv 1/4$ and $p_{e_1} : \Omega \to [\kappa, 1/2 - \kappa]$ with the mean $\mathbb{E}p_{e_1} = 1/4$ and covariance

$$C(y) := \left\langle \left(p(T_y\omega) - \frac{1}{4}\right)\left(p(\omega) - \frac{1}{4}\right)\right\rangle.$$

We assume that C is sufficiently strongly decaying, as $|y| \to +\infty$, so that

$$\hat{C}(k) := \sum_{y\in\mathbb{Z}} C(y) e^{\mathrm{i}\, ky}, \quad k \in \mathbb{R},$$

belongs to $C^\infty(\mathbb{R})$.

Set $p_{-e_1} = 1/2 - p_{e_1}$. The law of the resulting R.W.R.E. is identical to the law of $(X_n, Y_n)_{n\geq 0}$, considered over the probability space $\tilde{\mathcal{T}}\otimes\mathcal{T}_1$, where $\tilde{\mathcal{T}} := (\Omega\times\Xi, \mathcal{B}(\Omega)\otimes\mathcal{V}, \mathbb{P}\otimes R)$ for some probability space $(\Xi, \mathcal{V}, R)$ supporting $(Y_n)_{n\geq 1}$, a spatially homogeneous random walk on $\mathbb{Z}$ with non-random transition probabilities $p(y, y\pm 1) = 1/4$, $p(y,y) = 1/2$. $(X_n)_{n\geq 1}$ is a non-homogeneous in time R.W.R.E. over $\tilde{\mathcal{T}}\otimes\mathcal{T}_1$ such that for a given $(\omega,\xi)\in\Omega\times\Xi$ we have

$$Q[X_{n+1} = X_n + 1|X_0, \cdots, X_n] = p_{e_1}(T_{Y_n(\xi)}\omega)$$

and

$$Q[X_{n+1} = X_n - 1|X_0, \cdots, X_n] = p_{-e_1}(T_{Y_n(\xi)}\omega).$$

Denote by $\mathbf{E}^R$ the expectation operator relative to R. An elementary calculation shows that (recall that $\mathbf{M}$ is the expectation relative to Q)

$$\left\langle \mathbf{E}^R\mathbf{M}X_n^2\right\rangle = \frac{n}{2} + 2\sum_{1\leq i<j\leq n}\left\langle \mathbf{E}^R\left[\left(2p(T_{Y_i}\omega) - \frac{1}{2}\right)\left(2p(T_{Y_j}\omega) - \frac{1}{2}\right)\right]\right\rangle \tag{2.2}$$

$$= \frac{n}{2} + 8\sum_{j=1}^{n-1}(n-j)\mathbf{E}^R C(Y_j).$$

Denoting by $p_j(y)$ the probability that $Y_j = y$ we can rewrite the utmost right hand side of (2.2) as being equal to

$$\frac{n}{2} + 8n\sum_{j=1}^{n-1}\left(1 - \frac{j}{n}\right)\sum_{y\in\mathbb{Z}} C(y)p_j(y). \tag{2.3}$$

However (see (3.121), p. 137, of [1]),

$$p_j(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}(1+\cos k)\right]^j \mathrm{e}^{-\mathrm{i}ky}dk$$

so the expression in (2.3) equals

$$\frac{n}{2} + \frac{4n}{\pi}\sum_{j=1}^{n-1}\left(1 - \frac{j}{n}\right)\int_{-\pi}^{\pi}\left[\frac{1}{2}(1+\cos k)\right]^j \hat{C}(k)dk. \tag{2.4}$$

Substituting $k := k'/\sqrt{j}$ and using the fact that

$$\frac{1}{2}\left(1+\cos\left(\frac{k'}{\sqrt{j}}\right)\right) \sim 1 - \frac{(k')^2}{4j}$$

for sufficiently large j we conclude that for $n\gg 1$,

$$\frac{\left\langle \mathbf{E}^R\mathbf{M}X_n^2\right\rangle}{n^{3/2}} \sim \frac{4\hat{C}(0)}{\pi\sqrt{n}}\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\frac{1}{\sqrt{j}}\int_{-\infty}^{+\infty}\mathrm{e}^{-k^2/4}dk \tag{2.5}$$

$$\sim \frac{4\hat{C}(0)}{\pi}\int_0^1 (1-x)\frac{dx}{\sqrt{x}}\int_{-\infty}^{+\infty} \mathrm{e}^{-k^2/4}\,dk = \frac{32\hat{C}(0)}{3\sqrt{\pi}}.$$

The behavior of $(X_n)_{n\geq 1}$ is therefore superdiffusive, provided that $\hat{C}(0)\neq 0$.

3. The proof of Theorem 2.2

The generator of the environment chain (1.3) equals

$$LF(\omega) = \sum_{\mathbf{z}\in\Lambda} p_{\mathbf{z}}(\omega) D_{\mathbf{z}} F(\omega), \quad \forall\, F\in B(\Omega), \tag{3.1}$$

see Proposition 1, p. 120, of [4]. The measure $\mathbb{P}$ is invariant under the chain, i.e. $\int LF d\mathbb{P} = 0$ for all $F\in B(\Omega)$. The symmetric and anti-symmetric parts of the generator, with respect to $\mathbb{P}$, are given by

$$SF(\omega) = -\frac{1}{2}\sum_{\mathbf{z}\in\Lambda} D^*_{\mathbf{z}}(p_{\mathbf{z}}(\omega) D_{\mathbf{z}} F(\omega)) \tag{3.2}$$

and

$$AF(\omega) = \frac{1}{2}\sum_{\mathbf{z}\in\Lambda} q_{\mathbf{z}}(\omega) D_{\mathbf{z}} F(\omega) = -\frac{1}{2}\sum_{\mathbf{z}\in\Lambda} D^*_{\mathbf{z}}\left(q_{\mathbf{z}}(\omega) F(\omega)\right) \tag{3.3}$$

respectively, see [4]. Here

$$q_{\mathbf{z}}(\omega) := p_{\mathbf{z}}(\omega) - p_{-\mathbf{z}}(T_{\mathbf{z}}\omega). \tag{3.4}$$

Multiplying both sides of (1.5) by $E^{(p)}_\lambda$, integrating over Ω and using assumption (H) we conclude that

$$\sup_{1>\lambda>0} \lambda\|E^{(p)}_\lambda\|^2 < +\infty, \quad \sup_{1>\lambda>0} \|D_{\mathbf{z}} E^{(p)}_\lambda\|^2 < +\infty, \quad \forall\,\mathbf{z}\in\Lambda.$$

Hence there exists a sequence $\lambda_n\to 0$ as $n\to+\infty$ such that

$$e^{(p)}_{\mathbf{z}} = \lim_{n\to+\infty} D_{\mathbf{z}} E^{(p)}_{\lambda_n}, \quad \forall\,\mathbf{z}\in\Lambda,$$

in the weak L^2 sense. It can be shown, see e.g. Proposition 1, p. 86, and the following argument on p. 87 of [4], that there exists a random field

$$E(\mathbf{x},\omega) = (E^{(1)}(\mathbf{x},\omega),\cdots,E^{(d)}(\mathbf{x},\omega)), \quad (\mathbf{x},\omega)\in\mathbb{Z}^d\times\Omega,$$

that satisfies

(i) $E^{(p)}(\mathbf{x},\cdot)\in L^2$,
(ii) $\partial_{\mathbf{z}} E^{(p)}(\mathbf{x},\cdot) = U_{\mathbf{x}} e^{(p)}_{\mathbf{z}}$ for all $\mathbf{x}\in\mathbb{Z}^d$, $\mathbf{z}\in\Lambda$,
(iii) $E(\mathbf{0},\cdot) = \mathbf{0}$,

(iv) for any $K>0$ we have

$$\lim_{a\to+\infty}\sup_{|\mathbf{x}|\le aK}\frac{\|E^{(p)}(\mathbf{x},\cdot)\|_{L^2}}{a}=0. \tag{3.5}$$

Both (1.8) and (1.9) follow if we can establish that

$$\sum_{\mathbf{z}\in\Lambda}(p_{\mathbf{z}}e_{\mathbf{z}}^{(p)},e_{\mathbf{z}}^{(p)})_{L^2}=\sum_{\mathbf{z}\in\Lambda}(H_{\mathbf{z}}^{(p)},e_{\mathbf{z}}^{(p)})_{L^2},\quad \forall\, p=1,\cdots,d, \tag{3.6}$$

see e.g. the proof of Proposition 1.2.1 of [6], and the rest of our argument will be devoted to the proof of (3.6).

Note that thanks to (2.1), (3.2) and (3.3) we have

$$\frac{1}{2}\sum_{\mathbf{z}\in\Lambda}\int p_{\mathbf{z}}(\omega)e_{\mathbf{z}}^{(p)}(\omega)D_{\mathbf{z}}\phi(\omega)\mathbb{P}(d\omega) \tag{3.7}$$

$$+\frac{1}{2}\sum_{\mathbf{z}\in\Lambda}\int q_{\mathbf{z}}(\omega)e_{\mathbf{z}}^{(p)}(\omega)\phi(\omega)\mathbb{P}(d\omega)=\sum_{\mathbf{z}\in\Lambda}\int H_{\mathbf{z}}^{(p)}(\omega)D_{\mathbf{z}}\phi(\omega)\mathbb{P}(d\omega),$$

for any $\phi\in L^2$.

From (3.7) we conclude that for any test function $\phi:\mathbb{Z}^d\times\Omega\to\mathbb{R}$ such that $\phi(\cdot,\omega)$ is local for any $\omega\in\Omega$ and $\phi(\mathbf{x},\cdot)\in L^2$ for any $\mathbf{x}\in\mathbb{Z}^d$ we have

$$\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int p_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)\partial_{\mathbf{z}}\phi(\mathbf{x},\omega)\mathbb{P}(d\omega) \tag{3.8}$$

$$+\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)\phi(\mathbf{x},\omega)\mathbb{P}(d\omega)=\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int H_{\mathbf{z}}^{(p)}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}\phi(\mathbf{x},\omega)\mathbb{P}(d\omega).$$

We show that (3.6) is a consequence of (3.8).

Let $h:\mathbb{R}^d\to[0,+\infty)$ be any compactly supported C^∞-smooth density function, i.e. $\int_{\mathbb{R}^d}h(\mathbf{x})d\mathbf{x}=1$, and let $j_R(\cdot)$ be the indicator of the event $\sum_{\mathbf{z}\in\Lambda}|H_{\mathbf{z}}^{(p)}(\omega)|\le R$. We substitute $\phi_a(\mathbf{x},\omega):=E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x})j_R(\omega)$ into (3.8), where $h_a(\mathbf{x})=a^{-d}h(\mathbf{x}/a)$ for any $a>0$.

The first term on the left hand side of (3.8) then takes the form

$$\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int p_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)\partial_{\mathbf{z}}\left[E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x})\right]j_R(\omega)\,\mathbb{P}(d\omega). \tag{3.9}$$

Using the differentiation rule

$$\partial_{\mathbf{z}}(fg)(\mathbf{x})=f(\mathbf{x}+\mathbf{z})\partial_{\mathbf{z}}g(\mathbf{x})+g(\mathbf{x})\partial_{\mathbf{z}}f(\mathbf{x}) \tag{3.10}$$

we deduce that the expression in (3.9) equals

$$\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int p_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x}+\mathbf{z})j_R(\omega)\,\mathbb{P}(d\omega) \tag{3.11}$$

$$+\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int p_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)E^{(p)}(\mathbf{x},\omega)\partial_{\mathbf{z}}h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega).$$

Taking the first limit as $a\to+\infty$ (using the ergodic theorem) and then the second one as $R\to+\infty$, we conclude that the first term of (3.11) becomes equal to the expression appearing on the left hand side of (3.6). As for the second term of (3.11), note that $\partial_{\mathbf{z}}h_a(\mathbf{x})\sim a^{-d-1}\nabla h(\mathbf{x})\cdot\mathbf{z}$ for $a\gg 1$ uniformly in $\mathbf{x}$. Therefore this term is of the same order of magnitude as

$$\frac{1}{2a^{d+1}}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int p_{\mathbf{z}}(T_{\mathbf{x}}\omega)e^{(p)}_{\mathbf{z}}(T_{\mathbf{x}}\omega)E^{(p)}(\mathbf{x},\omega)(\nabla h)(\mathbf{x}/a)\cdot\mathbf{z}\,j_R(\omega)\,\mathbb{P}(d\omega)$$

$$=\frac{1}{2a^{d+1}}\sum_{\mathbf{x}\in a^{-1}\mathbb{Z}^d}\sum_{\mathbf{z}\in\Lambda}\int p_{\mathbf{z}}(T_{\mathbf{x}}\omega)e^{(p)}_{\mathbf{z}}(T_{a\mathbf{x}}\omega)E^{(p)}(a\mathbf{x},\omega)(\nabla h)(\mathbf{x})\cdot\mathbf{z}\,j_R(\omega)\,\mathbb{P}(d\omega)$$

Applying the Cauchy and Schwartz inequality we conclude that the right hand side of this equality can be estimated by

$$\frac{1}{2}|\Lambda|a^{-d}\max_{\mathbf{z}\in\Lambda}\left(|\mathbf{z}|\|e^{(p)}_{\mathbf{z}}\|_{L^2}\right)\sup_{|\mathbf{x}|\le a}\frac{\|E^{(p)}(\mathbf{x},\cdot)\|_{L^2}}{a}\sum_{\mathbf{y}\in a^{-1}\mathbb{Z}^d}|\nabla h(\mathbf{y})|,$$

which tends to 0 as $a\to+\infty$ by virtue of (3.5).

An analogous argument applied to the term on the right hand side of (3.8) yields that the respective limits, first as $a\to+\infty$ and then as $R\to+\infty$, produce the expression appearing on the right hand side of (3.6). What therefore remains to be shown is the fact that after taking the test function equal to $\phi_a(\cdot,\cdot)$ the second term on the left hand side of (3.8) vanishes upon taking the iterative limit procedure. Indeed, the corresponding expression equals

$$\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega)$$

$$=-\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int \partial^*_{\mathbf{z}}[\,q_{\mathbf{z}}(T_{\mathbf{x}}\omega)E^{(p)}(\mathbf{x},\omega)\,]E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega)$$

$$=-\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)E^{(p)}(\mathbf{x},\omega)\,\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega)$$

$$-\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)E^{(p)}(\mathbf{x},\omega)\,E^{(p)}(\mathbf{x}+\mathbf{z},\omega)\partial_{\mathbf{z}}h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega).$$

Using the relation $E^{(p)}(\mathbf{x}+\mathbf{z},\omega)=E^{(p)}(\mathbf{x},\omega)+\partial_{\mathbf{z}}E^{(p)}(\mathbf{x}+\mathbf{z},\omega)$ we therefore conclude that

$$\frac{3}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}E^{(p)}(\mathbf{x},\omega)E^{(p)}(\mathbf{x},\omega)h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega)\tag{3.12}$$

$$=-\frac{1}{2}\sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda}\int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)[E^{(p)}(\mathbf{x},\omega)]^2\,\partial_{\mathbf{z}}h_a(\mathbf{x})j_R(\omega)\,\mathbb{P}(d\omega)$$

and what remains to be proved is that the expression on right hand side of (3.12) vanishes as $a \to +\infty$ first and then $R \to +\infty$. This expression is of the same order of magnitude as

$$-\frac{1}{2a^{d+1}} \sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda} \int q_{\mathbf{z}}(T_{\mathbf{x}}\omega)[E^{(p)}(\mathbf{x},\omega)]^2\, \nabla_{\mathbf{x}} h\left(\frac{\mathbf{x}}{a}\right)\cdot \mathbf{z}\, j_R(\omega)\,\mathbb{P}(d\omega). \tag{3.13}$$

Since

$$\sum_{\mathbf{z}\in\Lambda} z_q\,(q_{\mathbf{z}},F)_{L^2} = 2\sum_{\mathbf{z}\in\Lambda} z_q\,(p_{\mathbf{z}},F)_{L^2} + \sum_{\mathbf{z}\in\Lambda} z_q\,(p_{\mathbf{z}},D_{\mathbf{z}}F)_{L^2},$$

thanks to the assumption (H), we can write

$$\sum_{\mathbf{z}\in\Lambda} z_q\, q_{\mathbf{z}} = \sum_{\mathbf{z}\in\Lambda} D^*_{\mathbf{z}} G^{(q)}_{\mathbf{z}},$$

where $G^{(q)}_{\mathbf{z}} = 2D^*_{\mathbf{z}} H^{(q)}_{\mathbf{z}} + z_q D^*_{\mathbf{z}} p^{(q)}_{\mathbf{z}}$. Hence, we can recast (3.13) in the form

$$-\frac{1}{2a^{d+1}} \sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda} \sum_{q=1}^{d} \int G^{(q)}_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}\left\{[E^{(p)}(\mathbf{x},\omega)]^2\,\partial_{x_q} h\left(\frac{\mathbf{x}}{a}\right)\right\} j_R(\omega)\,\mathbb{P}(d\omega) \tag{3.14}$$

$$= -\frac{1}{2a^{d+1}} \sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda} \sum_{q=1}^{d} \int G^{(q)}_{\mathbf{z}}(T_{\mathbf{x}}\omega)\partial_{\mathbf{z}}[E^{(p)}(\mathbf{x},\omega)]^2\,\partial_{x_q} h\left(\frac{\mathbf{x}}{a}+\mathbf{z}\right) j_R(\omega)\,\mathbb{P}(d\omega)$$

$$-\frac{1}{2a^{d+1}} \sum_{(\mathbf{x},\mathbf{z})\in\mathbb{Z}^d\times\Lambda} \sum_{q=1}^{d} \int G^{(q)}_{\mathbf{z}}(T_{\mathbf{x}}\omega)[E^{(p)}(\mathbf{x},\omega)]^2\,\partial_{\mathbf{z}}\partial_{x_q} h\left(\frac{\mathbf{x}}{a}\right) j_R(\omega)\,\mathbb{P}(d\omega)$$

The first term on the right hand side of (3.14) can be estimated by

$$(2R+L)|\Lambda| \times \sup_{|\mathbf{x}|\le a+2L} \frac{\|E^{(p)}(\mathbf{x},\cdot)\|_{L^2}}{a} \times \sup_{\mathbf{z}\in\Lambda}\sup_{q}\left\{\|e^{(p)}_{\mathbf{z}}\|_{L^2}\; a^{-d}\sum_{\mathbf{x}\in a^{-1}\mathbb{Z}^d} \partial_{x_q} h\left(\frac{\mathbf{x}}{a}+\mathbf{z}\right)\right\}.$$

Since $\partial_{\mathbf{z}}\partial_{x_q} h\left(\frac{\mathbf{x}}{a}\right) \sim a^{-1}\sum_{r=1}^{d} \partial^2_{x_r x_q} h\left(\frac{\mathbf{x}}{a}\right) z_p$ the second term on the right hand side of (3.14) can be estimated by

$$(2R+L)|\Lambda| \times \sup_{|\mathbf{x}|\le a}\left(\frac{\|E^{(p)}(\mathbf{x},\cdot)\|_{L^2}}{a}\right)^2 \times \sup_{r,q}\left\{a^{-d}\sum_{\mathbf{x}\in a^{-1}\mathbb{Z}^d} \partial^2_{x_r x_q} h\left(\frac{\mathbf{x}}{a}\right)\right\}.$$

Both these terms vanish as $a \to +\infty$ for any given R.

4. Certain sufficiency conditions for (H)

In this section we present three examples of conditions on the statistics of $p_{\mathbf{z}}$, $\mathbf{z}\in\Lambda$, that imply (H).

4.1. **Spectral condition.** Following [4], p. 116, we introduce

$$B_f(\mathbf{x}) := \langle U^{\mathbf{x}} f\, f\rangle\,, \qquad \mathbf{x}\in\mathbb{Z}^d,$$

where $f:\Omega\to\mathbb{R}$ is a square integrable, zero mean random variable. From the Herglotz theorem there exists a finite Borel measure $S_f(\cdot)$ on $(\mathbb{T}^d,\mathcal{B}(\mathbb{T}^d))$ such that

$$B_f(\mathbf{x}) = \int\limits_{\mathbb{T}^d} e^{i\,\mathbf{x}\cdot\mathbf{k}} S_f(d\mathbf{k}), \tag{4.1}$$

with $\mathbb{T}^d := [0,2\pi)^d$.

Proposition 4.1. *Condition* (H) *is equivalent* (*cf.* (2.7), *p.* 116, *of* [4])

(Sp)

$$\mathcal{E} := \sum_{p=1}^{d}\int\limits_{\mathbb{T}^d}\frac{S_{v^{(p)}}(d\mathbf{k})}{|\mathbf{k}|^2} < +\infty.$$

Recall that $v^{(p)}$, $p=1,\cdots,d$, *are the components of the local drift, cf.* (H).

Proof. First, suppose that (H) holds. Let $B^{(p)}_{\mathbf{z},\mathbf{z}'}(\mathbf{x}) = \left\langle U^{\mathbf{x}} H^{(p)}_{\mathbf{z}}\, H^{(p)}_{\mathbf{z}'}\right\rangle$ and $\mathbf{B}^{(p)}(\mathbf{x}) := [B^{(p)}_{\mathbf{z},\mathbf{z}'}(\mathbf{x})]_{\mathbf{z},\mathbf{z}'\in\Lambda}$. The matrix version of Herglotz's theorem asserts the existence of a positive definite, Hermite matrix–valued, finite Borel measure $\mathbf{S}^{(p)}(\cdot) = [S^{(p)}_{\mathbf{z},\mathbf{z}'}(\cdot)]_{\mathbf{z},\mathbf{z}'\in\Lambda}$ on $\mathbb{T}^d$ satisfying $\mathbf{B}^{(p)}(\mathbf{x}) = \int_{\mathbb{T}^d} e^{i\,\mathbf{x}\cdot\mathbf{k}}\mathbf{S}^{(p)}(d\mathbf{k})$. After a straightforward calculation one gets

$$\mathcal{E} = \sum_{p=1}^{d}\sum_{\mathbf{z},\mathbf{z}'\in\Lambda}\int\limits_{\mathbb{T}^d}\frac{1+e^{i\,(\mathbf{z}'-\mathbf{z})\cdot\mathbf{k}} - e^{i\,\mathbf{z}'\cdot\mathbf{k}} - e^{-i\,\mathbf{z}\cdot\mathbf{k}}}{|\mathbf{k}|^2}\, S^{(p)}_{\mathbf{z},\mathbf{z}'}(d\mathbf{k}) < +\infty$$

and (Sp) follows.

On the other hand, assuming that (Sp) holds, we use the spectral theorem to represent the field $U^{\mathbf{x}}v^{(p)}$, $\mathbf{x}\in\mathbb{Z}^d$, see [10], (4.11), p. 16. We know that there exists a random measure $\hat{v}^{(p)}(\cdot)$ such that

$$v^{(p)}(\tau_{\mathbf{x}}\omega) = \int\limits_{\mathbb{T}^d} e^{i\,\mathbf{x}\cdot\mathbf{k}}\,\hat{v}^{(p)}(d\mathbf{k}),$$

where $\left\langle \hat{v}^{(p)}(d\mathbf{k})\hat{v}^{(p)}(d\mathbf{k}')\right\rangle = \delta(\mathbf{k}-\mathbf{k}')S_{v^{(p)}}(d\mathbf{k})$. Set

$$H^{(p)}_{\mathbf{z}} := \int\limits_{\mathbb{T}^d}\frac{e^{i\,\mathbf{z}\cdot\mathbf{k}}-1}{\sum\limits_{\mathbf{z}'\in\Lambda}|e^{i\,\mathbf{z}'\cdot\mathbf{k}}-1|^2}\,\hat{v}^{(p)}(d\mathbf{k}).$$

One can check that

$$\sum_{\mathbf{z}\in\Lambda}\left\langle\left[H^{(p)}_{\mathbf{z}}\right]^2\right\rangle = \int\limits_{\mathbb{T}^d}\frac{S_{v^{(p)}}(d\mathbf{k})}{\sum\limits_{\mathbf{z}'\in\Lambda}|e^{i\,\mathbf{z}'\cdot\mathbf{k}}-1|^2}$$

$$= \int\limits_{\mathbb{T}^d}\frac{|\mathbf{k}|^2}{\sum\limits_{\mathbf{z}'\in\Lambda}|e^{i\,\mathbf{z}'\cdot\mathbf{k}}-1|^2}\times\frac{S_{v^{(p)}}(d\mathbf{k})}{|\mathbf{k}|^2} < +\infty,$$

by virtue of (Sp). □

4.2. **Mixing condition.** The following result holds.

Proposition 4.2. *With the notation of the previous section condition* (H) *holds provided that* $d \geq 3$ *and*

$$\sum_{p=1}^{d} \sum_{\mathbf{x} \in \mathbb{Z}^d} \frac{B_{v^{(p)}}(\mathbf{x})}{|\mathbf{x}|^{d-2}} < +\infty. \tag{4.2}$$

Remark 4.3. The above proposition states that in dimension $d \geq 3$ a sufficient rate of decorrelation of the local drift guarantees condition (H). Note that Example 2.3 shows that the situation in dimension $d = 2$ is quite different. One can assume any decorrelation rate (even finite dependence range for the environment) for the local drift, yet the behavior of the particle will be superdiffusive as soon as $\hat{C}(0) \neq 0$, see formula (2.5).

Proof. Let $\Lambda^s := \Lambda \cup (-\Lambda)$ and

$$\Lambda^+ := \{\mathbf{z} \in \Lambda^s : \text{ the last non-zero component of } \mathbf{z} \text{ is positive, or } \mathbf{z} = \mathbf{0}\}.$$

Let $\xi_n^\omega := T_{Y_n}(\omega)$, $n \geq 1$, where $(Y_n)_{n \geq 1}$ is a homogeneous random walk on $\mathbb{Z}^d$ with the (non-random) transition of probabilities $r(\mathbf{x}, \mathbf{x} + \mathbf{z}) := r_{\mathbf{z}}$, $\mathbf{z} \in \Lambda^s$, where $\sum_{\mathbf{z} \in \Lambda^s} r_{\mathbf{z}} = 1$ and $r_{-\mathbf{z}} = r_{\mathbf{z}}$, $\mathbf{z} \in \Lambda^+$. The chain $(\xi_n^\omega)_{n \geq 0}$ is Markovian, with the transition of probability operator $RF = \sum_{\mathbf{z} \in \Lambda^s} r_{\mathbf{z}} U^{\mathbf{z}} F$ and the generator

$$\mathcal{L}F = \sum_{\mathbf{z} \in \Lambda^s} r_{\mathbf{z}} D_{\mathbf{z}} F, \quad F \in B(\Omega).$$

The measure $\mathbb{P}$ is invariant and $-\mathcal{L}$ becomes a non-negative, self-adjoint operator when extended to L^2. We denote by $(\xi_n)_{n \geq 0}$ the chain obtained by randomization of the initial configuration ω under $\mathbb{P}$. Suppose that for any $p = 1, \cdots, d$ we can find $G_p \in L^2$ such that

$$v^{(p)} = (-\mathcal{L})^{1/2} G_p. \tag{4.3}$$

Hence,

$$(U^{\mathbf{x}} v^{(p)}, v^{(p)})_{L^2} = \sum_{\mathbf{z} \in \Lambda^s} r_{\mathbf{z}} (D_{\mathbf{z}} U^{\mathbf{x}} G_p, G_p)_{L^2}, \quad \mathbf{x} \in \mathbb{Z}^d. \tag{4.4}$$

A simple calculation using spectral measures of the expressions appearing on both sides of (4.4) shows that

$$S_{v^{(p)}}(d\mathbf{k}) = 2 \sum_{\mathbf{z} \in \Lambda^+} (1 - \cos(\mathbf{k} \cdot \mathbf{z})) S_{G_p}(d\mathbf{k}),$$

where $S_{G_p}(\cdot)$ is the spectral measure corresponding to G_p, cf. (4.1). Hence, condition (Sp) is satisfied. It therefore remains to verify (4.3). This condition is however equivalent to

$$\sum_{n=0}^{+\infty}(R^n v^{(p)}, v^{(p)})_{L^2} < +\infty. \tag{4.5}$$

Let $R^n F = \sum_{\mathbf{x}\in\mathbb{Z}^d} r(n,\mathbf{x})U^{\mathbf{x}}F$, $n \geq 0$. It is well known, see (3.179), p. 150, of [1], that

$$\sum_{n\geq 0} r(n,\mathbf{x}) \sim |\mathbf{x}|^{2-d}, \quad \text{for } |\mathbf{x}| \gg 1, \text{ when } d \geq 3. \tag{4.6}$$

(4.5) then easily follows from (4.2) and (4.6). □

4.3. **Finite cycle representation condition.** This is a generalization of the example presented on pp. 124-125 of [4]. We define a cycle C of length $n \geq 2$ as a sequence of points $(\mathbf{z}_0, \mathbf{z}_1, \cdots, \mathbf{z}_n) \subset \mathbb{Z}^d$ such that $\mathbf{z}_n = \mathbf{z}_0$ and the points corresponding to indices smaller than n are distinct. Let

$$p_C(\mathbf{x},\mathbf{y}) := \sum_{p=0}^{n-1} \mathbf{1}_{(\mathbf{z}_p,\mathbf{z}_{p+1})}(\mathbf{x},\mathbf{y}), \quad \mathbf{x},\mathbf{y}\in\mathbb{Z}^d.$$

Suppose also that $W : \Omega \to \mathbb{R}$ is a certain strictly positive random variable and $\|W\|_\infty < +\infty$. Set

$$p_{\mathbf{z}}(\omega) := \frac{1}{n\|W\|_\infty}\sum_{\mathbf{y}\in\mathbb{Z}^d} W(T_{\mathbf{y}}\omega) p_{C+\mathbf{y}}(0,\mathbf{z}) \quad \text{when } \mathbf{z} \neq \mathbf{0}, \tag{4.7}$$

and

$$p_{\mathbf{0}}(\omega) := 1 - \sum_{\mathbf{z}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} p_{\mathbf{z}}(\omega). \tag{4.8}$$

In this case $\Lambda = \{\mathbf{0}, \mathbf{z}_1 - \mathbf{z}_0, \cdots, \mathbf{z}_{n-1} - \mathbf{z}_{n-2}, \mathbf{z}_0 - \mathbf{z}_{n-1}\}$.

This model can be easily generalized to the case of a finite sum of cycles. Namely, let $M > 0$ be an integer, $C_1, \cdots, C_M$ be cycles of the corresponding lengths $n_1, \cdots, n_M$ and $W_m : \Omega \to \mathbb{R}$, $m = 1, \cdots, M$, be positive random variables that satisfy $\|W_m\|_\infty < +\infty$. We let

$$p_{\mathbf{z}}(\omega) := \frac{1}{M}\sum_{m=1}^{M}\sum_{\mathbf{y}\in\mathbb{Z}^d} \frac{W_m(T_{\mathbf{y}}\omega)}{n_m\|W_m\|_\infty} p_{C_m+\mathbf{y}}(0,\mathbf{z}) \quad \text{when } \mathbf{z}\neq\mathbf{0}, \tag{4.9}$$

and $p_{\mathbf{0}}$ is given by (4.8).

It is easy to verify that the model is two-fold stochastic, i.e. (2S) is satisfied. We check that (H) holds. It suffices only to check this condition for $M = 1$. In this case we obtain, after a simple calculation,

$$\mathbf{v}(\omega) = \frac{1}{n\|W\|_\infty}\sum_{p=0}^{n-1}(\mathbf{z}_{p+1} - \mathbf{z}_p) W(T_{-\mathbf{z}_p}\omega) = \frac{1}{n\|W\|_\infty}\sum_{p=1}^{n}\mathbf{z}_p[W(T_{-\mathbf{z}_{p-1}}\omega) - W(T_{-\mathbf{z}_p}\omega)].$$

Hence for any $F \in L^2$,

$$(v^{(q)}, F)_{L^2} = \frac{1}{n\|W\|_\infty} \sum_{p=1}^{n} \mathbf{z}_p^{(q)} (W, D^*_{\mathbf{z}_p - \mathbf{z}_{p-1}} F)_{L^2}$$

$$\leq \frac{1}{n} \max_p |\mathbf{z}_p| \sum_{\mathbf{z}\in\Lambda} \|D^*_{\mathbf{z}} F\|_{L^2}$$

and (H) follows.

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