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Driven Tracer Particle in One Dimensional Symmetric Simple Exclusion

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Abstract: Consider an infinite system of particles evolving in a one dimensional lattice according to symmetric random walks with hard core interaction. We investigate the behavior of a tagged particle under the action of an external constant driving force. We prove that the diffusively rescaled position of the test particle $\epsilon X(\epsilon^{-2}t)$, $t > 0$, converges in probability, as $\epsilon \to 0$, to a deterministic function $v(t)$. The function $v(\cdot)$ depends on the initial distribution of the random environment through a non-linear parabolic equation. This law of large numbers for the position of the tracer particle is deduced from the hydrodynamical limit of an inhomogeneous one dimensional symmetric zero range process with an asymmetry at the origin. An Einstein relation is satisfied asymptotically when the external force is small.

Introduction

The one dimensional, nearest neighbor symmetric simple exclusion process can be described as follows: particles evolve on the one dimensional lattice $\mathbb{Z}$ with an exclusion rule that prevents more than one particle occupying the same site. Each particle jumps after a mean one exponential time to the right or left with probability $1/2$. If the chosen site is already occupied, the jump is suppressed to conform to the exclusion rule. We add to this system an extra particle and refer to it as the tagged particle. This particle is subject to the same exclusion rule that forbids more than one particle at the same site and, in contrast with the other particles, experiences the action of a constant external driving force. In result, the tagged particle jumps with probability $1/2 < p \leq 1$ to the right and $q = 1 - p$ to the left.

Without the presence of the environment, the tagged particle would perform a simple asymmetric random walk. In particular, if X_t stands for its position at time t, $t^{-1}(X_t - X_0)$ would converge almost surely to $p - q$ as $t \uparrow \infty$. The presence of the symmetric environment affects dramatically the evolution of the tagged particle. Since the untagged particles behave as symmetric random walks, we expect an accumulation of particles at the right of the tagged particle and a rarefaction at the left. Thus the environment slows down the motion of the tagged particle and tends to confine it. In fact we prove in this article that if the initial configuration of the system is a Bernoulli measure with slowly varying density, then

$$\lim_{t\to\infty} \frac{X_t - X_0}{\sqrt{t}} = v \tag{0.1}$$

in probability, where v is a real number depending on the given density. An heuristic derivation of (0.1) can be found in Burlatsky et al. ([BMMO, BMOR]).

The diffusive scale $\sqrt{t}$ is peculiar to the nearest neighbor assumption that restrains the tagged particle to jump over the symmetric particles. In higher dimension or without the nearest neighbor assumption, one would expect the tagged particle to move in the scale t.

In the not driven case ($p = q$) Arratia [Ar] showed that $t^{-1/4}(X_t - X_0)$ converges in distribution, as $t \uparrow \infty$, to a Gaussian variable with variance

$$\frac{1-\alpha}{\alpha}\sqrt{\frac{2}{\pi}}\ . \tag{0.2}$$

A corresponding invariance principle, i.e. the convergence of the properly rescaled process $\epsilon(X_{\epsilon^{-4}t} - X_0)$ to a fractional Brownian motion of parameter $1/2$, is proven in [RV]. This behavior should be characteristic of every one dimensional nearest neighbor model [Spo]. The first results of this type were established by Harris ([H]) in the case of Brownian particles with hard core interaction in dimension 1.

In Sect. 6 we prove that if we start with a constant profile of density α then

$$\lim_{p-q\to 0} \frac{v}{p-q} = \frac{1-\alpha}{\alpha}\sqrt{\frac{2}{\pi}},$$

which is the Einstein relation between the *mobility* $v/(p-q)$ given by (0.1) and the diffusivity given by (0.2). This is in agreement with the heuristic results of [BMOR].

Einstein relations can be established for a large class of weakly asymmetric models (i.e. the asymmetry is rescaled with the parameter ϵ relating the microscopic and the macroscopic scales (cf. [LR])). If the asymmetry is strong (i.e. not rescaled in the macroscopic limit) rigorous results on the Einstein relations are rare, essentially because of the difficulty to compute the stationary state of the environment as seen from the particle.

Here is the idea of our approach. First we have to understand that this is a non-stationary problem: the tracer will start to push the particles in front and generate an inhomogeneous density profile that will evolve deterministically under a diffusive rescaling of space and time. The proper way to formulate the problem is thus to prove that

$$\epsilon(X_{\epsilon^{-2}t} - X_0) \to v(t),$$

where $v(t)$ is a deterministic function of the (macroscopic) time t. This suggests that the problem is basically a hydrodynamic limit (cf. [KL]) with a moving boundary. There is a natural map that transforms a one dimensional nearest neighbor exclusion process in a zero range process. This map transforms the moving boundary problem in a fixed boundary problem. Thus we need to prove the hydrodynamic limit for a zero range process with boundary conditions.

Herbert Spohn made us notice a connection between this problem and the evolution of the random interfaces in a 3-phase Potts model at zero temperature under a Glauber dynamics. For some particular initial conditions the triple point of intersection of the three phases evolves macroscopically exactly like (0.1).

1. Statements of the Results

Consider a family of indistinguishable particles moving according to continuous time, symmetric, nearest neighbor random walks on $\mathbb{Z}$ with an exclusion rule. We add a tagged particle that moves according to an asymmetric random walk, jumping with probability p to the right, probability q to the left, and that respects the exclusion rule. The configuration of the system is denoted by (X, ξ), where $X \in \mathbb{Z}$ is the position of the tagged asymmetric particle, and $\xi \in \{0,1\}^{\mathbb{Z}}$ is the configuration of all other particles. Clearly $\xi(X) = 0$, because that site is already occupied by the asymmetric particle. The system just described is a Markov process whose generator acts on local functions $F: \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{L}F(X,\xi) = (1/2) \sum_{z \neq X-1, X} & [F(X,\xi^{z,z+1}) - F(X,\xi)] \\ + \ & p(1-\xi(X+1))[F(X+1,\xi) - F(X,\xi)] \\ + \ & q(1-\xi(X-1))[F(X-1,\xi) - F(X,\xi)] \ , \end{aligned} \tag{1.1}$$

where $\xi^{z,z+1}$ is the configuration obtained from ξ, exchanging the occupation variables $\xi(z)$, $\xi(z+1)$.

To fix ideas set $p > 1/2$. Denote by $\mathbb{Z}_*$ the set of integers distinct from 0. For $0 \le \alpha \le 1$, denote by μ_α the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}_*}$ with density α:

$$\mu_\alpha\{\xi \ : \xi(x) = 1\} = \alpha \ ,$$

for every x in $\mathbb{Z}_*$. More generally, for a positive integer N and a profile $\kappa_0: \mathbb{R} \to [0,1]$, denote by $\mu^N_{\kappa_0(\cdot)}$ the Bernoulli product measure associated to κ_0:

$$\mu^N_{\kappa_0(\cdot)}\{\xi \ : \xi(x) = 1\} = \kappa_0(x/N)$$

for x in $\mathbb{Z}_*$ and by $P_{\mu^N_{\kappa_0(\cdot)}}$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{Z} \times \{0,1\}^{\mathbb{Z}})$ induced by the Markov process with generator $\mathcal{L}$ defined in (1.1) and the initial measure $\delta_0 \times \mu^N_{\kappa_0(\cdot)}$.

Before stating the theorem, we introduce some notation required to define the limit v_t. Fix a strictly positive profile κ_0. Denote by $\mathcal{H}: \mathbb{R} \to \mathbb{R}$, $\mathcal{F}: \mathbb{R} \to \mathbb{R}$ the functions defined by

$$\mathcal{H}(A) = \int_0^A \kappa_0(u)\, du \ , \quad \mathcal{F}(B) = \frac{1}{\kappa_0(\mathcal{H}^{-1}(B))} \ - \ 1. \tag{1.2}$$

Here $\mathcal{H}^{-1}$ stands for the inverse of the strictly increasing, absolutely continuous function $\mathcal{H}$.

Consider the non-linear parabolic equation with boundary condition on $\mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{cases} \partial_t \rho = (1/2)\Delta\Phi(\rho) \\ \rho(t,0) = 0 \\ \rho(0,\cdot) = \mathcal{F}_+(\cdot) \ , \end{cases} \tag{1.3}$$

where $\mathcal{F}_+$ stands for the restriction of $\mathcal{F}$ on $\mathbb{R}_+$ and $\Phi(\rho) = \rho/(1+\rho)$; and the nonlinear parabolic equation on $\mathbb{R}_+ \times \mathbb{R}$ with boundary condition at the origin

$$\begin{cases} \partial_t \rho = (1/2)\Delta\Phi(\rho) \\ p\Phi(\rho(t,0+)) = q\Phi(\rho(t,0-)) \\ \partial_u\Phi(\rho(t,0+)) = \partial_u\Phi(\rho(t,0-)) \\ \rho(0,\cdot) = \mathcal{F}(\cdot). \end{cases} \tag{1.4}$$

A precise definition of solutions of these differential equations is given in Sects. 3 and 4. In Sect. 6 we show that this equation can be transformed in a linear Stefan problem by a Lagrangian coordinate transformation. As consequence the original exclusion process with an asymmetric particle has a hydrodynamic behavior described by the solution of a Stefan problem.

Theorem 1.1. *Assume $p = 1$. Fix a profile $\kappa_0\colon \mathbb{R}_+ \to [0,1]$ such that $\sigma \le \kappa_0 \le 1-\sigma$ for some $\sigma > 0$. Then, for every $\delta > 0$,*

$$\lim_{N\to\infty} P_{\mu^N_{\kappa_0(\cdot)}}\Big[\Big|\frac{X_{tN^2}}{N} - v_t\Big| > \delta\Big] = 0\ , \tag{1.5}$$

where

$$v_t = \int_0^\infty \Big\{\mathcal{F}(u) - \rho(t,u)\Big\} du \tag{1.6}$$

and ρ is the solution of equation (1.3).

Theorem 1.2. *Assume $p < 1$. For $\alpha < 1$ define $\psi_\alpha(u) = \alpha\mathbf{1}\{u<0\} + (q\alpha/p)\mathbf{1}\{u>0\}$. Fix a profile $\kappa_0\colon \mathbb{R} \to [0,1]$ such that $\psi_\alpha \le \kappa_0 \le 1-\sigma$ for some $\sigma > 0$, $0 < \alpha < 1$. Then, for every $\delta > 0$, (1.5) holds provided v_t is given by (1.6) and ρ is the solution of Eq. (1.4).*

The integral defining v_t in (1.6) must be understood in the following sense: consider the sequence $\{H_n,\ n \ge 1\}$ of real functions defined by

$$H_n(u) = (1 - un^{-1})^+ . \tag{1.7}$$

It follows from the equation satisfied by ρ that $\int_0^{+\infty} H_n(u)\{\mathcal{F}(u) - \rho(t,u)\}du$ converges as $n \uparrow \infty$. This limit defines the right-hand side of (1.6).

In the case where the initial state is a Bernoulli product measure μ_α with a fixed density α, we can make more explicit computations:

Theorem 1.3. *If the initial state is μ_α, then*

$$\lim_{p-q\to 0} \frac{v_t}{p-q} = \frac{(1-\alpha)}{\alpha}\sqrt{\frac{2t}{\pi}} .$$

Theorems 1.1 and 1.2 are proven in Sect. 5. Theorem 1.3 and more asymptotic results are proven in Sect. 6.

We now explain why in Theorems 1.1 and 1.2 the asymmetric tagged particle moves at scale $\sqrt{t}$ and why the displacement is related to the solution of the differential equations (1.3), (1.4).

We start labeling all particles. The tagged asymmetric particle is labeled 0. For $j \ge 1$, we label the j^{th} particle at the right (left) of the tagged particle by j $(-j)$. For x in $\mathbb{Z}$, denote by $\eta(x)$ the number of holes between particle x and particle $x+1$. In this way we transform a configuration of $\{0,1\}^{\mathbb{Z}}$ with a particle at some site X into a configuration $\{\eta(x),\ x \in \mathbb{Z}\} \in \mathbb{N}^{\mathbb{Z}}$. Denote by $\mathcal{T}\colon \mathbb{Z}\times\{0,1\}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ the transformation just described. $\mathcal{T}$ induces a transformation on the space of functions (resp. probability measures) of $\mathbb{Z} \times \{0,1\}^{\mathbb{Z}}$ to the space of functions (resp. probability measures) of $\mathbb{N}^{\mathbb{Z}}$ still denoted by $\mathcal{T}$.

The dynamics of the process (X_t, ξ_t) induces a dynamics for η_t that can be informally described as follows. For every $x \neq -1$, if there is at least one particle at site x, at rate $1/2$ one of them jumps to site $x+1$ and, symmetrically, if there is at least one particle at site $x+1$, at rate $1/2$ one of them jumps to site x. The picture is slightly different between sites -1 and 0 due to the behavior of the asymmetric tagged particle. A particle jumps at rate q from site -1 to site 0 if there is a particle at -1 and a particle jumps at rate p from site 0 to site -1 if there is a particle at the origin.

This process is the so-called zero range process, with an asymmetry at the origin. The position at time t of the asymmetric tagged particle corresponds in the zero range model to the total number of jumps between 0 and t from 0 to -1 minus the total number of jumps in the same interval from -1 to 0:

$$X_t = \sum_{x \geq 0} \{\eta_0(x) - \eta_t(x)\}. \tag{1.8}$$

The right-hand side is to be understood in the same sense as the righ-hand side of (1.6) by the use of the functions (1.7) (with the limit in the L^2 sense) (cf. [RV]).

Since in the zero range process the jumps of particles over all bonds, except the bond $\{-1, 0\}$, are symmetric, we expect the process to have a diffusive hydrodynamic behavior, i.e., that for a large class of initial profiles, the process accelerated by N^2 is such that for all continuous functions with compact support G,

$$N^{-1} \sum_{x} G(x/N) \eta_{tN^2}(x) \tag{1.9}$$

converges in probability to $\int_{\mathbb{R}} G(u)\rho(t,u)du$, where ρ is the solution of a nonlinear heat equation. In particular, approximating $\mathbf{1}\{u > 0\}$ by the sequence defined in (1.7), it follows from (1.8) and (1.9) that

$$\frac{X_{tN^2}}{N} = N^{-1} \sum_{x \geq 0} \{\eta_0(x) - \eta_{tN^2}(x)\}$$

converges in probability to v_t given by (1.6).

2. The Case p=1

In the case where the asymmetric tagged particle jumps only to the right, the evolution of the medium on its left is irrelevant for its motion. For the corresponding zero range dynamics, $p = 1$ means that at rate 1 a particle at the origin jumps to -1 and no particle jumps from -1 to 0. We may therefore assume that there is at -1 an infinite reservoir or an absorption point to which particles from the origin jump at rate 1 and from which no particle jumps. Moreover, the position of the tagged particle at time t corresponds in the zero range process to the number of particles that left the system before time t.

Consider the zero–range process on $\mathbb{N}$ whose generator acts on cylinder functions as

$$L = L_b + \sum_{x \geq 0} \Big\{ L_{x,x+1} + L_{x+1,x} \Big\}, \tag{2.1}$$

where

$$L_{x,y} f(\eta) = (1/2) g(\eta(x))[f(\sigma^{x,y}\eta) - f(\eta)]$$

and

$$L_b f(\eta) = g(\eta(0))[f(\eta - \mathfrak{d}_0) - f(\eta)].$$

Here, for a site x, $\mathfrak{d}_x$ stands for the configuration with no particles but one at x, summation is performed site by site, $\sigma^{x,y}\eta$ is the configuration η with one particle less at x and one more at y: $\sigma^{x,y}\eta = \eta - \mathfrak{d}_x + \mathfrak{d}_y$ and $g(k) = \mathbf{1}\{k \geq 1\}$.

For $\alpha > 0$, denote by ν_α^+ the product measure on $\mathbb{N}^{\mathbb{N}}$ with marginals given by

$$\nu_\alpha^+\{\eta, \eta(x) = k\} = \frac{1}{1+\alpha}\Big(\frac{\alpha}{1+\alpha}\Big)^k. \tag{2.2}$$

It is easy to check that these measures are reversible with respect to the generators $L_{x,x+1} + L_{x+1,x}$ defined above and that $E_{\nu_\alpha^+}[\eta(x)] = \alpha$.

We need to introduce some terminology on weak solutions of non linear parabolic equations. Fix a bounded initial profile $\rho_0 : \mathbb{R}_+ \to \mathbb{R}$. A bounded function $\rho : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is said to be a weak solution of the partial differential equation (1.3) with initial condition ρ_0 in the layer $[0,T] \times \mathbb{R}_+$ if

(a) $\Phi(\rho(t,u))$ is absolutely continuous in the space variable and

$$\int_0^T ds \int_{\mathbb{R}_+} du\, e^{-u} \{\partial_u \Phi(\rho(s,u))\}^2 \ < \ \infty \ ,$$

(b) $\rho(t,0) = 0$ for almost every $0 \leq t \leq T$, and

(c) For every smooth function with compact support $G : \mathbb{R}_+ \to \mathbb{R}$ vanishing at the origin and for every $0 \leq t \leq T$,

$$\int du\, \rho(t,u) G(u) \ - \ \int du\, \rho_0(u) G(u) = -\,(1/2) \int_0^t ds \int_{\mathbb{R}_+} du\, G'(u) \partial_u \Phi(\rho(s,u)).$$

Uniqueness of weak solutions of (1.3) can be proved with similar methods to the ones presented in [ELS], we outline the argument in the appendix. The existence for special initial conditions ρ_0 follows from the tightness of the sequence $\mathbb{Q}_{\mu^N}$ defined below in Theorem 2.2.

We now describe the initial states considered in this section. Fix a sequence of probability measures $\{\mu^N,\ N \geq 1\}$ on $\mathbb{N}^{\mathbb{N}}$.We assume that

(H1) The sequence μ^N is bounded above (resp. below) by ν_α^+ (resp. ν_λ^+) for some $0 < \lambda < \alpha$.

(H2) There exists a bounded function $\rho_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each continuous function $G : \mathbb{R}_+ \to \mathbb{R}$ with compact support and each $\delta > 0$,

$$\lim_{N\to\infty} \mu^N \Big[\Big| N^{-1} \sum_x G(x/N)\eta(x) - \int du\, G(u)\rho_0(u) \Big| \geq \delta \Big] = 0.$$

The first assumption is needed in order to prove the two block estimates for zero range processes with bounded jump rate (cf. [KL]). The second one just imposes a hydrodynamic behavior at time 0.

For each probability measure μ on $\mathbb{N}^{\mathbb{N}}$, denote by $\mathbb{P}_\mu^N$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{N}^{\mathbb{N}})$ induced by the Markov process with generator (2.1) accelerated by N^2 and the initial measure μ. Expectation with respect to $\mathbb{P}_\mu^N$ is denoted by $\mathbb{E}_\mu^N$.

Theorem 2.1. *Fix a sequence of initial measures satisfying assumptions (H1) and (H2). For any continuous function $G: \mathbb{R}_+ \to \mathbb{R}$ with compact support and any $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu^N}\left[\left|N^{-1}\sum_x G(x/N)\eta_t(x) - \int du\, G(u)\rho(t,u)\right| \geq \delta\right] = 0,$$

where ρ is the unique solution of (1.3).

For each positive integer N and each configuration η, define the empirical distribution $\pi^N = \pi^N(\eta)$ as the positive Radon measure on $\mathbb{R}_+$ obtained by assigning a mass N^{-1} to each particle: $\pi^N = N^{-1}\sum_{z\geq 0}\eta(z)\delta_{z/N}$ and set $\pi^N_t = \pi^N(\eta_t)$. Fix $T > 0$. Theorem 2.1 follows from the convergence in distribution of the process $\{\pi^N_t,\ 0 \leq t \leq T\}$, stated below in Theorem 2.2, and some standard topology arguments (cf. Chap. IV of [KL]). To state the convergence in distribution of the empirical measure we need some notation. Denote by $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{R}_+)$ the space of positive Radon measures on $\mathbb{R}_+$ endowed with the vague topology, a metrizable topology. For each probability measure μ on $\mathbb{N}^{\mathbb{N}}$, denote by $\mathbb{Q}^N_\mu$ the probability measure on the path space $D([0,T],\mathcal{M}_+)$ induced by $\mathbb{P}^N_\mu$ and the empirical measure π^N.

Theorem 2.2. *The sequence $\mathbb{Q}^N_{\mu^N}$ converges to the probability measure concentrated on the absolutely continuous path $\pi(t,du) = \rho(t,u)du$ whose density is the solution of (1.3).*

Guo, Papanicolaou and Varadhan introduced in [GPV] a method, well known by now, to prove Theorem 2.2 provided one has a bound on the *entropy* and on the *Dirichlet form* of the system with respect to some invariant measure. These bounds are usually obtained computing the time derivative of the entropy of the distribution of particles at time t relative to the equilibrium distribution. In the present context, however, there is only one invariant measure: the trivial one $\delta_{\underline{0}}$ concentrated on the configuration $\underline{0}$ with no particles. Since all other probability measures on $\mathbb{N}^{\mathbb{N}}$ are orthogonal with respect to this one, the entropy of any reasonable measure with respect to $\delta_{\underline{0}}$ is infinite and the entropy method does not apply straightforwardly. To overcome this problem, we compute the relative entropy with respect to an inhomogeneous product measure that is not invariant but *close* to the invariant measure.

To obtain an estimate on the entropy and on the Dirichlet form, we first assume that there exists a parameter $\beta > 0$ for which the relative entropy $H(\mu^N|\nu^+_\beta)$ is bounded by $C_0 N$ for some finite constant C_0. Coupling arguments permit to remove this assumption. This is explained at the end of this section.

To deduce an estimate on the entropy of the system, we need to introduce a class of inhomogeneous product measures. For $x \geq 0$, define γ_x by $\gamma_x = \beta(1+x)/N$ for $0 \leq x \leq N-1$ and $\gamma_x = \beta$ for $x \geq N$. Denote by $\nu^N_{\gamma(\cdot)}$ the product measure on $\mathbb{N}^{\mathbb{N}}$ with marginals given by

$$\nu^N_{\gamma(\cdot)}\{\eta, \eta(x) = k\} = (1-\gamma_x)\gamma_x^k \tag{2.3}$$

for all $x \geq 0$ and $k \geq 0$.

A simple computation relying on the entropy inequality shows that the entropy of μ^N with respect to $\nu^N_{\gamma(\cdot)}$ is bounded by $C_1 N$ for some finite constant C_1 depending only on C_0, α and β: $H(\mu^N\,|\,\nu^N_{\gamma(\cdot)}) \leq C_1 N$ (cf. Remark V.1.2 in [KL]).

For each probability density f with respect to $\nu^N_{\gamma(\cdot)}$, define the Dirichlet form $D_\gamma(f)$ by

$$D_\gamma(f) = D_{\gamma,b}(f) \ + \ D_{\gamma,i}(f) = D_{\gamma,b}(f) \ + \ \sum_{x\geq 0} D_{x,x+1}(f) \ ,$$

$$\text{where} \quad D_{\gamma,b}(f) = (1/2)\int g(\eta(0))\left[\sqrt{f(\eta - \mathfrak{d}_0)} - \sqrt{f(\eta)}\right]^2 \ d\nu^N_{\gamma(\cdot)} \ , \tag{2.4}$$

$$D_{x,x+1}(f) = (1/2)\int g(\eta(x))\left[\sqrt{f(\eta + \mathfrak{d}_{x+1} - \mathfrak{d}_x)} - \sqrt{f(\eta)}\right]^2 \ d\nu^N_{\gamma(\cdot)}.$$

Proposition 2.3. *Let S^N_t be the semigroup associated to the generator L introduced in (2.1) accelerated by N^2. Denote by $f_t = f^N_t$ the Radon–Nikodym derivative of $\mu^N S^N_t$ with respect to $\nu^N_{\gamma(\cdot)}$. There exists a finite constant $C = C(\beta)$ such that*

$$\partial_t H(\mu^N S^N_t | \nu^N_{\gamma(\cdot)}) \ \leq \ -N^2 D_\gamma(f_t) \ + \ CN.$$

Proof. Denote by L^*_γ the adjoint operator of L with respect to $\nu^N_{\gamma(\cdot)}$. It is easy to check that f_t is the solution of the forward equation

$$\begin{cases} \partial_t f_t = N^2 L^*_\gamma f_t \\ f_0 = (d\mu^N)/(d\nu^N_{\gamma(\cdot)}). \end{cases} \tag{2.5}$$

Then by explicit calculation

$$\partial_t H(\mu^N S^N_t | \nu^N_{\gamma(\cdot)}) = \int N^2 L^*_\gamma f_t \ \log f_t \, d\nu^N_{\gamma(\cdot)} \ + \ \int N^2 L^*_\gamma f_t \, d\nu^N_{\gamma(\cdot)}$$

$$= \ \int f_t \, N^2 L \log f_t \, d\nu^N_{\gamma(\cdot)} = N^2 \int f_t (L \log f_t - \frac{L f_t}{f_t}) \, d\nu^N_{\gamma(\cdot)} \ + \ N^2 \int L f_t \, d\nu^N_{\gamma(\cdot)}. \tag{2.6}$$

Notice that the last term would vanish if $\nu^N_{\gamma(\cdot)}$ were an invariant measure.

Since for every a, $b > 0$, $a \log(b/a) - (b - a)$ is less than or equal to $-(\sqrt{b} - \sqrt{a})^2$, for every x, $y \geq 0$, we have that

$$f_t \, L_{x,y} \log f_t \ - \ L_{x,y} f_t \ \leq \ -(1/2) g(\eta(x)) \left[\sqrt{f_t(\eta + \mathfrak{d}_y - \mathfrak{d}_x)} - \sqrt{f_t(\eta)}\right]^2$$

$$f_t \, L_b \log f_t \ - \ L_b f_t \ \leq \ -g(\eta(0)) \left[\sqrt{f_t(\eta - \mathfrak{d}_0)} - \sqrt{f_t(\eta)}\right]^2 .$$

Recall the definition of the Dirichlet form $D_\gamma(\cdot)$ introduced in (2.4). The previous estimate shows that the first term on the rightmost expression of (2.6) is bounded above by $-2N^2 D_\gamma(f_t)$.

To estimate the term $N^2 \int L f_t d\nu^N_{\gamma(\cdot)}$, which corresponds to the price we are paying for not using an invariant distribution as a reference measure, let us write it explicitly:

$$N^2 \int L f_t \ d\nu^N_{\gamma(\cdot)} = N^2 \sum_{x \geq 0} \int \left(L_{x,x+1} f_t + L_{x+1,x} f_t\right) \ d\nu^N_{\gamma(\cdot)} + N^2 \int L_b f_t \ d\nu^N_{\gamma(\cdot)}. \tag{2.7}$$

Performing the change of variables $\xi = \eta - \mathfrak{d}_x + \mathfrak{d}_y$, the measures change as $d\nu^N_{\gamma(\cdot)}(\eta)/d\nu^N_{\gamma(\cdot)}(\xi) = \gamma_x g(\xi(y))/\gamma_y g(\eta(x))$. In particular, we have that

$$\int \big(L_{x,x+1}f_t + L_{x+1,x}f_t\big)\; d\nu^N_{\gamma(\cdot)}$$
$$= (1/2)\Big(\frac{\gamma_x}{\gamma_{x+1}} - 1\Big) \int g(\eta(x+1)) f_t(\eta)\, d\nu^N_{\gamma(\cdot)}$$
$$+\; (1/2)\Big(\frac{\gamma_{x+1}}{\gamma_x} - 1\Big) \int g(\eta(x)) f_t(\eta)\, d\nu^N_{\gamma(\cdot)}.$$

We may thus rewrite the right-hand side of (2.7) as

$$\begin{aligned} &(1/2)\sum_{x\geq 1} \frac{(\Delta_N\gamma)(x)}{\gamma_x} \int g(\eta(x)) f_t(\eta)\, d\nu^N_{\gamma(\cdot)} \\ &\quad +\; (N^2/2)\Big(\frac{\gamma_1}{\gamma_0} - 1\Big) \int g(\eta(0)) f_t(\eta)\, d\nu^N_{\gamma(\cdot)} \\ &\quad +\; N^2 \int g(\eta(0))[f_t(\eta - \mathfrak{d}_0) - f_t(\eta)]\, d\nu^N_{\gamma(\cdot)}. \end{aligned} \tag{2.8}$$

In this formula, $(\Delta_N\gamma)(x)$ stands for $N^2\{\gamma_{x+1} + \gamma_{x-1} - 2\gamma_x\}$. By definition of γ, $(\Delta_N\gamma)(x) = 0$ for all x except at $x = N-1$, where $(\Delta_N\gamma)(N-1) = N^2(\gamma_{N-2} - \gamma_{N-1})$, which is negative because γ is non decreasing. The first line of (2.8) is therefore negative. A change of variables $\xi = \eta - \mathfrak{d}_0$ permits to write the second term of the second line as

$$N^2 \int [\gamma_0 - g(\eta(0))] f_t(\eta)\, d\nu^N_{\gamma(\cdot)}.$$

The second line of (2.8) is therefore equal to

$$\beta N \;+\; (1/2)N^2\Big(\frac{\gamma_1}{\gamma_0} - 3\Big) \int g(\eta(0)) f_t(\eta)\, d\nu^N_{\gamma(\cdot)} \;\leq\; \beta N$$

because $\gamma_0 = \beta/N$, f is a density and $\gamma_1/\gamma_0 = 2$. This concludes the proof of the proposition. □

With the previous estimate on the entropy and on the Dirichlet form, we are in a position to apply the classical entropy method to prove the hydrodynamic behavior of the system (cf. Chapter V in [KL]). We just point out here the main difference coming from the absorption point at the origin.

Lemma 2.4. *For every* $0 \leq t \leq T$,

$$\lim_{N\to\infty} \mathbb{E}_{\mu^N}\Big[\int_0^t g(\eta_s(0)) ds\Big] = 0.$$

Proof. Recall that we denote by f_t the Radon–Nikodym derivative of $\mu^N S^N_t$ with respect to $\nu^N_{\gamma(\cdot)}$. Set $\overline{f_t} = t^{-1}\int_0^t f_s ds$. With this notation, the expectation in the statement writes

$$t \int \overline{f_t}(\eta) g(\eta(0)) d\nu^N_{\gamma(\cdot)}.$$

Adding and subtracting $\overline{f_t}(\eta - \mathfrak{d}_0)$ and changing variables, we obtain that this integral is equal to

$$t\int g(\eta(0))[\bar{f}_t(\eta) - \bar{f}_t(\eta - \mathfrak{d}_0)]d\nu^N_{\gamma(\cdot)} \;+\; t\gamma_0 .$$

The second term vanishes as $N \uparrow \infty$ because $\gamma_0 = \beta/N$. The first one, by Schwarz inequality and a change of variables, is bounded above by

$$\frac{t}{A}\{\|g\|_\infty + \gamma_0\} \;+\; tAD_{\gamma,b}(\bar{f}_t)$$

for every $A > 0$. Choosing $A = \sqrt{N}$, we conclude the proof of the lemma by virtue of Proposition 2.3 and the convexity of the Dirichlet form. □

Lemma 2.5. *For every* $0 \le t \le T$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\int_0^t ds\;\Phi(2\eta_s^{N\epsilon}(0))\Big] = 0.$$

Notice that in this last expression we multiply $\eta_t^{N\epsilon}(0)$ by 2 to obtain the density of particles on the box $[0, \epsilon N]$. The proof of Lemma 2.5 is performed in three steps. We first show that we may replace the cylinder function $g(\eta(0))$ by an average over a small macroscopic box around the origin. We then replace this average by $\Phi(2\eta^{\epsilon N}(0))$ and recall Lemma 2.4 to conclude.

Lemma 2.6. *For each* $0 \le t \le T$ *and smooth* $G: \mathbb{R}_+ \to \mathbb{R}$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu^N}\Big[\int_0^t ds\, G(s)\Big\{g(\eta_s(0)) - (N\epsilon)^{-1}\sum_{y=0}^{N\epsilon} g(\eta_s(y))\Big\}\Big] = 0.$$

Proof. Denote by $V(\eta_s)$ the expression inside braces in the previous formula:

$$V(\eta) = g(\eta(0)) - (\epsilon N)^{-1}\sum_{y=0}^{N\epsilon} g(\eta(y)). \tag{2.9}$$

Since $\bar{f}_t = t^{-1}\int_0^t f_s ds$, we may rewrite the expectation in the statement of the lemma as

$$t\int V(\eta)\bar{f}_t(\eta)\nu^N_{\gamma(\cdot)}(d\eta).$$

A change of variables $\xi = \eta - \mathfrak{d}_x$ gives that $\int V(\eta)\bar{f}_t(\eta)\nu^N_{\gamma(\cdot)}(d\eta)$ is equal to

$$(N\epsilon)^{-1}\sum_{x=0}^{N\epsilon}\sum_{y=0}^{x-1}\Big\{\gamma_y\int\Big\{\bar{f}_t(\eta + \mathfrak{d}_y) - \bar{f}_t(\eta + \mathfrak{d}_{y+1})\Big\}\nu^N_{\gamma(\cdot)}(d\eta)$$
$$+\;[\gamma_y - \gamma_{y+1}]\int \bar{f}_t(\eta + \mathfrak{d}_{y+1})\nu^N_{\gamma(\cdot)}(d\eta)\Big\}.$$

Since γ_x is increasing in x, the second term is negative.

On the other hand, rewriting the difference $\{a - b\} = \{\bar{f}_t(\eta + \mathfrak{d}_y) - \bar{f}_t(\eta + \mathfrak{d}_{y+1})\}$ as $\{\sqrt{a} - \sqrt{b}\}\{\sqrt{a} + \sqrt{b}\}$ and applying the Schwarz inequality, we bound the first term by

$$\frac{A}{2\epsilon N}\sum_{x=0}^{N\epsilon}\sum_{y=0}^{x-1}\gamma_y\int\Big\{\sqrt{\bar{f}_t(\eta+\mathfrak{d}_y)}-\sqrt{\bar{f}_t(\eta+\mathfrak{d}_{y+1})}\Big\}^2\nu^N_{\gamma(\cdot)}(d\eta)$$
$$+\ \frac{1}{A\epsilon N}\sum_{x=0}^{N\epsilon}\sum_{y=0}^{x-1}\gamma_y\int\Big\{\bar{f}_t(\eta+\mathfrak{d}_y)+\bar{f}_t(\eta+\mathfrak{d}_{y+1})\Big\}\nu^N_{\gamma(\cdot)}(d\eta)$$

for every $A > 0$. Changing variables back, keeping in mind that γ_x is a non decreasing function and inverting the order of summation, we show that this expression is bounded above by

$$\frac{A}{2}\sum_{y=0}^{N\epsilon-1}D_{y,y+1}(\bar{f}_t)\ +\ \frac{2\|g\|_\infty}{A}\epsilon N$$

for every positive A. Recalling Proposition 2.3 and taking $A = \sqrt{\epsilon}N$, we conclude the proof of the lemma. □

In view of Lemma 2.4, to conclude the proof of Lemma 2.5, it remains to replace the average of the cylinder function $g(\eta(x))$ by $\Phi(2\eta^{\epsilon N}(0))$ but this is the classical two blocks estimate. This concludes the proof of Theorem 2.2 under the assumption that the entropy of the initial state with respect to the product measure ν^+_β is bounded above by $C_0 N$ for some finite constant C_0. A coupling argument permits to remove Assumption (H2').

Consider a sequence μ^N satisfying assumptions (H1) and (H2). Fix $A > 0$ and let $\mu^{N,A}$ be the probability measure on $\mathbb{N}^{\mathbb{N}}$ defined by

$$\mu^{N,A} = \mu^N\Big|_{\Lambda_{AN}}\otimes\nu^+_\beta\Big|_{\Lambda^c_{AN}},$$

where $\Lambda_{AN} = \{0,\dots,AN\}$ and ν_Λ is the marginal of the probability measure ν on Λ.

Since $\nu^+_\lambda \le \mu^N \le \nu^+_\alpha$ and since all cylinder functions can be decomposed as the difference of two monotone functions (cf. [KL]), a simple computation and the explicit formula for the relative entropy give that

$$H(\mu^{N,A}|\nu^+_\beta)\ \le\ \frac{1}{2}\Big\{H(\nu^{+,AN}_\alpha|\nu^{+,AN}_\beta)\ +\ H(\nu^{+,AN}_\lambda|\nu^{+,AN}_\beta)\Big\}\ ,$$

where $\nu^{+,m}_\gamma$ is the marginal of ν^+_γ on $\{0,\dots,m\}$. In particular, the entropy $H(\mu^{N,A}|\nu^+_\beta)$ is bounded above by $C_0 N$ for some finite constant C_0 depending only on A, α and λ.

Let $\rho^A(t,u)$ denote the solution of (1.3) with initial condition $\rho^A_0(u) = \rho_0(u)\mathbf{1}\{u \le A\} + \beta\mathbf{1}\{u > A\}$. Investigating the time evolution of the integral $\int_{\mathbb{R}_+} du e^{-u}\rho^A(t,u)^2$ we obtain uniform in A a priori estimates that show that ρ^A converges to the unique solution of (1.3) with initial condition ρ_0.

Since the jump rate g is non decreasing, we may couple a zero range starting from μ^N with another one starting from $\mu^{N,A}$ and show that as $A \uparrow \infty$ both behave exactly in the same way on compact sets. This coupling, the hydrodynamic behavior of the empirical measure for a process starting from $\mu^{N,A}$ and the convergence of ρ^A to ρ, permit to extend Theorem 2.2 to the sequence of measures satisfying assumptions (H1) and (H2).

3. The Case $p < 1$

We turn in this section to the case where the asymmetric tagged particle jumps at rate p to the right and at rate q to the left. The corresponding zero range process has jumps at rate (1/2) over all bonds but $\{-1, 0\}$. From the origin, particles jump at rate p to -1 and from -1 particles jump at rate q to 0. Recall that to fix ideas we assumed $p > q$.

The purpose of this section is to deduce the hydrodynamic behavior of the just described space inhomogeneous process. Consider the zero–range process on $\mathbb{Z}$ with generator given by

$$L = \sum_{x \neq -1} \{L_{x,x+1} + L_{x+1,x}\} \; + \; 2pL_{0,-1} \; + \; 2qL_{-1,0} \;, \tag{3.1}$$

where $L_{x,y}$ is the generator defined just after (2.1). In contrast with the previous section, this system possesses a one parameter family of invariant measures. For each $\varphi < p^{-1}$, denote by $\overline{\nu}^i_\varphi$ the product measure on $\mathbb{N}^{\mathbb{Z}}$ with marginals given by

$$\overline{\nu}^i_\varphi\{\eta, \eta(x) = k\} = \frac{1}{Z(\varphi_x)} \frac{\varphi_x^k}{g(k)!} \;, \tag{3.2}$$

where $\varphi_x = p\varphi$ for $x \leq -1$ and $\varphi_x = q\varphi$ for $x \geq 0$. A direct computation shows that the Markov process with generator given by (3.1) is reversible with respect to these product measures.

Before stating the main result of this section, we introduce some terminology on weak solutions of non-linear parabolic equations. Fix a bounded function $\rho_0 \colon \mathbb{R} \to \mathbb{R}$. A bounded function $\rho \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is said to be a weak solution of the partial differential equation (1.4) with initial condition ρ_0 if

(a) $\Phi(\rho(t,u))$ is absolutely continuous in the space variable and for every $t > 0$,

$$\int_0^t ds \int_{\mathbb{R}} du \, e^{-|u|} \{\partial_u \Phi(\rho(s,u))\}^2 \; < \; \infty \;,$$

(b) $p\Phi(\rho(t, 0+)) = q\Phi(\rho(t, 0-))$ for almost every $t \geq 0$ and

(c) For every smooth function with compact support $G \colon \mathbb{R} \to \mathbb{R}$ and for every $t > 0$,

$$\int_{\mathbb{R}} du \, \rho(t,u) G(u) \; - \; \int_{\mathbb{R}} du \, \rho_0(u) G(u) = - \int_0^t ds \int_{\mathbb{R}} du \, G'(u) \partial_u \Phi(\rho(s,u)).$$

Since $\rho(t,u)$ is only a measurable function, requirement (b) must be understood as

$$\lim_{\epsilon \to 0} \int_0^t h(s) \Big\{ p\Phi\Big(\frac{1}{\epsilon} \int_0^\epsilon \rho(s,u) du\Big) \; - \; q\Phi\Big(\frac{1}{\epsilon} \int_{-\epsilon}^0 \rho(s,u) du\Big) \Big\} \, ds \; = 0 \tag{3.3}$$

for every $t \geq 0$ and any continuous function $h(t)$. The third property in (1.4) just states that there is conservation of the total mass at the origin.

Uniqueness of weak solutions of (1.4) is proved with similar techniques to the ones presented in [ELS](cf. Appendix). The existence for special initial conditions ρ_0 follows from the tightness of the sequence $\mathbb{Q}^N_{\mu^N}$ defined below.

For each probability measure μ on $\mathbb{N}^{\mathbb{Z}}$, denote by $\mathbb{P}^N_\mu$ the probability measure on the path space $D(\mathbb{R}_+, \mathbb{N}^{\mathbb{Z}})$ induced by the Markov process with generator (3.1) accelerated by N^2 and the initial measure μ. Expectation with respect to $\mathbb{P}^N_\mu$ is denoted by $\mathbb{E}^N_\mu$.

We now define the initial states considered in the first main theorem of this section. Fix a sequence of initial measures μ^N on $\mathbb{N}^{\mathbb{Z}}$, we assume that

(IS1) The sequence μ^N is bounded above (resp. below) by some invariant state $\overline{\nu}^i_\alpha$ (resp. $\overline{\nu}^i_\lambda$) for some $0 < \lambda < \alpha$.

(IS2) There exists a function $\rho_0 \colon \mathbb{R} \to \mathbb{R}_+$ such that for each continuous function $G \colon \mathbb{R} \to \mathbb{R}_+$ and each $\delta > 0$,

$$\lim_{N\to\infty} \mu^N \Big[\Big| N^{-1} \sum_x G(x/N)\eta(x) - \int du\, G(u)\rho_0(u) \Big| \geq \delta \Big] = 0.$$

Notice that it follows from assumption (IS1) that the function ρ_0 in (IS2) is necessarily bounded.

Theorem 3.1. *Consider a sequence of initial states μ^N satisfying assumptions (IS1), (IS2). For any continuous function $G \colon \mathbb{R} \to \mathbb{R}$ with compact support and any $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\mu^N} \left[\Big| N^{-1} \sum_x G(x/N)\eta_t(x) - \int du\, G(u)\rho(t,u) \Big| \geq \delta \right] = 0,$$

where ρ is the unique solution of (1.4).

Like in Sect. 3 (cf. also Chap. IV of [KL]), we deduce this result from the convergence in distribution of the empirical measure $\pi^N = \pi^N(\eta)$ defined as the positive Radon measure on $\mathbb{R}$ obtained by assigning a mass N^{-1} to each particle: $\pi^N = N^{-1}\sum_{z\in\mathbb{Z}} \eta(z)\delta_{z/N}$. Set $\pi^N_t = \pi^N(\eta_t)$ and denote by $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{R})$ the space of positive Radon measures on $\mathbb{R}$ endowed with the vague topology, a metrizable topology. Fix $T > 0$. For each probability measure μ on $\mathbb{N}^{\mathbb{Z}}$, denote by $\mathbb{Q}^N_\mu$ the probability measure on the path space $D([0,T], \mathcal{M}_+)$ induced by $\mathbb{P}^N_\mu$ and the empirical measure π^N.

Theorem 3.2. *The sequence $\mathbb{Q}^N_{\mu^N}$ converges to the probability measure concentrated on the absolutely continuous path $\pi(t,du) = \rho(t,u)du$ whose density is the solution of (1.4).*

Coupling arguments similar to the ones presented at the end of the previous section show that it is enough to prove Theorem 3.2 under the assumption that there exist a density $\beta > 0$ and a finite constant C_0 such that the entropy of μ^N with respect to $\overline{\nu}^i_\beta$ is bounded by $C_0 N$: $H(\mu^N | \overline{\nu}^i_\beta) \leq C_0 N$ for every $N \geq 1$. We therefore assume until the end of this section the existence of such constants β and C_0.

The main difference in the proof of the hydrodynamic limit of this model and the classical proof for space homogeneous systems resides in the behavior at the boundary $u = 0$. The next four lemmas solve this question. For a site x, a configuration η and a positive integer ℓ, denote by $M^{\pm}_\ell(x,\eta)$ the density of particles for the configuration η on a box of size ℓ at the right (left) of x:

$$M^+_\ell(x,\eta) = \frac{1}{\ell+1}\sum_{y=x}^{x+\ell} \eta(y)\,, \quad M^-_\ell(x,\eta) = \frac{1}{\ell+1}\sum_{y=x-\ell}^{x} \eta(y).$$

Lemma 3.3. *For every continuous function $H: [0,T] \to \mathbb{R}$,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\Big\{g(\eta_t(0)) - \Phi(M^+_{\epsilon N}(0,\eta_t))\Big\}\Big|\Big] = 0.$$

The same result holds if $g(\eta_t(0))$ is replaced by $g(\eta_t(-1))$ and $M^+_{\epsilon N}(0,\eta_t)$ by $M^-_{\epsilon N}(-1,\eta_t)$.

This result follows from the next lemma and the two blocks estimate.

Lemma 3.4. *For every continuous function $H: [0,T] \to \mathbb{R}$,*

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\Big\{g(\eta_t(0)) - (\epsilon N)^{-1}\sum_{x=0}^{\epsilon N} g(\eta_t(x))\Big\}\Big|\Big] = 0.$$

The same result holds if $g(\eta_t(0))$ is replaced by $g(\eta_t(-1))$ and the average over $\{0,\dots,\epsilon N\}$ is replaced by the average over $\{-\epsilon N,\dots,0\}$.

Proof. Recall from (2.9) the definition of $V(\eta_t)$. By the entropy inequality,

$$\begin{aligned}&\mathbb{E}_{\mu^N}\Big[\Big|\int_0^T ds\, H(s)V(\eta_s)\Big|\Big]\\ &\quad\le \frac{H(\mu^N|\overline{\nu}^i_\beta)}{NA} + \frac{1}{AN}\log \mathbb{E}_{\overline{\nu}^i_\beta}\Big[\exp\Big\{\Big|\int_0^T ds\, G(s)ANV(\eta_s)\Big|\Big\}\Big]\end{aligned}$$

for every $A > 0$. By assumption, the first term on the right-hand side is bounded by CA^{-1}. To prove the lemma it is therefore enough to show that the limit of the second one is less than or equal to 0 for every $A > 0$. Since $e^{|x|} \le e^x + e^{-x}$ and $\limsup_N N^{-1}\log\{a_N + b_N\} \le \max\{\limsup_N N^{-1}\log a_N, \limsup_N N^{-1}\log b_N\}$, replacing H by $-H$ we deduce that we only need to prove the previous statement without the absolute value in the exponent. By the Feynman–Kac formula and the variational formula for the largest eigenvalue of an operator,

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{AN}\log \mathbb{E}_{\overline{\nu}^i_\beta}\Big[\exp\Big\{\int_0^T ds\, G(s)ANV(\eta_s)\Big\}\Big]\\ &\quad\le \int_0^T dt\,\sup_f\Big\{\int H(t)V(\eta)f(\eta)\overline{\nu}^i_\beta(d\eta) + A^{-1}ND(f)\Big\}.\end{aligned} \tag{3.4}$$

In this formula, the supremum is taken over all densities f with respect to $\overline{\nu}^i_\beta$ and $D(f)$ is the Dirichlet form

$$D(f) = \int \sqrt{f}L\sqrt{f}\, d\nu^i_\beta.$$

We are now ready to integrate by parts the cylinder function V. The rest of the proof is similar to the proof of Lemma 2.6 and omitted for this reason. □

The same argument permits to deduce the following result.

Lemma 3.5. *For every continuous function $H: [0,T] \to \mathbb{R}$,*

$$\limsup_{N\to\infty} \mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\, H(t)\{pg(\eta_t(0)) - qg(\eta_t(-1))\}\Big|\Big] = 0.$$

The next result follows from Lemma 3.3 and Lemma 3.5.

Corollary 3.6. *For every continuous function* $H\colon[0,T]\to\mathbb{R}$,

$$\limsup_{N\to\infty}\mathbb{E}^N_{\mu^N}\Big[\Big|\int_0^T dt\,H(t)\Big\{p\Phi(M^+_{\epsilon N}(0,\eta_t))-q\Phi(M^-_{\epsilon N}(-1,\eta_t))\Big\}\Big|\Big]=0.$$

These technical lemmas permit to adapt the classical proof of the hydrodynamic behavior of reversible systems to the present context. Details are left to the reader.

Remark 3.7. In Sects 2 and 3 only the monotonicity and the boundness of the jump rate $g(\cdot)$ were used. The same arguments permit therefore to deduce the hydrodynamic behavior of a more general class of processes.

4. The Asymmetric Tagged Particle

We prove in this section Theorems 1.1 and 1.2 through the hydrodynamic behavior of the inhomogeneous zero range processes considered in the previous two sections.

We have seen in the first section that the displacement of the asymmetric tagged particle corresponds in the zero range process to the total flux of particles through the origin. For this reason, we start deducing the total flux through the origin from the hydrodynamic limit proved in the previous two sections.

Proposition 4.1. *In the case* $p=1$*, consider a sequence of probability measures* μ^N *satisfying assumptions (H1) and (H2). Then, for every* $t\geq 0$ *and* $\delta>0$,

$$\lim_{N\to\infty}\mathbb{P}^N_{\mu^N}\Big[\Big|N^{-1}\sum_{x\geq 0}\{\eta_t(x)-\eta_0(x)\}-\int_0^\infty du\,\{\rho(t,u)-\rho_0(u)\}\Big|>\delta\Big]=0\ , \quad (4.1)$$

where ρ *is the solution of (1.3). In the case* $p<1$*, consider a sequence of probability measures* μ^N *satisfying assumptions (IS1), (IS2). Then, for every* $t\geq 0$ *and* $\delta>0$ *(4.1) holds, where* ρ *is now the solution of (1.4).*

Proposition 4.1 follows from the hydrodynamic behavior of the inhomogeneous processes considered in Sects. 3, 4 and from the definition of the infinite sums appearing in (4.1).

Theorem 1.1 and 1.2 follow from Proposition 4.1 if we prove the following proposition:

Proposition 4.2. *Fix a sequence of initial states* $\mu^N_{\rho_0(\cdot)}$ *satisfying the assumptions of Theorem 1.1 or 1.2. The sequence* $\mathcal{T}\mu^N_{\rho_0(\cdot)}$ *satisfy assumptions (H1), (H2) in the case* $p=1$ *or (IS1), (IS2) in the case* $p<1$*, where* $\mathcal{T}$ *is the transformation defined in Sect. 1.*

Proof. We start with the case $p=1$. A simple computation shows that $\mathcal{T}$ transforms the Bernoulli product measure μ_ρ in the product measure $\nu^+_{(1-\rho)/\rho}$ defined by (2.2). Fix a profile $\rho_0\colon\mathbb{R}_+\to[0,1]$ for which there exists $\sigma>0$ such that $\sigma\leq\rho_0\leq 1-\sigma$. Recall that we denote by $\mu^N_{\rho_0(\cdot)}$ the inhomogeneous product measure associated to ρ_0. Let $\mathcal{T}\mu^N_{\rho_0(\cdot)}=\nu^N_{\rho_0(\cdot)}$. We shall now show that $\nu^N_{\rho_0(\cdot)}$ fulfills assumptions (H1), (H2).

We first claim that if μ is a product measure on $\{0,1\}^{\mathbb{N}_*}$ bounded above (resp. below) by μ^+_ρ for some $0<\rho<1$, then $\mathcal{T}\mu$ is bounded below (resp. above) by $\nu^+_{(1-\rho)/\rho}$. Here

μ_ρ^+ stands for the restriction on $\mathbb{N}$ of the measures μ_ρ. Notice that the inequalities are reversed by the application $\mathcal{T}$. To fix ideas assume that $\mu \le \mu_\rho^+$. For $x \ge 1$, denote by γ_x the probability of finding a particle at x for the probability μ so that $\gamma_x \le \rho$. For $j \ge 1$, denote by N_j the position of the $j^{\rm th}$ particle at the right of the origin. Since $\gamma_x \le \rho$ for every x and μ, μ_ρ are product measures, it is possible to couple μ and μ_ρ in such a way that $N_1^\mu \ge N_1^\rho$ and $N_{j+1}^\mu - N_j^\mu \ge N_{j+1}^\rho - N_j^\rho$ for all $j \ge 1$. In this formula, N_j^μ (resp. N_j^ρ) stands for the position of the $j^{\rm th}$ particle under the distribution μ (resp. μ_ρ). Applying the transformation $\mathcal{T}$ to this coupling measure, we construct a measure on $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with first marginal equal to $\mathcal{T}\mu$, second marginal equal to $\mathcal{T}\mu_\rho^+ = \nu^+_{(1-\rho)/\rho}$ and concentrated on configurations (η^1, η^2) below the diagonal. This shows that $\mathcal{T}\mu \ge \nu^+_{(1-\rho)/\rho}$, what concludes the proof of the claim. In particular, $\nu^+_{\sigma/(1-\sigma)} \le \nu^N_{\rho_0(\cdot)} \le \nu^+_{(1-\sigma)/\sigma}$ for every $N \ge 1$ and assumption (H1) is verified.

Notice, however, that the claim "$\mu^1 \le \mu^2$ implies $\mathcal{T}\mu^1 \ge \mathcal{T}\mu^2$" is not correct. Consider, for instance, the configuration ξ^1, ξ^2 such that

$$\xi^1(x) = 1 \quad \text{if and only if } x \ne 1,2,3 \text{ and } \quad \xi^2(x) = 1 \quad \text{if and only if } x \ne 1,3.$$

In this case the deterministic measures δ_{ξ^i} are such that $\delta_{\xi^1} \le \delta_{\xi^2}$ but it is not correct that $\delta_{\mathcal{T}\xi^1}$ is above $\delta_{\mathcal{T}\xi^2}$.

We turn now to the second assumption (H2). It follows from (1.2) that

$$\int_0^B \mathcal{F}(u)\, du = \mathcal{H}^{-1}(B) - B \tag{4.2}$$

for every $B > 0$. In order to check (H2), we just need to show that under $\nu^N_{\rho_0(\cdot)}$, $N^{-1}\sum_{x=0}^{[BN]} \eta(x)$ converges in probability to $\int_0^B \mathcal{F}(u)du$ for every $B > 0$. Fix a positive integer n. The following inequalities state that for the exclusion process the total number of sites in $\Lambda_n = \{0, \dots, n\}$ is equal to the total number of particles plus the total number of holes (that corresponds to the total number of particles for the zero range process):

$$\sum_{x=0}^n \xi(x) + \sum_{y=0}^{-1+\sum_{x=0}^n \xi(x)} \eta(y) \le n+1 \le \sum_{x=0}^n \xi(x) + \sum_{y=0}^{\sum_{x=0}^n \xi(x)} \eta(y).$$

The convergence of $N^{-1}\sum_{x=0}^{[BN]} \eta(x)$ follows from these inequalities, the fact that under the measure $\mu^N_{\rho_0(\cdot)}$, $N^{-1}\sum_{0\le x\le [nN]} \xi(x)$ converges to $\int_0^n \rho_0(u)\, du$ and identity (4.2). Details are left to the reader.

In exactly the same way, assumptions (IS1), (IS2) can be checked in the case $p < 1$. The only difference is that we assume in (IS1) that the sequence of initial measures is bounded below by an invariant measure $\overline{\nu}^i_\varphi$ which is inhomogeneous in space. This forces the initial profile ρ_0 to be bounded below by the function $\psi_\alpha(u) = (1-\alpha)\mathbf{1}\{u < 0\} + [1-(q/p)\alpha]\mathbf{1}\{u > 0\}$ for some $0 < \alpha < 1$. $\square$

5. Einstein Relation

We consider in this section initial profiles for which the solution of equation (1.4) is selfscaling. For two fixed densities ρ_- and ρ_+ consider, for instance, the initial condition $\rho_0(\cdot)$ given by

$$\rho_0(u) = \rho_+ \mathbf{1}\{u \geq 0\} \; + \; \rho_- \mathbf{1}\{y < 0\}.$$

The solution of (1.4) takes the form $\rho(t,u) = \varphi(u/\sqrt{t})$, where $\varphi(\cdot)$ is the solution of

$$\begin{cases} -z\varphi'(z) = \partial_z^2 \Phi(\varphi(z)) \, , \\ \dfrac{\varphi'(0+)}{(1+\varphi(0+))^2} = \dfrac{\varphi'(0-)}{(1+\varphi(0-))^2} \qquad \varphi(\pm\infty) = \rho_\pm . \\ p\Phi(\varphi(0+)) = q\Phi(\varphi(0-)). \end{cases} \tag{5.1}$$

It easy to see that in this case $v_t = v\sqrt{t}$, where v is given by

$$v \;=\; \int_0^{+\infty} \{\rho_+ - \varphi(y)\} \; dy.$$

Moreover, since $\rho_+ = \varphi(\infty)$, we may write the expression inside braces as $\int_{[y,\infty)} \partial_z \varphi(z) dz$. Performing an integration by parts and keeping in mind that φ is the solution of (5.1), we obtain that

$$v \;=\; \frac{\varphi'(0+)}{(1+\varphi(0+))^2} \; .$$

We now transform (5.1) in a linear equation through the following Lagrangian change of coordinates:

$$x(z) = \int_0^z (1+\varphi(y)) \; dy \quad m(x) = \frac{1}{1+\varphi(z(x))} \; .$$

We leave to the reader to check that this transformation is in fact the inverse of the transformation $\mathcal{T}$ described in (1.2). Moreover, a simple computation shows that $m(x)$ is the solution of the linear equation

$$\begin{cases} m''(x) \; = \; -(x+v)m'(x) \, , \\ -v = \dfrac{m'(0+)}{m(0+)} = \dfrac{m'(0-)}{m(0-)} \; , \\ p(1-m(0+)) = q(1-m(0-)) \; , \\ m(\pm\infty) = \alpha_\pm = \dfrac{1}{1+\rho_\pm}. \end{cases} \tag{5.2}$$

In fact (5.2) describes the selfscaling solution of the Stefan problem:

$$\begin{cases} \partial_t m^*(x,t) = \dfrac{1}{2} \partial_{xx} m^*(x,t) \, , \\ -v_t = \dfrac{\partial_x m^*(v_t+,t)}{m^*(v_t+,t)} = \dfrac{\partial_x m^*(v_t-,t)}{m^*(v_t-,t)} \; , \\ p\{1-m^*(v_t+,t)\} = q\{1-m^*(v_t-,t)\} \, , \\ m^*(x,0) = \alpha_+ \mathbf{1}\{x \geq 0\} + \alpha_- \mathbf{1}\{x < 0\} \, . \end{cases} \tag{5.3}$$

In other words, $m(x/\sqrt{t})$ is the macroscopic profile of density as seen from the tagged asymmetric particle.

The solution of (5.2) can be written as

$$m(x) = \begin{cases} A_+ + B_+ \int_0^x e^{-(1/2)y^2 - vy} \, dy & \text{for } x > 0 \, , \\ A_- + B_- \int_0^x e^{-(1/2)y^2 - vy} \, dy & \text{for } x < 0 \, , \end{cases}$$

where the parameters are related by the equations

$$p(1 - A_+) = q(1 - A_-) \, ; \quad -v = \frac{B_+}{A_+} = \frac{B_-}{A_-} \, ; \quad \alpha_\pm = A_\pm J(\pm v),$$

where $J(v) = 1 - v \int_0^{+\infty} e^{-(1/2)y^2 - vy} \, dy$.

It follows from the previous identities that the parameters p, α_+, α_- and v satisfy the equation

$$p\left(1 - \frac{\alpha_+}{J(v)}\right) = q\left(1 - \frac{\alpha_-}{J(-v)}\right) . \tag{5.4}$$

This equation was obtained heuristically by [BDMO]. In particular, we cannot write v as an explicit function of p, α_+, α_-, but we can study some asymptotic relations. We consider three distinct asymptotics.

We first investigate the case of a constant initial profile: $\alpha_+ = \alpha_- = \alpha$. In this case elementary computations give the identity

$$(p - q)\frac{1 - \alpha}{\alpha} = \frac{pJ(-v) - qJ(v) - (p - q)J(v)J(-v)}{J(v)J(-v)} \, . \tag{5.5}$$

For small asymmetry $p - q$, we have a small displacement v. Replacing in (5.5) $J(v)$ by its expansion for v small gives, for fixed α and small $p - q$, that

$$v = (p - q)\sqrt{\frac{2}{\pi}}\frac{1 - \alpha}{\alpha} \; + \; o(p - q).$$

This proves the validity of Einstein relation for small drifts.

In the case $\alpha_+ \neq \alpha_-$ one can expand around the equilibrium, i.e., for small $p(1 - \alpha_+) - q(1 - \alpha_-)$. The same expansions show that

$$v = \frac{p(1 - \alpha_+) - q(1 - \alpha_-)}{p\alpha_+ - q\alpha_-}\sqrt{\frac{2}{\pi}} \; + \; o\left(p(1 - \alpha_+) - q(1 - \alpha_-)\right) .$$

A third possible asymptotics is given when the initial profile is constant and the density $\alpha = \alpha_+ = \alpha_-$ is small. In this case, for a fixed drift $p - q$, the displacement v is very large. Asymptotically, for $|v|$ close to ∞, a simple computation shows that $J(v) \sim v^{-2}$, $J(-v) \sim ve^{v^2/2}\sqrt{2\pi}$. Using these expansions in (5.4) one obtains that

$$v \sim \sqrt{\frac{p - q}{\alpha_+}} \; + \; o\left(\frac{1}{\sqrt{\alpha_+}}\right) .$$

6. Appendix: Uniqueness

Case $p = 1$**.** This is an extension to infinite volume of an argument presented in [ELS2]. Fix a weak solution $\rho(t,u)$ of the differential Eq. (1.3). Since $\rho(t,\cdot)$ is in $L^1_{loc}(\mathbb{R}_+)$, we may define $R_t\colon \mathbb{R}_+^2 \to \mathbb{R}$ by

$$R_t(u,v) = \int_u^v \rho(t,w)\,dw. \tag{6.1}$$

Denote by $[\cdot,\cdot]$ the inner product in $L^2(\mathbb{R}_+^2)$. Fix a smooth function $H\colon \mathbb{R}_+^2 \to \mathbb{R}$ with compact support. Changing the order of summations we obtain that

$$[R_t, H] = \int_{\mathbb{R}_+} du\,\rho(t,w)h(w)\ , \tag{6.2}$$

where

$$h(w) = \int_0^w du \int_w^\infty dv\, H(u,v)\ -\ \int_w^\infty du \int_0^w dv\, H(u,v).$$

Notice that h is a smooth function with compact support that vanishes at the origin. Moreover, its derivative is given by

$$h'(w) = \int_0^\infty du\,\{H(w,u) - H(u,w)\}.$$

Therefore, in the virtue of (6.2), property (c) of weak solutions and a change of variables, for every smooth function H with compact support,

$$[R_t, H] = [R_0, H]\ +\ \int_0^t ds \int_{\mathbb{R}_+} du \int_{\mathbb{R}_+} dv\, H(u,v)\{\partial_v \Phi(\rho(s,v)) - \partial_u \Phi(\rho(s,u))\}.$$

In particular, we have that

$$R_t(u,v) - R_0(u,v) = \int_0^t ds\,\{\partial_v \Phi(\rho(s,v)) - \partial_u \Phi(\rho(s,u))\} \tag{6.3}$$

for almost all (u,v) in $\mathbb{R}_+^2$.

Consider now two solutions ρ^1, ρ^2 of Eq. (1.3), denote by R_t^1, R_t^2 the respective functions associated to ρ^1, ρ^2, through (6.1) and set $W_t = R_t^1 - R_t^2$, $\overline{\rho}_t = \rho_t^1 - \rho_t^2$. Denote by $[\cdot,\cdot]_e$ the inner product on $L^2(\mathbb{R}_+^2)$ associated to the measure $e^{-(u+v)}dudv$. In view of property (a) of weak solutions and identity (6.3), $R\colon [0,T] \to L^2(\mathbb{R}_+^2, e^{-(u+v)}dudv)$ is almost everywhere differentiable. Therefore,

$$\frac{d}{dt}[W_t, W_t]_e = 2\int du \int dv\, e^{-(u+v)} W_t(u,v)\{\partial_v \overline{\Phi}_t(v) - \partial_u \overline{\Phi}_t(u)\}\ ,$$

where $\overline{\Phi}_t(v)$ stands for $\Phi(\rho^1(t,v)) - \Phi(\rho^2(t,v))$. An integration by parts gives that the right-hand side is equal to

$$-2\int_{\mathbb{R}_+} e^{-u}\,\overline{\Phi}_t(u)\overline{\rho}(t,u)\ +\ 2\int du \int dv\, e^{-(u+v)} W_t(u,v)\overline{\Phi}_t(v) \tag{6.4}$$

because $\int du \exp\{-u\} = 1$. By Schwarz inequality, the second term is bounded above by

$$\|\Phi'\|_\infty [W_t, W_t]_e \; + \; \frac{1}{\|\Phi'\|_\infty} \int_{\mathbb{R}_+} du\, e^{-u} \big(\bar{\Phi}_t(u)\big)^2$$
$$\le \; \|\Phi'\|_\infty [W_t, W_t]_e \; + \; \int_{\mathbb{R}_+} du\, e^{-u} \bar{\Phi}_t(u) \bar{\rho}(t, u)$$

because Φ is an increasing function with a bounded first derivative. Adding this expression to the first term of (6.4), we obtain that the time derivative of $[W_t, W_t]_e$ is bounded above by

$$\|\Phi'\|_\infty [W_t, W_t]_e \; - \; \int_{\mathbb{R}_+} du\, e^{-u} \bar{\Phi}_t(u) \bar{\rho}(t, u) \; \le \; \|\Phi'\|_\infty [W_t, W_t]_e$$

because Φ is non decreasing. By the Gronwall inequality, we deduce that $[W_t, W_t]_e$ is bounded above by $[W_0, W_0]_e \exp\{\|\Phi'\|_\infty t\}$, which concludes the proof of the uniqueness of weak solutions of Eq. (1.3).

The case $p < 1$. The argument is similar to the one presented for $p = 1$. For $t \ge 0$, define $R_t : \mathbb{R}^2 \to \mathbb{R}_+$ as in (6.1). It can be shown that

$$R_t(u, v) - R_0(u, v) = \int_0^t ds\, \{\partial_v \Phi(\rho(s, v)) - \partial_u \Phi(\rho(s, u))\}$$

for almost all (u, v) in $\mathbb{R}^2$. Consider two solutions of Eq. (1.4). Denote by $m(du) = m(u)du$ the absolutely continuous measure with density $m(u) = p\mathbf{1}\{u < 0\} + q\mathbf{1}\{u > 0\}$ and fix a smooth function $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $\theta(0) = 0, \theta(u) = |u|$ for u large enough and $\int m(du) \exp\{-\theta(u)\} = 1$. Let $[\cdot, \cdot]_m$ stand for the inner product in $L^2(\mathbb{R}^2)$ with respect to the measure $m(du)m(dv) \exp\{-\theta(u) - \theta(v)\}$. Fix two solutions ρ^1, ρ^2 of Eq. (1.4), denote by R_t^1, R_t^2 the respective functions associated to ρ^1, ρ^2, through (6.1) and set $W_t = R_t^1 - R_t^2$. With the same arguments presented above one can show that $[W_t, W_t]_m$ is bounded above by $[W_0, W_0]_m \exp\{C(\theta, \|\Phi'\|_\infty)t\}$. In this deduction the use of the measure $m(du)$ instead of the Lebesgue measure is fundamental in the integration by parts performed in (6.4) for the boundary term to cancel.

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References

[A] Andjel, E.; Invariant measures for the zero-range processes. Ann. Probab. **10**, 525–547 (1982)
[Ar] Arratia, R: The motion of a tagged particle in the simple symmetric exclusion system in $\mathbb{Z}$. Ann.Prob. **11**, 362–373 (1983)
[BDMO] Burlatsky, S.F., De Coninck, J., Moreau, M., Oshanin, G. S: Dynamics of the shock front propagation in a one-dimensional hard core lattice gas. Preprint (1997)
[BMMO] Burlatsky, S. F., Mogutov, A. V., Moreau, M., Oshanin, G. S: Directed walk in a one–dimensional gas. Phys. Lett. A **166**, 230–234 (1992)
[BMOR] Burlatsky, S.F., Moreau, M., Oshanin, G. , Reinhardt, W.P: Motion of a driven tracer particle in a one-dimensional symmetric lattice gas. Phys.Rev. E, **54**, no. 4, 3165–3172 (1996)
[ELS] Eyink, G., Lebowitz, J.L. and Spohn, H.: Lattice gas models in contact with stochastic reservoirs: local equilibrium and relaxation to steady states. Commun. Math. Phys. **140**, 119–131 (1991)

[GPV] Guo, M.Z., Papanicolaou, G.C. and Varadhan, S.R.S.: Nonlinear diffusion limit for a system with nearest neighbor interactions. Comm. Math. Phys. **118**, 31–59 (1988)

[H] Harris, T.E: Diffusions with collisions between particles. J. Appl. Prob. **2**, 323–338 (1965)

[KL] Kipnis, C., Landim, C: *Hydrodynamical Limit of Interacting Particle Systems.*, Preprint 1996

[LM] Landim, C., Mourragui, M: Hydrodynamic limit of mean zero asymmetric zero range processes in infinite volume. Ann. Inst. H. Poincaré, Prob. et Stat. **33**, 65–82 (1997)

[LR] Lebowitz, J.L., Rost, H.: The Einstein relation for the displacement of a test particle in a random environment. Stoc. Proc. Appl. **54**, 183–196 (1994)

[RV] Rost, H. and Vares, M.E: Hydrodynamics of a one dimensional nearest neighbor model. Contemp. Math. **41**, 329–342 (1985)

[Spo] Spohn, H: *Large Scale Dynamics of Interacting Particles* Text and Monographs in Physics, New York: Springer Verlag, 1991

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