

Convection–diffusion equation with space–time ergodic random flow

C. Landim¹, S. Olla², H. T. Yau³

¹IMPA, Estrada Dona Castorina 110, CEP 22460 Rio de Janeiro, Brazil and CNRS URA 1378, Université de Rouen, F-76128 Mont Saint Aignan, France.

E-mail: landim@impa.br

²Département de Mathématiques, Université de Cergy–Pontoise, 2 Av. Adolphe Chauvin, Pontoise F-95.302 Cergy–Pontoise Cedex, France and Centre de Mathématiques Appliquées, Ecole Polytechnique, F-91128 Palaiseau Cedex, France.

E-mail: olla@v-cergy.fr

³Courant Institute, 251 Mercer Street, New York, NY 10012, USA.

E-mail: yau@cims.nyu.edu

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Abstract. We prove the homogenization of convection-diffusion in a time-dependent, ergodic, incompressible random flow which has a bounded stream matrix and a constant mean drift. We also prove two variational formulas for the effective diffusivity. As a consequence, we obtain both upper and lower bounds on the effective diffusivity.

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1 Introduction

Consider the density $u(\mathbf{x}, t)$ of a passive advected agent such as temperature, concentration of an impurity diffusing in an incompressible fluid. It satisfies the convection–diffusion equation

$$\partial_t u = \Delta u + \mathbf{F} \cdot \nabla u \quad (1.1)$$

with certain initial condition. The fluid velocity $\mathbf{F}(\mathbf{x}, t, \omega)$ is given by a random ergodic field on \mathbb{R}^{d+1} satisfying

$$\nabla_x \cdot \mathbf{F}(\mathbf{x}, t, \omega) = 0, \quad \langle \mathbf{F}(\mathbf{x}, t, \cdot) \rangle = \mathbf{v} \quad .$$

where $\langle \cdot \rangle$ denotes the expectation w.r.t. the random field.

We assume that $\mathbf{F} = \nabla_x \cdot \mathbf{H} + \mathbf{v}$; here \mathbf{v} is a constant vector and $\mathbf{H}(\mathbf{x}, t, \omega)$ is a bounded, skew-symmetric, and stationary random matrix in \mathbb{R}^{d+1} . We will make more precise assumptions on the regularity of the realization of \mathbf{H} in the next section. We study the effective diffusion on a macroscopic scale. Introducing a scale parameter ε , we will prove that

$$\lim_{\varepsilon \rightarrow 0} u(\varepsilon^{-1} \mathbf{x} - \varepsilon^{-2} t \mathbf{v}, \varepsilon^{-2} t, \omega) = \bar{u}(\mathbf{x}, t) \quad (1.2)$$

in probability with respect to ω , where \bar{u} is the (deterministic) solution of effective heat equation

$$\partial_t \bar{u} = \sum_{i,j} a_{i,j}^v \partial_{x_i} \partial_{x_j} \bar{u} \quad (1.3)$$

and a^v is the corresponding effective diffusion matrix. Indeed, we will prove a stronger statement concerning the convergence of the underlying stochastic process to a diffusion, *the invariance principle*. This problem arises in diffusions of contaminants in saturated porous media (cf. [BGW]) and in diffusion of particles deposited as sediment in convective flows (cf. [MC]).

Effective diffusions in periodic flows (cf. [BGW], [FP], [MM]) and time-independent random flow with zero mean drift (cf. [PV], [K], [Oe], [Os], [AM1], [FP]) have been extensively studied. Only some special cases of time-dependent random flows had been studied (cf. [K2], [AM2], [CX]). Some bounds on the mean-square displacement for general bounded time-dependent stream functions with zero mean drift are proven in [TKS]. We also prove two variational formulas for the effective diffusivity. As a consequence, we extend the results of [TKS] to flows with mean drift. Our technique follows the probabilistic approach of [KV] and is partly based on a perturbation argument of [LY].

2 Notations and results

Let $(\Omega, \mathcal{G}, \mu)$ be a probability space and $G = \{\tau_{\mathbf{x},t}; (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$ be a group of measure preserving transformations acting ergodically on Ω . Denote by $L^2(\mu)$ the space of square integrable functions and define on $L^2(\mu)$ the operators $\{T_{\mathbf{x},t}; (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$ given by

$$T_{\mathbf{x},t}f(\omega) = f(\tau_{\mathbf{x},t}\omega) \quad .$$

Assume that $T_{\mathbf{x},t}f(\omega)$ is jointly measurable in $\mathbb{R}^{d+1} \times \Omega$ for each measurable function f .

It follows from these assumptions that $\{T_{\mathbf{x},t}, (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$ is a group of strongly continuous unitary operators on $L^2(\Omega, \mathcal{G}, \mu)$ (cf. [JKO], p. 223).

For every f in $L^2(\mu)$, let $f(\mathbf{x}, t, \omega) = f(\tau_{\mathbf{x},t}\omega)$. Each function f in $L^2(\mu)$ defines in this way a stationary ergodic random field on \mathbb{R}^{d+1} . Reciprocally, given a stationary ergodic random field one can always find a probability space where such a representation is possible (cf. [Do]).

Denote by $D_t, D_i, 1 \leq i \leq d$ the infinitesimal generators of $\{T_{\mathbf{x},t}, (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$:

$$D_i = \frac{\partial}{\partial x_i} T_{\mathbf{x},t} \Big|_{\mathbf{x}=0, t=0}, \quad D_t = \frac{\partial}{\partial t} T_{\mathbf{x},t} \Big|_{\mathbf{x}=0, t=0} \quad . \quad (2.1)$$

By Corollary 1.1.6 of [EK] the infinitesimal generators are closed and densely defined.

Denote by $\mathcal{S}(\mathbb{R}^{d+1})$ the Schwartz space of C^∞ -functions that vanish at infinity faster than any polynomial as well as all its derivatives. For $f(\mathbf{x}, t)$ in $\mathcal{S}(\mathbb{R}^{d+1})$ and for $\varphi \in L^2(\Omega)$, define

$$\varphi_f(\omega) = \int_{\mathbb{R}^{d+1}} \varphi(\tau_{\mathbf{x},t}\omega) f(\mathbf{x}, t) \, d\mathbf{x} \, dt \quad . \quad (2.2)$$

A simple computation shows that $\langle \varphi_f^2 \rangle$ is bounded above by $\langle \varphi^2 \rangle \int f(\mathbf{x}, t)^2 \, d\mathbf{x} \, dt$ so that φ_f belongs to $L^2(\Omega)$. Denote by \mathcal{C} the space of all such functions: $\mathcal{C} = \{\varphi_f, \varphi \in L^2(\Omega), f \in \mathcal{S}(\mathbb{R}^{d+1})\}$. \mathcal{C} is dense in L^2 : Consider a sequence $\{f_\varepsilon, \varepsilon > 0\}$ of smooth functions that approximates the identity in $L^1(\mathbb{R}^{d+1})$ and denote by $\|\cdot\|_2$ the $L^2(\Omega)$ norm of Ω . If $S(f_\varepsilon)$ stands for the support of f_ε , we have that

$$\begin{aligned} \|\varphi_{f_\varepsilon} - \varphi\|_2 &= \left\| \int_{\mathbb{R}^{d+1}} \{\varphi(\tau_{\mathbf{x},t}\omega) - \varphi(\omega)\} f_\varepsilon(\mathbf{x}, t) \, d\mathbf{x} \, dt \right\|_2 \\ &\leq \sup_{(\mathbf{x}, t) \in S(f_\varepsilon)} \|T_{\mathbf{x},t}\varphi - \varphi\|_2 \end{aligned}$$

that converges to 0 as $\varepsilon \downarrow 0$ because the group $\{T_{\mathbf{x},t}, (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$ is strongly continuous. Moreover, \mathcal{C} is contained in the domain of the generators $D_t, D_i, 1 \leq i \leq d$ and a simple computation shows that

$$D_i\varphi_f = \varphi_{-\partial_{x_i}f}, \quad D_t\varphi_f = \varphi_{-\partial_t f} \quad (2.3)$$

for every f in $\mathcal{S}(\mathbb{R}^{d+1})$, φ in $L^2(\Omega)$ and $1 \leq i \leq d$. In particular, D_i and D_i , $1 \leq i \leq d$, map \mathcal{C} into \mathcal{C} and \mathcal{C} is a domain of essential self-adjointness for the operators $i\mathbf{D}$, iD_i .

Since $\{T_{\mathbf{x},t}, (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$ is a unitary group, by Theorem VIII.12 of [RS] there exists a projection valued measure $P_{\mathbf{k},h}$ on \mathbb{R}^{d+1} such that

$$\langle \psi, T_{\mathbf{x},t}\varphi \rangle = \int_{\mathbb{R}^{d+1}} e^{i(\mathbf{k}\cdot\mathbf{x}+ht)} d\langle \psi, P_{\mathbf{k},h}\varphi \rangle$$

for any $\psi, \varphi \in L^2(\Omega)$. For $\varphi \in L^2(\Omega)$ and f in $\mathcal{S}(\mathbb{R}^{d+1})$, denote by $\hat{R}_\varphi(d\mathbf{k}, dh)$ the spectral measure of φ : $\hat{R}_\varphi(d\mathbf{k}, dh) = d\langle \varphi, P_{\mathbf{k},h}\varphi \rangle$ and by $\hat{f}(\mathbf{k}, h)$ the Fourier transform of f :

$$\hat{f}(\mathbf{k}, h) = \int_{\mathbb{R}^{d+1}} e^{i(\mathbf{x}\cdot\mathbf{k}+th)} f(\mathbf{x}, t) d\mathbf{x} dt .$$

An elementary computation shows that

$$\varphi_f = \int_{\mathbb{R}^{d+1}} \hat{f}(\mathbf{k}, h) dP_{\mathbf{k},h}\varphi$$

for each f in $\mathcal{S}(\mathbb{R}^{d+1})$. In particular, the spectral measure $\hat{R}_{\varphi_f}(d\mathbf{k}, dh)$ is equal to $|\hat{f}(\mathbf{k}, h)|^2 \hat{R}_\varphi(d\mathbf{k}, dh)$.

Denote by \mathbf{D} the operator (D_1, \dots, D_d) and recall that (D_i, \mathbf{D}) maps \mathcal{C} into \mathcal{C} . We may therefore define on \mathcal{C} the \mathcal{H}_1 norm given by

$$\|\varphi\|_1^2 := \langle |\mathbf{D}\varphi|^2 \rangle = \int_{\mathbb{R}^{d+1}} |\mathbf{k}|^2 \hat{R}_\varphi(d\mathbf{k}, dh) .$$

In this formula $\langle \cdot \rangle$ stands for the expectation with respect to μ and the second identity follows from the spectral representation of the operators $\{T_{\mathbf{x},t}, (\mathbf{x}, t) \in \mathbb{R}^{d+1}\}$. Notice that only ‘‘space derivatives’’ are involved in this norm. Define \mathcal{H}_1 as the Hilbert space induced by \mathcal{C} and the inner product obtained from the norm $\|\cdot\|_1$.

Denote by \mathcal{C}_{-1} the subset of \mathcal{C} of all functions $\{\varphi_f, f \in \mathcal{S}(\mathbb{R}^{d+1})\}$ such that $S(\hat{f}) \cap C(\mathbf{0}, h_o) = \emptyset$ for some $h_o > 0$, where $S(\hat{f})$ stands for the support of \hat{f} and $C(\mathbf{0}, h_o)$ for the cylinder $[-h_o, h_o]^d \times \mathbb{R}$. Observe that \mathcal{C}_{-1} is not dense in L^2 . Define the \mathcal{H}_{-1} norm on \mathcal{C}_{-1} by

$$\|\varphi_g\|_{-1}^2 = \int_{\mathbb{R}^{d+1}} |\mathbf{k}|^{-2} \hat{R}_{\varphi_g}(d\mathbf{k}, dh) .$$

The right hand side is finite because $\hat{R}_{\varphi_g}(d\mathbf{k}, dh)$ is equal to $|\hat{g}(\mathbf{k}, h)|^2 \hat{R}_\varphi(d\mathbf{k}, dh)$ and $S(\hat{g}) \cap C(\mathbf{0}, h_o) = \emptyset$ so that we may replace $|k|^{-2}$ by a bounded function. Denote by \mathcal{H}_{-1} the closure of \mathcal{C}_{-1} with respect to this norm. Note that $\|f\|_{-1} < \infty$ imposes that f cannot be a constant with respect to translations in the space direction. Moreover,

it follows immediately from the definitions above that for any $\varphi \in \mathcal{C}_{-1}$,

$$\|\varphi\|_{-1}^2 = \sup_{\psi \in \mathcal{C}} \{2\langle \psi, \varphi \rangle - \|\psi\|_1^2\} \tag{2.4}$$

provided $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\Omega)$.

We make the following assumptions on the stream matrix $H_{i,j}(\omega)$:

- (i) $H_{i,j}(\omega)$ is skew-symmetric and a.e. bounded;
- (ii) $D_l H_{i,j}(\omega), l = 1, \dots, d$ exist and are bounded a.e.;
- (iii) $D_l D_h H_{i,j}(\omega), l, h = 1, \dots, d$ exist in $L^2(\mu)$.

Notice that we do not make any assumptions on the time derivatives of $H_{i,j}(\omega)$ (i.e. on $D_t H_{i,j}$). The assumption (i) is the major restriction we need for the proof of the homogenization with the perturbative method of section 4 below. The assumption (iii) is made to ensure the existence of the process as solution of a stochastic differential equation (cf. (2.5) below) and can be relaxed. The boundedness of the derivative of $H_{i,j}$ in assumption (ii) is needed in section 3 in order to construct a process on the Ω -space (“the environment as seen from the particle”).

We define the drift:

$$\mathbf{F} = \left\{ F_i = \sum_j D_j H_{i,j}, \quad i = 1, \dots, d \right\} .$$

Since \mathbf{H} is bounded (and therefore in $L^2(\Omega)$), the variational formula for the \mathcal{H}_{-1} norm and Schwarz inequality show that $\{F_i, 1 \leq i \leq d\}$ are in \mathcal{H}_{-1} . Moreover, it follows from assumptions (ii) and (iii) and the stationarity of the process that $F_i(\mathbf{x}, t, \omega) = F_i(\tau_{\mathbf{x},t}\omega)$ is bounded and Lipschitz in \mathbf{x} , $1 \leq i \leq d$, for almost every ω .

For each fixed realization ω of the environment, consider the diffusion process defined by the stochastic differential equation

$$\begin{cases} d\mathbf{y}^\omega(t) = \sqrt{2}d\mathbf{w}_t + \{\mathbf{v} + \mathbf{F}(\mathbf{y}^\omega(t), t, \omega)\}dt \\ \mathbf{y}^\omega(0) = \mathbf{0} \end{cases} \tag{2.5}$$

where \mathbf{w}_t is a standard Wiener process on \mathbb{R}^d and $\mathbf{v} \in \mathbb{R}^d$ is a constant vector. This diffusion is well defined because $F_i(\mathbf{x}, t, \omega)$ is bounded and Lipschitz in \mathbf{x} (cf. [SV]).

We are now in a position to state the main result of this article.

Theorem 2.1. *The law of the rescaled diffusion*

$$\mathbf{x}^\varepsilon(t) = \varepsilon(\mathbf{y}^\omega(\varepsilon^{-2}t) - \varepsilon^{-2}t\mathbf{v})$$

converges in probability (with respect to ω) to the law of a Brownian motion with a symmetric diffusion matrix a^y characterized by the quadratic form

$$\mathbf{e} \cdot a^y \mathbf{e} = \|\mathbf{e}\|^2 + 2 \sup_{\varphi \in \mathcal{C}_{-1}} \{2\langle \mathbf{F} \cdot \mathbf{e}, \varphi \rangle - \|A\varphi\|_{-1}^2 - \|\varphi\|_1^2\}$$

where $A = \mathbf{v} \cdot \mathbf{D} + D_t + \mathbf{F} \cdot \mathbf{D}$.

We present in Proposition 5.1 an alternative variational formula for the diffusion coefficient.

In order to connect the above result with the homogenization of the parabolic equation (1.1), we need to assume some regularity in the time dependence, i.e.

- (iv) There exist $0 < \gamma \leq 1$ and $C(\omega)$ a.s.-finite, such that for any (x, s, t)

$$|F_i(\mathbf{x}, t, \omega) - F_i(\mathbf{x}, s, \omega)| \leq C(\omega)|t - s|^\gamma \quad \omega - a.e.$$

Under condition (iv) The parabolic linear equation

$$\begin{cases} \partial_t u_\varepsilon(\mathbf{x}, t) = \Delta u_\varepsilon(\mathbf{x}, t) - \mathbf{v} \cdot \nabla u_\varepsilon(\mathbf{x}, t) - \mathbf{F}(\mathbf{x}, t, \omega) \cdot \nabla u_\varepsilon(\mathbf{x}, t) \\ u_\varepsilon(\mathbf{x}, 0) = u_0(\varepsilon \mathbf{x}) \end{cases} \quad (2.6)$$

has a unique solution (for a.e. ω) (cf. [F]).

Then the next result follows from Theorem 2.1.

Theorem 2.2. *For any $\mathbf{x} \in \mathbb{R}^d$, $t > 0$ and $\delta > 0$*

$$\lim_{\varepsilon \rightarrow 0} \mu \{ |u_\varepsilon(\varepsilon^{-1} \mathbf{x} - \varepsilon^{-2} t \mathbf{v}, \varepsilon^{-2} t, \omega) - \bar{u}(\mathbf{x}, t)| \geq \delta \} = 0 \quad , \quad (2.7)$$

where \bar{u} is the solution of the effective heat equation

$$\begin{cases} \partial_t \bar{u} = \sum_{i,j} a_{i,j} \partial_{x_i} \partial_{x_j} \bar{u} \\ \bar{u}(\mathbf{x}, 0) = u_0(\mathbf{x}) \end{cases} \quad (2.8)$$

3 Probabilistic convergence theorem

To the diffusion process defined by (2.5) one can associate a canonical Markov process on Ω by

$$\begin{cases} \eta(t) = \tau_{\mathbf{y}^\omega(t), t} \omega \\ \eta(0) = \omega \end{cases} \quad .$$

Consider the linear operator L defined on $L^2(\mu)$ by

$$L = \mathbf{D}^2 + \mathbf{v} \cdot \mathbf{D} + \mathbf{F} \cdot \mathbf{D} + D_t := L_0 + \mathbf{F} \cdot \mathbf{D} ,$$

where $\mathbf{D}^2 = \sum_{1 \leq i \leq d} D_i^2$. Since D_t, \mathbf{D} map \mathcal{C} into \mathcal{C} , \mathcal{C} is contained in the domain of each of the operators entering in the definition of L and thus contained also in the domain $\text{Dom } L$ of L . Moreover, it follows from the definition of the diffusion $y^\omega(t)$ given in (2.5) that $\varphi_f(\eta(t)) - \varphi_f(\eta(0)) - \int_0^t L\varphi_f(\eta(s))ds$ is a mean zero martingale for each φ_f in \mathcal{C} .

Since the operators $D_t, \{D_i, 1 \leq i \leq d\}$ are anti-symmetric, L is a dissipative operator: $\|(\lambda - L)f\|_2 \geq \lambda\|f\|_2$ for every $\lambda > 0$. Since \mathcal{C} is dense in $L^2(\mu)$, to prove that L is closable and that the closure (still denoted by L to keep notation simple) is a generator of a strongly continuous group in $L^2(\mu)$ it remains to check that the range $\mathcal{R}(\lambda - L)$ of $\lambda - L$ is dense in $L^2(\mu)$ for some $\lambda > 0$ (cf. Theorem 1.2.12 in [EK]). We prove in fact a stronger result from which it follows that \mathcal{C} is a core for L .

Proposition 3.1. *For any $\lambda > 0$ and $g \in L^2$ there exists a sequence $u_n \in \mathcal{C}$ such that*

$$\lim_{n \rightarrow \infty} \|(\lambda - L)u_n - g\|_2 = 0 .$$

Proof. For any $f \in \mathcal{S}(\mathbb{R}^{d+1}), g = (\lambda - \Delta - \mathbf{v} \cdot \nabla - \partial_t)^{-1}f \in \mathcal{S}(\mathbb{R}^{d+1})$. Since $(\lambda - L_0)\varphi_g = \varphi_f, \mathcal{C}$ forms a core for L_0 .

By Theorem 1.7.1 of [EK], in order to prove the proposition, we just need to show that $\mathbf{F} \cdot \mathbf{D}$ is a relatively bounded perturbation of L_0 , i.e., that there exist $0 \leq \alpha < 1$ and $\beta > 0$ such that

$$\|\mathbf{F} \cdot \mathbf{D}u\|_2 \leq \alpha\|L_0u\|_2 + \beta\|u\|_2$$

for all $u \in \mathcal{C}$. The domain condition stated in [EK], $\text{Dom}(\mathbf{F} \cdot \mathbf{D}) \supset \text{Dom}(L_0)$ is a consequence of this bound.

Since \mathbf{F} is bounded by assumption (ii),

$$\langle |\mathbf{F} \cdot \mathbf{D}u|^2 \rangle \leq \|\mathbf{F}\|_\infty^2 \langle |\mathbf{D}u|^2 \rangle = -\|\mathbf{F}\|_\infty^2 \langle u, \mathbf{D}^2u \rangle .$$

By Schwarz inequality, the previous expression is bounded above by

$$\|\mathbf{F}\|_\infty^2 \left\{ \frac{\alpha}{2} \langle |\mathbf{D}^2u|^2 \rangle + \frac{1}{2\alpha} \langle u^2 \rangle \right\}$$

for any $\alpha > 0$. Since $L_0 = \mathbf{D}^2 + M$, where M is antisymmetric, and \mathbf{D}^2 and M commute, $\|L_0u\|_2$ is bounded below by $\|\mathbf{D}^2u\|_2$. Therefore,

$$\langle |\mathbf{F} \cdot \mathbf{D}u|^2 \rangle \leq \|\mathbf{F}\|_\infty^2 \left\{ \frac{\alpha}{2} \langle |L_0u|^2 \rangle + \frac{1}{2\alpha} \langle u^2 \rangle \right\}$$

for any $\alpha > 0$. This shows that $\mathbf{F} \cdot \mathbf{D}$ is a relatively bounded perturbation of L_0 because $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. \square

It follows from this proposition and the fact that $\varphi_f(\eta(t)) - \varphi_f(\eta(0)) - \int_0^t L\varphi_f(\eta(s))ds$ is a martingale for each φ_f in \mathcal{C} that L is the generator of the Markov process $\eta(t)$.

The probability measure μ is an ergodic invariant measure for this process. In fact this is a consequence of the ergodicity of μ with respect to the space-time translation of the group G . Indeed let $\varphi \in \mathcal{C}$ such that $L\varphi = 0$. Multiplying this equation by φ and integrating we obtain

$$\langle (\mathbf{D}\varphi)^2 \rangle = 0$$

which implies $\mathbf{D}\varphi = 0$ a.e. Since $L\varphi = 0$ and $\mathbf{D}\varphi = 0$, we have that $D_t\varphi = 0$. So φ is invariant with respect to the action of G . By the ergodicity of this action φ is constant μ a.e.

To state the main step in the proof of Theorem 2.1, we need to point out some elementary properties of the generator L and the space \mathcal{C}_{-1} . We first claim that $L\mathcal{C}_{-1} \subset \mathcal{H}_{-1} \cap L^2(\mu)$. Observe that for each φ_f in \mathcal{C} we have $L_0\varphi_f = \varphi_g$ where $g = \Delta f - \mathbf{v} \cdot \nabla f - \partial_t f$. Since the support of the Fourier transform of g is contained in the support of the Fourier transform of f , we have that $L_0\mathcal{C}_{-1} \subset \mathcal{C}_{-1}$. On the other hand, from the variational formula for the norm \mathcal{H}_1 , the explicit formula for \mathbf{F} and assumption (ii), $\mathbf{F} \cdot \mathbf{D}\varphi_f \in \mathcal{H}_{-1} \cap L^2(\mu)$ for each f in $\mathcal{S}(\mathbb{R}^{d+1})$.

Proposition 3.2. *For any $g \in \mathcal{H}_{-1}$ there exists a sequence $u_n \in \mathcal{C}_{-1}$ such that*

$$\lim_{n \rightarrow \infty} \|Lu_n - g\|_{-1} = 0. \tag{3.1}$$

The proof of this proposition is the main content of section 4.

Corollary 3.3. *The sequence u_n defined in Proposition 3.2 converges strongly to some u_0 in \mathcal{H}_1 .*

Proof. Since u_n belongs to \mathcal{C}_{-1} and $\mathcal{C}_{-1} \subset \mathcal{C}$, u_n belongs to \mathcal{C} . On the other hand, $\mathbf{D}^2\mathcal{C}_{-1} \subset \mathcal{C}_{-1}$ and $\|\mathbf{D}^2\varphi\|_{-1} = \|\varphi\|_1$ for each φ in \mathcal{C}_{-1} . Finally, since $L = \mathbf{D}^2 + A$, where A is an antisymmetric operator,

$$\|L\varphi\|_{-1}^2 = \|\mathbf{D}^2\varphi\|_{-1}^2 + \|A\varphi\|_{-1}^2 \geq \|\mathbf{D}^2\varphi\|_{-1}^2$$

for each φ in \mathcal{C}_{-1} . In conclusion,

$$\|u_n - u_m\|_1 = \|\mathbf{D}^2(u_n - u_m)\|_{-1} \leq \|L(u_n - u_m)\|_{-1} .$$

Since, by Proposition 3.2 the right hand side vanishes as $n, m \uparrow \infty$, the corollary is proved. \square

In our route to prove Theorem 2.1, we need the following lemma proved independently by [SVY] and [L].

Proposition 3.4. *Let $g \in \mathcal{H}_{-1} \cap L^2$. Then*

$$E_\mu \left(\sup_{0 \leq t \leq T} \left| \int_0^t g(\eta(s)) ds \right|^2 \right) \leq 16T \|g\|_{-1}^2$$

Proof. A simple computation with functions belonging to \mathcal{C}_{-1} shows that $(\lambda - S)^{-1}$ is a bounded operator in \mathcal{H}_{-1} for each $\lambda > 0$. In particular, if u_λ stands for $u_\lambda = (\lambda - \mathbf{D}^2)^{-1}g$, by Schwarz inequality,

$$E_\mu \left(\sup_{0 \leq t \leq T} \left| \int_0^t g(\eta(s)) ds \right|^2 \right) \leq 2E_\mu \left(\sup_{0 \leq t \leq T} \left| \int_0^t \lambda u_\lambda(\eta(s)) ds \right|^2 \right) + 2E_\mu \left(\sup_{0 \leq t \leq T} \left| \int_0^t \mathbf{D}^2 u_\lambda(\eta(s)) ds \right|^2 \right) .$$

Multiplying both sides of the equation $\lambda u_\lambda - Su_\lambda = g$ by u_λ and applying Schwarz inequality we obtain that $\lambda \langle u_\lambda^2 \rangle$ is bounded above by $\|g\|_{-1}^2$. Therefore, since μ is an invariant measure, by Schwarz inequality,

$$E_\mu \left(\sup_{0 \leq t \leq T} \left| \int_0^t \lambda u_\lambda(\eta(s)) ds \right|^2 \right) \leq \lambda T^2 \|g\|_{-1}^2 .$$

Since by hypothesis g belongs to $L^2(\mu)$, $(\lambda - S)u_\lambda = g$ is in $L^2(\mu)$. In particular, $Su_\lambda = \lambda u_\lambda - g$ also belongs to $L^2(\mu)$. There exists therefore a sequence $\{\varphi_n, n \geq 1\}$ in \mathcal{C} such that $\|\varphi_n - u_\lambda\|_{L^2} \rightarrow 0$ and $\|\mathbf{D}^2 \varphi_n - \mathbf{D}^2 u_\lambda\|_{L^2} \rightarrow 0$. From these estimates we obtain that

$$\lim_{n \rightarrow \infty} \langle |D\varphi_n|^2 \rangle = \lim_{n \rightarrow \infty} \langle \varphi_n, S\varphi_n \rangle = \langle u_\lambda, Su_\lambda \rangle \leq \|g\|_{-1}^2 .$$

Let L^* the adjoint operator of L . It is easy to check that $\mathcal{C} \subset \mathcal{D}(L^*)$ and that $L^* = \mathbf{D}^2 - \mathbf{v} \cdot \mathbf{D} - \mathbf{F} \cdot \mathbf{D} - D_t$ on \mathcal{C} . By Ito's formula we have

$$\begin{aligned} \int_0^t \mathbf{D}^2 \varphi_n(\eta(s)) ds &= \frac{1}{2} \int_0^t L\varphi_n(\eta(s)) ds + \frac{1}{2} \int_0^t L^* \varphi_n(\eta(s)) ds \\ &= M_{\varphi_n}(t) + M_{\varphi_n}^*(t), \end{aligned} \tag{3.2}$$

where $M_{\varphi_n}(t)$ and $M_{\varphi_n}^*(t)$ are respectively a forward and a backward martingale such that

$$E(M_{\varphi_n}(t)^2) = E(M_{\varphi_n}^*(t)^2) = \frac{t}{2} \langle |D\varphi_n|^2 \rangle .$$

Observe that in (3.2) the boundary terms cancel exactly. Applying Doob’s inequality,

$$E \left(\sup_{0 \leq t \leq T} \left| \int_0^t S\varphi_n(\eta(s)) ds \right|^2 \right) \leq 8T \langle |D\varphi_n|^2 \rangle \leq 8T \|g\|_{-1}^2 .$$

Recollecting the previous bounds, we obtain that

$$E \left(\sup_{0 \leq t \leq T} \left| \int_0^t g(\eta(s)) ds \right|^2 \right) \leq (16 + 2\lambda T) T \|g\|_{-1}^2 .$$

Letting $\lambda \downarrow 0$ we conclude the proof of the proposition. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. By (2.5),

$$\mathbf{x}^\varepsilon(t) \cdot \mathbf{e} = \sqrt{2\varepsilon} \mathbf{w}_{\varepsilon^{-2}t} \cdot \mathbf{e} + \varepsilon \int_0^{t\varepsilon^{-2}} \mathbf{F}(\eta(s)) \cdot \mathbf{e} ds .$$

We start proving the relative compactness of the process $x^\varepsilon(t)$. Since the first term on the right hand side is just a standard Brownian motion, we only have to consider the second. Let $g^\varepsilon = \mathbf{F} \cdot \mathbf{e}$. By assumption (ii), $g^\varepsilon \in \mathcal{H}_{-1}$. Compactness follows therefore from Proposition 3.4.

We now prove that the limiting process is indeed a diffusion process. We will use the central limit theorem for martingales in order to conclude the argument.

Since g^ε belongs to \mathcal{H}_{-1} , by Proposition 3.2, there exists a sequence $\{u_n^\varepsilon, n \geq 1\}$ in \mathcal{C}_{-1} such that Lu_n^ε converges to g^ε in \mathcal{H}_{-1} . Let $f_n = g^\varepsilon - Lu_n^\varepsilon$. By Ito’s formula,

$$\begin{aligned} \varepsilon \int_0^{t\varepsilon^{-2}} g^\varepsilon(\eta(s)) ds &= \varepsilon \int_0^{t\varepsilon^{-2}} (Lu_n^\varepsilon)(\eta(s)) ds + \varepsilon \int_0^{t\varepsilon^{-2}} f_n(\eta(s)) ds \\ &= \varepsilon \{u_n^\varepsilon(\eta(t\varepsilon^{-2})) - u_n^\varepsilon(\eta(0))\} + \varepsilon M_n(t\varepsilon^{-2}) \\ &\quad + \varepsilon \int_0^{t\varepsilon^{-2}} f_n(\eta(s)) ds , \end{aligned} \tag{3.3}$$

where $M_n(t)$ is a martingale with quadratic variation

$$\int_0^t \left\{ Lu_n^\varepsilon(\eta(s))^2 - 2u_n^\varepsilon(\eta(s))Lu_n^\varepsilon(\eta(s)) \right\} ds = 2 \int_0^t \left| \mathbf{D}u_n^\varepsilon(\eta(s)) \right|^2 ds .$$

Since u_n^ε belongs to $L^2(\mu)$, the first term in (3.3), $\varepsilon\{u_n^\varepsilon(\eta(t\varepsilon^{-2})) - u_n^\varepsilon(\eta(0))\}$ converges to 0 in L^2 as $\varepsilon \downarrow 0$. By Proposition 3.2, $\|f_n\|_{-1}$ vanishes as $n \uparrow \infty$. On the other hand, f_n belongs to $L^2(\mu)$ because $g^\varepsilon \in L^2(\mu)$ and u_n^ε are in \mathcal{C}_{-1} . In particular, by Proposition 3.4,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} E \left[\left(\varepsilon \int_0^{t\varepsilon^{-2}} f_n(\eta(s)) ds \right)^2 \right] = 0 .$$

By the central limit theorem for martingales and the ergodic theorem (cf. [O] for example), as $\varepsilon \rightarrow 0$, the martingale converges (in law, for a.e. ω) to a martingale with constant quadratic variation given by $\|u_n^\varepsilon\|_1^2$, namely a Brownian motion with variance $\|u_n^\varepsilon\|_1^2$. By Corollary 3.3, as $n \uparrow \infty$, the variance converges to $\|u_0^\varepsilon\|_1^2$. Observe that the joint variation of $\varepsilon M^{u_n^\varepsilon}(t\varepsilon^{-2})$ and $\varepsilon \mathbf{w}_{\varepsilon^{-2}t} \cdot \mathbf{e}$ is given by

$$\varepsilon^2 \int_0^{\varepsilon^2} \mathbf{e} \cdot \mathbf{D}u_0^\varepsilon(\eta(s)) ds$$

so by the ergodic theorem they are asymptotically orthogonal as $\varepsilon \rightarrow 0$. We have then obtained that the finite dimensional distributions of $\mathbf{x}^\varepsilon(t) \cdot \mathbf{e}$ converge to those of a Brownian motion with diffusion coefficient given by

$$\mathbf{e} \cdot a^\mathbf{v} \mathbf{e} = \|\mathbf{e}\|^2 + 2\langle \|\mathbf{D}u_0^\varepsilon\|^2 \rangle .$$

By polarization we have the convergence of the finite dimensional distributions of $\mathbf{x}^\varepsilon(t)$ to a Brownian motion with diffusion matrix

$$a_{i,j}^\mathbf{v} = \delta_{i,j} + 2\langle \mathbf{D}u_0^{\mathbf{e}_i} \cdot \mathbf{D}u_0^{\mathbf{e}_j} \rangle . \tag{3.4}$$

The diffusivity shall be characterized by variational principles in Proposition 5.1. □

4 A perturbation argument

We prove in this section Proposition 3.2. To keep notation simple, for a in \mathbb{R} , let $J = \mathbf{F} \cdot \mathbf{D} = \mathbf{D} \cdot \mathbf{H}\mathbf{D}$, $M = \mathbf{v} \cdot \mathbf{D} + D_t$, $S = \mathbf{D}^2$, $L_a = S + aM$ and $A = M + J$.

We first claim that (3.1) can be explicitly solved if L is replaced by L_a .

Proposition 4.1. *For any ψ in \mathcal{H}_{-1} there exists a sequence $u_n \in \mathcal{C}_{-1}$ such that*

$$\lim_{n \rightarrow \infty} \|L_a u_n - g\|_{-1} = 0 . \quad (4.1)$$

Proof. Since \mathcal{C}_{-1} is dense in \mathcal{H}_{-1} , it is enough to prove the proposition for ψ in \mathcal{C}_{-1} . Assume thus that $\psi = \varphi_f$ for some φ in $L_2(\mu)$ and some f in $\mathcal{S}(\mathbb{R}^{d+1})$ whose Fourier transform \hat{f} has support $S(\hat{f})$ is such that $S(\hat{f}) \cap C(\mathbf{0}, h_o) = \emptyset$ for some $h_o > 0$.

Fix $\lambda > 0$ and $\varepsilon > 0$ and denote by $g_{\lambda, \varepsilon}$ the function $(\lambda - \Delta - a\beta v \cdot \nabla - a\partial_t - \varepsilon\partial_t^2)^{-1} f$ that belongs to $\mathcal{S}(\mathbb{R}^{d+1})$. It follows from (2.3) that $(\lambda - L_a - \varepsilon D_t^2)\varphi_{g_{\lambda, \varepsilon}} = \varphi_f$. To conclude the proof, it remains to show that $\lambda\varphi_{g_{\lambda, \varepsilon}}$ and $\varepsilon D_t^2\varphi_{g_{\lambda, \varepsilon}}$ converges to 0 in \mathcal{H}_{-1} as $\varepsilon \downarrow 0$ and then $\lambda \downarrow 0$.

By definition,

$$\begin{aligned} & \|\varepsilon D_t^2 u_{\lambda, \varepsilon}\|_{-1}^2 \\ &= \int \frac{\varepsilon^2 h^4}{|\mathbf{k}|^2 |\lambda + |\mathbf{k}|^2 - i(\mathbf{v} \cdot \mathbf{k} + h) + \varepsilon h^2|^2} |\hat{f}(\mathbf{k}, h)|^2 \hat{R}_\varphi(d\mathbf{k}, dh) . \end{aligned}$$

The integrand is bounded by $C_\lambda |\hat{f}(\mathbf{k}, h)|^2 / |\mathbf{k}|^2$, for some finite constant C_λ depending only on λ . Since φ_f belongs to \mathcal{C}_{-1} , by the dominated convergence theorem, for any fixed λ , $\|\varepsilon D_t^2 u_{\lambda, \varepsilon}\|_{-1}$ vanishes as $\varepsilon \downarrow 0$. On the other hand,

$$\|\lambda u_{\lambda, \varepsilon}\|_{-1}^2 = \int \frac{\lambda^2}{|\mathbf{k}|^2 |\lambda + |\mathbf{k}|^2 - i(\mathbf{v} \cdot \mathbf{k} + h) + \varepsilon h^2|^2} |\hat{f}(\mathbf{k}, h)|^2 \hat{R}_\varphi(d\mathbf{k}, dh) .$$

By the same reasons, $\|\lambda u_{\lambda, \varepsilon}\|_{-1}$ vanishes as $\varepsilon \downarrow 0$ and then $\lambda \downarrow 0$, what concludes the proof of the proposition. \square

Lemma 4.2. *For each a in \mathbb{R} , JL_a^{-1} is a bounded operator in \mathcal{H}_{-1} and $\|JL_a^{-1}\|_{\mathcal{H}_{-1}} \leq \|\mathbf{H}\|_\infty$.*

Proof. Because \mathbf{H} is bounded a.e., from the variational characterization (2.4) of the \mathcal{H}_{-1} norm,

$$\|Ju\|_{-1} \leq \|\mathbf{H}\|_\infty \|u\|_1$$

for every u in \mathcal{C}_{-1} . On the other hand, since M is antisymmetric,

$$\|L_a u\|_{-1}^2 = \|u\|_1^2 + a^2 \|Mu\|_{-1}^2 \geq \|u\|_1^2 .$$

In particular,

$$\|JL_a^{-1}v\|_{-1} \leq \|\mathbf{H}\|_\infty \|v\|_{-1}$$

for any v in the set $\{L_a u, u \in \mathcal{C}_{-1}\}$. Since by Proposition 4.1 $\{L_a u, u \in \mathcal{C}_{-1}\}$ is dense in \mathcal{H}_{-1} , the lemma is proved. \square

We now prove that for the δ small enough the set $\{(L_a + \delta J)u, u \in \mathcal{C}_{-1}\}$ is dense in \mathcal{H}_{-1} .

Lemma 4.3. *Fix a in \mathbb{R} . For any $\delta < \|\mathbf{H}\|_\infty^{-1}$ and ψ in \mathcal{H}_{-1} , there exists a sequence $\{u_n, n \geq 1\}$ in \mathcal{C}_{-1} such that*

$$\lim_{n \rightarrow \infty} \|(L_0 + \delta J)u_n - \psi\|_{-1} = 0 . \tag{4.2}$$

Proof. Fix a in \mathbb{R} . Since the operator JL_0^{-1} is bounded in \mathcal{H}_{-1} by a finite constant $\|\mathbf{H}\|_\infty$, for $\delta < \|\mathbf{H}\|_\infty^{-1}$, we may define the bounded operator $(1 + \delta JL_0^{-1})^{-1}$ by the serie $\sum_{n \geq 0} (\delta JL_0^{-1})^n$. Let $g = (1 + \delta JL_0^{-1})^{-1} \psi$. By Proposition 4.1 there exists a sequence $\{u_n, n \geq 1\}$ in \mathcal{C}_{-1} such that $\lim_{n \rightarrow \infty} \|L_a u_n - g\|_{-1} = 0$. Therefore,

$$\begin{aligned} \|(L_a + \delta J)u_n - \psi\|_{-1} &= \|(1 + \delta JL_0^{-1})(L_0 u_n - g)\|_{-1} \\ &\leq C(\delta) \|L_0 u_n - g\|_{-1} . \end{aligned}$$

for some finite constant $C(\delta)$. Since the right hand side vanishes as $n \uparrow \infty$, the lemma is proved. \square

Proof of Proposition 3.2. We have to prove that Lemma 4.3 is valid with $a, \delta = 1$. Fix $\delta < \|\mathbf{H}\|_\infty^{-1}$. Formally we have

$$L = S + A = \frac{1}{\delta} (1 - (1 - \delta)S(S + \delta A)^{-1})(S + \delta A) .$$

We first give a rigorous sense to the second identity. Since A is anti-symmetric, $\|u\|_1 \leq \|(S + \delta A)u\|_{-1}$ for all u in \mathcal{C}_{-1} . Thus if $v = (S + \delta A)u$ for some u in \mathcal{C}_{-1} ,

$$\begin{aligned} \|S(S + \delta A)^{-1}v\|_{-1} &= \|(S + \delta A)^{-1}v\|_1 = \|u\|_1 \\ &\leq \|(S + \delta A)u\|_{-1} = \|v\|_{-1} . \end{aligned}$$

Hence the operator $S(S + \delta A)^{-1}$ restricted to the set $\{(S + \delta A)u, u \in \mathcal{C}_{-1}\}$ is bounded by 1 in \mathcal{H}_{-1} . By Lemma 4.3, with $a = \delta$, $\{(S + \delta A)u, u \in \mathcal{C}_{-1}\}$ is dense in \mathcal{H}_{-1} . In particular, $S(S + \delta A)^{-1}$ is bounded by 1 in \mathcal{H}_{-1} . We may therefore define the bounded operator $(1 - (1 - \delta)S(S + \delta A)^{-1})^{-1}$ by a power serie as in Lemma 4.3.

Let $\psi = \delta(1 - (1 - \delta)S(S + \delta A)^{-1})^{-1}g$. By Lemma 4.3 there exists a sequence $u_n \in \mathcal{C}_{-1}$ such that

$$\lim_{n \rightarrow \infty} \|(S + \delta A)u_n - \psi\|_{-1} = 0 .$$

By definition of this sequence,

$$\begin{aligned} \|Lu_n - g\|_{-1} &= \left\| \frac{1}{\delta} (1 - (1 - \delta)S(S + \delta A)^{-1}) [(S + \delta A)u_n - \psi] \right\|_{-1} \\ &\leq C(\delta) \|(S + \delta A)u_n - \psi\|_{-1} \end{aligned}$$

for some finite constant $C(\delta)$. This concludes the proof of the proposition because the right hand side vanishes as $n \uparrow \infty$. \square

5 Variational formulae and bounds for the effective diffusion

We provide here two variational formulae for the effective diffusivity. They are straightforward consequences of Proposition 3.2. Variational formulae had been useful in estimating the effective diffusion coefficients (cf. [FG1]).

From (3.4), up to $2\|\mathbf{e}\|^2$ the diffusivity along the direction \mathbf{e} is given by the \mathcal{H}_1 norm of $u_0^\mathbf{e}$ where $u_0^\mathbf{e} = L^{-1}\mathbf{F} \cdot \mathbf{e}$. Formally,

$$\begin{aligned} \langle |\mathbf{D}u_0^\mathbf{e}|^2 \rangle &= \langle L^{-1}\mathbf{F} \cdot \mathbf{e}, (-S)L^{-1}\mathbf{F} \cdot \mathbf{e} \rangle \\ &= \sup_{\varphi \in \mathcal{C}_{-1}} \{ 2\langle \mathbf{F} \cdot \mathbf{e}, \varphi \rangle - \langle L\varphi(-S)^{-1}L\varphi \rangle \} . \end{aligned}$$

Since $\langle L\varphi, (-S)^{-1}L\varphi \rangle$ is equal to $\|A\varphi\|_{-1}^2 + \|\varphi\|_1^2$, we obtain a variational formula for the diffusion coefficient. A rigorous argument is given in the following result.

Proposition 5.1. *The following variational formulas for the diffusion coefficient hold:*

$$\langle |\mathbf{D}u_0^\mathbf{e}|^2 \rangle = \sup_{\varphi \in \mathcal{C}_{-1}} \{ 2\langle \mathbf{F} \cdot \mathbf{e}, \varphi \rangle - \|A\varphi\|_{-1}^2 - \|\varphi\|_1^2 \} , \quad (5.1)$$

$$\langle |\mathbf{D}u_0^\mathbf{e}|^2 \rangle = \inf_{\varphi \in \mathcal{C}_{-1}} \{ \|\mathbf{F} \cdot \mathbf{e} - A\varphi\|_{-1}^2 + \|\varphi\|_1^2 \} . \quad (5.2)$$

Proof. To keep notation simple, let $g = \mathbf{F} \cdot \mathbf{e}$ and recall that we denote by $L^* = S - A$ the adjoint of L . By Proposition 3.2 there exists sequences $\{u_n, n \geq 1\}$ and $\{u_n^*, n \geq 1\}$ in \mathcal{C}_{-1} such that

$$Lu_n = g + f_n , \quad L^*u_n^* = g + f_n^*$$

and $\|f_n\|_{-1}, \|f_n^*\|_{-1}$ vanish as $n \uparrow \infty$. For each $n \geq 1$, let $\varphi_n^+ = (1/2) \times (u_n + u_n^*)$, $\varphi_n^- = (1/2)(u_n - u_n^*)$. With this definition,

$$S\varphi_n^+ + A\varphi_n^- = g + \frac{1}{2}(f_n + f_n^*) ,$$

$$S\varphi_n^- + A\varphi_n^+ = \frac{1}{2}(f_n - f_n^*) .$$

Since the sequence u_n converges in \mathcal{H}_1 to u_0^ϵ ,

$$\langle |\mathbf{D}u_0^\epsilon|^2 \rangle = \lim_{n \rightarrow \infty} \|u_n\|_1^2 .$$

Rewrite u_n as $\varphi_n^+ + \varphi_n^-$ and notice that

$$\langle \varphi_n^+, S\varphi_n^- \rangle = \langle \varphi_n^+, -A\varphi_n^+ \rangle + (1/2)\langle \varphi_n^+, f_n - f_n^* \rangle .$$

Since A is antisymmetric the first expression on the right hand side vanishes. On the other hand the second converges to 0 as $n \uparrow \infty$ because φ_n^+ is uniformly bounded in \mathcal{H}_1 and $\|f_n\|_{-1}, \|f_n^*\|_{-1}$ vanish as $n \uparrow \infty$. Therefore,

$$\lim_{n \rightarrow \infty} \|u_n\|_1^2 = \lim_{n \rightarrow \infty} \{ \|S\varphi_n^+\|_{-1}^2 + \|\varphi_n^-\|_1^2 \} .$$

Since $S\varphi_n^+ = g - A\varphi_n^- + (1/2)(f_n + f_n^*)$, by the property of the sequence f_n, f_n^* ,

$$\lim_{n \rightarrow \infty} \|S\varphi_n^+\|_{-1}^2 = \lim_{n \rightarrow \infty} \|g - A\varphi_n^-\|_{-1}^2 .$$

We have therefore proved that

$$\langle |\mathbf{D}u_0^\epsilon|^2 \rangle = \lim_{n \rightarrow \infty} \{ \|g - A\varphi_n^-\|_{-1}^2 + \|\varphi_n^-\|_1^2 \} .$$

In particular,

$$\langle |\mathbf{D}u_0^\epsilon|^2 \rangle \geq \inf_{\varphi \in \mathcal{C}_{-1}} \{ \|g - A\varphi\|_{-1}^2 + \|\varphi\|_1^2 \} .$$

In contrast, recall that

$$\langle |\mathbf{D}u_0^\epsilon|^2 \rangle = \lim_{n \rightarrow \infty} \{ \|\varphi_n^+\|_1^2 + \|\varphi_n^-\|_1^2 \} .$$

For each fixed n , we may rewrite the right hand side as $2\|\varphi_n^+\|_1^2 + 2\|\varphi_n^-\|_1^2 - \|\varphi_n^+\|_1^2 - \|S\varphi_n^-\|_{-1}^2$. We have already seen that $\|S\varphi_n^-\|_{-1}^2 = \|A\varphi_n^+\|_{-1}^2 + o_n(1)$ and that $\|\varphi_n^-\|_1^2 = -\langle \varphi_n^-, A\varphi_n^+ \rangle + o_n(1)$. Since A is antisymmetric and $A\varphi_n^- = g - S\varphi_n^+ + (1/2)(f_n + f_n^*)$,

$$\lim_{n \rightarrow \infty} \|\varphi_n^-\|_1^2 = \lim_{n \rightarrow \infty} 2\langle g - S\varphi_n^+, \varphi_n^+ \rangle .$$

In conclusion,

$$\begin{aligned} \langle |\mathbf{D}u_0^\epsilon|^2 \rangle &= \lim_{n \rightarrow \infty} \{ 2\langle g, \varphi_n^+ \rangle - \|\varphi_n^+\|_1^2 - \|A\varphi_n^+\|_{-1}^2 \} \\ &\leq \sup_{\varphi \in \mathcal{C}_{-1}} \{ 2\langle g, \varphi \rangle - \|\varphi\|_1^2 - \|A\varphi\|_{-1}^2 \} . \end{aligned}$$

Up to this point we proved that

$$\begin{aligned} \inf_{\varphi \in \mathcal{C}_{-1}} \{ \|\mathbf{F} \cdot \mathbf{e} - A\varphi\|_{-1}^2 + \|\varphi\|_1^2 \} &\leq \langle |\mathbf{D}u_0^\mathbf{e}|^2 \rangle \\ &\leq \sup_{\varphi \in \mathcal{C}_{-1}} \{ 2\langle \mathbf{F} \cdot \mathbf{e}, \varphi \rangle - \|A\varphi\|_{-1}^2 - \|\varphi\|_1^2 \} . \end{aligned} \quad (5.3)$$

To conclude the proof of the proposition, it remains to check the reversed inequalities. By the variational formula for the \mathcal{H}_{-1} norm,

$$\|A\varphi\|_{-1}^2 = \sup_{\psi \in \mathcal{C}} \{ 2\langle \psi, A\varphi \rangle - \|\psi\|_1^2 \} .$$

Hence,

$$\begin{aligned} &\sup_{\varphi \in \mathcal{C}_{-1}} \{ 2\langle \mathbf{F} \cdot \mathbf{e}, \varphi \rangle - \|A\varphi\|_{-1}^2 - \|\varphi\|_1^2 \} \\ &= \sup_{\varphi \in \mathcal{C}_{-1}} \inf_{\psi \in \mathcal{C}} \{ 2\langle \mathbf{F} \cdot \mathbf{e} + A\psi, \varphi \rangle + \|\psi\|_1^2 - \|\varphi\|_1^2 \} . \end{aligned}$$

Since $\sup_{\varphi} \inf_{\psi} B(\varphi, \psi) \leq \inf_{\psi} \sup_{\varphi} B(\varphi, \psi)$, replacing ψ by $-\psi$, we have that

$$\begin{aligned} &\sup_{\varphi \in \mathcal{C}_{-1}} \{ 2\langle \mathbf{F} \cdot \mathbf{e}, \varphi \rangle - \|A\varphi\|_{-1}^2 - \|\varphi\|_1^2 \} \\ &\leq \inf_{\psi \in \mathcal{C}} \{ \|\mathbf{F} \cdot \mathbf{e} - A\psi\|_{-1}^2 + \|\psi\|_1^2 \} \\ &\leq \inf_{\psi \in \mathcal{C}_{-1}} \{ \|\mathbf{F} \cdot \mathbf{e} - A\psi\|_{-1}^2 + \|\psi\|_1^2 \} \end{aligned}$$

because $\mathcal{C}_{-1} \subset \mathcal{C}$. This estimate in addition to (5.3) concludes the proof. \square

The well known bound (cf. [TKS]) states that

$$\mathbf{e} \cdot a(v)\mathbf{e} - |e|^2 = 2\langle |\mathbf{D}u_0^\mathbf{e}|^2 \rangle \leq 2\langle |\mathbf{H}|^2 \rangle .$$

This follows by dropping the term $\|A\varphi\|_{-1}^2$ in (5.1) and thus

$$\langle |\mathbf{D}u_0^\mathbf{e}|^2 \rangle \leq \|\mathbf{F} \cdot \mathbf{e}\|_{-1}^2 \leq \langle |\mathbf{H}|^2 \rangle .$$

In dimension 2 and for time independent fields we obtain a more interesting bound on the effective diffusivity on the direction orthogonal to the mean drift \mathbf{v} . For $d = 2$, choose a direction \mathbf{e} such that $\mathbf{v} \cdot \mathbf{e} = 0$. In this case \mathbf{F} can be written as $\mathbf{F} = \mathbf{D}^\perp H$ for a bounded smooth stream function H . Then, as a result of what proved in the previous sections, there exists $u_0^\mathbf{e}$ such that

$$\mathbf{D}^2 u_0^\mathbf{e} + \mathbf{e}_\perp \cdot (|\mathbf{v}|\mathbf{D}u_0^\mathbf{e} - \mathbf{D}H) + \mathbf{D}^\perp H \cdot \mathbf{D}u_0^\mathbf{e} = 0$$

in \mathcal{H}_{-1} . Then testing this equation against u_0^e and H one obtains the relations

$$\langle |\mathbf{D}u_0^e|^2 \rangle = \langle H\mathbf{e}_\perp \cdot \mathbf{D}u_0^e \rangle = \frac{1}{|\mathbf{v}|} \langle \mathbf{D}H \cdot (\mathbf{I} + \mathbf{H})\mathbf{D}u_0^e \rangle$$

so by Schwarz inequality

$$\mathbf{e} \cdot a(v\mathbf{e}_\perp)\mathbf{e} - |\mathbf{e}|^2 = \langle |\mathbf{D}u_0^e|^2 \rangle \leq \frac{1}{|\mathbf{v}|^2} (1 + \|H\|_\infty)^2 \langle |\mathbf{D}H|^2 \rangle .$$

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