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Homogenization of a bond diffusion in a locally ergodic random environment

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Abstract

We consider a nearest neighbors random walk on $\mathbb{Z}$. The jump rate from site x to site $x+1$ is equal to the jump rate from $x+1$ to x and is a bounded, strictly positive random variable $\eta(x)$. We assume that $\{\eta(x)\}_{x\in\mathbb{Z}}$ is distributed by a *locally ergodic* probability measure. We prove that, under diffusive scaling of space and time, the random walk converges in distribution to the diffusion process on $\mathbb{R}$ with infinitesimal generator $\mathrm{d}/\mathrm{d}X(a(X)\mathrm{d}/\mathrm{d}X)$, for a certain homogenized diffusion function $a(X)$, independent of η. The main tools of the proof are a local ergodic result and the explicit solution of the corresponding Poisson equation.

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1. Introduction

Homogenization and other invariance principles for random walks in random environments have been widely studied in the case when the distribution of the random environment is invariant and ergodic with respect to space translations (e.g. cf. Papanicolaou and Varadhan, 1979; Anshelevich and Vologodskii, 1981; Anshelevich et al., 1982; Kunnemann, 1983; Kozlov, 1985; Kipnis and Varadhan, 1986; De Masi

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et al., 1989). In these cases, the macroscopic process is given by a Brownian motion with an effective constant diffusion matrix.

We are interested here in the study of homogenization in random environments which are *locally ergodic* with respect to translations. Typically, these are distributions depending on parameters that are slowly varying functions in the space variable, i.e. parameters that change smoothly in the macroscopic scale and are almost constant in the microscopic one. Examples can be given by inhomogeneous product measures or by local Gibbs measures, like those appearing in hydrodynamic limits for interacting particles systems (cf. Kipnis and Landim, 1999). In fact, it is the search of invariance principles for tagged particles in interacting particles systems in non-equilibrium that initially motivates us in this research. For some result in this direction we refer to Siri (1998) and Grigorescu (1999).

We are not aware of any invariance principle result in this non-translation invariant context, except for the *locally periodic* (non-random) cases studied in Bensoussan et al. (1978). Some weaker form of the homogenization problem for elliptic equations in random environment has been treated in Bourgeat et al. (1994). We will discuss at the end of this section about the difference of our result with respect to the one contained in Bourgeat et al. (1994).

We consider a random walk on $\mathbb{Z}$ with random jump rates. The model is described as follows: one particle performs a random walk on $\mathbb{Z}$ to the nearest neighbors. The random environment is given by the jump rates across each bond: to any nearest neighbor bond $(x, x \pm 1)$, $x \in \mathbb{Z}$, we associate the corresponding jump rate $\eta_{\pm 1}(x)$; moreover, we suppose that $\eta_1(x) = \eta_{-1}(x+1)$ and we denote both with $\eta(x)$. We assume $\eta(x)$ bounded and strictly positive, i.e. there exist two constants c^+, c^-, such that $0 < c^- \leqslant \eta(x) \leqslant c^+ < +\infty$.

Let us indicate with $\Omega \doteq [c^-, c^+]^{\mathbb{Z}}$ the space of environments and with $\eta = \{\eta(x)\}_{x\in\mathbb{Z}}$ any fixed configuration in Ω. We denote with $\langle\cdot\rangle_m$ the expectation according to any measure m on the configurations space. Let $\{\tau_z, z \in \mathbb{Z}\}$ be the group of translations on Ω defined by $\tau_z\eta(x) = \eta(x+z)$, $\forall x \in \mathbb{Z}$.

For each fixed configuration $\eta \in \Omega$, let $(x_t^\eta)_{t\geqslant 0}$ be the random walk performed by the particle, with $x_0^\eta = 0$. The corresponding infinitesimal generator is given, for any function f on $\mathbb{Z}$, by

$$L^\eta f(x) = -\nabla^*[\eta(x)\nabla f(x)], \tag{1}$$

where we have denoted $\nabla f(x) = f(x+1) - f(x)$ and $\nabla^* f(x) = f(x-1) - f(x)$, $\forall x \in \mathbb{Z}$.

The aim of this paper is to consider the rescaled process $(X_t^{\eta,\varepsilon})_{t\geqslant 0}$, defined by $X_t^{\eta,\varepsilon} = \varepsilon x_{\varepsilon^{-2}t}^\eta$ and to study the limit as $\varepsilon \to 0$, in order to establish its diffusive behavior.

If m is a measure on Ω, ergodic with respect to the translations group, then $X_t^{\eta,\varepsilon}$ converges in law to a Brownian motion with diffusion coefficient given by $2\langle\eta(0)^{-1}\rangle_m^{-1}$ (cf. Anshelevich and Vologodskii, 1981; Anshelevich et al., 1982; Kunnemann, 1983; Kipnis and Varadhan, 1986; De Masi et al., 1989). This convergence can be proved almost everywhere with respect to m.

We generalize this result to *locally ergodic* distributions on the random environments space Ω. We say that a sequence of probability distributions $\{\mu_\varepsilon\}_{\varepsilon>0}$ is locally ergodic if there exists a family of ergodic probability distributions $\{\bar{\mu}_X, X \in \mathbb{R}\}$ such that, for any bounded measurable function $f(X,\eta)$ on $\mathbb{R} \times \Omega$ continuous in X and local in η, we have the convergence in μ_ε-probability:

$$\varepsilon \sum_z g(\varepsilon z) f(\varepsilon z, \tau_z \eta) \underset{\varepsilon \to 0}{\longrightarrow} \int g(X) \langle f(X, \cdot) \rangle_{\bar{\mu}_X} \, \mathrm{d}X, \tag{2}$$

for any bounded continuous function g with compact support on $\mathbb{R}$.

We need some minimal assumptions on the regularity of the family of measures $\{\bar{\mu}_X, X \in \mathbb{R}\}$ with respect to X.

(a) For any bounded local function $f(\eta)$ on Ω, the function $\hat{f}(X) = \langle f \rangle_{\bar{\mu}_X}$ is locally integrable on $\mathbb{R}$.

(b) Let us define the function a on $\mathbb{R}$ by $a(X) \doteq \langle \eta(0)^{-1} \rangle_{\bar{\mu}_X}^{-1}$. We assume that $a \in C^2(\mathbb{R})$, with bounded derivatives.

Our main result can then be formulated as follows:

Theorem 1.1. *The law of the rescaled process* $(X_t^{\eta,\varepsilon})_{t \geqslant 0}$ *converges, in probability with respect to* μ_ε, *to the distribution of the diffusion process on* $\mathbb{R}$ *with infinitesimal generator*

$$Lf(X) = \frac{\mathrm{d}}{\mathrm{d}X}\left(a(X) \frac{\mathrm{d}}{\mathrm{d}X} f(X) \right), \quad \forall f \in C^2(\mathbb{R}). \tag{3}$$

The structure of the paper is as follows. In Section 2, we recall some basic estimates on the random walk which are uniformly valid for any realization of the environment η. In particular, the Nash–Aronson estimate will guarantee the tightness of the rescaled process (following the same argument as in Section 9 of Papanicolaou and Varadhan, 1979). In Section 3, we show that the *space* local ergodicity assumption (2) implies a local ergodicity in time. Then in Section 4, we identify the diffusion and the drift functions of the limit process. It is in particular in this last section that we use the one-dimensionality of the process. In fact, we need the explicit expression of the correctors in order to prove their sublinear growth (cf. formulas (8) and (9)). We have not found yet a successful strategy in order to deal with this point of the proof in more dimensions, i.e. without using the explicit form of the correctors.

Observe that Bourgeat et al. (1994) do not make any local ergodicity assumption, and a more abstract result is stated for the homogenization of an elliptic equation in random field.

2. Uniform estimates on the random walk and tightness

We recall here some estimates for random walks with inhomogeneous rates. These estimates depend only on the bounds of the jump rates, so they are valid for our random walk, uniformly with respect to the realization of the environment.

The following proposition says that a particle performing a random walk with uniformly bounded jump rates, essentially does not move away from a suitable box throughout a finite time interval. This will be useful in the next section to prove the local ergodic result.

Proposition 2.1. *Let* $(x_t)_{t\geqslant 0}$ *be a random walk on* $\mathbb{Z}$, *with strictly positive jump rates which are uniformly bounded by a constant* $c^+>0$. *Then,* $\forall\ 0\leqslant S<T,\ h>2c^+(T-S)$,

$$\mathbb{P}\left[\sup_{S\leqslant t\leqslant T}|x_t-x_S|>h\right]\leqslant \exp\left(-\frac{h}{2}\log\frac{h}{2c^+(T-S)}+\frac{h}{2}-c^+(T-S)\right).$$

Proof. Let us observe that the process $(x_t-x_0)_{t\geqslant 0}$ can be written as the difference of two bistochastic Poisson processes: $(N_t^-)_{t\geqslant 0}$ counting the jumps to the left and $(N_t^+)_{t\geqslant 0}$ the jumps to the right. Let us also denote with $(N_t)_{t\geqslant 0}$ the Poisson process with constant rate c^+; then

$$\mathbb{P}\left[\sup_{S\leqslant t\leqslant T}|x_t-x_S|>h\right]$$

$$\leqslant \mathbb{P}\left[\sup_{S\leqslant t\leqslant T}[(N_t^+-N_S^+)+(N_t^--N_S^-)]>h\right]\leqslant \mathbb{P}\left[N_{T-S}>\frac{h}{2}\right].$$

If N is a Poisson random variable, with parameter λ, then, for any $\beta\geqslant 0$,

$$\mathbb{P}[N>h]=\mathbb{P}[\mathrm{e}^{\beta N}>\mathrm{e}^{\beta h}]\leqslant \mathrm{e}^{-\beta h}\mathbb{E}[\mathrm{e}^{\beta N}]=\mathrm{e}^{-\beta h}\mathrm{e}^{\lambda(\mathrm{e}^{\beta}-1)},$$

from which, if $h>\lambda$,

$$\log\mathbb{P}[N>h]\leqslant \inf_{\beta\geqslant 0}[-\beta h+\lambda(\mathrm{e}^{\beta}-1)]=-h\log\frac{h}{\lambda}+h-\lambda,$$

and the thesis follows. □

Finally, we recall the Nash–Aronson estimate for the transition kernel of a Markov semigroup generated by a divergence form operator. Such an estimate can be found in the literature in the most general cases. For diffusion processes, also with time-dependent coefficients, see for example Aronson (1967), Fabes and Stroock (1987). In the discrete set up (random walks on $\mathbb{Z}^d$) the case of time-independent jump rates is widely developed by Carlen et al. (1987) and Stroock and Zheng (1997), while the one with time-dependent rates is treated by Giacomin et al. (2001). Here we recall the estimate only in our very particular case of time-independent random walk (cf. Stroock and Zheng, 1997).

Theorem 2.2 (Nash–Aronson estimate). *Let* $(x_t)_{t\geqslant 0}$ *be a uniformly elliptic random walk on* $\mathbb{Z}$ *with bounded, time-independent jump rates. Then, there exists a constant* $C_0>1$, *depending only on the bounds, such that,* $\forall x,y\in\mathbb{Z}$,

$$p_t(x,y)\leqslant\frac{C_0}{1\vee\sqrt{t}}\exp\left(-\frac{|y-x|}{1\vee\sqrt{t}}\right),$$

where $p_t(x,y)$ *defines the transition kernel of the process.*

As observed in Papanicolaou and Varadhan (1979), one of the immediate consequences of the Nash–Aronson estimate in this context is the tightness of the distribution of the rescaled process $(X_t^{\eta,\varepsilon})_{t\geqslant 0}$. In fact, it easily follows from this estimate that, for any $\eta \in \Omega$,

$$\mathbb{E}_\eta^0(|X_t^{\eta,\varepsilon} - X_s^{\eta,\varepsilon}|^4) \leqslant C(t-s)^2$$

and consequently

$$\mathbb{E}_{\mu_\varepsilon}^0(|X_t^{\eta,\varepsilon} - X_s^{\eta,\varepsilon}|^4) \leqslant C(t-s)^2.$$

Therefore, all we have to do is to identify the limit process on $\mathscr{C}([0,\infty),\mathbb{R})$. In the following two sections we will show that, under the local ergodicity condition (2), this limit process is unique and is the one described in Theorem 1.1.

3. A local ergodic result

In this section, we show that the local ergodicity assumption implies a *local ergodicity in time* property for the environment as seen from the particle.

Consider the Markov process $(\xi_t)_{t\geqslant 0}$, describing the evolution of the environment as seen from the particle, i.e. given by

$$\begin{cases} \xi_t = \tau_{x_t^\eta}\eta, \\ \xi_0 = \eta, \end{cases} \tag{4}$$

for any fixed initial configuration $\eta \in \Omega$.

It is easy to see that every translation invariant measure on Ω is stationary for the process ξ_t, and that ergodicity with respect to translations implies ergodicity for this process. The difficulty here is that our measure μ_ε is not space translation invariant, and, as a consequence, it is not stationary in time for the process ξ_t.

Proposition 3.1. *Let $\phi : \mathbb{R}\times\Omega \to \mathbb{R}$ be a bounded function, smooth in the first argument and local in the second one, such that $\langle \phi(X,\cdot)\rangle_{\bar{\mu}_X} = 0, \forall X \in \mathbb{R}$. Then, $\forall T > 0$,*

$$\lim_{\varepsilon\to 0} \mathbb{E}_{\mu_\varepsilon}^0 \left[\sup_{0\leqslant t\leqslant T} \left(\varepsilon^2 \int_0^{\varepsilon^{-2}t} \phi(\varepsilon x_r, \tau_{x_r}\eta)\,\mathrm{d}r \right)^2 \right] = 0.$$

Proof. Let us divide the interval $[0,\varepsilon^{-2}t]$ into subintervals of fixed length k (suppose $k > 1$). Using Schwarz inequality, we get

$$\mathbb{E}_{\mu_\varepsilon}^0 \left[\sup_{0\leqslant t\leqslant T} \left(\varepsilon^2 \int_0^{\varepsilon^{-2}t} \phi(\varepsilon x_r, \tau_{x_r}\eta)\,\mathrm{d}r \right)^2 \right]$$

$$\leqslant T\varepsilon^2 k \sum_{i=0}^{[T/\varepsilon^2 k]-1} \mathbb{E}_{\mu_\varepsilon}^0 \left[\left(\frac{1}{k} \int_{ik}^{(i+1)k} \phi(\varepsilon x_r, \tau_{x_r}\eta)\,\mathrm{d}r \right)^2 \right]. \tag{5}$$

By Nash–Aronson estimate (cf. Theorem 2.2) and the homogeneous Markov property, the right-hand side of (5) is bounded above by

$$T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}} \mathrm{e}^{-|y|/\sqrt{ik}} \mathbb{E}^y_{\mu_\varepsilon}\left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon x_r, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2\right] + T\varepsilon^2 k\|\phi\|_\infty^2.$$

Let us now define the set of trajectories

$$A_k \doteq \left\{(x_\cdot)\in\mathscr{D}([0,\infty),\mathbb{Z}) : \sup_{0\leqslant r\leqslant k}|x_r - x_0| \leqslant k^\alpha\right\}, \quad \text{with } \alpha > 1 \text{ fixed.}$$

By Proposition 2.1, $\mathbb{P}^y_{\mu_\varepsilon}(A_k^{\mathrm{c}}) \to 0$ as $k\to\infty$, uniformly in $y\in\mathbb{Z}$ and $\varepsilon > 0$. Then, since ϕ is bounded,

$$\begin{aligned}
&T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}}\, \mathbb{E}^y_{\mu_\varepsilon}\left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon x_r, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k^{\mathrm{c}}}\right]\\
&\quad\leqslant T\|\phi\|_\infty^2 \varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}} \mathbb{P}^y_{\mu_\varepsilon}(A_k^{\mathrm{c}})\\
&\quad\leqslant CT^2\|\phi\|_\infty^2 \sup_{y\in\mathbb{Z}}\sup_{\varepsilon>0} \mathbb{P}^y_{\mu_\varepsilon}(A_k^{\mathrm{c}}) \to 0
\end{aligned}$$

as $k\to\infty$.

On the other hand, since $\phi(X,\eta)$ is smooth in X, one can easily show that

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{\varepsilon\to 0} T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}}\, \mathbb{E}^y_{\mu_\varepsilon}\left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon x_r, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right]\\
&\quad= \lim_{k\to\infty}\lim_{\varepsilon\to 0} T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}}\, \mathbb{E}^y_{\mu_\varepsilon}\left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right].
\end{aligned}$$

Moreover, the following lemma holds:

Lemma 3.2.

$$\begin{aligned}
&\lim_{\varepsilon\to 0} T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}}\\
&\quad\times\left(\mathbb{E}^y_{\mu_\varepsilon}\left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right]\right.\\
&\quad\left. -\mathbb{E}^y_{\bar{\mu}_{\varepsilon y}}\left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right]\right) = 0.
\end{aligned}$$

As a consequence of the above lemma, we are left to prove that

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0} T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}}\, \mathbb{E}^{y}_{\bar\mu_{\varepsilon y}} \left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2\right] = 0. \tag{6}$$

Now, since the measure $\bar\mu_{\varepsilon y}$ is translation invariant,

$$\mathbb{E}^{y}_{\bar\mu_{\varepsilon y}} \left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2\right] = \mathbb{E}^{0}_{\bar\mu_{\varepsilon y}} \left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \xi_r)\,\mathrm{d}r\right)^2\right],$$

so the left-hand side of (6) can be rewritten as

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0} T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \varepsilon\sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik\varepsilon^2}}\, \mathrm{e}^{-|\varepsilon y|/\sqrt{ik\varepsilon^2}}\, \mathbb{E}^{0}_{\bar\mu_{\varepsilon y}} \left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \xi_r)\,\mathrm{d}r\right)^2\right]$$

$$\leqslant \lim_{k\to\infty} T\int_0^T \mathrm{d}s \int \mathrm{d}X \frac{C_0}{\sqrt{s}} \mathrm{e}^{-|X|/\sqrt{s}} \mathbb{E}^{0}_{\bar\mu_X} \left[\left(\frac{1}{k}\int_0^k \phi(X, \xi_r)\,\mathrm{d}r\right)^2\right].$$

By the ergodic theorem in L^2 this last quantity is equal to zero. □

Proof of Lemma 3.2. Define

$$f(X,\eta) = \mathbb{E}^0_\eta \left[\left(\frac{1}{k}\int_0^k \phi(X, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right].$$

Then

$$f(\varepsilon y, \tau_y\eta) = \mathbb{E}^0_{\tau_y\eta} \left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r+y}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right]$$

$$= \mathbb{E}^y_\eta \left[\left(\frac{1}{k}\int_0^k \phi(\varepsilon y, \tau_{x_r}\eta)\,\mathrm{d}r\right)^2 \mathbf{1}_{A_k}\right].$$

Observe that $f(X,\eta)$ is continuous in X and local in η, so by the local ergodicity assumption, for any bounded continuous function g with compact support on $\mathbb{R}$:

$$\varepsilon\sum_{y\in\mathbb{Z}} g(\varepsilon y)\left(\langle f(\varepsilon y, \tau_y\eta)\rangle_{\mu_\varepsilon} - \langle f(\varepsilon y, \tau_y\eta)\rangle_{\bar\mu_{\varepsilon y}}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

From this, after some elementary steps (by approximating the exponential function with a compact support function), we obtain

$$T\varepsilon^2 k \sum_{i=1}^{[T/\varepsilon^2 k]-1} \sum_{y\in\mathbb{Z}} \frac{C_0}{\sqrt{ik}}\, \mathrm{e}^{-|y|/\sqrt{ik}} \left(\langle f(\varepsilon y, \tau_y\eta)\rangle_{\mu_\varepsilon} - \langle f(\varepsilon y, \tau_y\eta)\rangle_{\bar\mu_{\varepsilon y}}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad \square$$

4. Proof of the main result

Proof of Theorem 1.1. Consider the function on $\mathbb{Z} \times \Omega$

$$\chi(x,\eta) = \operatorname{sign}(x) \sum_{z=x\wedge 0}^{(x-1)\vee -1} \left(1 - \frac{a(\varepsilon z)}{\eta(z)}\right), \tag{7}$$

with $\chi(0,\eta)=0$, and observe that it solves the Poisson equation

$$-L^\eta \chi(x,\eta) = \nabla^*[\eta(x) - a(\varepsilon x)],$$

for each fixed $\eta \in \Omega$.

Since $L^\eta x = -\nabla^* \eta(x)$, there exists a martingale $(\mathscr{M}_t^\eta)_{t\geqslant 0}$, such that

$$\begin{aligned} X_t^{\eta,\varepsilon} &= \varepsilon \mathscr{M}^\eta_{\varepsilon^{-2}t} - \varepsilon \int_0^{\varepsilon^{-2}t} \nabla^* \eta(x_s^\eta)\,\mathrm{d}s \\ &= \varepsilon \mathscr{M}^\eta_{\varepsilon^{-2}t} + \varepsilon \int_0^{\varepsilon^{-2}t} L^\eta \chi(x_s^\eta,\eta)\,\mathrm{d}s - \varepsilon \int_0^{\varepsilon^{-2}t} \nabla^* a(\varepsilon x_s^\eta)\,\mathrm{d}s. \end{aligned}$$

Let us consider the martingale $\mathscr{N}_t^\eta = \chi(x_t^\eta,\eta) - \int_0^t L^\eta \chi(x_s^\eta,\eta)\,\mathrm{d}s$. Then, $X_t^{\eta,\varepsilon}$ can be written as

$$X_t^{\eta,\varepsilon} = \varepsilon(\mathscr{M}^\eta_{\varepsilon^{-2}t} - \mathscr{N}^\eta_{\varepsilon^{-2}t}) - \varepsilon \int_0^{\varepsilon^{-2}t} \nabla^* a(\varepsilon x_s^\eta)\,\mathrm{d}s + \varepsilon \chi(x^\eta_{\varepsilon^{-2}t},\eta).$$

Denoting $f(x) = x - \chi(x,\eta)$, the square integrable martingale $\varepsilon(\mathscr{M}^\eta_{\varepsilon^{-2}t} - \mathscr{N}^\eta_{\varepsilon^{-2}t})$ has quadratic variation

$$\begin{aligned} A_t^{\eta,\varepsilon} &\doteq \varepsilon^2 \int_0^{\varepsilon^{-2}t} \left(L^\eta f^2(x_s^\eta) - 2 f(x_s^\eta) L^\eta f(x_s^\eta)\right)\,\mathrm{d}s \\ &= \varepsilon^2 \int_0^{\varepsilon^{-2}t} (\eta(x_s^\eta)[f(x_s^\eta+1) - f(x_s^\eta)]^2 + \eta(x_s^\eta-1)[f(x_s^\eta) - f(x_s^\eta-1)]^2)\,\mathrm{d}s \\ &= \varepsilon^2 \int_0^{\varepsilon^{-2}t} (\eta(x_s^\eta)[1 - \nabla\chi(x_s^\eta,\eta)]^2 + \eta(x_s^\eta-1)[1 - \nabla\chi(x_s^\eta-1,\eta)]^2)\,\mathrm{d}s \\ &= \varepsilon^2 \int_0^{\varepsilon^{-2}t} \left(\frac{a^2(\varepsilon x_s^\eta)}{\eta(x_s^\eta)} + \frac{a^2(\varepsilon(x_s^\eta-1))}{\eta(x_s^\eta-1)}\right)\,\mathrm{d}s. \end{aligned}$$

Then, by Proposition 3.1 we have

$$\lim_{\varepsilon\to 0} \mathbb{E}^0_{\mu_\varepsilon}\left[\sup_{0\leqslant t\leqslant T}\left(A_t^{\eta,\varepsilon} - \int_0^t 2a(X_r^{\eta,\varepsilon})\,\mathrm{d}r\right)^2\right] = 0.$$

Observing that $\varepsilon^{-1}\nabla^* a(\varepsilon x_s^\eta)$ behaves in the limit like $-a'(\varepsilon x_s^\eta)$, by the smoothness hypotheses on a, we also obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}^0_{\mu_\varepsilon}\left[\sup_{0\leqslant t\leqslant T}\left(\varepsilon\int_0^{\varepsilon^{-2}t}\nabla^* a(\varepsilon x_s^\eta)\,\mathrm{d}s+\int_0^t a'(X_r^{\eta,\varepsilon})\,\mathrm{d}r\right)^2\right]=0.$$

About the boundary term, we have

$$\mathbb{E}^0_{\mu_\varepsilon}[(\varepsilon\chi(x^\eta_{\varepsilon^{-2}t},\eta))^2]=\mathbb{E}^0_{\mu_\varepsilon}\left[\varepsilon^2\left(\operatorname{sign}(x^\eta_{\varepsilon^{-2}t})\sum_{z=x^\eta_{\varepsilon^{-2}t}\wedge 0}^{(x^\eta_{\varepsilon^{-2}t}-1)\vee -1}\left(1-\frac{a(\varepsilon z)}{\eta(z)}\right)\right)^2\right], \tag{8}$$

which, using the Nash–Aronson estimate, can be bounded by

$$\sum_{y\in\mathbb{Z}\setminus\{0\}}\left\langle\left(\varepsilon\sum_{z=y\wedge 0}^{(y-1)\vee -1}\left(1-\frac{a(\varepsilon z)}{\eta(z)}\right)\right)^2\right\rangle_{\mu_\varepsilon}\frac{\varepsilon C_0}{\sqrt{t}}\,\mathrm{e}^{-\varepsilon|y|/\sqrt{t}}. \tag{9}$$

By the local ergodicity assumption

$$\varepsilon\sum_{z=\varepsilon^{-1}Y\wedge 0}^{(\varepsilon^{-1}Y-1)\vee -1}\left(1-\frac{a(\varepsilon z)}{\eta(z)}\right)\underset{\varepsilon\to 0}{\to}\operatorname{sign}(Y)\int_0^Y\left\langle\left(1-\frac{a(X)}{\eta(0)}\right)\right\rangle_{\bar{\mu}_X}\mathrm{d}X=0,$$

in μ_ε-probability.

This implies that (9) converges to 0 as $\varepsilon\to 0$.

We have then identified the limit process as the diffusion on $\mathbb{R}$ with diffusion coefficient $2a(X)$ and drift $a'(X)$, and this concludes the argument. □

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