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Equilibrium Fluctuations for Interacting Ornstein-Uhlenbeck Particles*

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Abstract: We study the hydrodynamic density fluctuations of an infinite system of interacting particles on $\mathbb{R}^d$. The particles interact between them through a two body superstable potential, and with a surrounding fluid in equilibrium through a random viscous force of Ornstein-Uhlenbeck type. The stationary initial distribution is the Gibbs measure associated with the potential and with a given temperature and fugacity. We prove that the time-dependent density fluctuation field converges in law, under diffusive scaling of space and time, to the solution of a linear stochastic partial differential equation driven by white noise.

1. Introduction

Consider an aqueous suspension of particles in equilibrium at temperature $T = 1/\beta$. Let the interaction between the particle be given by a two body potential V. We assume that V is superstable, positive and smooth with compact support. The interaction with the fluid is modeled by an Ornstein-Uhlenbeck type force (linear viscosity plus white noise) such that the particle velocities are maintained in equilibrium, i.e. distributed by a Maxwellian distribution of temperature T (cf. [14, 20]). The dynamics of this model is given by the solution of the following infinite system of stochastic differential equations:

$$\begin{aligned} dx_j(t) &= v_j(t)dt, \\ dv_j(t) &= -\sum_{i\neq j} \nabla V(x_j(t) - x_i(t))dt - \gamma v_j(t)dt + \sqrt{\frac{2\gamma}{\beta}} dw_j(t). \end{aligned} \tag{1.1}$$

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Here $\{w_j(t)\}$ are independent standard Wiener processes, and $\gamma > 0$ is the friction coefficient. We set here the mass of each particle equal to 1. We refer to this system as the interacting Ornstein-Uhlenbeck particles.

Consider the grand canonical Gibbs measures $\mu_{z,\beta}$ corresponding to the formal Hamiltonian

$$\mathfrak{H} = \sum_j \frac{v_j^2}{2} + \frac{1}{2}\sum_{i \neq j} V(x_j - x_i) \tag{1.2}$$

with fugacity z and inverse temperature β.

As in the case of the corresponding deterministic Hamiltonian system ($\gamma = 0$), the proof of the existence of the dynamics given by (1.1) for a *wide* set of initial configurations is a challenging problem. In [8], J. Fritz proves the existence and uniqueness of the solution of (1.1) for a class of initial configurations that has probability 1 for any grand canonical Gibbs measure. The results contained in [8] are limited to dimension $d \leq 2$. For our purposes it is enough to consider the existence of the *equilibrium dynamics*: for a given $\mu_{z,\beta}$ there exists a set of initial configurations and a set of realizations of the Wiener processes $\{w_j\}$ such that they have full measure and for which (1.1) has a non-exploding solution (see Sect. 2 for a precise definition). We prove this existence theorem in Appendix A (Sect. 7), by approximation with finite local dynamics. The proof, that has the advantage that it works in any dimension, follows the classical approach of Lanford ([11, 13]) adapted to our stochastic case. A certain care should be done in this proof for existence of the dynamics when dealing with random evolutions. In fact an approximation by local dynamics defined using reflection on hard walls would bring difficulties concerning the existence of these local dynamics (cf. the paper of R. Lang with the addition of T. Shiga [12], where a similar problem arises in the context of the interacting Brownian motions). In order to avoid such problems, we apply an idea of J. Fritz ([9, 10]: define special *smooth* local dynamics such that the stationary Gibbs measures stay unmodified (cf. Eqs. (7.1)).

The purpose of this paper is to study the macroscopic behavior of this system in equilibrium. Let us consider the positions of the particles under diffusive rescaling:

$$x_j^\varepsilon(t) = \varepsilon x_j(\varepsilon^{-2}t), \tag{1.3}$$

where ε is a scaling parameter. We are interested in the macroscopic limit as $\varepsilon \to 0$.

The empirical distribution of the particles is a random positive measure on $\mathbb{R}^d$ defined by

$$n_t^\varepsilon(G) = \varepsilon^d \sum_j G(x_j^\varepsilon(t)),$$

where G is a smooth function with compact support. If the particles are distributed (in the original microscopic coordinates x_j) by the equilibrium Gibbs measure $\mu_{z,\beta}$, by the law of large numbers, we have

$$n_t^\varepsilon(G) \to \rho \int G(q)dq \qquad \mu_{z,\beta} - a.s.,$$

where $\rho = \rho(z, \beta)$ is the average density of particles per unit volume for $\mu_{z,\beta}$.

Then we consider the density fluctuation field

$$\xi_t^\varepsilon(G) = \varepsilon^{-d/2}\left(n_t^\varepsilon(G) - \rho\int G(q)dq\right) .$$

In equilibrium, the positions of the particles are distributed by $\mu_{z,\beta}$, and because of the mixing properties of this Gibbs measure one can show (cf. [1]) that $\xi_t^\varepsilon(J)$, for a fixed t, converges in law to a Gaussian field ξ on $\mathbb{R}^d$ with mean zero and covariance

$$< \xi(f)\xi(g) >= \chi\int f(q)g(q)\, dq, \tag{1.4}$$

where $\chi = \chi(z,\beta)$ is the compressibility of $\mu_{z,\beta}$.

We prove in this paper that ξ_t^ε, as a distribution valued process, converges in law to the solution ξ_t of the stochastic linear partial differential equation:

$$\partial_t\xi = D(\rho)\Delta\xi \ + \sqrt{\frac{2\rho}{\gamma\beta}}\ \nabla\cdot j, \tag{1.5}$$

where $\{j_i\}_{i=1,\dots,d}$ are δ-correlated space–time white noises, i.e. d-vector Gaussian fields on $\mathbb{R}^{d+1}$ with covariance

$$< j_i(q,t)j_h(q',s) >= \delta(q-q')\delta(t-s)\delta_{i,h}.$$

The diffusion coefficient $D(\rho)$ is identified as the derivative of the thermodynamic pressure as a function of the density, as in [17].

Equation (1.5) should be intended in the weak sense, i.e. for any smooth test function G,

$$\xi_t(G) - \xi_0(G) = \int_0^t \xi_s(D\Delta G)\, ds \ + \ M_t(\nabla G),$$

where $M_t(\nabla G)$ is a continuous martingale with constant quadratic variation given by $\frac{2\rho t}{\gamma\beta}\int|\nabla G(q)|^2 dq$. Since (1.5) is obtained in equilibrium, the initial condition ξ_0 is distributed by the Gaussian field with covariance given by (1.4), which is in fact the invariant law for the evolution given by (1.5). This implies the following identity for the bulk diffusion coefficient:

$$D = \frac{\rho}{\gamma\beta\chi}. \tag{1.6}$$

If we consider the solution of (1.1) in a long time scale, the velocities will relax to equilibrium. If we suppress them we obtain a closed evolution on the positions given by

$$dx_j(t) = -\frac{1}{\gamma}\sum_{i\neq j}\nabla V(x_j(t) - x_i(t))\, dt \ + \sqrt{\frac{2}{\gamma\beta}}\, dw_j(t). \tag{1.7}$$

We refer to this system as the interacting Brownian particles. In [20] Spohn proved the convergence of the density fluctuation field ξ_t^ε for (1.7). The limit equation is still given by (1.5). The equivalence of the bulk diffusion in the two systems could also be seen from the hydrodynamic limit out of equilibrium (cf. [24, 17]).

As noticed first by Rost [19], the main point in the proof of a theorem of equilibrium fluctuations for a system with conserved quantities is the so-called Boltzmann-Gibbs principle. This principle is an estimate of the space-time variance of the difference between a non-conserved quantity (typically the flux of the density fluctuation field) and its linearization along the conserved quantities (here the density fluctuation field itself). This principle has been proven in various reversible models (in [2] for zero-range models, in [6] for speed-change exclusion models and in [20] for the interacting Brownian particle system (1.7)). In all these papers the approach to the proof of the Boltzmann-Gibbs principle is the following: one defines an Hilbert space of functions of the configurations, with a scalar product related to the static variance of these functions with respect to the stationary measure of the process. Then one needs to identify the subspace that is invariant under the action of the semigroup of the generator of the process with the one dimensional subspace generated by the density fluctuation field. This identification follows by a strong version of the equivalence of the canonical and grand canonical Gibbs measures. One of the difficulties in this approach is the selfadjoint extension of the generator of the process in this new Hilbert space (observe that all these results concern reversible models).

In [4], C. C. Chang introduced a different, and simpler, approach to the proof of the Boltzmann-Gibbs principle. In Chang's approach, by using the same strong form of the equivalence of ensembles, the problem is reduced to the estimate of the space-time variance in the canonical measure in a fixed finite volume. Then this estimate can be done by standard finite dimensional analysis (cf. [7] Chapter 11, for a clear exposition of the method). Besides its simplicity, this approach has two other advantages: it avoids problems in defining the dynamics in a different Hilbert space, and it can be extended more easily to non-reversible models (cf. [5]).

Our proof follows the direction of this second approach. The difficulty here is due to the fact that the generator of our Markov process is degenerate in the directions of the positions of the particles (noise acts only on the velocities). This poses a problem even in the finite dimensional analysis needed to estimate the variance in the canonical measure on a finite box. We solve this problem by relating the (degenerate) generator of our process to the strictly elliptic and symmetric generator of the interacting Brownian particles process (cf. Proof of Proposition 4.2). Then we need an estimate of the spectral gap of this symmetric operator, that we prove in Appendix B.

For the strong form of the equivalence of ensembles, which is in fact a local central limit theorem for the Gibbs measures, we use the results by Spohn contained in [20] (recently improved in [3], cf. Sect. 2).

The paper is organized as follows: in Sect. 2 we give a precise definition of the model and of the dynamics in equilibrium. In Sect. 3 we study the evolution equations of the fluctuation field. In Sect. 4 we prove the *Boltzmann-Gibbs principle* for our system. In Sect. 5 we prove the tightness of the distribution of the fluctuation field as a stochastic process with values in a certain negative Sobolev space. In Sect. 6, by using the Boltzmann-Gibbs principle and the compactness result, we prove the convergence to the solution of (1.5). In Appendix A we prove the existence of the dynamics in equilibrium. In Appendix B we prove a lower bound on the spectral gap of the generator of the interacting Brownian motions (cf. (1.7)) in a finite box. We use this bound in order to prove some estimates on the derivatives of the solutions of the Poisson equation related to the generator of the interacting Brownian motions. We need these estimates in the proof of the Boltzmann-Gibbs principle.

2. The Dynamics in Equilibrium

We assume that $V : \mathbb{R}^d \to \mathbb{R}$ is a pair potential satisfying the following assumptions:

(i) V is central (i.e. depends only on $|q|$) and symmetric: $V(q) = (-q)$.
(ii) V is positive: $V \geq 0$, and $V(0) > 0$.
(iii) V has finite range: $V(q) = 0$ if $|q| \geq R$.
(iv) V is two times continuously differentiable.

Notice that the positivity implies immediately the lower regularity condition (LR) of [16]. Furthermore observe that (ii) is equivalent to superstability (cf. condition (SS) in [16]).

The configuration space Ω is defined as the set of locally finite labeled configurations of particles $\omega = \{(x_i, v_i), i \in \mathbb{N}\}$, where $x_i = x_i(\omega)$ and $v_i = v_i(\omega)$ have values in $\mathbb{R}^d$, and the sequence $x_i = x_i(\omega)$ has no accumulation points. We will use the notation $\omega^x = \{x_i(\omega)\}$. Let Ω be equipped with the weak topology: $\lim_n \omega_n = \omega$ means that $\lim_n x_i(\omega_n) = x_i(\omega)$ and $\lim_n v_i(\omega_n) = v_i(\omega)$. We denote by $\mathcal{B}$ the corresponding Borel σ-algebra of subsets of Ω.

A grand canonical Gibbs state for V at temperature β^{-1} and activity (or fugacity) z is a probability measure μ on $(\Omega, \mathcal{B})$ that distribute ω^x according to a grand canonical Gibbsian point field with pair interacting potential βV and activity z, while velocities are independent of positions and are identically distributed as independent Gaussian variables of zero mean and variance β^{-1}. This means that the μ-conditional probability to find n particles in a finite volume region $\Lambda \subset \mathbb{R}^d$ with configuration $\{(x_1, v_1), \ldots, (x_n, v_n)\}$, conditioned to the configuration outside Λ, i.e. $\omega_{\Lambda^c} = \{(x_i(\omega), v_i(\omega)), x_i(\omega) \in \Lambda^c, i \in \mathbb{N}\}$, has density with respect to the Lebesgue measure on $(\Lambda \times \mathbb{R}^d)^n$ given by the DLR equations:

$$\mu_\Lambda((x_1, v_1), \ldots, (x_n, v_n)|\omega_{\Lambda^c}) = \frac{1}{Z_\Lambda(\omega^x_{\Lambda^c})} \frac{z^n}{n!} \exp\left[-\beta \mathfrak{H}_n((x_1, v_1), \ldots, (x_n, v_n); \omega^x_{\Lambda^c})\right], \tag{2.1}$$

where

$$\mathfrak{H}_n((x_1, v_1), \ldots, (x_n, v_n); \omega^x_{\Lambda^c}) = \sum_{j=1}^n \left\{ \frac{v_j^2}{2} + \frac{1}{2} \sum_{i=1}^n V(x_j - x_i) + \sum_{y_i \in \omega^x_{\Lambda^c}} V(x_j - y_i) \right\}$$

and $Z_\Lambda(\omega^x_{\Lambda^c})$ is the corresponding normalization.

In order to simplify notations we fix the values $\gamma = \beta = 1$, and, since the activity $z \in (0, z_0)$ is fixed, we will omit. Writing the explicit dependence on z and we write $\mu = \mu_{z,\beta}$, and $\rho = \rho(z, \beta)$ for the corresponding density of particles.

In the following we will assume that

$$0 < z < z_0 = 0.28 \left(e \int (1 - e^{-V(q)})\, dq \right)^{-1}. \tag{2.2}$$

We need this condition on the activity in order to apply Spohn's results on the equivalence of ensembles (cf. [20]). As a consequence we are in the low fugacity regime where the grand canonical Gibbs measure is unique. A recent paper by Cancrini and Tremoulet (cf. [3]) permits to improve (2.2) substituting the constant 0.28 with 1/3, that is the sufficient condition for the exponential L^2-mixing of the Gibbs measure needed by Spohn in Lemma 4 in [20].

We give now a precise definition of the *dynamics in equilibrium* associated to (1.1) and to a grand canonical Gibbs measure μ.

Let $\mathbb{W}$ be the probability measure induced on $\beth = \mathcal{C}([0,\infty),\mathbb{R}^d)^{\mathbb{N}}$ by the infinite independent Wiener processes $\{w_i(\cdot), i \in \mathbb{N}\}$. On $\Omega \times \beth$ we define the product measure $\mathbb{P}_\mu = \mu \otimes \mathbb{W}$, and we denote $\mathbb{E}_\mu$ the corresponding expectation. On $\Omega \times \beth$ we also define the increasing filtration $\{\mathcal{F}_t\}_{t>0}$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{w_i(s), s \le t, i \in \mathbb{N}\}$ and $\mathcal{B}$.

Given an initial configuration $\omega(0)$, a solution of (1.1) is a $\mathcal{F}_t$-adapted continuous stochastic process $\{\omega(t)\}$ with values in Ω, which satisfies

$$\begin{aligned} x_i(\omega(t)) &= x_i(\omega(0)) + \int_0^t v_i(\omega(s))\, ds, \\ v_i(\omega(t)) &= v_i(\omega(0)) - \int_0^t \sum_{j\neq i} \nabla V(x_i(\omega(s)) - x_j(\omega(s))\, ds \\ &\qquad - \int_0^t v_i(\omega(s)) ds + \sqrt{2}\, w_i(t). \end{aligned} \tag{2.3}$$

In Appendix A it is proven that there exists a set $M \subset \Omega \times \beth$ such that $\mathbb{P}_\mu(M) = 1$, and such that if $(\omega(0), \{w_i(\cdot)\}_{i\in\mathbb{N}}) \in M$, then (2.3) has a solution. This way it is possible to define a strongly continuous semigroup of contraction operators P^t on $L^2(\Omega, \mathcal{B}, \mu)$. A straightforward calculation shows that the generator of this process (1.1) can be written as a sum of a symmetric operator and an antisymmetric one (with respect to μ) :

$$\begin{aligned} L &= A + S, \\ A &= \{\mathfrak{H}, \cdot\} = \sum_j \left(v_j \cdot \nabla_{x_j} - \sum_{i \neq j} \nabla V(x_j - x_i) \cdot \nabla_{v_j} \right), \\ S &= \sum_j \left(\Delta_{v_j} - v_j \cdot \nabla_{v_j} \right). \end{aligned} \tag{2.4}$$

Observe that the antisymmetric operator A is given by the Poisson brackets with the Hamiltonian $\mathfrak{H}$, i.e. is the generator of the corresponding deterministic Hamiltonian dynamics.

3. Time Evolution of the Fluctuation Field

For every configuration $\omega \in \Omega$ and $\varepsilon > 0$, the density fluctuation field is the distribution on $\mathcal{S}'(\mathbb{R}^d)$ (the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of smooth rapidly decreasing functions) defined by

$$\xi^\varepsilon(G)(\omega) = \varepsilon^{-d/2} \left(\varepsilon^d \sum_j G(\varepsilon x_j(\omega)) - \rho \int G(q)\, dq \right),$$

where $G \in \mathcal{S}(\mathbb{R}^d)$.

Defining $G_\varepsilon(q) = \varepsilon^{d/2} G(\varepsilon q)$, observe that $\xi^\varepsilon(G)(\omega) = \xi^1(G_\varepsilon)(\omega)$. It is easy to see that $\xi^\varepsilon(G) \in L^2(\Omega, \mu)$. In fact we have the following bound:

Lemma 3.1. *There exists a constant $B = B(\rho) < \infty$ such that for any G:*

$$\sup_{\varepsilon>0}\left\langle \xi^\varepsilon(G)^2\right\rangle \le B\|G\|_{L^2}^2. \tag{3.1}$$

Proof. Using the 2-point correlation function ρ_2 of the grand canonical Gibbs measure μ, and the translation invariance of μ:

$$\begin{aligned}\left\langle \xi^\varepsilon(G)^2\right\rangle &= \iint G_\varepsilon(q_1)G_\varepsilon(q_2)(\rho_2(q_1,q_2)-\rho^2)\,dq_1\,dq_2\\ &\le \iint \frac{1}{2}\left(G_\varepsilon(q_1)^2+G_\varepsilon(q_2)^2\right)|\rho_2(q_1,q_2)-\rho^2|\,dq_1\,dq_2\\ &= \left(\int G_\varepsilon(q)^2\,dq\right)\int|\rho_2(0,q')-\rho^2|\,dq' = \|G\|_{L^2}^2\int|\rho_2(0,q')-\rho^2|\,dq'\,.\end{aligned}$$

This last quantity is finite by Lemma 4.4.8 of [15]. □

In order to simplify notations, we will denote $\xi_t^\varepsilon = \xi^\varepsilon(\omega(\varepsilon^{-2}t))$. For a fixed arbitrary $T > 0$, we denote by P^ε the probability distribution, under $\mathbb{P}_\mu$, of $\{\xi_t^\varepsilon, 0 \le t \le T\}$ in $\mathcal{C}\left([0,T],\mathcal{S}'(\mathbb{R}^d)\right)$.

Consider now a smooth test function G with compact support. By a simple calculation we have

$$\begin{aligned}\xi_t^\varepsilon(G)-\xi_0^\varepsilon(G) &= \int_0^{\varepsilon^{-2}t}\varepsilon^{d/2+1}\sum_j \nabla G(\varepsilon x_j(s))\cdot v_j(s)\,ds\\ &= \sqrt{2}\varepsilon^{d/2+1}\int_0^{\varepsilon^{-2}t}\sum_j\nabla G(\varepsilon x_j(s))\cdot dw_j(s)+\varepsilon^2\int_0^{\varepsilon^{-2}t}\gamma_\varepsilon(\omega(s))\,ds\\ &\quad -\varepsilon^{d/2+1}\sum_j\left(\nabla G(\varepsilon x_j(\varepsilon^{-2}t))\cdot v_j(\varepsilon^{-2}t)-\nabla G(x_j^\varepsilon(0))\cdot v_j(0)\right),\end{aligned} \tag{3.2}$$

where

$$\gamma_\varepsilon(\omega) = \varepsilon^{d/2}\sum_j\left(\sum_{\alpha,\sigma=1}^d \partial_\alpha\partial_\sigma G(\varepsilon x_j)v_j^\alpha v_j^\sigma - \varepsilon^{-1}\sum_{i\ne j}\nabla G(\varepsilon x_j)\cdot\nabla V(x_j-x_i)\right) \tag{3.3}$$

with q^α the α^{th} component of q and $\partial_\alpha \doteq \partial_{q^\alpha}$.

It is easy to see that variance of the last two terms in the rhs of (3.2) converges to 0 as $\varepsilon \to 0$. In fact, by stationarity we have

$$\begin{aligned}&\mathbb{E}_\mu\left(\left[\varepsilon^{d/2+1}\sum_j\left(\nabla G(\varepsilon x_j(\varepsilon^{-2}t))\cdot v_j(\varepsilon^{-2}t)-\nabla G(x_j^\varepsilon(0))\cdot v_j(0)\right)\right]^2\right)\\ &\le\ 2\varepsilon^2 < (\varepsilon^{d/2}\sum_j\nabla G(\varepsilon x_j)\cdot v_j)^2 >_\mu\end{aligned}$$

and by the independence of the velocities:

$$< (\varepsilon^{d/2} \sum_j \nabla G(\varepsilon x_j) \cdot v_j)^2 >_\mu = \varepsilon^d < \sum_j |\nabla G(\varepsilon x_j)|^2 >_\mu = \rho \int |\nabla G(q)|^2 \, dq. \tag{3.4}$$

The main step we need to do, in order to obtain (1.5) as a macroscopic equation, is to prove that:

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu \left(\left[\int_0^t (\gamma_\varepsilon(\omega(\varepsilon^{-2} s)) - D\xi_s^\varepsilon(\Delta G)) \, ds \right]^2 \right) = 0. \tag{3.5}$$

For any continuous function f on $\mathbb{R}^d$ with compact support, let us define the local function

$$\Upsilon_{\alpha,\sigma}(f)(\omega) = \sum_{i,j} (x_i^\sigma - x_j^\sigma) V_\alpha(x_i - x_j) \int_0^1 f(\lambda x_i + (1-\lambda) x_j) d\lambda, \tag{3.6}$$

where $V_\alpha(q) = \partial_{q^\alpha} V(q)$.

Lemma 3.2. *Let f be a continuous function with compact support such that $\int f(q)\, dq = 0$. Then there exists a constant $C = C(\rho)$ such that*

$$< (\Upsilon_{\alpha,\sigma}(f))^2 >_\mu \le C \, \|f\|_{L^2}^2.$$

Proof. This is basically (4.13) in [20]. Since $\int f(q)\, dq = 0$, we have

$$\begin{aligned} < \Upsilon_{\alpha,\sigma}(f) > &= \int_0^1 d\lambda \iint (q_1^\sigma - q_2^\sigma) V_\alpha(q_1 - q_2) \\ &\quad \times f(\lambda q_1 + (1-\lambda) q_2) \rho_2(q_1, q_2) \, dq_1 \, dq_2 \\ &= \int_0^1 d\lambda \int dz \, z^\sigma V_\alpha(z) \rho_2(0, z) \int dq_2 \, f(\lambda z + q_2) = 0. \end{aligned}$$

Let us denote by $\tilde{\rho}_4$ the truncated 4-point correlation function of μ, i.e.

$$\tilde{\rho}_4(q_1, q_2, q_3, q_4) = \rho_4(q_1, q_2, q_3, q_4) - \rho_2(q_1, q_2)\rho_2(q_3, q_4).$$

In order to simplify notations, let us define

$$g(q_1, q_2, \lambda) = (q_1^\sigma - q_2^\sigma) V_\alpha(q_1 - q_2) f(\lambda q_1 + (1-\lambda) q_2) .$$

Then, by using the Jensen inequality, we have

$$\begin{aligned} &< (\Upsilon_{\alpha,\sigma}(f))^2 >_\mu \\ &\quad = < (\Upsilon_{\alpha,\sigma}(f) - < \Upsilon_{\alpha,\sigma}(f) >)^2 >_\mu \\ &\quad \le \int_0^1 d\lambda \left\langle \left(\sum_{(i,j)} g(x_i, x_j, \lambda) - \iint g(q_1, q_2, \lambda) \rho_2(q_1, q_2) \, dq_1 \, dq_2 \right)^2 \right\rangle_\mu \\ &\quad = \int_0^1 d\lambda \iiiint dq_1 dq_2 dq_3 dq_4 \, g(q_1, q_2, \lambda) \, g(q_3, q_4, \lambda) \, \tilde{\rho}_4(q_1, q_2, q_3, q_4). \end{aligned} \tag{3.7}$$

By a linear change of variables in the quadruple integration and translation invariance, the right-hand side of (3.7) is equal to

$$\int_0^1 d\lambda \iiiint z_1^\sigma V_\alpha(z_1)\, z_3^\sigma V_\alpha(z_3)\; f(\lambda z_1 + z_2)\; f(\lambda z_3 + z_2 + z_4) \\ \times\; \rho_4(z_1, 0, z_3 + z_4, z_4)\; dz_1\; dz_2\; dz_3\; dz_4,$$

which is bounded by

$$C_V \left(\int f(z)^2\, dz \right) \iiint dz_1\; dz_3\; dz_4\; |\tilde\rho_4(z_1, 0, z_3 + z_4, z_4)|.$$

This last integral is finite by Thm. 4.4.8 of [15]. □

In the following $G_{\alpha,\sigma}(q) = \partial_{q^\alpha}\partial_{q^\sigma} G(q)$. Defining $G^\varepsilon_{\alpha,\sigma}(q) = \varepsilon^{d/2} G_{\alpha,\sigma}(\varepsilon q)$, then using the symmetry of V we can rewrite

$$\varepsilon^{d/2-1} \sum_j \sum_{i\neq j} \nabla G(\varepsilon x_j) \cdot \nabla V(x_j - x_i)) = \frac{\varepsilon^{d/2-1}}{2} \sum_j \sum_{i\neq j} (\nabla G(\varepsilon x_j) - \nabla G(\varepsilon x_i)) \cdot \\ \nabla V(x_j - x_i) = \frac{1}{2} \sum_{\alpha,\sigma} \Upsilon_{\alpha,\sigma}(G^\varepsilon_{\alpha,\sigma}).$$

Let $h(q)$ be a positive continuous function with support in $B(0, 1/2)$, the ball centered at the origin and of radius $1/2$, and with total integral equal to 1. Since $\|G^\varepsilon_{\alpha,\sigma} - G^\varepsilon_{\alpha,\sigma} * h\|_{L^2}$ converges to 0 as $\varepsilon \to 0$, by Lemma 3.2 we have

$$\lim_{\varepsilon\to 0} < (\Upsilon_{\alpha,\sigma}(G^\varepsilon_{\alpha,\sigma} - G^\varepsilon_{\alpha,\sigma} * h))^2 >_\mu \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.8}$$

Let $\{\tau_q, q \in \mathbb{R}^d\}$ the shift operator on Ω, defined by $\tau_q\omega = \{(x_j(\omega)+q, v_j(\omega)), j \in \mathbb{N}\}$. By (3.8), Schwarz inequality and stationarity, we can substitute in (3.5) γ_ε with

$$\tilde\gamma_\varepsilon(\omega) = \sum_{\alpha,\sigma} \varepsilon^{d/2} \int G_{\alpha,\sigma}(\varepsilon q)\phi_{\alpha,\sigma}(\tau_q\omega)\, dq,$$

where $\phi_{\alpha,\sigma}$ is the local function given by

$$\phi_{\alpha,\sigma}(\omega) \;=\; \sum_j h(x_j) v_j^\alpha v_j^\sigma - \frac{1}{2}\Upsilon_{\alpha,\sigma}(h). \tag{3.9}$$

Taking the expectation with respect to μ and using the translation invariance we have

$$< \phi_{\alpha,\sigma} >_\mu \\ = \rho\delta_{\alpha,\sigma} - \frac{1}{2} \iint dq_1 dq_2 \rho_2(q_1, q_2)(q_1^\sigma - q_2^\sigma) V_\alpha(q_1 - q_2) \int_0^1 h(\lambda q_1 + (1-\lambda) q_2)\, d\lambda \\ = \rho\delta_{\alpha,\sigma} - \frac{1}{2} \int dq \rho_2(q, 0) q^\sigma V_\alpha(q) = \delta_{\alpha,\sigma} \left(\rho - \frac{1}{2d} \int dq\; \rho_2(q, 0) q \cdot \nabla V(q) \right)$$

and observe that, by the virial theorem (cf. [18, 23]), this last quantity is equal to $\delta_{\alpha,\sigma} P(\rho)$, where P is the thermodynamic pressure expressed as a function of the density ρ. Consequently $D = P'(\rho)$.

The limit (3.5) is a direct consequence of the Boltzmann-Gibbs principle that we enunciate and prove in the next section.

4. Boltzmann-Gibbs Principle

In this section we will consider the following set of smooth local functions on Ω. We say that a measurable function $\phi : \Omega \to \mathbb{R}$ is in $\mathcal{E}$ if it can be written as

$$\phi(\omega) = H\left(\sum_i f_1(x_i(\omega), v_i(\omega)), \dots, \sum_i f_m(x_i(\omega), v_i(\omega))\right),$$

where $\{f_k(q, v), k = 1, \dots, m\}$ are smooth functions on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support in q and growth at most linearly in v (i.e. $|f(q, v)| \le C(1+|v|)$), and $H(y_1, \dots, y_m) \in C^\infty(\mathbb{R}^m)$ such that there exists a constant $c < \infty$ such that $|H(y_1, \dots, y_m)| \le e^{c \sum_{k=1}^m |y_k|}$.

If Λ is a finite region containing all the q-supports of f_k, then there exists a finite constant C such that for any configuration ω,

$$|\phi(\omega)| \le \exp\left[C \sum_{x_j(\omega)\in\Lambda} (1 + |v_j(\omega)|)\right]. \tag{4.1}$$

By (4.1) and by superstability estimate (cf. [16]), it follows that $\phi \in L^p(\mu)$ for any $p < \infty$. We denote its expectation by

$$\widehat{\phi}(\rho) = <\phi>_{\mu_\rho}$$

as a function of the density ρ. Observe that $\widehat{\phi}(\rho)$ is a smooth function of ρ. In fact, always by the superstability estimates, $\widehat{\phi}(\rho)$ is a smooth function of the activity z, which is a smooth function of the density ρ in the ranges defined by (2.2) (cf. [15], Theorem 4.2.3, p.76).

Let us define

$$\Phi(\omega) = \phi(\omega) - \widehat{\phi}(\rho) - \frac{d\widehat{\phi}}{d\rho}(\rho)\left(\sum_j h(x_j) - \rho\right),$$

where $h : \mathbb{R}^d \to \mathbb{R}_+$ is a positive smooth function with support in $B(0, 1/2)$, the ball centered at the origin and of radius $1/2$, and with total integral $\int h(q)dq = 1$.

The main result of this section is contained in the following proposition:

Proposition 4.1 (Boltzmann-Gibbs principle). *Let $\phi \in \mathcal{E}$ and Φ be defined as above. Then for any smooth function G with compact support on $\mathbb{R}^d$,*

$$\lim_{\varepsilon\to 0} \mathbb{E}_\mu\left[\left(\varepsilon^{d/2}\int_0^t ds \int dq\ G(\varepsilon q)\Phi(\tau_q\omega(\varepsilon^{-2}s))\right)^2\right] = 0.$$

We will make wide use of the following estimate:

Lemma 4.1. *There exists a finite constant C, depending only on V and ρ, such that for any G in $L^2(\mathbb{R}^d)$ and for any local function $\psi(\omega)$ in $L^2(\mu)$ such that $<\psi>= 0$ and whose support is contained in a finite set $\Lambda \subset \mathbb{R}^d$, we have:*

$$\left\langle\left(\int G(q)\psi(\tau_q\omega)\, dq\right)^2\right\rangle \le C|\Lambda| <\psi^2> \int G(q)^2\, dq. \tag{4.2}$$

Proof. This is a consequence of the exponential decay of the correlations, cf. Lemma 4 in [20], and the proof is very close to Lemma 9 in [20].

$$
\begin{aligned}
&\left\langle \left(\int G(q)\psi(\tau_q\omega)\, dq \right)^2 \right\rangle \\
&= \iint G(q)G(q') < \psi(\tau_q\omega)\psi(\tau_{q'}\omega) > \ dq\ dq' \\
&\le \frac{1}{2} \iint \left(G(q)^2 + G(q')^2 \right)\ | < \psi(\tau_q\omega)\psi(\tau_{q'}\omega) > |\ dq\ dq' \\
&= \int dq\ G(q)^2 \int dq' | < \psi(\tau_{q-q'}\omega)\psi(\omega) > |.
\end{aligned}
$$

By Lemma 4 in [20] there exists c and α positive constants, depending only on V and ρ, such that

$$
| < \psi(\tau_q\omega)\psi(\omega) > |\ \le < \psi^2 >_\mu \min\{1, c|\bar{\Lambda}| e^{-\alpha d(\tau_q\bar{\Lambda},\bar{\Lambda})} e^{|\bar{\Lambda}| e^{-\alpha d(\tau_q\bar{\Lambda},\bar{\Lambda})}}\},
$$

where d is the distance on $\mathbb{R}^d$, and $\bar{\Lambda} = \{q \in \mathbb{R}^d : d(q,\Lambda) \le R\}$, ($R$ being the radius of the support of the interaction V). Then performing the integration in q' one obtains (4.2). □

Proposition 4.1 will be proven in several steps. In the first one, we will condition Φ on the positions configuration that we denote by ω^x. We will use then the following lemma:

Lemma 4.2. *Let Ψ be a local function on $L^2(\mu)$ such that its μ-expectation conditioned on the positions configuration ω^x is*

$$
< \Psi|\omega^x >= 0 \qquad \mu - a.s.
$$

Then for any smooth function G with compact support on $\mathbb{R}^d$,

$$
\lim_{\varepsilon\to 0} \mathbb{E}_\mu \left[\left(\varepsilon^{d/2} \int_0^t ds \int dq\ G(\varepsilon q)\Psi(\tau_q\omega(\varepsilon^{-2}s)) \right)^2 \right] = 0.
$$

Proof. Observe that the operator S, the symmetric part of the generator of the process defined by (2.4), has a spectral gap equal to 1. Since $< \Psi|\omega^x >= 0$, $S^{-1}\Psi(\omega)$ is a function in $L^2(\mu)$. Recall we have defined $\varepsilon^{d/2}G(\varepsilon q) = G_\varepsilon(q)$. It follows that $S^{-1}\int dq\ G_\varepsilon(q)\Psi(\tau_q\omega) \in L^2(\mu)$. Then (cf. Theorem 2.2 in [22]):

$$
\begin{aligned}
&\mathbb{E}_\mu \left[\left(\int_0^t ds \int dq\ G_\varepsilon(q)\Psi(\tau_q\omega(\varepsilon^{-2}s)) \right)^2 \right] \\
&\quad \le 8t\varepsilon^2 \left\langle \left(\int dq\ G_\varepsilon(q)\Psi(\tau_q\omega) \right) (-S)^{-1} \left(\int dq\ G_\varepsilon(q)\Psi(\tau_q\omega) \right) \right\rangle.
\end{aligned}
$$

By the spectral gap of S this last quantity is bounded by:

$$
\le 8t\varepsilon^2 \left\| \int dq\ G_\varepsilon(q)\Psi(\tau_q\omega) \right\|^2_{L^2(\mu)}.
$$

By Lemma 4.1 this is bounded by

$$8t\varepsilon^2 \|G\|_{L^2}^2 C(\Psi).$$

□

As a consequence of Lemma 4.2, we only need to prove Proposition 4.1 for the function

$$\tilde{\Phi}(\omega^x) = < \Phi|\omega^x > .$$

Observe that $\tilde{\Phi}$ is still a local smooth function satisfying (4.1).

For l large, let Λ be a centered box of size l. The μ canonical expectation of $\tilde{\Phi}$ conditioned on the configuration of the positions of the particles outside Λ (denoted by $\omega^x_{\Lambda^c}$), and on the number of particles in Λ is defined by:

$$\Gamma_\Lambda \tilde{\Phi}(\omega^x) \; = < \tilde{\Phi}|N_\Lambda(\omega), \omega^x_{\Lambda^c} > .$$

By DLR equations, it can be defined for every $\omega^x_{\Lambda^c}$ and it does not depend on ρ. Always by DLR equations, one can see that $\Gamma_\Lambda \tilde{\Phi}$ is smooth on the set of configurations where there are no particles on the border of Λ. Since this set has μ full measure, we have that Γ_Λ is smooth μ-a.s.

Again by the DLR equations one has

$$\Gamma_\Lambda \left(\sum_j h(x_j) \right) = \frac{N_\Lambda(\omega)}{|\Lambda|}, \tag{4.3}$$

consequently

$$\Gamma_\Lambda \tilde{\Phi}(\omega^x) = \Gamma_\Lambda \tilde{\phi}(\omega^x) - \widehat{\phi}(\rho) - \frac{d\widehat{\phi}}{d\rho}(\rho) \left(\frac{N_\Lambda(\omega)}{|\Lambda|} - \rho \right),$$

Recall the thermodynamic relation $\frac{d\rho}{d(log z)} = \chi$. Then $\frac{d\widehat{\phi}}{d\rho}(\rho) = \frac{z}{\chi} \frac{d <\phi>_{\mu_z}}{dz}$. It is proven in [20], Eq. (8.5), that, if $z(\rho)$ satisfies (2.2), one has the following strong equivalence of ensembles:

$$\lim_{\Lambda \uparrow \mathbb{R}^d} |\Lambda| < (\Gamma_\Lambda \tilde{\Phi})^2 > = \; 0. \tag{4.4}$$

Indeed in [3] an even stronger statement is proved under a weaker condition than (2.2).

Then, using Lemma 4.1, (4.4) implies

$$\lim_{\Lambda \uparrow \mathbb{R}^d} \lim_{\varepsilon \to 0} < \left(\varepsilon^{d/2} \int dq \; G(\varepsilon q) \Gamma_\Lambda \tilde{\Phi}(\tau_q \omega^x) \right)^2 >= 0.$$

(this is basically Lemma 9 in [20]). So we can recenter $\tilde{\Phi}$ around its canonical expectation $\Gamma_\Lambda \tilde{\Phi}$, and Proposition 4.1 will follow from the following one:

Proposition 4.2. *For any smooth function G with compact support on $\mathbb{R}^d$ and for any finite box Λ,*

$$\lim_{\varepsilon\to 0}\mathbb{E}_\mu\left[\left(\varepsilon^{d/2}\int_0^t ds\int dq\; G(\varepsilon q)(\tilde{\Phi}-\Gamma_\Lambda\tilde{\Phi})(\tau_q\omega^x(\varepsilon^{-2}s))\right)^2\right]=0.$$

Proof. We denote by

$$f(x_1,\ldots,x_{N_\Lambda};\omega^x_{\Lambda^c})=(\tilde{\Phi}-\Gamma_\Lambda\tilde{\Phi})(\omega^x)$$

and we will consider f as a function of the positions of the particles inside Λ^n, for any fixed exterior configuration $\omega^x_{\Lambda^c}$ and any fixed number of particles $N_\Lambda=n$ inside Λ. Since the set of configurations with particles on the boundary of Λ have null measure with respect to μ, we can consider only configurations $\omega^x_{\Lambda^c}$ without particle on the boundary of Λ, and the dependence of f on $\omega^x_{\Lambda^c}$ is smooth on this set of configurations. Furthermore, for any such $\omega^x_{\Lambda^c}$ fixed, f is a smooth function of the positions $(x_1,\ldots,x_n)$ on the interior of Λ^n,

Then, for any $\omega^x_{\Lambda^c}$ and $N_\Lambda=n$ fixed, we consider the elliptic operator on Λ^n,

$$L^W_{n,\omega^x_{\Lambda^c}}=\sum_{j=1}^n\left(\Delta_{x_j}-\sum_{i\neq j}(\nabla V)(x_j-x_i)\cdot\nabla_{x_j}\right)\tag{4.5}$$

with Neumann boundary conditions. Then, by the properties of f, a smooth function $u(x_1,\ldots,x_n;\omega^x_{\Lambda^c})$ solution of

$$-L^W_{n,\omega^x_{\Lambda^c}}u\;=\;f(x_1,\ldots,x_n;\omega^x_{\Lambda^c})\tag{4.6}$$

exists. We consider now u as a local function of ω^x (i.e. as a function of $\{x_1,\ldots,x_{N_\Lambda},\omega^x_{\Lambda^c}\}$).

By Lemma 9.1 in Appendix C, there exist c_1, c_2 finite constant independent of n, $\omega^x_{\Lambda^c}$ and u such that

$$\Gamma_\Lambda\left(\sum_{k=1}^n|\nabla_{x_k}u|^2\right)(n,\omega^x_{\Lambda^c})\le c_1 n\exp\left\{c_2(n+N_{\partial^+_R\Lambda}(\omega^x_{\Lambda^c}))\right\}\Gamma_\Lambda\left(f^2\right)(n,\omega^x_{\Lambda^c}),$$

where $\partial^+_R\Lambda=\Lambda(l+R)\setminus\Lambda$. Integrating with respect to μ we obtain

$$\left\langle\sum_{x_k\in\Lambda}|\nabla_{x_k}u|^2\right\rangle\le c_1\left\langle N_\Lambda e^{c_2 N_{\Lambda(l+R)}}f^2\right\rangle,\tag{4.7}$$

and the left-hand side is bounded by (4.1) and superstability estimates.

The trick now is to relate $L^W_{n,\omega^x_{\Lambda^c}}$ to the generator of our process L (cf. (2.4)). Define the local function

$$F(\omega)=\sum_{x_j\in\Lambda}v_j\cdot\nabla_{x_j}u.$$

By (4.7), $\mathcal{F} \in L^2(\mu)$, and by the results contained in Appendix C it is in the domain of the generator L. Then we apply L to $\mathcal{F}$ and we obtain

$$L\mathcal{F}(\omega) = \sum_{x_j \in \Lambda} \sum_k (v_k \cdot \nabla_{x_k})(v_j \cdot \nabla_{x_j})u - \sum_{x_j \in \Lambda} \left(\sum_{i \neq j} \nabla V(x_j - x_i) \right) \cdot \nabla_{x_j} u - \mathcal{F}. \tag{4.8}$$

The first term on the rhs of (4.8) can be rewritten as

$$\sum_{x_j \in \Lambda} \sum_k \sum_{\alpha,\sigma}^{d} v_k^\alpha v_j^\sigma \partial_{x_k^\alpha} \partial_{x_j^\sigma} u = \sum_{x_j \in \Lambda} \Delta_{x_j} u + \Psi_1(\omega)$$

where, by Lemmas 9.2 and 9.3, $\Psi_1(\omega)$ is a local function in $L^2(\mu)$ such that

$$< \Psi_1 | \omega^x > = 0 \qquad \mu - a.s.$$

Observe that also $\mathcal{F}$ has the same property. In conclusion we can write

$$(\tilde{\Phi} - \Gamma_\Lambda \tilde{\Phi})(\omega^x) = L^W_{N_\Lambda, \omega^x_{\Lambda^c}} u = L\mathcal{F}(\omega) + \Psi(\omega),$$

where $\Psi(\omega)$ is a local function in $L^2(\mu)$ such that

$$< \Psi | \omega^x > = 0 \qquad \mu - a.s.$$

So we can apply Lemma 4.2, and we are left to prove

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu \left[\left(\varepsilon^{d/2} \int_0^t ds \int dq \ G(\varepsilon q)(L\mathcal{F})(\tau_q \omega(\varepsilon^{-2} s)) \right)^2 \right] = 0. \tag{4.9}$$

Integrating in time we can rewrite

$$\begin{aligned} &\varepsilon^{d/2} \int_0^t ds \int dq \ G(\varepsilon q)(L\mathcal{F})(\tau_q \omega(\varepsilon^{-2} s)) \\ &= \varepsilon^{d/2+2} \int dq \ G(\varepsilon q)(\mathcal{F}(\tau_q \omega(\varepsilon^{-2} t)) - \mathcal{F}(\tau_q \omega(0))) + \mathcal{M}_t^\varepsilon, \end{aligned} \tag{4.10}$$

where $\mathcal{M}_t^\varepsilon$ is a martingale such that

$$\begin{aligned} \mathbb{E}\left[(\mathcal{M}_t^\varepsilon)^2 \right] &= t\varepsilon^{d+2} < \sum_k \left(\int dq \ G(\varepsilon q) \partial_{v_k} \mathcal{F}(\tau_q \omega) \right)^2 > \\ &= t\varepsilon^{d+2} \iint dq \ dq' \ G(\varepsilon q) \ G(\varepsilon q') < \sum_k \partial_{v_k} \mathcal{F}(\tau_q \omega) \partial_{v_k} \mathcal{F}(\tau_{q'} \omega) >, \end{aligned}$$

and by the Schwarz inequality this is bounded by

$$t\varepsilon^{d+2}\iint dqdq'|G(\varepsilon q)||G(\varepsilon q')| < \left(\sum_k (\partial_{v_k}F(\tau_q\omega))^2\right)^{1/2}\left(\sum_k (\partial_{v_k}F(\tau_{q'}\omega))^2\right)^{1/2} >$$

$$= t\varepsilon^2 < \left(\varepsilon^{d/2}\int dq|G(\varepsilon q)|\left(\sum_k (\partial_{v_k}F(\tau_q\omega))^2\right)^{1/2}\right)^2 > .$$

Since

$$\sum_k (\partial_{v_k}F(\omega))^2 = \sum_{x_k\in\Lambda}(\nabla_{x_k}u)^2,$$

then, by (4.7), this is a local function in $L^1(\mu)$, and by Lemma 4.1, $\mathbb{E}\left[(\mathcal{M}_t^\varepsilon)^2\right]$ converges to 0 as $\varepsilon \to 0$. It follows by Lemma 4.2 that the variance of the first term on the rhs of (4.10) goes to 0 as $\varepsilon \to 0$. □

5. Tightness

For any $k \geq 0$ and $f, g \in \mathcal{C}^\infty(\mathbb{R}^d)$ consider the scalar product

$$(g, f)_k = \int_{\mathbb{R}^d} g(q)\left(|q|^2 - \Delta\right)^k f(q)\,dq \tag{5.1}$$

and denote by $\mathcal{H}_k$ the corresponding closure. For any positive k we denote by $\mathcal{H}_{-k}$ its dual space with respect to the $L^2(\mathbb{R}^d) \equiv \mathcal{H}_0$ scalar product.

It is convenient to represent the scalar product $(\cdot,\cdot)_k$ in the ON base of the Hermite polynomials, which are the eigenfunctions of $|q|^2 - \Delta$. Let $\vec{n}$ be a multi-index of $(\mathbb{Z}^+)^d$ and $|\vec{n}| = \sum_{i=1}^d n(i)$. We denote by $\lambda_{n(i)} = 2n(i) + 1$ for $n(i) \in \mathbb{Z}^+$ and $\lambda_{\vec{n}} = \sum_{i=1}^d \lambda_{n(i)}$. Define $h_{\vec{n}}(q) = \prod_{i=1}^d h_{n(i)}(q_i)$, where h_m is the m^{th} normalized Hermite polynomial of order m in $\mathbb{R}$. We have then for every $k \geq 0$ and $f \in L^2$,

$$\|f\|_k^2 = \int_{\mathbb{R}^d} f(q)(|q|^2-\Delta)^k f(q)\,dq = \sum_{\vec{n}\in(\mathbb{Z}^+)^d}\lambda_{\vec{n}}^k\left(\int_{\mathbb{R}^d} f(q)h_{\vec{n}}(q)\,dq\right)^2.$$

This is valid also for negative k. So the $\mathcal{H}_{-k}$-norm of a distribution ξ on $\mathbb{R}^d$ can be written as

$$\|\xi\|_{-k}^2 = \sum_{\vec{n}\in(\mathbb{Z}^+)^d}\lambda_{\vec{n}}^{-k}\,\xi(h_{\vec{n}})^2. \tag{5.2}$$

Observe that for $k' > k$, the injection J of $\mathcal{H}_{-k}$ in $\mathcal{H}_{-k'}$ is compact. In fact it can be approximated by the finite range operators $J_m\xi = \sum_{|\vec{n}|\leq m}\xi(h_{\vec{n}})h_{\vec{n}}$, and it is easy to see that the operator norm of the difference is bounded by

$$\|J - J_m\| \leq (2m+d)^{-(k'-k)}.$$

Recall that for a fixed arbitrary $T > 0$, we have denoted by P^ε the distribution, under $\mathbb{P}_\mu$, of $\{\xi_t^\varepsilon = \xi^\varepsilon(\omega(\varepsilon^{-2}t)), 0 \leq t \leq T\}$ in $\mathcal{C}\left([0,T], \mathcal{S}'(\mathbb{R}^d)\right)$.

Proposition 5.1. *For any $k > d + 1$ and every $T > 0$, the sequence of probability measures $\{P^\varepsilon, \varepsilon \in (0, 1]\}$ has support in $\mathcal{C}([0, T], \mathcal{H}_{-k})$ and is relatively compact in this space.*

By the compactness of the injections $\mathcal{H}_{-k} \hookrightarrow \mathcal{H}_{-k'}$ for $k < k'$, and standard compactness arguments, Proposition 5.1 is a consequence of the following proposition.

Proposition 5.2. *For any $k > d + 1$ and every $T > 0$, we have that*

(i)

$$\sup_{\varepsilon \in (0,1)} \mathbb{E}_\mu \left(\sup_{t \in [0,T]} ||\xi_t^\varepsilon||_{-k}^2 \right) < +\infty.$$

(ii) For any $R > 0$,

$$\lim_{\delta \to 0} \limsup_{\varepsilon \to 0} \mathbb{P}_\mu \left(\sup_{\substack{t,s \in [0,T] \\ |t-s| \leq \delta}} ||\xi_t^\varepsilon - \xi_s^\varepsilon||_{-k} > R \right) = 0.$$

To prove Proposition 5.2, we need the following key estimate:

Lemma 5.3. *Let $G \in \mathcal{S}(\mathbb{R}^d)$. Then there exists a constant $B = B(\rho, T) < \infty$ such that*

$$\sup_{\varepsilon \in (0,1)} \mathbb{E}_\mu \left(\sup_{t \in [0,T]} \xi_t^\varepsilon(G)^2 \right) \leq B \int \left(G(q)^2 + |\nabla G(q)|^2 \right) \, dq.$$

Proof. Let us define $\mathcal{F}_\varepsilon(\omega) = \sum_j \nabla G(\varepsilon x_j) \cdot v_j$. By (1.1), we have

$$\xi_t^\varepsilon(G) = \xi_0^\varepsilon(G) + \varepsilon^{d/2+1} \int_0^{\varepsilon^{-2}t} \mathcal{F}_\varepsilon(\omega(s)) \, ds. \tag{5.3}$$

Then

$$\mathbb{E}_\mu \left(\sup_{t \in [0,T]} \xi_t^\varepsilon(G)^2 \right) \leq 2 \left\langle \xi^\varepsilon(G)^2 \right\rangle + 2\mathbb{E}_\mu \left(\sup_{t \in [0,T]} \left[\varepsilon^{d/2+1} \int_0^{\varepsilon^{-2}t} \mathcal{F}_\varepsilon(\omega(s)) \, ds \right]^2 \right).$$

By Lemma 3.1, there exists a constant B' such that $\left\langle \xi^\varepsilon(G)^2 \right\rangle \leq B' \|G\|_{L^2}^2$.

Since $\mathcal{F}_\varepsilon = -S\mathcal{F}_\varepsilon$, by Theorem 2.2 of [22]

$$\begin{aligned} \mathbb{E}_\mu \left(\sup_{t \in [0,T]} \left[\varepsilon^{d/2+1} \int_0^{\varepsilon^{-2}t} \mathcal{F}_\varepsilon(\omega(s)) \, ds \right]^2 \right) &\leq 8T\varepsilon^d < \mathcal{F}_\varepsilon (-S)^{-1} \mathcal{F}_\varepsilon > \\ &= 16T\varepsilon^d < \sum_j |\nabla G(\varepsilon x_j)|^2 > \\ &= 16T\rho \|\nabla G\|_{L^2}^2 . \quad \square \end{aligned}$$

Proof of Proposition 5.2. We start by proving (i). By (5.2) and Lemma (5.3) we have

$$\begin{aligned}\mathbb{E}_\mu\left[\sup_{t\in[0,T]}||\xi_t^\varepsilon||_{-k}^2\right] &\le \sum_{\vec{n}}\lambda_{\vec{n}}^{-k}\mathbb{E}_\mu\left[\sup_{t\in[0,T]}\xi_t^\varepsilon(h_{\vec{n}})^2\right]\\ &\le B\sum_{\vec{n}}\lambda_{\vec{n}}^{-k}(1+\|\nabla h_{\vec{n}}\|_{L^2}^2) \qquad (5.4)\\ &\le B\sum_{\vec{n}}\lambda_{\vec{n}}^{-k}(1+\|h_{\vec{n}}\|_1^2)\\ &\le B\sum_{\vec{n}}\lambda_{\vec{n}}^{-k}(1+\lambda_{\vec{n}}).\end{aligned}$$

This series converges provided $k > d+1$ and therefore (i) is proven.

We are left with the proof of (ii). Always by (5.2) we have

$$\mathbb{E}_\mu\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}||\xi_t^\varepsilon-\xi_s^\varepsilon||_{-k}^2\right] \le \sum_{\vec{n}\in\mathbb{N}^d}\lambda_{\vec{n}}^{-k}\mathbb{E}_\mu\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}\big(\xi_t^\varepsilon(h_{\vec{n}})-\xi_s^\varepsilon(h_{\vec{n}})\big)^2\right]. \qquad (5.5)$$

For every $R\ge 1$,

$$\begin{aligned}&\sum_{\vec{n}\in\mathbb{N}^d:|\vec{n}|\ge R}\lambda_{\vec{n}}^{-k}\mathbb{E}_\mu\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}\big(\xi_t^\varepsilon(h_{\vec{n}})-\xi_s^\varepsilon(h_{\vec{n}})\big)^2\right]\\ &\quad\le 4\sum_{\vec{n}\in\mathbb{N}^d:|\vec{n}|\ge R}\lambda_{\vec{n}}^{-k}\mathbb{E}_\mu\left[\sup_{t\in[0,T]}(\xi_t^\varepsilon(h_{\vec{n}}))^2\right].\end{aligned}$$

By (5.4), since $k > d+1$,

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\sum_{\vec{n}\in\mathbb{N}^d:|\vec{n}|\ge R}\lambda_{\vec{n}}^{-k}\mathbb{E}_\mu\left[\sup_{t\in[0,T]}(\xi_t^\varepsilon(h_{\vec{n}}))^2\right]=0. \qquad (5.6)$$

Consequently, in order to prove (ii), we only need to show that for every $\vec{n}$,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\mathbb{E}_\mu\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\le\delta}}(\xi_t^\varepsilon(h_{\vec{n}})-\xi_s^\varepsilon(h_{\vec{n}}))^2\right]=0. \qquad (5.7)$$

By (2.3)

$$\xi_t^\varepsilon(h_{\vec{n}})-\xi_s^\varepsilon(h_{\vec{n}}) = I_1^\varepsilon(t,s)+I_2^\varepsilon(t,s)+I_3^\varepsilon(t,s),$$

where

$$I_1^\varepsilon(t,s)=\sqrt{2}\varepsilon^{d/2+1}\int_{\varepsilon^{-2}s}^{\varepsilon^{-2}t}\sum_j\nabla h_{\vec{n}}(\varepsilon x_j(\tau))\cdot dw_j(\tau),$$

$$I_2^\varepsilon(t,s) = \int_{\varepsilon^{-2}s}^{\varepsilon^{-2}t} \gamma_\varepsilon(\omega(u))\, du,$$

where γ_ε is given by

$$\gamma_\varepsilon(\omega) = \varepsilon^{d/2} \sum_j \left(\sum_{\alpha,\sigma} \partial_\alpha \partial_\sigma h_{\vec{n}}(\varepsilon x_j)\; v_j^\alpha v_j^\sigma \, - \varepsilon^{-1} \sum_{i \neq j} \nabla h_{\vec{n}}(\varepsilon x_j) \cdot \nabla V(x_i - x_j) \right)$$

and

$$I_3^\varepsilon(t,s) = -\left(\eta_\varepsilon(\omega(\varepsilon^{-2}t)) - \eta_\varepsilon(\omega(\varepsilon^{-2}s)) \right),$$

where we have defined

$$\eta_\varepsilon(\omega) = \varepsilon^{\frac{d}{2}+1} \sum_j \nabla h_{\vec{n}}(\varepsilon x_j(\omega)) \cdot v_j(\omega).$$

Observe that, by Lemma 3.2, γ_ε is uniformly bounded in $L^2(\mu)$, then by the Cauchy-Schwarz inequality and stationarity

$$\mathbb{E}_\mu \left[\sup_{\substack{t,s\in[0,T] \\ |t-s|\leq\delta}} I_2^\varepsilon(t,s)^2 \right] \leq \delta \int_0^T \mathbb{E}_\mu \left(\gamma_\varepsilon(\omega(\varepsilon^{-2}\tau))^2 \right) \, d\tau \, \leq \delta T \, < \gamma_\varepsilon^2 > \leq \delta T c(\vec{n}) \tag{5.8}$$

with $c(\vec{n})$ a constant independent of ε and δ. It follows that this term vanishes to 0 as δ goes to 0.

For what concerns $I_1^\varepsilon(t,s)$, we will prove that for any $R > 0$,

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \mathbb{P}_\mu \left(\sup_{\substack{t,s\in[0,T] \\ |t-s|\leq\delta}} |I_1^\varepsilon(t,s)| > R \right) = 0.$$

This follows if we prove

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \frac{1}{\delta} \mathbb{P}_\mu \left(\sup_{s\leq t\leq s+\delta} |I_1^\varepsilon(t,s)| > R \right) = 0.$$

By stationarity this is equivalent to prove

$$\lim_{\delta\to 0} \frac{1}{\delta} \limsup_{\varepsilon\to 0} \mathbb{P}_\mu \left(\sup_{0\leq t\leq\delta} |I_1^\varepsilon(t,0)| > R \right) = 0.$$

Let us write $I_1^\varepsilon(t,s)$ as the difference $M_t^\varepsilon - M_s^\varepsilon$, where M_t^ε is a martingale with quadratic variation given by

$$A_\varepsilon(t) = 2\varepsilon^d \int_0^t \sum_j (\nabla h_{\vec{n}})^2 \, (\varepsilon x_j(\varepsilon^{-2}u))\, du.$$

Observe that $\sup_\varepsilon < A_\varepsilon(t) > \leq Ct\rho n$. For any $\ell > 0$ define the stopping time

$$\tau_{\varepsilon,\ell} = \inf\{t : A_\varepsilon(t) > \ell\}$$

and let $\widehat{M}_t^{\varepsilon,\ell} = M^\varepsilon_{t\wedge\tau_{\varepsilon,\ell}}$. Then for any fixed ℓ, $\widehat{M}_t^{\varepsilon,\ell}$ is a martingale with quadratic variation bounded by ℓ, and by a standard estimate

$$\mathbb{P}_\mu\left(\sup_{0\leq t\leq\delta}|\widehat{M}_t^{\varepsilon,\ell}| \geq R\right) \leq 2e^{-\frac{R^2}{2\delta\ell}}. \tag{5.9}$$

Then for any fixed ℓ and any $R > 0$,

$$\mathbb{P}_\mu\left(\sup_{0\leq t\leq\delta}|M_t^\varepsilon| > R\right) \leq 2e^{-\frac{R^2}{2\delta\ell}} + \mathbb{P}_\mu\left(\tau_{\varepsilon,\ell} < \delta\right)$$

and

$$\mathbb{P}_\mu\left(\tau_{\varepsilon,\ell} < \delta\right) \leq \mathbb{P}_\mu\left(A_\varepsilon(\delta) > \ell\right) \leq \frac{C\rho n\delta}{\ell}.$$

It follows that for any $R > 0$,

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\frac{1}{\delta}\mathbb{P}_\mu\left(\sup_{0\leq t\leq\delta}|M_t^\varepsilon| \geq R\right) = 0.$$

In order to study $I_3^\varepsilon(t,s)$, observe that

$$\mathbb{E}_\mu\left[\sup_{\substack{t,s\in[0,T]\\|t-s|\leq\delta}} I_3^\varepsilon(t,s)^2\right] \leq 2\mathbb{E}_\mu\left(\sup_{t\in[0,T]}\left[\eta_\varepsilon(\omega(\varepsilon^{-2}t))\right]^2\right).$$

Then the evolution equations for η_ε say

$$\begin{aligned}\eta_\varepsilon(\omega(\varepsilon^{-2}t)) = e^{-\varepsilon^{-2}t}\eta_\varepsilon(\omega(0)) &+ \int_0^t e^{-\varepsilon^{-2}(t-\tau)}\gamma_\varepsilon(\omega(\varepsilon^{-2}\tau))\,d\tau\\ &+ \int_0^t e^{-\varepsilon^{-2}(t-\tau)}\,dM_\tau^\varepsilon.\end{aligned}$$

For the first term of the rhs of the above expression, observe that $<\eta_\varepsilon^2> \to 0$ as $\varepsilon \to 0$ (cf. (3.4)). About the second term, by Schwarz inequality we have

$$\begin{aligned}&\mathbb{E}_\mu\left(\sup_{t\in[0,T]}\left[\int_0^t e^{-\varepsilon^{-2}(t-\tau)}\gamma_\varepsilon(\omega(\varepsilon^{-2}\tau))\,d\tau\right]^2\right)\\ &\leq T < \gamma_\varepsilon^2 > \sup_{t\in[0,T]}\int_0^t e^{-2\varepsilon^{-2}(t-\tau)}d\tau \underset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

About the martingale term, by Doob's inequality:

$$\mathbb{E}_\mu\left(\sup_{t\in[0,T]}\left[\int_0^t e^{-\varepsilon^{-2}(t-\tau)}dM_\tau^\varepsilon\right]^2\right) \leq 8\left(\int_0^T e^{-2\varepsilon^{-2}(T-\tau)}\,d\tau\right)\rho\int(\nabla h_{\vec{n}})^2(q)\,dq$$

that converges to 0 as $\varepsilon \to 0$. $\square$

6. The Macroscopic Equation

As a consequence of the results contained in Sects. 3 and 4 we have that for any test function $G \in \mathcal{C}^2(\mathbb{R}^d)$ with compact support

$$\lim_{\varepsilon \to 0} \mathbb{E}_\mu \left(\sup_{0 \le t \le T} \left| \xi_t^\varepsilon(G) - \xi_0^\varepsilon(G) - \int_0^t \xi_s^\varepsilon(D\Delta G)\, ds - M_t^\varepsilon(G) \right|^2 \right) = 0,$$

where the martingale M_t^ε is given by the stochastic integral,

$$M_t^\varepsilon(G) = \int_0^{\varepsilon^{-2}t} \sqrt{2}\varepsilon^{d/2+1} \sum_j (\nabla G)(\varepsilon x_j(s)) \cdot dw_j(s).$$

The quadratic variation of M_t^ε is given by

$$A_t^\varepsilon = 2\varepsilon^2 \int_0^{\varepsilon^{-2}t} \varepsilon^d \sum_j |(\nabla G)(\varepsilon x_j(s))|^2\, ds.$$

It is easy to see that A_t^ε converges uniformly for $t \in [0, T]$ to $2\rho t \int |(\nabla G)(q)|^2\, dq$ in $L^1(\mathbb{P}_\mu)$ and a.s., i.e.

$$\mathbb{E}_\mu \left(\sup_{0 \le t \le T} \left| A_t^\varepsilon - 2\rho t \int |(\nabla G)(q)|^2\, dq \right| \right) = 0.$$

Consequently the law of $M_t^\varepsilon(G)$ in $\mathcal{C}([0, T], \mathbb{R})$ converges to a brownian motion $M_t(G)$ with variance given by $2\rho t \int |(\nabla G)(q)|^2\, dq$.

By the results in Sect. 5, the laws P^ε of ξ_t^ε are tight in $\mathcal{C}([0, T], \mathcal{H}_{-k})$ for any $k > d + 1$. It follows that any limit point P of P^ε is concentrated on the solutions of the equation

$$\xi_t(G) - \xi_0(G) = \int_0^t \xi_s(D\Delta G)\, ds + M_t(G) \qquad \forall G \in \mathcal{C}_c^2(\mathbb{R}^d). \tag{6.1}$$

Then the conditions of Theorem 11.0.2 in [7] are satisfied and Holley-Stroock theory can be applied to identify ξ_t as the corresponding *generalized Ornstein-Uhlenbeck process*. Since these limits are obtained in the stationary state μ, for each fixed time $t \in [0, T]$ the marginal distribution of ξ_t is the law of the centered Gaussian field on $\mathbb{R}^d$ with covariance (cf. [1])

$$\mathbb{E}_P(\xi_t(G)\xi_t(F)) = \chi \int G(q)F(q)\, dq.$$

This permits to identify P as the distribution of the stationary solution of (6.1), and D with ρ/χ. We summarize the final result in the following theorem.

Theorem 6.1. *Let $k > d + 1$ and $T > 0$. The law ξ_t^ε on $\mathcal{C}([0, T], \mathcal{H}_{-k})$ converges to the law of the Gaussian process with covariance given by*

$$\mathbb{E}_P(\xi_t(G)\xi_s(F)) = \chi \int G(q)(e^{|t-s|D\Delta}F)(q)\, dq$$

with $D = \frac{\rho}{\chi}$.

7. Appendix A: Existence of the Dynamics

We prove here the existence of the *equilibrium dynamics* for the system of stochastic differential equations defined by (2.3).

The idea is to approximate the infinite dynamics by some local dynamics that have μ as invariant measure.

For that, let us introduce a once continuously differentiable function $a : \mathbb{R}^d \to [0, 1]$ with compact support $\Lambda \subset \mathbb{R}^d$ such that $|\nabla a(q)| \leq 1$ for all $q \in \mathbb{R}^d$. For every cutoff a, we consider the following system of stochastic differential equations:

$$\begin{aligned} dx_i(t) &= a(x_i(t))v_i(t)dt, \\ dv_i(t) &= -\,a(x_i(t))\sum_{j\neq i}\nabla V(x_j(t)-x_i(t))dt + \nabla a(x_i(t))dt \\ &\quad - a(x_i(t))v_i(t)dt + \sqrt{2a(x_i(t))}dw_i(t) \end{aligned} \tag{7.1}$$

with initial conditions $x_i(0) = x_i(\omega)$ and $v_i(0) = v_i(\omega)$.

Observe that particles outside Λ are frozen, while in the region where $a = 1$ our particles follow the original equations of motion.

The generator of the Markov process associated to (7.1) can be written as

$$L_\Lambda = \sum_i e^{\mathfrak{H}_\Lambda(\omega)}\left[\partial_{v_i}a(x_i)e^{-\mathfrak{H}_\Lambda(\omega)}\partial_{v_i} + \partial_{v_i}a(x_i)e^{-\mathfrak{H}_\Lambda(\omega)}\partial_{x_i} - \partial_{x_i}a(x_i)e^{-\mathfrak{H}_\Lambda(\omega)}\partial_{v_i}\right],$$

where we have defined

$$\mathfrak{H}_\Lambda(\omega) = \sum_{x_j(\omega)\in\Lambda}\left[\frac{v_j(\omega)^2}{2} + \frac{1}{2}\sum_{x_i(\omega)\in\Lambda}V(x_j(\omega)-x_i(\omega)) + \sum_{x_k(\omega)\in\Lambda^c}V(x_j(\omega)-x_k(\omega))\right].$$

For any $m \in \mathbb{N}$ and any fixed configuration $\omega^x_{\Lambda^c}$, the canonical Gibbs measure $\mu(\,\cdot\,|N_\Lambda = m, \omega^x_{\Lambda^c})$ is invariant for (7.1). Consequently μ is also invariant.

We shall write $\omega^a(t) = (x^a(t), v^a(t))$ for any solution of (7.1). Since for almost every configuration with respect to μ there are only finitely many particles in Λ, the global existence of these dynamics for μ-a.e. initial configuration does not pose any problem.

Step 1. We prove first some bounds on the velocities and the densities that are valid with $\mathbb{P}_\mu$-probability 1.

In the following, we shall write B_L for the ball centered at the origin and of radius 2^L. Moreover, we shall denote $B(x, r)$ the ball centered at x and of radius r.

Proposition 7.1. *Let $T \in \mathbb{R}^+$. For any constant $C > 0$ we have*

$$\mathbb{P}_\mu\left(\exists L_0 > 0 :\ \forall L\ \geq L_0\ \sup_a\ \sup_{i:x_i\in B_L}\ \sup_{0\leq t\leq T}|v_i^a(t)| \leq CL\right) = 1. \tag{7.2}$$

Proof. Let us fix $L > 2T/C$ and consider

$$\mathbb{P}_\mu\left(\sup_a \sup_{i:x_i(\omega(0))\in B_L} \sup_{0\le t\le T} |v_i^a(t)| > CL\right)$$
$$\le \mathbb{E}_\mu\left(N_{B_L}(\omega(0))\mathbb{P}_\mu\left(\sup_a \sup_{0\le t\le T} |v_1^a(t)| > CL \Big| N_{B_L}(\omega(0))\right)\right). \qquad (7.3)$$

Since

$$\sup_a \sup_{0\le t\le T} |v_1^a(t)| \le |v_1^a(0)| + \sqrt{2}\sup_{0\le t\le T}|w_1(t)| + \int_0^T \Big|\sum_{j\ne 1}\nabla V(x_1^a(s)-x_j^a(s))\Big|\, ds$$
$$+\ T + \int_0^T |v_1^a(s)|\, ds,$$

(7.3) is bounded above by

$$< N_{B_L}\mathbf{1}_{[|v_1|>CL/8]} >_\mu + C_1\rho 2^{Ld}\mathbb{E}\left(\sup_{0\le t\le T}|w_1(t)| > \frac{CL}{8}\right)$$
$$+\mathbb{E}_\mu\left(N_{B_L}(\omega(0))\mathbb{P}_\mu\left(\int_0^T \Big|\sum_{j\ne 1}\nabla V(x_1^a(s)-x_j^a(s))\Big|\, ds > \frac{CL}{8}\Big|\ N_{B_L}(\omega(0))\right)\right)$$
$$+\mathbb{E}_\mu\left(N_{B_L}(\omega(0))\mathbb{P}_\mu\left(\int_0^T |v_1^a(s)|\, ds \ge \frac{CL}{8}\Big|\ N_{B_L}(\omega(0))\right)\right), \qquad (7.4)$$

where C_1 is a constant depending only on the dimension d. Since, under μ, v_1 is Gaussian distributed and is independent from the positions,

$$< N_{B_L}\mathbf{1}_{[|v_1|>CL/8]} >_\mu \le C_1\rho 2^{Ld}e^{-C_2L^2}.$$

The second term involving $w_1(t)$, by a standard estimate on Brownian motion, satisfies a similar bound.

Using the Chebichef inequality, the third term of (7.4), is bounded, for any $\alpha > 0$, by

$$\mathbb{E}_\mu\left(N_{B_L}\exp\left\{\alpha\left[\int_0^T \Big|\sum_{j\ne 1}\nabla V(x_1^a(s)-x_j^a(s))\Big|\, ds\right]^2\right\}\right)e^{-\frac{\alpha C^2L^2}{16}}.$$

By the Schwarz and Jensen inequalities this is bounded by

$$\mathbb{E}_\mu\left(N_{B_L}\frac{1}{T}\int_0^T \exp\left\{\alpha T^2\left[\sum_{j\ne 1}\nabla V(x_1^a(s)-x_j^a(s))\right]^2\right\}ds\right)e^{-\alpha C_3L^2},$$

where C_3 depends only on C and on T. Then by stationarity this is equal to

$$\left\langle N_{B_L} \exp\left\{\alpha T^2 \left[\sum_{j\neq 1} \nabla V(x_1 - x_j)\right]^2\right\}\right\rangle e^{-\alpha C_3 L^2}.$$

Since the range of the interaction is R, posing $\eta = \alpha\|\nabla V\|_\infty T^2$, this last quantity is bounded by

$$\begin{aligned}\left\langle N_{B_L} \exp\left[\eta N^2_{B(x_1,R)}\right]\right\rangle e^{-\alpha C_3 L^2} &\leq \left\langle N^2_{B_L}\right\rangle^{1/2} \left\langle \exp\left[2\eta N^2_{B(x_1,R)}\right]\right\rangle^{1/2} e^{-\alpha C_3 L^2} \\ &\leq C_5 \rho 2^{2Ld} \left\langle \exp\left[2\eta N^2_{B(0,1\vee 2R)}\right]\right\rangle^{1/2} e^{-\alpha C_3 L^2}. \quad (7.5)\end{aligned}$$

By superstability estimates $\left\langle \exp\left[2\eta N^2_{B(0,1\vee 2R)}\right]\right\rangle$ is finite for η small enough. Then we can correspondingly choose α small and we obtain that (7.5) is bounded by $C_6 \rho 2^{Ld}$ $e^{-\alpha C_3 L^2}$.

We treat in a similar way the 4$^{\text{th}}$ term in (7.4) and, for α small enough, we can bound it by

$$\left\langle N_{B_L} e^{\alpha T^2 |v_1|^2}\right\rangle e^{-\alpha C_3 L^2} \leq C_7 \rho 2^{Ld} e^{-\alpha C_3 L^2}.$$

Then (7.2) follows by Borel-Cantelli lemma. □

Proposition 7.2.

$$\mu\left(\exists l > 0, \forall L : \sup_{x\in B_L} N_{B(x,L)} \leq l L^d\right) = 1. \quad (7.6)$$

Proof. This is a consequence of Ruelle's superstability estimates (cf. [16, 13]). □

Step 2. To prove the existence of limiting solutions when the cutoff is removed, we have to compare the solution corresponding to different cutoffs a.

Let A_L denote the set of all once continuously differentiable $a : \mathbb{R}^d \to [0, 1]$ with compact support such that $|\nabla a(q)| \leq 1 \ \ \forall q \in \mathbb{R}^d$ and such that $a(q) = 1$ for all $|q| \leq 2^L$.

For a and $\bar{a}$ in A_L, let us consider $\omega^a(t) = (x^a(t), v^a(t))$ and $\omega^{\bar{a}}(t) = (x^{\bar{a}}(t), v^{\bar{a}}(t))$ the corresponding solutions to (7.1) with the same initial value $\omega(0) = (x_i, v_i)_{i\in\mathbb{N}}$.

For some $r > 0$ and $t > 0$ let us define

$$\Gamma_1(a, \bar{a}; r, t) \doteq \sum_{i:|x_i|\leq r} \sup_{0\leq s\leq t} |x_i^a(s) - x_i^{\bar{a}}(s)|$$

and

$$\Gamma_2(a, \bar{a}; r, t) \doteq \sum_{i:|x_i|\leq r} \sup_{0\leq s\leq t} |v_i^a(s) - v_i^{\bar{a}}(s)|.$$

We prove that

Lemma 7.3. *For any $T > 0$ and any $r > 0$,*

$$\lim_{L\to+\infty} \sup_{a,\bar{a}\in A_L} \Gamma_1(a,\bar{a};r,T) + \Gamma_2(a,\bar{a};r,T) = 0 \qquad \mathbb{P}_\mu - a.e.$$

Proof. We recall that by Proposition 7.1, for a given $C > 0$, with $\mathbb{P}_\mu$ probability one there exists a real L_0 such that for any $L \geq L_0$, any $t \leq T$ and any i such that $x_i \in B_L$ we have $|x_i^a(t) - x_i| \leq CLt$ and $|x_i^{\bar{a}}(t) - x_i| \leq CLt$. So, take $L \geq L_0$ in the following.

Fix $r_0 > 0$, and take $L > L_0$ sufficiently large such that $r_0 + R + 2CLT \leq 2^L$. Then

$$\sup_{i:|x_i|\leq r_0} \sup_{0\leq t\leq T} |x_i^a(t)| \leq 2^L \qquad \sup_{i:|x_i|\leq r_0} \sup_{0\leq t\leq T} |x_i^{\bar{a}}(t)| \leq 2^L.$$

By a simple computation, if $|x_i(0)| \leq r_0$,

$$|x_i^a(t) - x_i^{\bar{a}}(t)| \leq \int_0^t |v_i^a(s) - v_i^{\bar{a}}(s)|\, ds,$$

which implies immediately

$$\Gamma_1(a,\bar{a};r_0,t) \leq \int_0^t \Gamma_2(a,\bar{a};r_0,s)\, ds.$$

Then, always for $|x_i(0)| \leq r_0$,

$$|v_i^a(t) - v_i^{\bar{a}}(t)| \leq \int_0^t \left(C_V \sum_{x_j\in B(x_i,R+2CLT)} |x_j^a(s) - x_j^{\bar{a}}(s)| \right) ds + \int_0^t |v_i^a(s) - v_i^{\bar{a}}(s)|\, ds,$$

where $C_V = \|\Delta V\|_\infty$. Therefore,

$$\Gamma_2(a,\bar{a};r_0,t) \leq \int_0^t \left(C_V \sup_{x\in B_L} N_{B(x,R+2CLT)} \Gamma_1(a,\bar{a};r_1,s) + \Gamma_2(a,\bar{a};r_0,s) \right) ds,$$

where $r_1 = r_0 + R + 2CLT \leq 2^L$. By Proposition 7.2 with $\mathbb{P}_\mu$ probability one, there exists a finite l such that $\sup_{x\in B_L} N_{B(x,R+2CLT)} \leq C_1 l L^d T^d$.

So we have proven that, with $\mathbb{P}_\mu$ probability one, there exist two finite constants $l, L_0 > 0$ such that for any $L \geq L_0$, any $t \leq T$ and any $a, \bar{a} \in A_L$,

$$\Gamma_1(a,\bar{a};r_0,t) + \Gamma_2(a,\bar{a};r_0,t) \leq C' l T^d L^d \int_0^t (\Gamma_1(a,\bar{a};r_1,s) + \Gamma_2(a,\bar{a};r_1,s))\, ds.$$

We finish the proof by iterating the above relation for a number of times given by

$$h_L = \left[\frac{2^L - r_0}{R + 2CTL} \right]$$

and we obtain

$$\begin{aligned}&\Gamma_1(a,\bar{a};r_0,T)+\Gamma_2(a,\bar{a};r_0,T)\\&\le \left(C' l L^d T^d\right)^{h_L}\int_0^T dt_1\cdots\int_0^{t_{h_L}} dt_{h_L}\left(\Gamma_1(a,\bar{a};2^L,t_{h_L})+\Gamma_2(a,\bar{a};2^L,t_{h_L})\right).\end{aligned}$$

Since $\Gamma_1(a,\bar{a};2^L,t_{h_L})\le 2^L$ and $\Gamma_2(a,\bar{a};2^L,t_{h_L}))\le 2CL$ we have

$$\sup_{a,\bar{a}}\Gamma_1(a,\bar{a};r_0,T)+\Gamma_2(a,\bar{a};r_0,T)\le \frac{C_1 2^L\left(C' l T^d L^d\right)^{h_L}}{h_L!}T^{h_L+1}$$

that converges to 0 as $L\to\infty$. □

Step 3. Conclusion. Now, we are able to prove the existence of limiting solutions to (1.1). Let us consider a sequence of a partial solution $\omega_n(t)$ for $n\in\mathbb{N}$ of (7.1) with a common initial value $\omega_n(0)=(x_i,v_i)$. The corresponding cutoff function a_n is taken in A_n, i.e. $a_n(q)=1$ if $|q|\le 2^n$. By Lemma 7.3, with P_μ probability 1, ω_n converges to some limit $\omega(t)$ for each $t<T$. Elementary considerations following the argument of the previous lemma show that this limit is a solution of (1.1) and it does not depend on the particular cutoff chosen.

8. Appendix B: Spectral Gap for Interacting Brownian Particles

We prove here the spectral gap bound we used in Sect. 4. In this Appendix Λ is a fixed centered cube of sidelength $2l$ ($|\Lambda|=(2l)^d$). We fix the number of particles in the box Λ to be equal to n, and an arbitrary configuration $\omega^x_{\Lambda^c}$ outside Λ. Let $m=m(\omega^x_{\Lambda^c})$ be the number of particles of the outside configuration $\omega^x_{\Lambda^c}$ that are at distance less than or equal to R, the radius of the range of the interaction V.

Recall that $\Gamma_\Lambda(\cdot)(n,\omega^x_{\Lambda^c})$ is the corresponding canonical Gibbs expectation.

Theorem 8.1. *Let $f(x_1,\dots,x_n)$ be a $\mathcal{C}^1(\Lambda^n)$ function such that $\Gamma_\Lambda(f)(n,\omega^x_{\Lambda^c})=0$. Then*

$$\Gamma_\Lambda(f^2)(n,\omega^x_{\Lambda^c})\le 8dl^2 n e^{4(2n+m)\|V\|_\infty}\sum_{i=1}^n\Gamma_\Lambda\left(\left|\nabla_{x_i}f\right|^2\right)(n,\omega^x_{\Lambda^c}).\qquad(8.1)$$

Proof. We denote by $\mu_\Lambda(d\mathbf{x})=\mu_\Lambda(dx_1,\dots,dx_n|n,\omega^x_{\Lambda^c})$ the canonical measure corresponding to $\Gamma_\Lambda(\cdot)(n,\omega^x_{\Lambda^c})$. Observe that we can rewrite the canonical variance of f as

$$\begin{aligned}&\Gamma_\Lambda(f^2)(n,\omega^x_{\Lambda^c})\\&=\frac{1}{2}\iint_{\Lambda^n\times\Lambda^n}\left(f(\mathbf{x})-f(\mathbf{x}')\right)^2\mu_\Lambda(d\mathbf{x})\mu_\Lambda(d\mathbf{x}')\\&=\frac{1}{2Z^2}\iint_{\Lambda^n\times\Lambda^n}\left(f(\mathbf{x})-f(\mathbf{x}')\right)^2 e^{-\mathcal{H}_n(\mathbf{x},\omega^x_{\Lambda^c})-\mathcal{H}_n(\mathbf{x}',\omega^x_{\Lambda^c})}d\mathbf{x}\,d\mathbf{x}',\qquad(8.2)\end{aligned}$$

where we denoted

$$\mathcal{H}_n(\mathbf{x},\omega^x_{\Lambda^c})=\sum_{(i,j)}^n V(x_i-x_j)+\sum_{i=1}^n\sum_{y_j\in\omega^x_{\Lambda^c}}V(x_i-y_j)$$

and $Z=Z_\Lambda(n,\omega^x_{\Lambda^c})$ is the corresponding canonical partition function.

For each couple of configurations $\mathbf{x}, \mathbf{x}'$ we will choose a particular piecewise differentiable path that will connect these two configurations. The choice of this transformation follows a simple rule: we move each particle one by one along the coordinates axis. So, in order to simplify notation, it is convenient to consider $\mathbf{x}, \mathbf{x}'$ as points in $[-l, l]^{nd}$ and write $\mathbf{x} = \{y_1, \dots, y_{nd}\}$ and $\mathbf{x}' = \{y'_1, \dots, y'_{nd}\}$. Then defining

$$\xi_\alpha(t) = y_\alpha + t(y'_\alpha - y_\alpha) \qquad t \in [0, 1) \quad \alpha = 1, \dots, nd$$

we can rewrite the difference

$$f(\mathbf{x}) - f(\mathbf{x}') = \sum_{\alpha=1}^{nd} \int_0^1 \frac{d}{dt} f(y'_1, \dots, y'_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \dots, y_{nd})\, dt$$

and by the Schwarz inequality

$$\begin{aligned} &\big(f(\mathbf{x}) - f(\mathbf{x}')\big)^2 \\ &\quad \le 2nd \sum_{\alpha=1}^{nd} \int_0^1 \left| f_\alpha(y'_1, \dots, y'_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \dots, y_{nd})\right|^2 |\dot\xi_\alpha(t)|^2 \sqrt{1-t}\, dt, \end{aligned} \tag{8.3}$$

where we have denoted by f_α the partial derivative of f with respect to y_α and $\dot\xi_\alpha(t) = y'_\alpha - y_\alpha$. Since $|\dot\xi_\alpha(t)| \le 2l$ we have that the right-hand side of (8.3) is bounded by

$$8l^2 nd \sum_{\alpha=1}^{nd} \int_0^1 \left| f_\alpha(y'_1, \dots, y'_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \dots, y_{nd})\right|^2 \sqrt{1-t}\, dt. \tag{8.4}$$

Since the interaction V is bounded, for any $\xi_\alpha \in [-l, l]$ we have the uniform bound

$$\begin{aligned} &\mathcal{H}_n(\mathbf{x}, \omega^x_{\Lambda^c}) + \mathcal{H}_n(\mathbf{x}', \omega^x_{\Lambda^c}) \\ &\quad \ge -4(2n+m)\|V\|_\infty + \mathcal{H}_n(y'_1, \dots, y'_{\alpha-1}, \xi_\alpha, y_{\alpha+1}, \dots, y_{nd}, \omega^x_{\Lambda^c}) \\ &\qquad + \mathcal{H}_n(y_1, \dots, y_{\alpha-1}, y'_\alpha, y'_{\alpha+1}, \dots, y'_{nd}, \omega^x_{\Lambda^c}). \end{aligned} \tag{8.5}$$

To prove this, observe that for any $\theta, \eta \in \Lambda$ we can rewrite

$$\begin{aligned} &\mathcal{H}_n(\mathbf{x}, \omega^x_{\Lambda^c}) + \mathcal{H}_n(\mathbf{x}', \omega^x_{\Lambda^c}) \\ &\quad = \mathcal{H}_n(x_1, \dots, x_{i-1}, \theta, x'_{1+1}, \dots, x'_n, \omega^x_{\Lambda^c}) \\ &\qquad + \mathcal{H}_n(x'_1, \dots, x'_{i-1}, \eta, x_{1+1}, \dots, x_n, \omega^x_{\Lambda^c}) \\ &\qquad + \sum_{k=1}^{i-1} \left\{ (V(x_k - x_i) - V(x_k - \theta)) + \big(V(x'_k - x'_i) - V(x'_k - \eta)\big) \right\} \\ &\qquad + \sum_{k=i+1}^{n} \left\{ (V(x_k - x_i) - V(x_k - \eta)) + \big(V(x'_k - x'_i) - V(x'_k - \theta)\big) \right\} \\ &\qquad + \sum_{y_j \in \omega^x_{\Lambda^c}} \left\{ \big(V(x_i - y_j) - V(\theta - y_j)\big) + \big(V(x'_i - y_j) - V(\eta - y_j)\big) \right\}. \end{aligned}$$

Then, by an appropriate choice of θ, η, (8.5) follows.

Now, by the simple change of variable $y_\alpha \to \xi_\alpha(t)$, we have

$$\iint_{\Lambda^n \times \Lambda^n} dy_1 \dots dy_{nd} dy'_1 \dots dy'_{nd} \left| f_\alpha(y'_1, \dots, y'_{\alpha-1}, \xi_\alpha(t), y_{\alpha+1}, \dots, y_{nd}) \right|^2$$
$$\times e^{-\mathcal{H}_n(y'_1,\dots,y'_{\alpha-1},\xi_\alpha(t),y_{\alpha+1},\dots,y_{nd},\omega^x_{\Lambda^c}) - \mathcal{H}_n(y_1,\dots,y_{\alpha-1},y'_\alpha,y'_{\alpha+1},\dots,y'_{nd},\omega^x_{\Lambda^c})}$$

$$= \int_{-l}^{l} dy_1 \dots \int_{-l}^{l} dy_{\alpha-1} \int_{-l}^{l} dy'_\alpha \int_{-l}^{l} dy'_{\alpha+1} \dots \int_{-l}^{l} dy'_{nd} e^{-\mathcal{H}_n(y_1,\dots,y_{\alpha-1},y'_\alpha,y'_{\alpha+1},\dots,y'_{nd},\omega^x_{\Lambda^c})}$$
$$\times \int_{-l}^{l} dy'_1 \cdots \int_{-l}^{l} dy'_{\alpha-1} \int_{-l}^{l} dy_{\alpha+1} \cdots \int_{-l}^{l} dy_{nd} \int_{-(1-t)l+ty'_\alpha}^{(1-t)l+ty'_\alpha} \frac{d\xi_\alpha}{1-t}$$
$$\times \left| f_\alpha(y'_1, \dots, y'_{\alpha-1}, \xi_\alpha, y_{\alpha+1}, \dots, y_{nd}) \right|^2 e^{-\mathcal{H}_n(y'_1,\dots,y'_{\alpha-1},\xi_\alpha,y_{\alpha+1},\dots,y_{nd},\omega^x_{\Lambda^c})}$$

$$\leq \frac{Z}{1-t} \int_{\Lambda^n} d\mathbf{x} \, |f_\alpha(\mathbf{x})|^2 \, e^{-\mathcal{H}_n(\mathbf{x},\omega^x_{\Lambda^c})}. \tag{8.6}$$

Putting together (8.2), (8.4), (8.5) and (8.6), we obtain (8.1). □

9. Appendix C: Regularity of the Solution of the Poisson Equation

We prove here the regularity of the solution of Eq. (4.6), as a consequence of the spectral gap bound of the previous section. Let Λ be again the centered box of sidelength $2l$, and let Λ_R be the centered box of sidelength $2(l+R)$. In Λ_R we consider configurations with $n+m$ particles such that there are n particles in Λ (whose positions will be denoted by $\{x_1, \dots, x_n\}$) and m particles in $\Lambda_R \setminus \Lambda$ (whose positions will be denoted by $\omega^x_{\Lambda^c} = \{y_1, \dots, y_m\}$). We do not consider configurations with particles on the boundary $\partial\Lambda$. Let $g(x_1, \dots, x_n; y_1, \dots, y_m)$ be a smooth function of these configurations such that the canonical expectation $\Gamma_\Lambda(g)(n, \omega^x_{\Lambda^c}) = 0$. Recall we have defined the elliptic operator

$$L^W_{n,\omega^x_{\Lambda^c}} = \sum_{j=1}^{n} \left(\Delta_{x_j} - (\nabla_{x_j} \mathfrak{H}_\Lambda) \cdot \nabla_{x_j} \right),$$

where

$$\nabla_{x_j} \mathfrak{H}_\Lambda = \sum_{i \neq j}^{n} \nabla V(x_j - x_i) + \sum_{i}^{m} \nabla V(x_j - y_i).$$

Let $u(x_1, \dots, x_n; y_1, \dots, y_m)$ be the solution of the equation

$$L^W_{n,\omega^x_{\Lambda^c}} u(x_1, \dots, x_n; y_1, \dots, y_m) = g(x_1, \dots, x_n; y_1, \dots, y_m), \quad (x_1, \dots, x_n) \in \Lambda^n \tag{9.1}$$

with Neumann boundary conditions on $\partial\Lambda^n$. The position of the exterior particles $\omega^x_{\Lambda^c} = (y_1, \dots, y_m) \in (\Lambda_R \setminus \Lambda)^m$ should be considered as exterior parameters in Eq. (9.1).

Lemma 9.1.

$$\Gamma_\Lambda\left(\sum_{j=1}^n |\nabla_{x_j} u|^2\right)(n,\omega^x_{\Lambda^c}) \le c_1 n e^{c_2(n+m)}\Gamma_\Lambda(g^2)(n,\omega^x_{\Lambda^c}), \tag{9.2}$$

where $c_1 = 8dl^2$ *and* $c_2 = 8\|V\|_\infty$.

Proof. Multiplying (9.1) by u and integrating respect to the canonical Gibbs measure we obtain

$$\begin{aligned}
&\Gamma_\Lambda\left(\sum_{j=1}^n |\nabla_{x_j} u|^2\right)(n,\omega^x_{\Lambda^c})\\
&\quad= \Gamma_\Lambda\left(u(-L^W_{n,\omega^x_{\Lambda^c}} u)\right)(n,\omega^x_{\Lambda^c})\\
&\quad= \Gamma_\Lambda(ug)(n,\omega^x_{\Lambda^c}) \le \left(\Gamma_\Lambda\left(u^2\right)(n,\omega^x_{\Lambda^c})\right)^{1/2}\left(\Gamma_\Lambda\left(g^2\right)(n,\omega^x_{\Lambda^c})\right)^{1/2}.
\end{aligned}$$

By the spectral gap bound (8.1),

$$\Gamma_\Lambda\left(u^2\right)(n,\omega^x_{\Lambda^c}) \le c_1 n e^{c_2(n+m)}\Gamma_\Lambda\left(\sum_{j=1}^n |\nabla_{x_j} u|^2\right)(n,\omega^x_{\Lambda^c}).$$

Inserting this in the previous inequality, we obtain the bound (9.2). □

Lemma 9.2. *There exists a constant* c_3 *such that*

$$\sum_{j,k=1}^n \Gamma_\Lambda\left(|\nabla_{x_j}\nabla_{x_k} u|^2\right)(n,\omega^x_{\Lambda^c}) \le c_3 n^2 e^{2c_2(n+m)}\Gamma_\Lambda\left(n^2 g^2 + \sum_{k=1}^n |\nabla_{x_k} g|\right)(n,\omega^x_{\Lambda^c}). \tag{9.3}$$

Proof. This is just a standard elliptic regularity argument. Fix a $k = 1,\dots,n$. Let $\{\mathbf{e}_1,\cdots,\mathbf{e}_d\}$ be the canonical base in $\mathbb{R}^d$. The function $u_{k,\alpha} = \mathbf{e}_\alpha\cdot\nabla_{x_k} u$ satisfies the equation

$$L^W_{n,\omega^x_{\Lambda^c}} u_{k,\alpha} = \mathbf{e}_\alpha\cdot\nabla_{x_k} g - \sum_{j=1}^n V''(x_j - x_k) u_{j,\alpha}.$$

Defining $g_{k,\alpha}$, the right-hand side of this equation, we observe that

$$\begin{aligned}
\sum_{\alpha=1}^d \Gamma_\Lambda(g^2_{k,\alpha}) &\le 2\Gamma_\Lambda(|\nabla_{x_k} g|^2) + 2\|V''\|^2_\infty n\Gamma_\Lambda\left(\sum_{j=1}^n |\nabla_{x_j} u|^2\right)\\
&\le 2\Gamma_\Lambda(|\nabla_{x_k} g|^2) + 2c_1\|V''\|^2_\infty n^2 e^{c_2(n+m)}\Gamma_\Lambda(g^2).
\end{aligned}$$

By reapplying the same argument as in the proof of Lemma 9.1, we have

$$\Gamma_\Lambda\left(\sum_{j=1}^n |\nabla_{x_j} u_{k,\alpha}|^2\right) \le c_1 n e^{c_2(n+m)}\Gamma_\Lambda(g^2_{k,\alpha}).$$

and (9.3) follows from these last two inequalities. □

Lemma 9.3.

$$\sum_{k=1}^{m}\sum_{j=1}^{n}\Gamma_\Lambda\left(|\nabla_{x_j}\nabla_{y_k}u|^2\right)(n,\omega^x_{\Lambda^c}) \le c_3 n^2 e^{2c_2(n+m)}\Gamma_\Lambda\left(nmg^2+\sum_{k=1}^{m}|\nabla_{y_k}g|^2\right)(n,\omega^x_{\Lambda^c}). \tag{9.4}$$

Proof. The proof follows the same argument as in the one of Lemma 9.1, starting with the equations for $\mathbf{e}_\alpha \cdot \nabla_{y_k} u$. □

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