EINSTEIN RELATION FOR RANDOM WALKS IN RANDOM ENVIRONMENTS

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ABSTRACT. We consider a tracer particle performing a nearest neighbor random walk on $\mathbb{Z}^d$ in dimension $d \geq 3$ with random jump rates. This kind of a walk models the motion of a charged particle under a constant external electric field. We assume that the jump rates admit only two values $0 < \gamma_- < \gamma_+ < +\infty$, representing the lower and upper conductivities. We prove the existence of the mobility coefficient and that it equals to the diffusivity coefficient of the particle in zero external field.

1. INTRODUCTION

Consider a particle moving in a random medium, which can be constituted either by the molecules of a fluid in thermal equilibrium, or by atoms in a fixed periodic or random lattice. The trajectory $\mathbf{X}(t)$ of this particle, in a *large space-time scale*, can be regarded as a centered Brownian motion whose mean square displacement is proportional to time. The diffusivity of a Brownian particle is defined as a matrix $\mathbf{D} = [D_{p,q}]$, where for each $t > 0$

$$D_{p,q} := t^{-1}\mathbb{E}[X_p(t)X_q(t)], \quad p, q = 1, \ldots, d. \tag{1.1}$$

The mobility σ is defined in the following way. Suppose that the moving particle is electrically charged and an exterior uniform electric field $\mathbf{E} = E\mathbf{l}$ is applied in a given direction $\mathbf{l}$, represented by a unit vector in $\mathbb{R}^d$. In the corresponding *stationary state*, the particle will pick up a mean velocity $\mathbf{v}(E)$ corresponding to the magnitude E of the field. The limit vector

$$\lim_{E\to 0+} \frac{\mathbf{v}(E)}{E} =: \sigma \tag{1.2}$$

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defines *the mobility* of the particle. The Einstein relation, established in [5], states that $\sigma = \beta \mathbf{D}\mathbf{I}$, with $\beta^{-1} = k_B T$, T the fluid temperature and k_B the Boltzmann constant. A heuristic derivation of this relation can be found in section 8.8 of [22].

A rigorous derivation of the Einstein relation for a physically realistic model is a challenging problem. It usually requires proving the existence of a stationary state for a perturbed process and then showing good properties of relaxation towards this stationary state. For purely mechanical systems even the existence of a stationary state can be controversial (see [3] for a counter-example). For very special environments, where no re-collisions are possible, the Einstein relation can be proven (cf. [2]). In general, for models in which the *environment dynamics* is sufficiently strongly mixing (i.e. it has the spectral gap property) the Einstein relation can be established by classical perturbative methods (cf. [12] for a general result in that direction).

The problem becomes much more difficult for dynamics with conserved quantities (like the simple exclusion model), or with static environments (cf. [10], [7], [4]). In these cases the relaxation to equilibrium is slow. Recently M. Loulakis (cf. [13]) proved the Einstein relation for the tagged particle in the symmetric simple exclusion in dimension 3 or higher. The method of Loulakis uses the duality properties of the dynamics and transience estimates. Roughly speaking the transience property helps in limiting the effect of re-collisions.

In the present paper we consider a particle motion, modelled by a continuous time nearest neighbor random walk in $\mathbb{Z}^d$ in a *static* random environment. The dynamics can be described as follows. The particle located at given time t at site x waits for an exponential random time of unit intensity and performs a jump from site x to a neighboring site $x+e$ with the probability

$$c(x, x+e) := \frac{\gamma(\{x, x+e\})}{\sum_{|e'|=1} \gamma(\{x, x+e'\})},$$

where $\gamma(\{x, x+e\}) = \gamma(\{x+e, x\}), x, e \in \mathbb{Z}^d, |e| = 1$ are independent identically distributed random variables with values in the interval $[\gamma_-, \gamma_+] \subset R_+$. Both here and in what follows $|\cdot|$ defines the Euclidean norm in $\mathbb{R}^d$. This type of a walk is sometimes called *the random walk among random conductances*, see [21]. In this paper we consider only the case where the i.i.d. random variables $\gamma(\{x, x+e\})$ take only two possible values γ_- and γ_+, with $0 < \gamma_- < \gamma_+ < \infty$. A degenerate version of this model has been discussed in the physics literature in the context of random walks on an infinite percolation cluster. In that case $\gamma(\{x, x+e\})$ can take only the two values 0, or 1.

It can be shown, see part i) of theorem 2.1 below, that $t^{-1}\mathbf{X}(t)$ converges to 0, as $t \to +\infty$ almost surely (jointly with respect to the realization of the environment and the random jumps of the walk). In addition, (see part ii) of theorem 2.1) the laws of $t^{-1/2}\mathbf{X}(t)$ converge weakly to a centered normal distribution $N(0, \mathbf{D})$, with $\mathbf{D}$ *the effective diffusivity matrix*.

For a given direction $\mathbf{l}$ and $\alpha \in \mathbb{R}$ we can consider the perturbed process $\left(\mathbf{X}^{(\alpha)}(t)\right)_{t\geq 0}$ that corresponds to the motion under an external forcing field. The jump rates are now given by

$$c^{(\alpha)}(x, x+e) := \mathrm{e}^{\alpha \mathbf{l}\cdot e} c(x, x+e). \tag{1.3}$$

The degenerate case of this model describing biased random walks on the supercritical percolation cluster can be found in the theoretical physics literature, see [8]. We also add here that the results concerning the law of large numbers for biased random walks, with jumps rates as in (1.3), in the degenerate case have been recently obtained in [23, 1].

Coming back to the situation of nondegenerate rates considered here it has been shown in [11] that *the environment process as seen from the particle* (see section 3.1 for its precise definition) has a unique stationary measure whose properties guarantee the existence of the mean velocity $\mathbf{v}(\alpha) = \lim_{t\to+\infty} t^{-1}\mathbf{X}^{(\alpha)}(t)$.

In the main theorem of this paper, see theorem 2.3 below, we prove the existence of the mobility coefficient (1.2) for the particle and establish the Einstein relation between the mobility and the effective diffusivity if the dimension is $d \geq 3$. In order to prove the Einstein relation, we adopt an appropriate modification of the method of Loulakis (cf. [13]). In fact the two-values assumption on the environments permits us to establish a duality property similar to the one used in [13]. We believe that a generalization of our result to n-valued, i.i.d. environments should be possible, provided a similar duality property can be established. The notion of a generalized duality has been introduced recently by Nagahata in [17].

However, there are some important differences between the results in [13] and those obtained in the present article. In [13] there is no proof of the existence of a steady state, the definition of mobility used there (see theorem 1, p. 351 in [13]) is weaker than the one established in this paper. Also technically there is a difference between working in a static and a dynamic environment. The example of such a difference is quite clearly manifested in lemma 4.2 below, where we establish the estimate (4.17). An analogue of this estimate in the simple exclusion case is provided by inequality (20) of [13], where the Dirichlet form corresponding to the particle exchange (which corresponds to the temporal evolution of the environment), not to the shift of the environment (as in (4.17)), appears as an upper bound.

At the beginning of section 4 we present a guide throughout the basic steps of the proof. The first step is to control the distance between the perturbed and unperturbed steady states in terms of the magnitude of the perturbation, see proposition 4.1. As in [13], by the use of the entropy inequality and estimating the entropy of the perturbed process relative to the unperturbed one, the problem is reduced to an estimate of the principal eigenvalue of the generator perturbed by a potential of order α. This is done using the variational formula for this eigenvalue and the crucial inequality (4.20). We point out here that establishing this inequality is essentially the only reason for which we need the special duality features of the environment. The second step of the proof is based on some local approximation of the correctors for the homogenization problem. This approximation is proved in theorem 4.5.

We note also that the Einstein relation in the one-dimensional case can be proved even for a more general ergodic random environment via explicit calculations. This was done in [7] in the continuous space case and can be adapted, with practically no change, to the present model.

Finally, we add that we are not aware of any other result for static models (i.e. when the environment does not evolve in time) in $d \geq 2$, with the possible exception of some periodic ones (see [18]), where the validity of the Einstein relation has been established.

2. PRELIMINARIES AND THE FORMULATION OF THE MAIN RESULTS

2.1. **The description of the model.** We denote by $\mathbb{B}^d$ the set of bonds on $\mathbb{Z}^d$, i.e. the set consisting of all unordered pairs $\{x, x+e\}$, where $x, e \in \mathbb{Z}^d$ and $|e| = 1$. Let Ω be the compact metric space $\{0,1\}^{\mathbb{B}^d}$. By $\mathcal{B}(\Omega)$ we denote the Borel σ–algebra of Ω. In fact, for any metric space $\mathbb{X}$ we denote by $\mathcal{B}(\mathbb{X})$ the σ–algebra of Borel sets. In addition, if $\mathcal{A}$ is any σ–algebra of subsets of $\mathbb{X}$ we denote by $B_b(\mathcal{A})$ the set of all bounded and $\mathcal{A}$–measurable real valued functions.

For any subset $A \subset \mathbb{B}^d$ let $\Omega_A := \{0,1\}^A$. Let also $B(A)$ be the set of all functions $F : \Omega_A \to \mathbb{R}$. When $\eta \in \Omega$ we denote by η_A the restriction of η to the set A. Let $0 < \gamma_- < \gamma_+ < +\infty$ be fixed and $\gamma : \{0,1\} \to \{\gamma_-, \gamma_+\}$ be given by $\gamma(0) = \gamma_-$, $\gamma(1) = \gamma_+$. Let $C(\Omega)$

denote the space of all real valued continuous functions on Ω. By $C_0(\Omega)$ we denote the space of all *local functions* $F : \Omega \to \mathbb{R}$, i.e. those for which there exists a finite set $A \subset \mathbb{B}^d$ and a function $G \in B(A)$ such that $F(\eta) = G(\eta_A)$.

Let $\rho \in [0,1]$ and B_ρ be the Bernoulli probability measure on $\{0,1\}$ given by $B_\rho[\{1\}] = \rho$, $B_\rho[\{0\}] = 1-\rho$. Denote by $\bar{\gamma} := (1-\rho)\gamma_- + \rho\gamma_+$. Let $\mu_\rho := B_\rho^{\bigotimes \mathbb{B}^d}$ be the product measure given on $\mathcal{B}(\Omega)$. By $\mu_\rho^A := B_\rho^{\bigotimes A}$ we denote the respective product measure induced on $(\Omega_A, \mathcal{B}(\Omega_A))$.

Suppose that $R > 0$ is a certain integer. Denote by Λ_R the set of those bonds $b = \{v, w\}$ that satisfy $|v|, |w| < R$. Let us fix $s < t$ and $\mathbf{l} \in \mathbb{S}^{d-1} := [\mathbf{m} \in \mathbb{R}^d : |\mathbf{m}| = 1]$. Let $\mathcal{V}_s^t$ be the σ-algebra generated by bonds b having non-empty intersection with the slab $[x \in \mathbb{Z}^d : s \le \mathbf{l} \cdot x \le t]$ and that do not intersect the half-lattice $\mathbb{H} := [x \in \mathbb{Z}^d : x \cdot \mathbf{l} > t]$. For a fixed $s \in \mathbb{R}$ we let $\mathcal{V}_s^+ := \bigvee_{t: s<t} \mathcal{V}_s^t$ and for a fixed $t \in \mathbb{R}$ we let $\mathcal{V}_t^- := \bigvee_{s:s<t} \mathcal{V}_s^t$.

For a given $\eta \in \Omega$, $\mathbf{l} \in \mathbb{S}^{d-1}$ and $\alpha \in \mathbb{R}$ we consider a continuous time nearest neighbor random walk on $\mathbb{Z}^d$, starting at 0, with the generator

$$L_\eta^{(\alpha)} f(x) := \sum_{|e|=1} c^{(\alpha)}(x, e; \eta) \partial_e f(x), \quad f \in C_0(\mathbb{Z}^d), \tag{2.1}$$

where $C_0(\mathbb{Z}^d)$ is the space of compactly supported functions of $\mathbb{Z}^d$, $\partial_e f(x) := f(x+e) - f(x)$ and

$$c^{(\alpha)}(x, e; \eta) := \mathrm{e}^{\alpha \mathbf{l} \cdot e} \frac{\gamma(x, e; \eta)}{Z(x, \eta)},$$

with $\gamma(x, e; \eta) := \gamma(\eta(x, x+e))$, $Z(x, \eta) := \sum_{|e|=1} \gamma(x, e; \eta)$. When $\alpha = 0$ the generator of the walk can be rewritten (regardless of the direction $\mathbf{l}$) in the following form

$$L_\eta f(x) := -Z^{-1}(x, \eta) \sum_{i=1}^{d} \partial_i^* \left[\gamma_i(x; \eta) \partial_i f(x)\right], \quad f \in C_0(\mathbb{Z}^d). \tag{2.2}$$

Here $\partial_i f := \partial_{e_i} f$, $\partial_i^* f := \partial_{-e_i} f$, $\gamma_i(x; \eta) := \gamma(x, e_i; \eta)$, where $e_1, \ldots, e_d$ is the canonical basis in $\mathbb{Z}^d$. Let $\mathcal{D} := D([0, +\infty); \mathbb{Z}^d)$ be equipped with the standard σ–algebra $\mathcal{M}$ and the filtration $(\mathcal{M}_t)$. The transition probabilities of the walk, its path measures on $\mathcal{D}$ and the respective expectations shall be denoted correspondingly by $p_\eta^{(\alpha)}(t, x, y)$, $P_{x,\eta}^{(\alpha)}$, $E_{x,\eta}^{(\alpha)}$, $x, y \in \mathbb{Z}^d$. We shall always assume that the random walk is defined over the canonical path space $\mathcal{T}_{x,\eta}^{(\alpha)} := (\mathcal{D}, \mathcal{M}, P_{x,\eta}^{(\alpha)})$ by the formula $(\mathbf{X}(t))_{t \ge 0}$, for $\mathbf{X} \in \mathcal{D}$. The subscript x shall be suppressed when the walk starts at 0.

Suppose that ν is a Borel probability measure on Ω. We shall consider the process $(\mathbf{X}(t))_{t\ge 0}$ over $\mathcal{T}_\nu^{(\alpha)} := (\mathcal{D}\times\Omega, \mathcal{M}\otimes\mathcal{B}(\Omega), Q_\nu^{(\alpha)})$, where $Q_\nu^{(\alpha)}$ is the semi-product measure $Q_\nu^{(\alpha)}(d\mathbf{X}, d\eta) = P_\eta^{(\alpha)}(d\mathbf{X})\otimes\nu(d\eta)$. Let $E_\nu^{(\alpha)}$ denote the expectation w.r.t. this measure. We remark here that according to our convention introduced above $P_\eta^{(\alpha)} = P_{0,\eta}^{(\alpha)}$, so the process in question starts at $\mathbf{0}$ with $Q_\nu^{(\alpha)}$ probability 1. In case that ν is the product Bernoulli measure μ_ρ we shall suppress the initial measure from the notation. We shall also suppress the superscript α from the notation when $\alpha = 0$.

2.2. The statements of the main results.

We consider first the trajectory process when no external field is present, i.e. $\alpha = 0$. Then, using arguments from homogenization theory (see section 3.2 below) one can show the following.

Theorem 2.1. *Suppose that the trajectory process $(\mathbf{X}(t))_{t\ge 0}$ is considered over the product probability space $\mathcal{T}$. Then,*

(i)

$$\lim_{t\to+\infty}\frac{\mathbf{X}(t)}{t} = 0, \quad Q - a.s.,$$

(ii) the laws of the r.v. $t^{-1/2}\mathbf{X}(t)$, converge weakly, as $t\to+\infty$, to a zero mean Gaussian r. v. whose co-variance matrix we shall denote by $\mathbf{D} = [D_{p,q}]$.

Our second principal result concerns the law of large numbers for the trajectory process describing the particle motion in the presence of a constant external field of magnitude α acting in the direction $\mathbf{l}$, see (2.1).

Theorem 2.2. *For each $\alpha\in\mathbb{R}$ we have*

$$\mathbf{v}(\alpha) := \lim_{t\uparrow+\infty}\frac{\mathbf{X}(t)}{t}, \quad Q^{(\alpha)} - a.s. \tag{2.3}$$

We note here that the version of this theorem for the walks in discrete time has been proved by L. Shen in [20].

Our theorem concerning the Einstein relation can be stated as follows.

Theorem 2.3. *Suppose that $d\ge 3$. Then, the function $\alpha\to\mathbf{v}(\alpha)$ defined by (2.3) is differentiable at 0 and*

$$\mathbf{v}'(0) = \mathbf{D}\mathbf{l}. \tag{2.4}$$

3. THE PROOFS OF THEOREMS 2.1 AND 2.2

3.1. **The environment process.** For any $y \in \mathbb{Z}^d$ we define a shift operator $T_y : \mathbb{B}^d \to \mathbb{B}^d$ via $T_y\{x, x+e\} := \{x+y, x+y+e\}$. With the help of T_x, we define the shift operator on Ω, which we also denote T_x, via $T_x(\eta)(b) := \eta(T_x(b))$, $b \in \mathbb{B}^d$. For any function $F : \Omega \to \mathbb{R}$ we let $D_x F := F \circ T_x - F$ and $D_p F := D_{e_p} F$, $p = 1, \ldots, d$.

Let $\mathcal{L}^{(\alpha)} : C(\Omega) \to C(\Omega)$ be a linear bounded operator given by

$$\mathcal{L}^{(\alpha)} F(\eta) := \sum_{|e|=1} c^{(\alpha)}(e;\eta) D_e F(\eta), \quad F \in C(\Omega), \tag{3.1}$$

with $c^{(\alpha)}(e;\eta) := c^{(\alpha)}(0,e;\eta)$. It is the generator of the Ω–valued, Markov process, that we shall call *the environment process*, given by

$$\zeta_\eta(t;\mathbf{X}) := T_{\mathbf{X}(t)}(\eta) \tag{3.2}$$

and defined over $\mathcal{T}_\eta^{(\alpha)}$. Here $(\mathbf{X}(t))_{t\ge 0}$ is the canonical trajectory process defined in the previous section. The transition of probability semigroup for $(\zeta_\eta(t))_{t\ge 0}$ is given by the formula

$$P_\alpha^t F(\eta) := \sum_{x \in \mathbb{Z}^d} p_\eta^{(\alpha)}(t,0,x) F(T_x \eta), \quad F \in C(\Omega),\ \eta \in \Omega. \tag{3.3}$$

For any Borel probability measure ν on Ω we denote by $\mathbb{P}_\nu^{(\alpha)}$ the path measure on the space $\mathcal{D}_\Omega := \mathcal{D}([0,+\infty);\Omega)$ corresponding to the process starting with the initial distribution ν. In case that $\nu = \delta_\eta$ the corresponding measure shall be denoted by $\mathbb{P}_\eta^{(\alpha)}$. Let $\mathbb{P}^{(\alpha)}$ denote the path measure of the environment process starting from the product Bernoulli measure μ_ρ.

We define *the equilibrium measure*

$$\bar\mu_0(d\eta) := \bar Z^{-1} Z(\eta)\, \mu_\rho(d\eta), \tag{3.4}$$

with $Z(\eta) := Z(0,\eta)$ and the normalizing factor $\bar Z := \int Z d\mu_\rho = 2d\bar\gamma$. We point out that when $\alpha \neq 0$ the measure $\bar\mu_0(d\eta)$ is not invariant under $\left(P_\alpha^t\right)_{t\ge 0}$ so the process $\zeta(t;\eta,\mathbf{X}) := \zeta_\eta(t;\mathbf{X})$ considered over probability space $\mathcal{T}_{\bar\mu_0}^{(\alpha)}$ is not stationary.

When $\alpha = 0$ the generator (3.1) (we omit in that case α from the notation for both the generator and semigroup) can be rewritten in the form

$$\mathcal{L}F := -\frac{1}{Z} \sum_{p=1}^{d} D_p^* \left(\gamma_p D_p F\right), \qquad F \in C(\Omega). \tag{3.5}$$

Here D_p^* is the adjoint to D_p w.r.t. the scalar product of $L^2(\bar{\mu}_0)$ (in fact $D_p^* = D_{-e_p}$). The measure $\bar{\mu}_0$ is then invariant, reversible and ergodic under the semigroup defined by (3.3). Indeed, note that for any $F, G \in C(\Omega)$ we have

$$\int G\,\mathcal{L}F d\bar{\mu}_0 = -\bar{Z}^{-1}\sum_{p=1}^{d}\int \gamma_p D_pF D_pG d\mu_\rho = \int F\mathcal{L}G d\bar{\mu}_0. \tag{3.6}$$

Here $\gamma_p(\eta) := \gamma_p(0;\eta)$. This shows invariance and reversibility of $\bar{\mu}_0$ under the Markovian dynamics governed by the generator $\mathcal{L}$. To prove ergodicity note that $\mathcal{L}F = 0$ implies, using (3.6) with $G = F$, that $D_pF = 0$ for all $p = 1, \ldots, d$ (thanks to positivity of γ_p). This in turn shows that F is constant because the product measure μ_ρ is ergodic under the action of the group T_x, $x \in \mathbb{Z}^d$. Invariance and reversibility of $\bar{\mu}_0$ imply that the semigroup (P^t) extends to a C_0–continuous semigroup of self-adjoint operators over $L^2(\bar{\mu}_0)$. The Dirichlet form that corresponds to the semigroup equals

$$\mathcal{E}(F,F) := -\int F\,\mathcal{L}F d\bar{\mu}_0 = \sum_{p=1}^{d}\int c_p(D_pF)^2\,d\bar{\mu}_0, \qquad F \in L^2(\bar{\mu}_0).$$

Here $c_p(\eta) := c(0, e_p; \eta)$, $p = 1, \ldots, d$.

3.2. **The proof of theorem 2.1 – homogenization.**

Here we deal with the motion of an unperturbed tracer in the equilibrium environment, i.e. when $\alpha = 0$. We recall that in such a case we suppress index α from the notation.

The position of the tracer at time t in the direction e_p is given by the formula

$$X_p(t) = \mathbf{X}(t)\cdot e_p = \int_0^t u_p(\zeta(s))\,ds + M_t^{(p)}, \tag{3.7}$$

where the random vector $\mathbf{u} := (u_1, \ldots, u_d)$, called *a local drift*, is given by

$$u_p := -Z^{-1}D_p^*\gamma_p, \quad p = 1, \ldots, d. \tag{3.8}$$

The process $\{M_t^{(p)}\}_{p=1}^d$, $t \geq 0$ is a square integrable vector valued martingale with the quadratic variation given by

$$\left\langle M^{(p)}, M^{(q)}\right\rangle_t = 2\delta_{p,q}\int_0^t d_p(\zeta(s))\,ds, \quad p, q = 1, \ldots, d, \tag{3.9}$$

where $d_p(\eta) := Z^{-1}(\eta)[\gamma_p(\eta) + \gamma_p(T_{-e_p}\eta)]$. Since $\bar{\mu}_0$ is ergodic and invariant for $(\zeta(t))_{t\geq 0}$ we can use the individual ergodic theorem to conclude that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t u_p(\zeta(s))\,ds = \int u_p d\bar{\mu}_0,\ Q - a.s.$$

On the other hand, the law of large numbers applied to a square integrable martingale $\{M_t^{(p)}\}_{p=1}^d$, see theorem 7.8.2 of [6], (it suffices only to check that $\sum_{n\geq 1} n^{-2} E\left(M_{n+1}^{(p)} - M_n^{(p)}\right)^2 < +\infty$) allows us to conclude that $t^{-1}\, M_t^{(p)} \to 0$, Q–a.s. Hence part i) of the theorem follows.

To show part ii) note first that

$$\left|\int u_p F d\bar\mu_0\right| = \bar Z^{-1}\left|\int \gamma_p D_p F d\mu_\rho\right| \leq C\mathcal{E}_{\mathcal{L}}^{1/2}(F,F), \quad \forall\, F \in L^2(\bar\mu_0) \tag{3.10}$$

for some positive constant C. We also note that the position $\mathbf{X}(t)$ of the random walker is an antisymmetric functional w.r.t. the time reversal operation, i.e. $\mathbf{X}(t) = \Theta_t((\zeta(s))_{0\leq s\leq t})$ for some measurable $\Theta_t : \mathcal{D}([0,t],\Omega) \to \mathbb{Z}^d$ satisfying $\Theta_t \circ R_t = -\Theta_t$, where $R_t(\xi)(s) := \xi(t-s)$, $s \in [0,t]$, $\xi \in \mathcal{D}([0,t],\Omega)$. This fact can be argued in exactly the same way as it is done on pp. 817-818 of [4]. Reversibility and ergodicity of the equilibrium measure $\bar\mu_0$, estimate (3.10), formula (3.7) and antisymmetry of $\mathbf{X}(t)$ are the hypotheses under which theorem 2.2 of [4] holds. The central limit theorem asserted in part ii) follows from the conclusion of the aforementioned theorem, see also [10].

Let us describe in more details the limiting co-variance matrix appearing in part ii) of theorem 2.1. By the time reversibility of $\bar\mu_0$ and antisymmetry of $\mathbf{X}(t)$ w.r.t. the time reversal (cf. (2.48) of [4])

$$E_{\bar\mu_0}\left(M_t^{(p)}M_t^{(q)}\right) = E_{\bar\mu_0}\left[\left(\int_0^t u_p(\zeta(s))\,ds\right)\left(\int_0^t u_q(\zeta(s))\,ds\right)\right] + E_{\bar\mu_0}\left[X_p(t)X_q(t)\right]. \tag{3.11}$$

As we recall from section 2.1 $E_{\bar\mu_0}$ denotes the expectation w.r.t. the semi-product measure $Q_{\bar\mu_0}(d\mathbf{X}, d\eta) = P_\eta(d\mathbf{X}) \otimes \bar\mu_0(d\eta)$. On the other hand, by (3.9), we have

$$E_{\bar\mu_0}\left(M_t^{(p)}M_t^{(q)}\right) = t\delta_{p,q}\int d_p(\eta)\;\bar\mu_0(d\eta) = \frac{t}{d}\,\delta_{p,q}. \tag{3.12}$$

So the asymptotic variance is given by

$$D_{p,q} = \frac{1}{d}\,\delta_{p,q} - \lim_{t\to\infty}\frac{1}{t}E_{\bar\mu_0}\left[\left(\int_0^t u_p(\zeta(s))\,ds\right)\left(\int_0^t u_q(\zeta(s))\,ds\right)\right] \tag{3.13}$$

In order to compute the second term on the right hand side of (3.13) we introduce the Hilbert space $\mathcal{H}_+$, the completion of the subspace $\mathcal{H}_+^0$ of $C(\Omega)$ consisting of those F for which $\int F d\bar\mu_0 = 0$ in the norm $\|F\|_+ := \mathcal{E}(F,F)^{1/2}$. We denote by $(\cdot,\cdot)_+$ the scalar product that corresponds to the norm $\|\cdot\|_+$. The dual of $\mathcal{H}_+$ will be denoted by $\mathcal{H}_-$. It is also a Hilbert space and we shall denote its scalar product by $(\cdot,\cdot)_-$. The operator $\mathcal{L}$ extends to a unitary isomorphism mapping $\mathcal{H}_+$ onto $\mathcal{H}_-$. The norm of $\Psi \in \mathcal{H}_-$ can be characterized via the following variational principle

$$\|\Psi\|_-^2 = (\Psi, (-\mathcal{L})^{-1}\Psi)_{L^2(\bar{\mu}_0)} = \sup_{F \in \mathcal{H}_+^0} [2(\Psi, F)_{L^2(\bar{\mu}_0)} - \|F\|_+^2].$$

Here $(-\mathcal{L})^{-1}$ is understood as the inverse operator to the extension $\mathcal{L} : \mathcal{H}_+ \to \mathcal{H}_-$. Then, according to [4], p. 804 the asymptotic variance of (3.13) equals

$$D_{p,q} = \frac{1}{d}\,\delta_{p,q} - 2(u_p, u_q)_- \tag{3.14}$$

Let $\lambda > 0$ and $\chi_\lambda^{(p)} \in C(\Omega)$ be the unique solution of the resolvent equation

$$(\lambda - \mathcal{L})\chi_\lambda^{(p)} := u_p, \quad p = 1, \ldots, d. \tag{3.15}$$

The following proposition gathers some useful properties of u_p and $\chi_\lambda^{(p)}$.

Proposition 3.1. *For each* $p = 1, \ldots, d$ *we have:*

i) $\sup_{\lambda \in (0,1]} \lambda \|\chi_\lambda^{(p)}\|_\infty < +\infty$.

ii) $\lim_{\lambda \to 0+} \lambda \|\chi_\lambda^{(p)}\|_- = 0$.

iii) $\chi_\lambda^{(p)}$ *and* $D_q\chi_\lambda^{(p)}$ *converge in* $\mathcal{H}_+$ *and* $L^2(\bar{\mu}_0)$*, respectively, as* $\lambda \to 0+$*. Denote by* $\chi_q^{(p)}$, $\Phi_q^{(p)}$ *their corresponding limits.*

iv) The functional $(u_p, \cdot)_{L^2(\bar{\mu}_0)}$ *has a continuous extension from* $\mathcal{H}_+^0$ *to* $\mathcal{H}_+$*, which we denote by the same symbol* (*i.e.* $u_p \in \mathcal{H}_-$)*. We have*

$$-\mathcal{L}\chi^{(p)} = u_p. \tag{3.16}$$

The limiting variance $\mathbf{D} = [D_{p,q}]$ *of theorem* 2.1 *equals*

$$D_{p,q} = 2\left[\frac{1}{2d}\,\delta_{p,q} + \left(\Phi_p^{(q)}, c_p\right)_{L^2(\bar{\mu}_0)}\right]. \tag{3.17}$$

Proof. To obtain *i)* we use the representation $\chi_\lambda^{(p)} = \int_0^\infty e^{-\lambda t} P^t u_p dt$ and the fact that $\|P^t u_p\|_\infty \leq \|u_p\|_\infty$ for all $t \geq 0$. Multiplying both sides of (3.15) by $\chi_\lambda^{(p)}$ and integrating w.r.t. $\bar{\mu}_0$ we obtain that $\sup_{\lambda>0} \|\chi_\lambda^{(p)}\|_+ < +\infty$. Since $\mathcal{L}$ is an isometry between $\mathcal{H}_+$ and $\mathcal{H}_-$ we have $\sup_{\lambda>0} \|\mathcal{L}\chi_\lambda^{(p)}\|_- < +\infty$. This, in conjunction with the fact that $u_p \in \mathcal{H}_-$ (see (3.10)), is equivalent with *ii)*, see p. 79 of [16]. Part *iii)* is a consequence of proposition 2.6

of [15]. The possibility of extending the functional $(u_p,\cdot)_{L^2(\bar\mu_0)}$ to the entire $\mathcal{H}_+$ follows from estimate (3.10). The results of parts *ii)* and *iii)* imply (3.16). Finally, since

$$(u_p,u_q)_- = -(u_p,\chi_q)_{L^2(\bar\mu_0)} \overset{(3.8)}{=} -\bar Z^{-1}\int \gamma_p\Phi_p^{(q)}d\mu_\rho = -(c_p,\Phi_p^{(q)})_{L^2(\bar\mu_0)}$$

we conclude (3.17) from (3.14). □

3.3. **The law of large numbers for additive functionals of local functions - the proof of theorem 2.2.**

We fix a direction $\mathbf{l}\in\mathbb{S}^{d-1}$ and assume that $\alpha\neq 0$. Below we formulate a result proven in [11] that asserts the existence of a steady state $\bar\mu_\alpha$ for the environment process corresponding to the perturbed trajectory. This measure is equivalent to $\bar\mu_0$ when restricted to the σ–algebra that can be associated with the "forward bonds" in the direction pointed by the drift $\mathbf{l}$, i.e. $\mathcal{V}_{-N}^+$ for any $N\geq 1$. Also, we assert a version of the strong law of large numbers that holds w.r.t. $Q^{(\alpha)}$.

To make the statement of the result precise we need some notation. Let $(\theta_t)_{t\geq 0}$ be the semi-dynamical system defined by the temporal shifts on $\mathcal{D}_\Omega$, i.e. $\theta_t\xi(\cdot):=\xi(\cdot+t)$, $\xi\in\mathcal{D}_\Omega$. For any $a\in\mathbb{R}$ we denote by $\mathcal{O}_a^+$ the smallest sub-σ–algebra of $\mathcal{B}(\mathcal{D}_\Omega)$ generated by mappings $\xi\to F(\xi(t))$, $\xi\in\mathcal{D}_\Omega$, where F is $\mathcal{V}_a^+$–measurable and $t\geq 0$. Note that each θ_t is $\mathcal{O}_a^+$ to $\mathcal{O}_a^+$–measurable, i.e. $\theta_t^{-1}(A)\in\mathcal{O}_a^+$, when $A\in\mathcal{O}_a^+$.

Theorem 3.2. *There exists a Borel probability measure $\bar\mu_\alpha$ on Ω satisfying the following conditions*

1) *it is invariant*

$$\int P_\alpha^t F d\bar\mu_\alpha = \int F d\bar\mu_\alpha, \quad \forall\, t\geq 0,\ F\in C(\Omega), \tag{3.18}$$

2) *for an arbitrary $N\geq 0$, $\bar\mu_\alpha$ is equivalent with $\bar\mu_0$, when restricted to $\mathcal{V}_{-N}^+$, i.e.*

$$\bar\mu_0(A)=0 \quad \textit{iff} \quad \bar\mu_\alpha(A)=0 \textit{ for all } A\in\mathcal{V}_{-N}^+, \tag{3.19}$$

3) *it is ergodic, i.e. if $F\in C(\Omega)$ is such that $P_\alpha^t F=F$ for all $t\geq 0$ we have $F=$const $\bar\mu_\alpha$-a.s.,*

4) *the law of large numbers holds, i.e. for any $N\geq 0$ and $F\in B_b(\mathcal{O}_{-N}^+)$ we have*

$$\lim_{T\uparrow+\infty}\frac{1}{T}\int_0^T F(\theta_t\xi)dt = \int F\, d\mathbb{P}_{\bar\mu_\alpha}^{(\alpha)}, \quad \textit{for } \mathbb{P}^{(\alpha)}-\textit{a.s. } \xi\in\mathcal{D}_\Omega, \tag{3.20}$$

5) $\bar{\mu}_\alpha$ *is unique, i.e. any other Borel measure on* Ω *satisfying conditions* $1) - 4)$ *listed above coincides with* $\bar{\mu}_\alpha$.

To conclude the law of large numbers asserted in theorem 2.2 we write

$$X_p(t) = \mathbf{X}(t)\cdot e_p = \int_0^t u_p^{(\alpha)}(\zeta(s))\,ds + M_t^{(p,\alpha)}, \tag{3.21}$$

In this case the local drift equals $\mathbf{u}^{(\alpha)} := (u_1^{(\alpha)},\ldots,u_d^{(\alpha)})$, where

$$u_p^{(\alpha)}(\eta) = Z^{-1}(\eta)\left[e^{\alpha l_p}\gamma_p(\eta) - e^{-\alpha l_p}\gamma_p(T_{-e_p}\eta)\right], \quad \eta\in\Omega. \tag{3.22}$$

The corresponding vector valued martingale $\{M_t^{(p,\alpha)}\}_{p=1}^d$ has the quadratic variation

$$\left\langle M^{(p,\alpha)}, M^{(q,\alpha)}\right\rangle_t = 2\delta_{p,q}\int_0^t d_p^{(\alpha)}(\zeta(s))\,ds, \quad p,q=1,\ldots,d, \tag{3.23}$$

where $d_p^{(\alpha)}(\eta) := Z^{-1}(\eta)[\mathrm{e}^{\alpha l_p}\gamma_p(\eta) + \mathrm{e}^{-\alpha l_p}\gamma_p(T_{-e_p}\eta)]$. The law of large numbers for a square integrable martingale implies again that $t^{-1}M_t^{(p,\alpha)}\to 0$, as $t\to+\infty$, $Q^{(\alpha)}$–a.s. To conclude the assertion of theorem 2.2 it suffices therefore to apply part 4) of theorem 3.2 with $F = u_p^{(\alpha)}$ for each $p=1,\ldots,d$. We obtain then that

$$\mathbf{v}(\alpha) = \int \mathbf{u}^{(\alpha)}\,d\bar{\mu}_\alpha. \tag{3.24}$$

Remark 3.3. It follows from the proof of theorem 3.1 of [11] that the component of the mean velocity $\mathbf{v}(\alpha)$ in the direction $\mathbf{l}$ is non-zero. □

4. The proof of theorem 2.3

Recall that according to (3.24) the velocity of the particle $\mathbf{v}(\alpha)$ is the expectation of the function $\mathbf{u}^{(\alpha)}$ with respect to the stationary measure. By (3.22) we can rewrite

$$u_p^{(\alpha)}(\eta) = \frac{l_p}{d} + u_p(\eta) + \alpha F(\eta) + O(\alpha^2), \tag{4.1}$$

where

$$F(\eta) := \frac{l_p}{Z(\eta)}\left(2\bar{\gamma} - \frac{Z(\eta)}{d} + \tilde{\gamma}_p(\eta) + \tilde{\gamma}_p(T_{-e_p}\eta)\right) \tag{4.2}$$

and $\tilde{\gamma}_p(\eta) := \gamma_p(\eta) - \bar{\gamma}$. Note that F is local and satisfies $\int F d\bar{\mu}_0 = 0$. In fact, we have $\int F d\bar{\mu}_\alpha \to 0$, as $\alpha\to 0$. This is a consequence of the following proposition:

Proposition 4.1. *For any local function F we have*

$$\left|\int F\, d\bar{\mu}_\alpha - \int F\, d\bar{\mu}_0\right| \le c_1|\alpha|. \tag{4.3}$$

The constant $c_1 > 0$ may depend on F but it does not depend on α.

Proof. We prove the result only in the case $\alpha > 0$. By considering $-F$ in place of F we can immediately conclude inequality (4.3) also for $\alpha < 0$. Recall our convention of omitting the superscript in relevant expressions when $\alpha = 0$. Let F be a local function such that $\int F\, d\bar{\mu}_0 = 0$. Applying the entropy inequality, see e.g. [9] p. 347, we get

$$\int \left(\alpha \int_0^T F(\zeta(s))\, ds\right) P_\eta^{(\alpha)}(d\mathbf{X}) \le \log \int \exp\left\{\alpha \int_0^T F(\zeta(s))\, ds\right\} P_\eta(d\mathbf{X}) + h_{T,\eta}(\alpha). \tag{4.4}$$

Here $h_{T,\eta}(\alpha)$ is the relative entropy of $P_\eta^{(\alpha)}$ w.r.t. P_η on interval $[0,T]$, i.e. for $\Psi_T(\alpha) := dP_\eta^{(\alpha)}/dP_\eta\Big|_{[0,T]}$, $h_{T,\eta}(\alpha) := \int \Psi_T(\alpha) \log \Psi_T(\alpha) dP_\eta$. A straightforward calculation using proposition 2.6, p. 320 of [9] shows that $h_{T,\eta}(\alpha) \le c_2\alpha^2 T$ for some deterministic $c_2 > 0$ independent of T. Using (4.4), (3.20) and Jensen's inequality we obtain that

$$\begin{aligned} \alpha \int F\, d\bar{\mu}_\alpha &= \lim_{T\to\infty} \int\int \left[\frac{1}{T}\int_0^T \alpha F(\zeta(s))\, ds\right] P_\eta^{(\alpha)}(d\mathbf{X})\, \bar{\mu}_0(d\eta) \\ &\le \limsup_{T\to+\infty} \frac{1}{T} \log \int\int \exp\left\{\int_0^T \alpha F(\zeta(s))\, ds\right\} P_\eta(d\mathbf{X})\, \bar{\mu}_0(d\eta) + \frac{h_T(\alpha)}{T} \end{aligned} \tag{4.5}$$

with $h_T(\alpha) := \int h_{T,\eta}(\alpha) \bar{\mu}_0(d\eta) \le c_2\alpha^2$T. For any bounded function F on Ω we denote by,

$$\lambda_0(\mathcal{L} + F) := \sup_{\|\phi\|_{L^2(\bar{\mu}_0)} \le 1} ((\mathcal{L} + F)\phi, \phi)_{L^2(\bar{\mu}_0)} \tag{4.6}$$

the supremum of the $L^2(\bar{\mu}_0)$–spectrum of $\mathcal{L} + F$. Applying the Feynman-Kac formula, we conclude from (4.5) that

$$\alpha \int F\, d\bar{\mu}_\alpha \le \lambda_0(\mathcal{L} + \alpha F) + c_2\alpha^2\ . \tag{4.7}$$

Using lemma 4.4, see section 4.3 below, we obtain that $\lambda_0(\mathcal{L} + \alpha F) \le c_3\alpha^2$ and (4.3) follows. □

What remains to be proved to claim (2.4) is that

$$\lim_{\alpha\to 0} \frac{1}{\alpha} \int u_p\, d\bar{\mu}_\alpha = 2\sum_{q=1}^{d} (\Phi_p^{(q)}, c_p)_{L^2(\bar{\mu}_0)} l_q. \tag{4.8}$$

The rigorous argument is given in section 4.5. Let us indicate however how the equality (4.8) can be obtained. Recall that χ_p is the corrector obtained in proposition 3.1. If it were a local function then by stationarity of $\bar{\mu}_\alpha$ we would have $\int \mathcal{L}^{(\alpha)}\chi_p d\bar{\mu}_\alpha = 0$. With this and (3.16) we could write

$$\begin{aligned}\frac{1}{\alpha}\int u_p\, d\bar{\mu}_\alpha &= \frac{1}{\alpha}\int \left(\mathcal{L}^{(\alpha)} - \mathcal{L}\right)\chi_p\, d\bar{\mu}_\alpha \\ &= \sum_{q=1}^{d} \ell_q \int Z^{-1}\left(\gamma_q D_q\chi_p + \gamma_q \circ T_{-e_q} D_q\chi_p \circ T_{-e_q}\right)\, d\bar{\mu}_\alpha + O(\alpha)\end{aligned} \tag{4.9}$$

and (4.8) would follow by proposition 4.1. Unfortunately χ_p is not local, we only know that $D_q\chi_p$ is in $L^2(\bar{\mu}_0)$. So in order to justify the passage to the limit, as $\alpha \to 0$, we need to approximate $u_p = -\mathcal{L}\chi_p$ by $-\mathcal{L}G$, where G is local. This approximation can be made with the help of theorem 4.5, see section 4.4. According to the theorem for any $\varepsilon > 0$ one can choose a local G in such a way that $H := \mathcal{L}G - \mathcal{L}\chi_p$ satisfies $\|H\|_- < \varepsilon$. Since G is local we can repeat the argument used in (4.9), this time with a local function G taking place of χ_p. In such a case the application of proposition 4.1 is fully justified. To deal with the remainder term we need to show that there exists a constant $c_4 > 0$ independent of the choice of H, for which $\limsup_{\alpha\to 0} |\alpha^{-1}\int H d\bar{\mu}_\alpha| < c_4\|H\|_-$. This is established in (4.42).

We end this section with the comment on the use of the duality structure of $L^2(\mu_\rho)$. It is applied to show the estimate of $\lambda_0(\mathcal{L} + \alpha F)$ mentioned in the proof of proposition 4.1 above, see lemma 4.4 of section 4.3. To show this lemma we introduce an auxiliary Glauber dynamics and obtain with its help the estimate (4.20), see lemma 4.3 below. The desired upper bound on the principal eigenvalue is a simple consequence of the variational principle (4.25).

4.1. **The duality structure of $L^2(\mu_\rho)$.** We adopt the notation of [19]. Recall

$$\xi_b(\eta) := \frac{\eta(b) - \rho}{\sqrt{\rho(1-\rho)}}, \quad b \in \mathbb{B}^d,\ \eta \in \Omega.$$

Suppose that $Z \subseteq \mathbb{B}^d$. Denote by $\mathcal{F}_n(Z)$ the family of all subsets of Z of cardinality n. Let also $\mathcal{F}(Z) := \bigcup_{n\geq 1} \mathcal{F}_n(Z)$. We shall omit writing the set Z if it equals $\mathbb{B}^d$. For $A \in \mathcal{F}$ we let

$$\xi_A(\eta) := \begin{cases} \prod_{b\in A} \xi_b(\eta), & \text{if } A \neq \phi, \\ \mathbf{1}, & \text{if } A = \phi. \end{cases}$$

The functions ξ_A, $A \subseteq \mathbb{E}^d$ form an orthonormal basis of $L^2(\mu_\rho)$ and

$$L^2(\mu_\rho) = \bigoplus_{n=0}^{+\infty} H_n,$$

where $H_n := span\{\xi_A : A \in \mathcal{F}_n\}$.

4.2. **The Glauber dynamics.** Let us fix an integer $R > 0$ and consider a Markovian dynamics on Ω_{Λ_R} given by the generator

$$\mathcal{G}_{\Lambda_R} F(\eta) := \sum_{b \in \Lambda_R} \left(\frac{1}{\rho}\right)^{\eta(b)} \left(\frac{1}{1-\rho}\right)^{1-\eta(b)} [F(\eta^b) - F(\eta)], \quad \forall\, F \in B(\Lambda_R). \tag{4.10}$$

Here

$$\eta^b(b') = \begin{cases} \eta(b'), & \text{if } b' \neq b, \\ 1 - \eta(b), & \text{if } b' = b \end{cases}$$

and $B(\Lambda_R)$ denotes the family of all functions $F : \Omega_{\Lambda_R} \to \mathbb{R}$. The corresponding Dirichlet form is given by

$$\mathcal{E}^g_{\Lambda_R}(F, F) := \sum_{b \in \Lambda_R} \int [F(\eta^b) - F(\eta)]^2 \mu_\rho^{\Lambda_R}(d\eta), \quad \forall\, F \in B(\Lambda_R). \tag{4.11}$$

It is well known that this form satisfies *the spectral gap estimate*

$$\mathcal{E}^g_{\Lambda_R}(F, F) \geq \|F\|^2_{L^2(\mu_\rho^{\Lambda_R})}, \quad \forall\, F \in L^2_0(\mu_\rho^{\Lambda_R}), \tag{4.12}$$

where $L^2_0(\mu_\rho^{\Lambda_R}) := [f \in L^2(\mu_\rho^{\Lambda_R}) : \int f d\mu_\rho^{\Lambda_R} = 0]$. If $F \in L^2_0(\mu_\rho^{\Lambda_R})$ then there is a unique $G \in L^2_0(\mu_\rho^{\Lambda_R})$ (which is obviously also bounded) satisfying

$$F = -\mathcal{G}_{\Lambda_R} G. \tag{4.13}$$

We also note, after a direct calculation, that, if

$$F = \sum_A \hat{F}_A \xi_A \tag{4.14}$$

then

$$\int [F(\eta^b) - F(\eta)]^2 \mu_\rho(d\eta) = \frac{1}{(1-\rho)\rho} \sum_{b \in A} \hat{F}_A^2. \tag{4.15}$$

Let

$$\mathcal{E}^{sh}(F, F) := \frac{1}{2d} \sum_{|e|=1} \int (D_e F)^2 d\mu_\rho, \quad F \in C(\Omega) \tag{4.16}$$

be the Dirichlet form of the environment process corresponding to the symmetric simple random walk on the random lattice. A crucial estimate of the Glauber form by the Dirichlet form (4.16) is provided by the following lemma.

Lemma 4.2. *Suppose that* $d \geq 3$. *Then, for any integer* $R > 0$ *there exists a constant* $c_5 > 0$, *depending on* R, *such that*

$$\mathcal{E}^g_{\Lambda_R}(F,F) \leq c_5 \mathcal{E}^{sh}(F,F), \quad \forall\, F \in B(\Lambda_R). \tag{4.17}$$

Proof. By (4.15),

$$\mathcal{E}^g_{\Lambda_R}(F,F) = \frac{1}{(1-\rho)\rho} \sum_{b \in \Lambda_R} \sum_{A \ni b} \hat{F}^2_A. \tag{4.18}$$

Define $\tau_e(A) := [\tau_e(b) : b \in A]$ and suppose that F, given by (4.14), belongs to H_n for some n. We have then

$$\mathcal{E}^{sh}(F,F) = \frac{1}{2d} \sum_{|A|=n} \sum_{|e|=1} (\hat{F}_{\tau_e(A)} - \hat{F}_A)^2, \quad F \in H_n.$$

This is a Dirichlet form of the process $\xi(t) := \tau_{X_t}(A)$, $t \geq 0$, where $(X_t)_{t\geq 0}$ is a symmetric, simple, random walk on $\mathbb{Z}^d$ starting at 0. The state space of this process is $\mathcal{F}_n$ – the family of sets of cardinality n. The transition of probability from set A to a set B (both of cardinality n) in time t for this process equals

$$P(t,A,B) = \sum_{y \in \mathbb{Z}^d} \delta(B, \tau_y(A)) p(t,0,y),$$

where $p(t,x,y)$ is the transition of probability of the symmetric simple random walk. In fact, only one term of this sum could possibly be non-zero, corresponding to the eventual value of $y = r(A,B)$ such that $B = \tau_y(A)$.

Let g the Green function of the simple symmetric random walk. Then the Green function of $\xi(t)$ is given by $G(A,B) = g(0, r(A,B))$ if A is a parallel translation of B, $G(A,B) = 0$, otherwise. According to lemma 3.1 p. 984 of [19] we have the following bound stemming from transience of the process $(\xi(t))_{t\geq 0}$

$$\hat{F}^2_A \leq c_6 \sup_{B \in \mathcal{F}_n} G(A,B)\, \mathcal{E}_{sh}(F,F) \leq c_6 g(0,0) \mathcal{E}_{sh}(F,F) \tag{4.19}$$

and summing over $A \ni b$ and $b \in \Lambda_R$ we obtain (4.17). $\square$

4.3. The eigenvalue estimate.

Lemma 4.3. *Suppose that Ψ is local, supported in Λ_R for a certain $R > 0$ and such that $\int \Psi d\mu_\rho = 0$. Then there exists a constant $c_7 > 0$ depending on Ψ, $\rho \in (0,1)$ such that*

$$\left|\int \Psi\phi^2 d\mu_\rho\right| \le c_7 \|\phi\|_{L^2(\mu_\rho)} \mathcal{E}^{sh}(\phi,\phi)^{1/2}, \quad \forall\, \phi \in L^2(\mu_\rho). \tag{4.20}$$

Proof. Let $\phi = \sum_A \hat\phi_A \xi_A$. The expression under the absolute value on the left hand side of (4.20) equals

$$\sum_{A,A'} \hat\phi_A\hat\phi_{A'} \int \Psi\xi_A\xi_{A'} d\mu_\rho = \sum_{A,A'} \hat\phi_A\hat\phi_{A'} \int \Psi\xi_{A\cap\Lambda_R}\xi_{A'\cap\Lambda_R} d\mu_\rho \int \xi_{A\cap\Lambda_R^c}\xi_{A'\cap\Lambda_R^c} d\mu_\rho \tag{4.21}$$

$$= \sum_{B\in\mathcal{F}(\Lambda_R^c)} \sum_{B_1,B_2\in\mathcal{F}(\Lambda_R)} \hat\phi_{B\cup B_1}\hat\phi_{B\cup B_2} \int \Psi\xi_{B_1}\xi_{B_2} d\mu_\rho = \sum_{B\in\mathcal{F}(\Lambda_R^c)} \int \Psi\phi_B^2 d\mu_\rho,$$

where for any fixed $B \in \mathcal{F}(\Lambda_R^c)$

$$\phi_B := \sum_{B_1\in\mathcal{F}(\Lambda_R)} \hat\phi_{B\cup B_1}\xi_{B_1}.$$

Suppose that $G \in B(\Lambda_R)$ is such that $\mathcal{G}_{\Lambda_R} G = \Psi$. We can write then that the utmost right hand side of (4.21) equals

$$\sum_{B\in\mathcal{F}(\Lambda_R^c)} \int G\mathcal{G}_{\Lambda_R}(\phi_B^2) d\mu_\rho = \frac{1}{\rho(1-\rho)} \sum_{B\in\mathcal{F}(\Lambda_R^c)} \sum_{b\in\Lambda_R} \int G(\eta)[\phi_B^2(\eta^b) - \phi_B^2(\eta)] d\mu_\rho. \tag{4.22}$$

The absolute value of the expression on the right hand side of (4.22) can be estimated by

$$\frac{2\|G\|_\infty}{\rho(1-\rho)} \sum_{B\in\mathcal{F}(\Lambda_R^c)} \sum_{b\in\Lambda_R} \|\phi_B\|_{L^2(\mu_\rho)} \left(\int [\phi_B(\eta^b) - \phi_B(\eta)]^2 d\mu_\rho\right)^{1/2}. \tag{4.23}$$

Using the result of lemma 4.2 we can further estimate (4.23) by

$$\frac{2c_5\|G\|_\infty}{\rho(1-\rho)} \sum_{B\in\mathcal{F}(\Lambda_R^c)} \|\phi_B\|_{L^2(\mu_\rho)} \mathcal{E}^{sh}(\phi_B,\phi_B)^{1/2}.$$

Using Cauchy-Schwartz inequality we can bound this expression by

$$\frac{2c_5\|G\|_\infty}{\rho(1-\rho)} \left(\sum_{B\in\mathcal{F}(\Lambda_R^c)} \|\phi_B\|^2_{L^2(\mu_\rho)}\right)^{1/2} \left(\sum_{B\in\mathcal{F}(\Lambda_R^c)} \mathcal{E}^{sh}(\phi_B,\phi_B)\right)^{1/2},$$

Note however that

$$\sum_{B\in\mathcal{F}(\Lambda_R^c)} \|\phi_B\|^2_{L^2(\mu_\rho)} = \sum_{B\in\mathcal{F}(\Lambda_R^c)} \sum_{B_1\in\mathcal{F}(\Lambda_R)} \hat\phi^2_{B\cup B_1} \le \|\phi\|^2_{L^2(\mu_\rho)}.$$

and

$$\sum_{B\in\mathcal{F}(\Lambda_R^c)}\mathcal{E}^{sh}(\phi_B,\phi_B)=\frac{1}{2d}\sum_{B\in\mathcal{F}(\Lambda_R^c)}\sum_{B_1\in\mathcal{F}(\Lambda_R)}\sum_{|e|=1}(\hat{\phi}_{\tau_e(B\cup B_1)}-\hat{\phi}_{B\cup B_1})^2=\mathcal{E}^{sh}(\phi,\phi).$$

□

Let $\mathcal{C}_0$ be the space of all $F\in C_0(\Omega)$ such that $\int Fd\bar{\mu}_0=0$.

Lemma 4.4. *Suppose that $F\in\mathcal{C}_0$ and $d\geq 3$. Then, there exists a constant $c_3>0$, depending only on F and γ_-, such that*

$$\lambda_0(\mathcal{L}+\alpha F)\leq c_3\alpha^2. \tag{4.24}$$

Proof. Suppose without any loss of generality that $\alpha>0$, otherwise we would consider $-F$ instead of F. We have

$$\begin{aligned}\lambda_0(\mathcal{L}+\alpha F)&=\sup_{\|\phi\|_{L^2(\bar{\mu}_0)}\leq 1}\left[\alpha\int F\phi^2\,d\bar{\mu}_0-\frac{1}{2}\sum_{i=1}^d\int c_i(\eta)(D_i\phi)^2\,d\bar{\mu}_0\right]\\&=\sup_{\|\phi\|_{L^2(\bar{\mu}_0)}\leq 1}\left[\alpha\int ZF\phi^2\,d\mu_\rho-\frac{1}{2}\sum_{i=1}^d\int \gamma_i(\eta)(D_i\phi)^2\,d\mu_\rho\right]\end{aligned} \tag{4.25}$$

On the other hand since $\int ZFd\mu_\rho=0$ and $F\in C_0(\Omega)$ there exists Λ_R such that $ZF\in B(\Lambda_R)$ for a sufficiently large positive integer R and using (4.20) we can estimate the right hand side of (4.25) by

$$\sup_\phi\left[c_7\alpha\mathcal{E}^{sh}(\phi,\phi)^{1/2}-\gamma_-\mathcal{E}^{sh}(\phi,\phi)\right]\leq c_3\alpha^2$$

for some constant $c_3>0$ depending only on F and γ_-. □

4.4. **Localization.** Let $c_*:=\inf c_p$, $c^*:=\sup c_p$. Our first principal observation is the following analogue of theorem 4.2 of [14].

Theorem 4.5. *For any $\varepsilon>0$, $p=1,\ldots,d$, there exist $G,H\in\mathcal{C}_0$ such that*

i)

$$u_p=-\mathcal{L}G+H. \tag{4.26}$$

In addition, $H\in\mathcal{H}_-$ and $\|H\|_-<\varepsilon$.

ii) We have

$$ZH=\sum_{q=1}^d D_q^*K_q,\quad \textit{where } K_q\in\mathcal{C}_0,\ q=1,\ldots,d. \tag{4.27}$$

Moreover, there exists a constant $c_8 > 0$*, depending only on* $c^*, \bar{Z}$*, such that one can choose* $K = (K_1, \dots, K_d)$*, for which*

$$\|K\|_{L^2(\mu_\rho)} \le c_8 \|H\|_-. \tag{4.28}$$

Proof of i). The proof relies on the following lemma.

Lemma 4.6. *Let us fix* $\lambda > 0$ *and* $p \in \{1, \dots, d\}$*. Then, for any* $\varepsilon > 0$ *there exists* $F \in C_0(\Omega)$ *such that*

$$\|\chi_\lambda^{(p)} - F\|_\infty < \varepsilon. \tag{4.29}$$

Proof. Let us fix $R > 0$ and let $\chi_{\lambda,R}^{(p)}$ be the unique solution of the Dirichlet boundary value problem

$$\begin{cases} \lambda \chi_{\lambda,R}^{(p)}(x;\eta) - L_\eta \chi_{\lambda,R}^{(p)}(x;\eta) = u_p(x;\eta), & x \in \Box_R \\ \chi_{\lambda,R}^{(p)}(x;\eta) = 0, & x \in \partial\Box_R, \end{cases} \tag{4.30}$$

where L_η is given by (2.2). We denote here by $\Box_R$ the set of all vertices $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ for which $\max_{i=1,\dots,d} |x_i| \le R$. Then $\delta\chi_\lambda^{(p)}(x;\eta) := \chi_\lambda^{(p)}(x;\eta) - \chi_{\lambda,R}^{(p)}(x;\eta)$ satisfies the Dirichlet boundary value problem

$$\begin{cases} \lambda \delta\chi_\lambda^{(p)}(x;\eta) - L_\eta \delta\chi_\lambda^{(p)}(x;\eta) = 0, & x \in \Box_R \\ \delta\chi_\lambda^{(p)}(x;\eta) = \chi_\lambda^{(p)}(x;\eta), & x \in \partial\Box_R. \end{cases} \tag{4.31}$$

A standard bound on Green's function of the penalized Dirichlet boundary value problem, see Appendix A, yields that

$$|\delta\chi_\lambda^{(p)}(x;\eta)| \le c_A^{(1)} \|\chi_\lambda^{(p)}\|_\infty \exp\{-c_A^{(2)} R^\delta\}, \quad \forall\, x \in \Box_{R/2}, \tag{4.32}$$

where a parameter $\delta \in (0,1)$ while deterministic constants $c_A^{(1)}, c_A^{(2)} > 0$ depend only on δ, λ, d, c_*, c^*. The proof of the lemma follows if we choose $F(\eta) := \chi_{\lambda,R}^{(p)}(0;\eta) \in C_0(\Omega)$. $\Box$

Returning to the proof of theorem 4.5 we choose $G \in \mathcal{C}_0$, such that $\|G - \chi_\lambda^{(p)}\|_\infty + \|DG - D\chi_\lambda^{(p)}\|_\infty < \varepsilon/2$. Note that

$$\|\mathcal{L}\chi_\lambda^{(p)} - \mathcal{L}G\|_-^2 = \frac{1}{2}\sum_{|e|=1} \int c(e;\eta) |D_e G(\eta) - D_e \chi_\lambda^{(p)}(\eta)|^2 \bar{\mu}_0(d\eta) \le \varepsilon^2/4. \tag{4.33}$$

We have therefore

$$u_p = -\mathcal{L}G + u_p + \mathcal{L}\chi_\lambda^{(p)} - \mathcal{L}\chi_\lambda^{(p)} + \mathcal{L}G = -\mathcal{L}G + \lambda\chi_\lambda^{(p)} - \mathcal{L}\chi_\lambda^{(p)} + \mathcal{L}G.$$

Set $H := \lambda\chi_\lambda^{(p)} + \mathcal{L}\chi_\lambda^{(p)} - \mathcal{L}G$. The conclusion of part i) of the theorem follows from (4.33) and part i) of proposition 3.1, provided that λ is chosen in such a way that $\lambda\|\chi_\lambda^{(p)}\|_- < \varepsilon/2$.

Proof of ii). Note that (4.27) follows easily from i) since, according to (4.26) and (3.5) we have

$$ZH = -\sum_{q=1}^{d} D_q^*\left(\gamma_q D_q G\right) - D_p^*\gamma_p.$$

Denote by $L^2_{div}(\mu_\rho)$ the space of those square integrable, centered, divergenceless random vectors $L = (L_1, \ldots, L_d)$, i.e. the fields that satisfy

$$\sum_{q=1}^{d}\int L_q D_q\phi\, d\mu_\rho = 0, \quad \text{for all } \phi \in C(\Omega).$$

Let $K_q^{(0)} := \gamma_q D_q G + \gamma_p\delta_{p,q}$ and $K^{(0)} = (K_1^{(0)}, \ldots, K_d^{(0)})$. From Hodge decomposition $K^{(0)} = K_{pot}^{(0)} + K_{div}^{(0)}$, where $K_{pot}^{(0)}$ is a potential field and $K_{div}^{(0)}$ is divergenceless. Denoting $c^* := \sup c_p$ we can write

$$\|H\|_- \ge \sup_{\|D\phi\|^2_{L^2(\bar\mu_0)} \le 2(c^*)^{-1}} (H, \phi)_{L^2(\bar\mu_0)}$$

$$= \bar{Z}^{-1} \sup_{\|D\phi\|^2_{L^2(\bar\mu_0)} \le 2(c^*)^{-1}} \int ZH\phi d\mu_\rho = \sqrt{2}(c^*)^{-1/2}\bar{Z}^{-1}\|K_{pot}^{(0)}\|_{L^2(\mu_\rho)}.$$

Since $C_{div}(\Omega)$, the space of all divergenceless local vector fields, is L^2–dense in $L^2_{div}(\mu_\rho)$, see lemma B.1 we can find $F = (F_1, \ldots, F_d) \in C_{div}(\Omega)$ such that $\|F - K_{div}^{(0)}\|_{L^2(\mu_\rho)} < \|H\|_-$. Then, the field $K := K^{(0)} - F$ satisfies the conclusions of part ii) of the theorem. □

4.5. **The Proof of (4.8).** Again, with no loss of generality we assume that $\alpha > 0$. Let $\varepsilon > 0$ be chosen arbitrarily. As a rule all the constants appearing in the following shall not depend on ε and α. Suppose that G, H are as in the statement of theorem 4.5. We can write then that

$$\int u_p d\bar\mu_\alpha = -\int \mathcal{L}G d\bar\mu_\alpha + \int H d\bar\mu_\alpha. \tag{4.34}$$

Denoting the first and second terms on the right hand side of (4.34) by $I(\alpha)$, $I\!I(\alpha)$ respectively we can write that

$$I(\alpha) = \int (\mathcal{L}^{(\alpha)}G - \mathcal{L}G) d\bar\mu_\alpha - \int \mathcal{L}^{(\alpha)} G d\bar\mu_\alpha. \tag{4.35}$$

Since $\bar{\mu}_\alpha$ is a steady state we conclude that the last term on the right hand side of (4.35) vanishes. Using (3.1) we conclude that

$$I(\alpha)=\alpha\sum_{q=1}^{d}l_q\int Z^{-1}(\eta)[\gamma_q(\eta)D_qG(\eta)+\gamma_q(T_{-e_q}\eta)D_qG(T_{-e_q}\eta)]\bar{\mu}_\alpha(d\eta)+o(\alpha) \tag{4.36}$$

$$=\alpha\sum_{q=1}^{d}l_q\int\Gamma_q(\eta)\bar{\mu}_\alpha(d\eta)+2\alpha\sum_{q=1}^{d}l_q\int Z^{-1}\gamma_qD_qGd\bar{\mu}_0+o(\alpha),$$

where

$$\Gamma_q(\eta):=Z^{-1}(\eta)[\gamma_q(\eta)D_qG(\eta)+\gamma_q(T_{-e_q}\eta)D_qG(T_{-e_q}\eta)]-2\sum_{q=1}^{d}l_q\int Z^{-1}\gamma_qD_qGd\bar{\mu}_0.$$

Note that $\Gamma_q\in\mathcal{C}_0$ so by proposition 4.1, the first term on the utmost right hand side of (4.36) is of order of magnitude $O(\alpha^2)$. We also have

$$\left|\sum_{q=1}^{d}l_q\int Z^{-1}\gamma_qD_qGd\bar{\mu}_0-\sum_{q=1}^{d}l_q\int\chi^{(p)}\mathcal{L}\chi^{(q)}d\bar{\mu}_0\right|$$

$$\le\sum_{q=1}^{d}|l_q|\left|\int Z^{-1}\gamma_q(D_qG-D_q\chi^{(p)})d\bar{\mu}_0\right|$$

$$\le\left(\sum_{q=1}^{d}\|Z^{-1}\gamma_q\|^2_{L^2(\bar{\mu}_0)}\right)^{1/2}\left(\sum_{q=1}^{d}\|D_qG-D_q\chi^{(p)})\|^2_{L^2(\bar{\mu}_0)}\right)^{1/2}\le c_9\|u_q-\mathcal{L}G\|_-$$

for some constant $c_9>0$ not depending on the choice of G and H. The utmost right hand side of (4.5) is less than $c_9\varepsilon$ by virtue of part i) of theorem 4.5. We have proved therefore that

$$\limsup_{\alpha\to 0}\left|\frac{I(\alpha)}{\alpha}-2\sum_{q=1}^{d}l_q\int Z^{-1}\gamma_qD_qGd\bar{\mu}_0\right|\le c_9\varepsilon. \tag{4.37}$$

To estimate $II(\alpha)$ we choose $A\in(\|H\|_-^{-1},2\|H\|_-^{-1})$. The constants appearing in what follows shall not depend on A. Repeating the argument leading up to (4.7) we conclude that

$$\alpha A\left|\int H\,d\bar{\mu}_\alpha\right|\le\lambda_0(\mathcal{L}+\alpha AH)+c_2\alpha^2,$$

where, as we recall, the constant c_2 does not depend on the choice of H. Hence

$$\frac{1}{\alpha}\left|\int H\,d\bar{\mu}_\alpha\right|\le\frac{1}{\alpha^2A}\lambda_0(\mathcal{L}+\alpha AH)+\frac{c_2}{A}\le\frac{1}{\alpha^2A}\lambda_0(\mathcal{L}+\alpha AH)+c_2\|H\|_-. \tag{4.38}$$

Using once more the variational principle to calculate $\lambda_0(\mathcal{L}+\alpha AH)$ we get

$$\lambda_0(\mathcal{L}+\alpha AH)=\sup_{\|\phi\|_{L^2(\bar\mu_0)}\le 1}\left[\int \alpha AH\phi^2\,d\bar\mu_0-\frac{1}{2}\bar{Z}^{-1}\sum_{|e|=1}\int\gamma(e)(D_e\phi)^2\,d\mu_\rho\right]. \tag{4.39}$$

By virtue of the representation (4.27) we can rewrite the expression on the right hand side of (4.39) in the following form

$$\bar{Z}^{-1}\sup_{\|\phi\|_{L^2(\bar\mu_0)}\le 1}\left[\sum_{q=1}^{d}\int \alpha AK_qD_q\phi^2\,d\mu_\rho-\sum_{p=1}^{d}\int\gamma_p(D_p\phi)^2\,d\mu_\rho\right] \tag{4.40}$$

$$\le \bar{Z}^{-1}\sup_{\|\phi\|_{L^2(\mu_\rho)}\le 1}\left\{\alpha A\left[\sum_{q=1}^{d}\int(\phi+\phi\circ T_{e_q})^2K_q^2d\mu_\rho\right]^{1/2}\|\phi\|_+-\gamma_-\|\phi\|_+^2\right\}$$

$$\le \bar{Z}^{-1}\alpha A\sup\left\{\left(\int\phi^2J^2d\mu_\rho\right)^{1/2}\|\phi\|_+:\|\phi\|_{L^2(\mu_\rho)}\le 1,\|\phi\|_+\le\gamma_-^{-1}\alpha A\left(\int\phi^2J^2d\mu_\rho\right)^{1/2}\right\}$$

$$+\bar{Z}^{-1}\alpha A\sup\left\{\left(\int\phi^2\widehat{J}^2d\mu_\rho\right)^{1/2}\|\phi\|_+:\|\phi\|_{L^2(\mu_\rho)}\le 1,\|\phi\|_+\le\gamma_-^{-1}\alpha A\left(\int\phi^2\widehat{J}^2d\mu_\rho\right)^{1/2}\right\}$$

Here $J^2:=\sum_q K_q^2$, $\widehat{J}^2:=\sum_q(K_q\circ T_{-e_q})^2$. We deal with the two terms appearing on the utmost right hand side of (4.40) in the same fashion so we only show how to estimate the first one. The term in question can be estimated by

$$(\bar{Z}\gamma_-)^{-1}\alpha^2A^2\sup\left\{\int\phi^2J^2d\mu_\rho:\|\phi\|_{L^2(\mu_\rho)}\le 1,\ \|\phi\|_+\le\gamma_-^{-1}\alpha A\|J\|_\infty.\right\}$$

Note that

$$\int\phi^2J^2d\mu_\rho=\|J\|^2_{L^2(\mu_\rho)}\|\phi\|^2_{L^2(\mu_\rho)}+\int\phi^2\widetilde{J}d\mu_\rho, \tag{4.41}$$

where $\widetilde{J}:=J^2-\|J\|^2_{L^2(\mu_\rho)}$. Since $\|\phi\|_{L^2(\mu_\rho)}\le 1$ we can estimate the first term on the right hand side of (4.41), with the help of (4.28) by $c_8^2\|H\|_-^2$. To estimate the second term on the right hand side of (4.41) we use once more lemma 4.3 and obtain that it is bounded by $c_7\gamma_-^{-1}\alpha A\|J\|_\infty$. Summarizing, we have just shown that

$$\frac{1}{\alpha^2A}\lambda_0(\mathcal{L}+\alpha AH)\le(\bar{Z}\gamma_-)^{-1}Ac_8^2\|H\|_-^2+c_7(\bar{Z}\gamma_-^2)^{-1}\alpha A^2\|J\|_\infty$$

$$\le 2(\bar{Z}\gamma_-)^{-1}c_8^2\|H\|_-+c_7(\bar{Z}\gamma_-^2)^{-1}\alpha A^2\|J\|_\infty.$$

The last inequality holds because $A\|H\|_- < 2$. The constant c_7 may depend on the choice of H but upon taking the limit as $\alpha \to 0$ we obtain

$$\limsup_{\alpha\to 0} \frac{II(\alpha)}{\alpha} \le [c_2 + 2(\bar{Z}\gamma_-)^{-1}c_8^2]\|H\|_- < [c_2 + 2(\bar{Z}\gamma_-)^{-1}c_8^2]\varepsilon, \tag{4.42}$$

where the constants c_2, c_8 are independent of the choice of H. Hence, equality (4.8) follows.

APPENDIX A. PROOF OF (4.32).

Let $\delta \in (0,1)$ and $R \ge 1$ be fixed. Let τ_R denote the exit time of a walker from the box $\square_R$. We have

$$\delta\chi_\lambda^{(p)}(x;\eta) = E_{x,\eta}[\mathrm{e}^{-\lambda\tau_R}\chi_\lambda^{(p)}(\mathbf{X}(\tau_R);\eta)] \le \|\chi_\lambda^{(p)}\|_\infty E_{x,\eta}[\mathrm{e}^{-\lambda\tau_R}].$$

Note that

$$E_{x,\eta}[\mathrm{e}^{-\lambda\tau_R}] \le E_{x,\eta}[\mathrm{e}^{-\lambda R^\delta},\, \tau_R \ge R^\delta] + E_{x,\eta}[\,\tau_R < R^\delta\,] \le \mathrm{e}^{-\lambda R^\delta} + E_{x,\eta}[\,\tau_R < R^\delta\,] \tag{A.1}$$

According to Girsanov theorem, see [9], proposition 2.6, p. 320, we can rewrite the second term on the utmost right hand side of (A.1) in the form

$$E_x\left\{\tau_R < R^\delta,\, \exp\left\{-\sum_{0\le s\le R^\delta}\log[2dp_\eta(\mathbf{X}(s-),\mathbf{X}(s))]\mathbf{1}_{[\mathbf{X}(s-)\ne\mathbf{X}(s)]}\right\}\right\}. \tag{A.2}$$

For any $x, y \in \mathbb{Z}^d$

$$p_\eta(x,y) := \begin{cases} 0, & |x-y| > 1 \\ c(y-x;\eta), & |x-y| = 1 \end{cases}$$

and E_x denotes the expectation w.r.t. the path measure P_x corresponding to the symmetric simple random walk in $\mathbb{Z}^d$ with unit intensity. Let A_m be the event consisting of those paths which have exactly m jumps in time interval $[0, R^\delta]$. Obviously, since the rate of jumps equals 1 we have $P_x[A_m] = \mathrm{e}^{-R^\delta}R^{\delta m}/m!$. Hence, for a certain constant $c_A^{(3)} > 0$ we can estimate the expression in (A.2) from above by

$$\sum_{m=0}^{\infty} \mathrm{e}^{c_A^{(3)}m}\, P_x\left[\tau_R < R^\delta,\, A_m\right]. \tag{A.3}$$

Note however that if $x \in \square_{R/2}$ we must have $P_x\left[\tau_R < R^\delta,\, A_m\right] = 0$ for all $m \le R/2$. The expression in (A.3) equals therefore

$$\sum_{m\ge R/2} \mathrm{e}^{c_A^{(3)}m}\, P_x\left[\tau_R < R^\delta,\, A_m\right] \le \sum_{m\ge R/2} \mathrm{e}^{c_A^{(3)}m - R^\delta}\frac{R^{m\delta}}{m!}$$

$$\leq \frac{1}{[R/2]!}\exp\{(\mathrm{e}^{c_A^{(3)}}-1)R^\delta\} \leq c_A^{(4)}\mathrm{e}^{-c_A^{(5)}R}$$

for some constants $c_A^{(4)}$, $c_A^{(5)} > 0$ and all $R \geq 1$. Summarizing, the left hand side of (A.1) can be therefore estimated by $c_A^{(1)}\mathrm{e}^{-c_A^{(2)}R^\delta}$ for some constants $c_A^{(1)}$, $c_A^{(2)} > 0$ and (4.32) follows. □

Appendix B. Density argument.

Lemma B.1. *The space $C_{div}(\Omega)$ of local, divergence free fields is L^2-dense in $L^2_{div}(\mu_\rho)$.*

Proof. Choose an arbitrary $\varepsilon > 0$ and suppose that $F = (F_1, \ldots, F_d) \in L^2_{div}(\mu_\rho)$. Let $\hat{F} = (\hat{F}_1, \ldots, \hat{F}_d)$ be the random spectral measure corresponding to F. We have

$$F_p(T_x\eta) = \int\limits_{\mathbb{T}^d} \mathrm{e}^{i\,x\cdot k}\hat{F}_p(dk;\eta).$$

For any $\lambda > 0$ set

$$H^{(\lambda)}_{p,q}(\eta) := \int\limits_{\mathbb{T}^d} \frac{(\mathrm{e}^{i\,k_q}-1)\hat{F}_p(dk;\eta) - (\mathrm{e}^{i\,k_p}-1)\hat{F}_q(dk;\eta)}{\sum_{r=1}^d |\mathrm{e}^{i\,k_r}-1|^2+\lambda}.$$

Obviously $H^{(\lambda)}_{p,q} = -H^{(\lambda)}_{q,p}$ for all $p, q = 1, \ldots, d$. Since $\sum_q D_q^* F_q \equiv 0$ we have

$$F^{(\lambda)}_p := \sum_q D_q^* H^{(\lambda)}_{p,q} = \int\limits_{\mathbb{T}^d} \frac{\sum_{r=1}^d |\mathrm{e}^{i\,k_r}-1|^2}{\sum_{r=1}^d |\mathrm{e}^{i\,k_r}-1|^2+\lambda}\,\hat{F}_p(dk).$$

Let $F^{(\lambda)} = (F^{(\lambda)}_1, \ldots, F^{(\lambda)}_d)$. Choosing $\lambda > 0$ sufficiently small we obtain that $\|F - F^{(\lambda)}\|_{L^2(\mu_\rho)} < \varepsilon/2$. Selecting suitable local $\widetilde{H}_{p,q}$ satisfying $\widetilde{H}^{(\lambda)}_{p,q} = -\widetilde{H}^{(\lambda)}_{q,p}$ for all $p, q = 1, \ldots, d$ we can guarantee that

$$\sum_{p=1}^d \|\sum_{q=1}^d D_q^*(H^{(\lambda)}_{p,q} - \widetilde{H}_{p,q})\|^2_{L^2(\mu_\rho)} < \varepsilon^2/4.$$

Note that $G := (G_1, \ldots, G_d)$, where $G_p := \sum_q D_q^* \widetilde{H}_{p,q} \in C_{div}(\Omega)$ and $\|F - G\|_{L^2(\mu_\rho)} < \varepsilon$. □

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