Inverse source problem with multi-frequency data

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Outline

- Inverse source problem.
- Stability estimates.
- Numerical illustration.
- Future works.

Inverse source problem

We consider in $\mathbb{R}^d$ the homogeneous Helmholtz equation

$$\begin{cases} \Delta u + k^2 u &= S(x), \\ \partial_r u - iku &= o(r^{\frac{1-d}{2}}), \end{cases} \tag{1}$$

where k is the wavenumber of the radiated scalar field ψ.

Assume that $S(x)$ has a compact support in $B_{R_0}(0) \subset \Omega$, and

$$\operatorname{dist}(B_{R_0}, \Gamma) > d_0.$$

Determine the source S that generates the field u_k everywhere outside the source region Ω for $k \in [0, k_0]$.

Define the Green function in the whole space

$$\Phi(k,r) \;=\; \begin{cases} -\frac{i}{4}H_0^{(1)}(kr) & \text{if } d=2, \\ \frac{e^{ikr}}{4\pi r} & \text{if } d=3. \end{cases}$$

There is a unique solution u to the system (1)

$$u(x) \;=\; \int_{\mathbb{R}^d} \Phi(k,|x-y|)S(y)dy.$$

Let $\nu(x)$ be the outward normal derivative at $x \in \Gamma$.

The functions u and $\frac{\partial u}{\partial \nu}$ on Γ determine uniquely u in the whole space $\mathbb{R}^d$.

For a fixed wavenumber k, we define the radiation operators $L_k^{(1)}$, $L_k^{(2)}$ from $L^2(\Gamma) \to L^2(\Gamma)$ as

$$\begin{aligned} L_k^{(1)}(S)(x) &= \int_{B_{R_0}} \Phi(k,|x-y|)S(y)dy, \quad \text{for } x \in \Gamma \\ L_k^{(2)}(S)(x) &= \int_{B_{R_0}} \frac{\partial \Phi(k,|x-y|)}{\partial \nu(x)} S(y)dy, \quad \text{for } x \in \Gamma. \end{aligned}$$

The M-isp is to find $S(x) \in L^2(B_{R_0})$ such that the following linear equations are satisfied simultaneously for $k \in [0, k_0]$

$$L_k^{(1)}(S)(x) = u_k(x), \quad L_k^{(2)}(S)(x) = \frac{\partial u_k}{\partial \nu}(x) \quad \text{for } x \in \Gamma. \tag{2}$$

It should be pointed out that the inverse problem with single frequency data is not well-posed:

(i) The solution to (2) is not unique. This is due to the existence of the non-radiating source, whose radiating field vanishes identically outside the support volume B_{R_0}.

(ii) The problem is severely ill-posed for low wavenumbers, that is, an infinitesimal noise in the measurement will give rise to large errors in the reconstruction solution.
In fact, it can be shown that the singular values of forward maps $L_k^{(1)}$ and $L_k^{(2)}$ decay exponentially for low k.

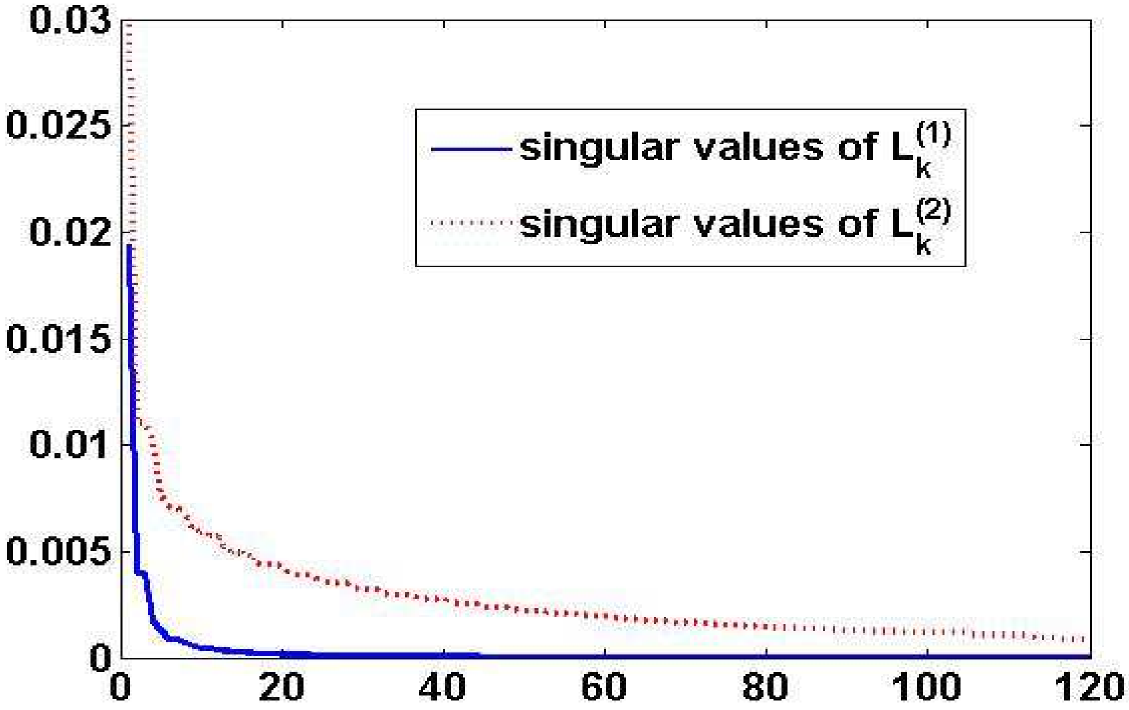

Figure 1: The first 120 singular values of $L_k^{(1)}$ and $L_k^{(2)}$ when $k = 1$.

Stability estimates

Let $\xi \in \mathbb{R}^d$ such that $|\xi| = k$.
Multiplying the equation (1) by $e^{-i\xi.x}$ and integrating over Ω, we obtain

$$\widehat{S}(\xi) \;=\; \int_\Gamma e^{-i\xi.x}\big(\frac{\partial u_k}{\partial \nu} + i\xi.\nu u_k\big)d\sigma_x \quad |\xi| = k \in [0, k_0].$$

By collecting the measurements u_k, $\frac{\partial u_k}{\partial \nu}$ on Γ for $k \in [0, k_0]$, the Fourier transform of S on B_{k_0} can be reconstructed directly.

The M-isp is to determine S from the knowledge of $\widehat{S}(\xi)$ for $\xi \in B_{k_0}$.

Theorem 0.1. *Let $(k_j)_j$ be a bounded sequence of positive reals. The measurements $\left(u_{k_j}\right)_j$ determine uniquely the source function S.*

Theorem 0.2. *Let S be a function in $C^1(\overline{B}_c)$. Assume that*

$$\epsilon = \|\frac{\partial u_k}{\partial \nu}\|_{C^0(B_1, L^1(\Gamma))} + \|k u_k\|_{C^0(B_1, L^1(\Gamma))} < 1$$

and

$$\|S\|_{C^1(\overline{B}_c)} \leq M,$$

where M is a positive constant. The following statements hold:

A) If $c \leq \epsilon$, Then

$$\|S\|_{C^0(\overline{B}_c)} \quad \leq \quad M\epsilon.$$

B) If $c \geq \frac{1}{2}\epsilon^{\frac{1}{1+2d}}$, then

$$\|S\|_{C^0(\overline{B}_c)} \quad \leq \quad C\Big(M\big(1+2\sqrt{c}\big)^{\frac{1}{3}}\big(4c\big)^{\frac{2}{3}}+1\Big)\frac{1}{\big(\ln(\epsilon^{-1})\big)^{\frac{1}{3}}},$$

where C is a constant that only depends on the dimension d.

Theorem 0.3. *Let S be a function in $C^1(\overline{B}_1)$. Assume that*

$$\epsilon = \|\frac{\partial u_k}{\partial \nu}\|_{L^1(B_c, L^1(\Gamma))} + \|k\psi\|_{L^1(B_c, L^1(\Gamma))} < 1,$$

and

$$\|S\|_{C^1(\overline{B}_1)} \leq M,$$

where M is a positive constant. The following statements hold:

A) *If $c \geq \epsilon^{-4} + 1$, Then*

$$\|S\|_{C^0(\overline{B}_1)} \quad \leq \quad C\Big(1 + 2M(1 + \frac{1}{c}) + (\frac{1}{16c})^{\frac{d+1}{2}}\Big)\epsilon,$$

where the constant C only depends on the dimension d.

B) *If $c < \epsilon^{-4} + 1$, then*

$$\|S\|_{C^0(\overline{B}_1)} \quad \leq \quad C\Big(1 + 2M(1 + \frac{1}{c})(\frac{1}{4c} + \frac{1}{2\sqrt{c}})^{\frac{1}{6}}\Big)\frac{1}{\big(\ln(\epsilon^{-1})\big)^{\frac{1}{6}}},$$

where C is a constant that only depends on the dimension d.

Numerical illustration

Assume that at $k = k_m$, the source function has been recovered with $S = S_m$. Then at a higher wavenumber $k = k_{m+1} := k_m + \delta k_m$, where $\delta k_m > 0$ is the increment, the Landweber iteration is applied to solve (2) with $k = k_{m+1}$.

In the example, the support volume of the source function is $[-0.3, 0.3] \times [-0.3, 0.3]$, which lies in the domain Ω such that $dist(B_{R_0}, \Gamma) = 0.05$.
The measurements $\{u_k, \frac{\partial u_k}{\partial \nu}\}$ are made on Γ for $k \in [k_{min}, k_{max}]$.
We set $k_{min} = 1$ in the following numerical example.
Assume that the true source

$$S(x_1, x_2) = 1.1e^{-200((x_1-0.01)^2)+(x_2-0.12)^2} - 100(x_2^2 - x_1^2)e^{-90(x_1^2+x_2^2)}.$$

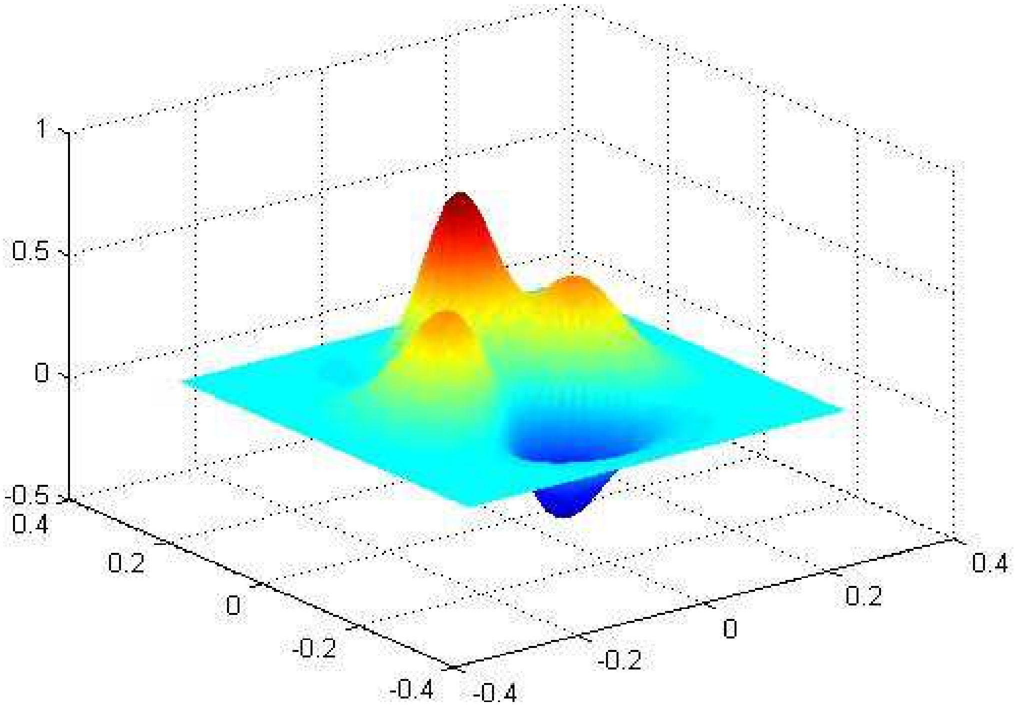

Figure 2: Real source function.

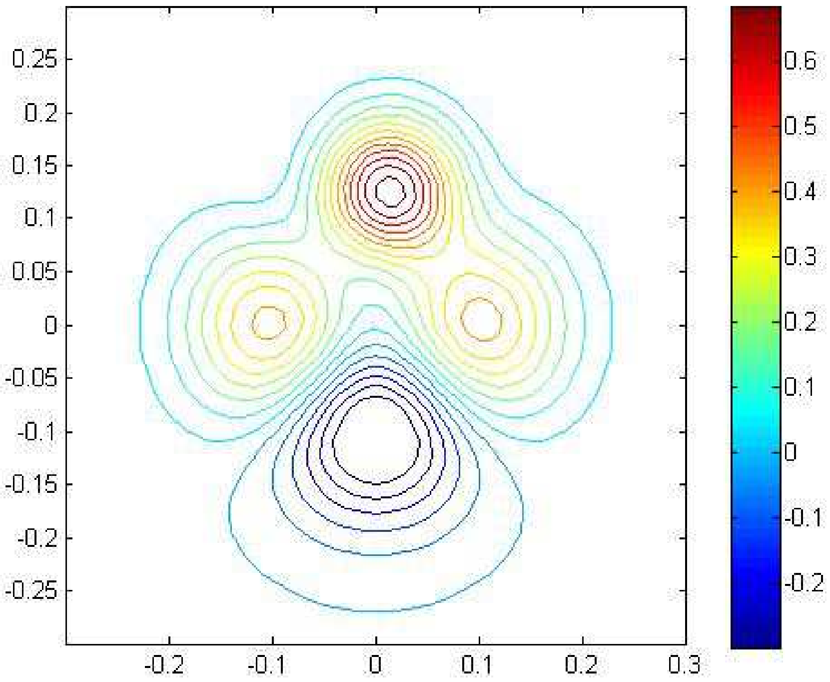

Figure 3: The contour plot of the real source function.

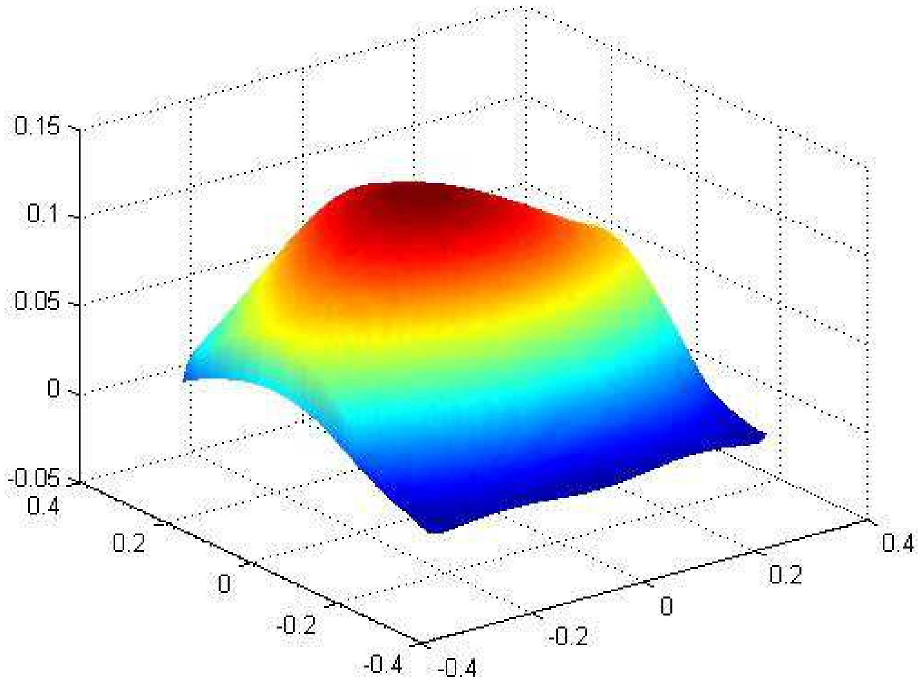

Figure 4: Reconstruction at k=9.

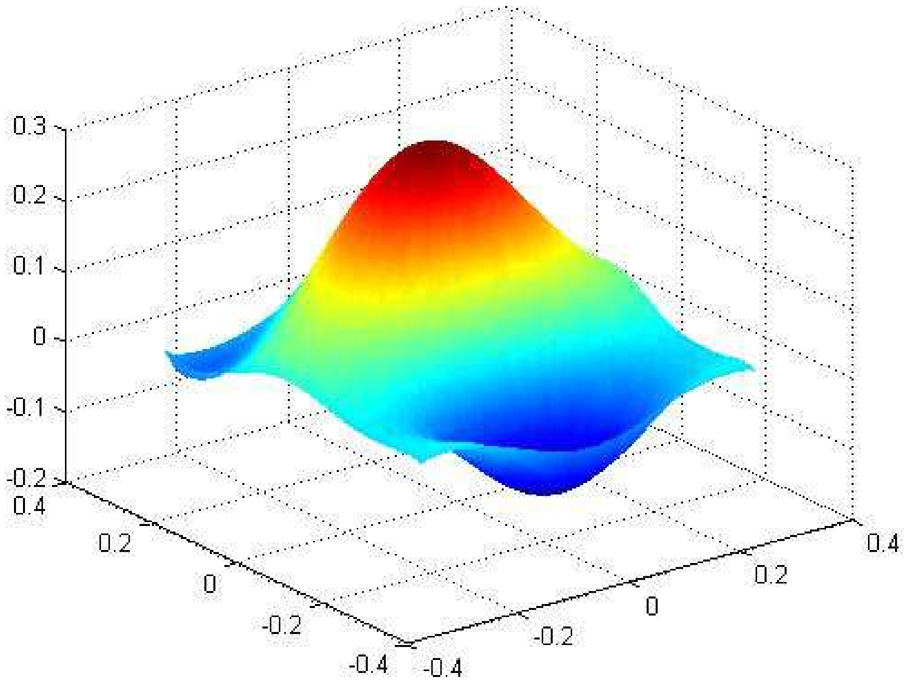

Figure 5: Reconstruction at k=17.

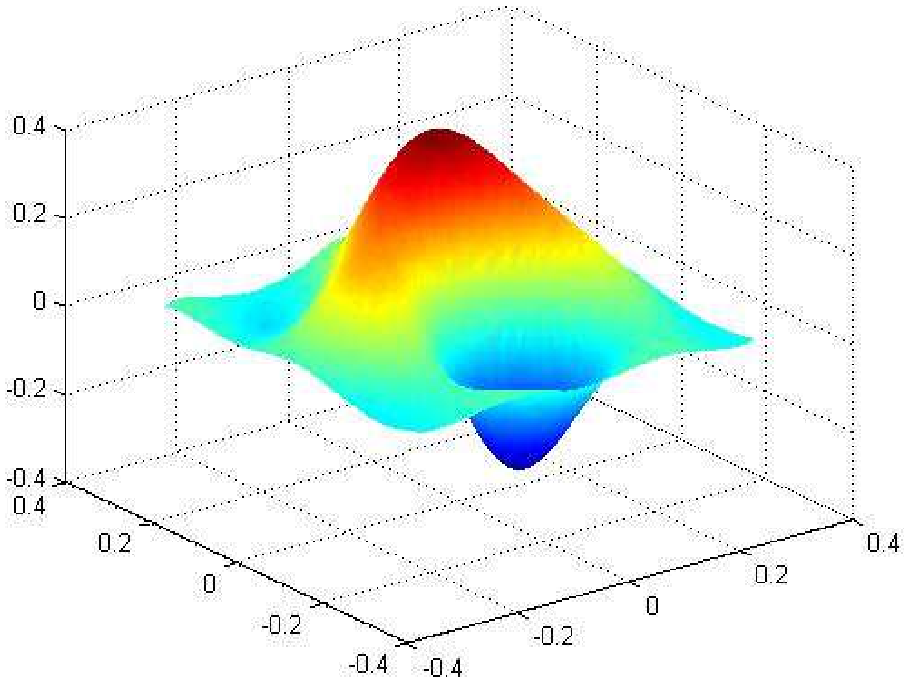

Figure 6: Reconstruction at k=25.

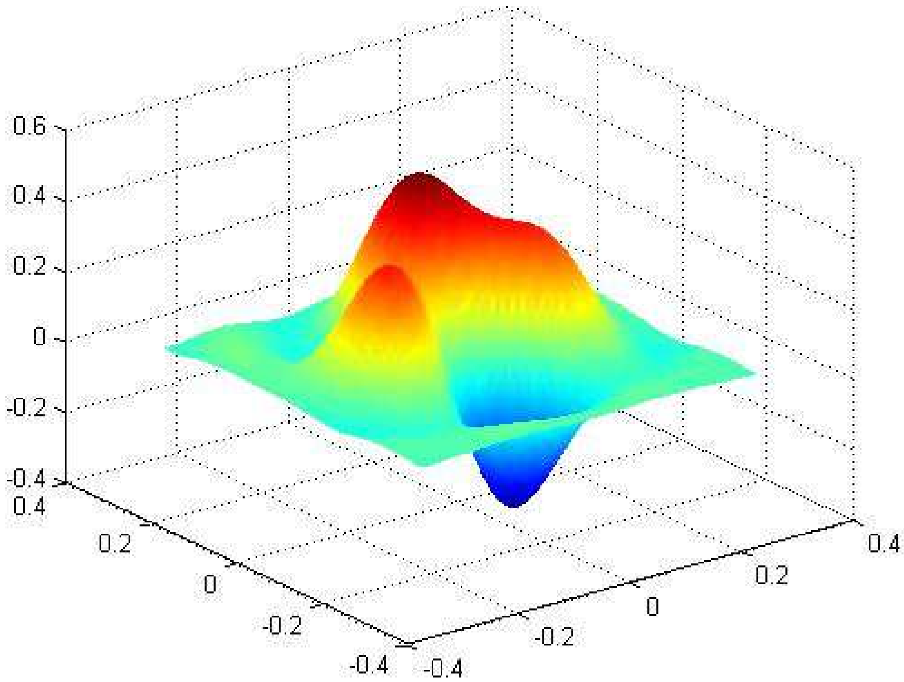

Figure 7: Reconstruction at k=33.

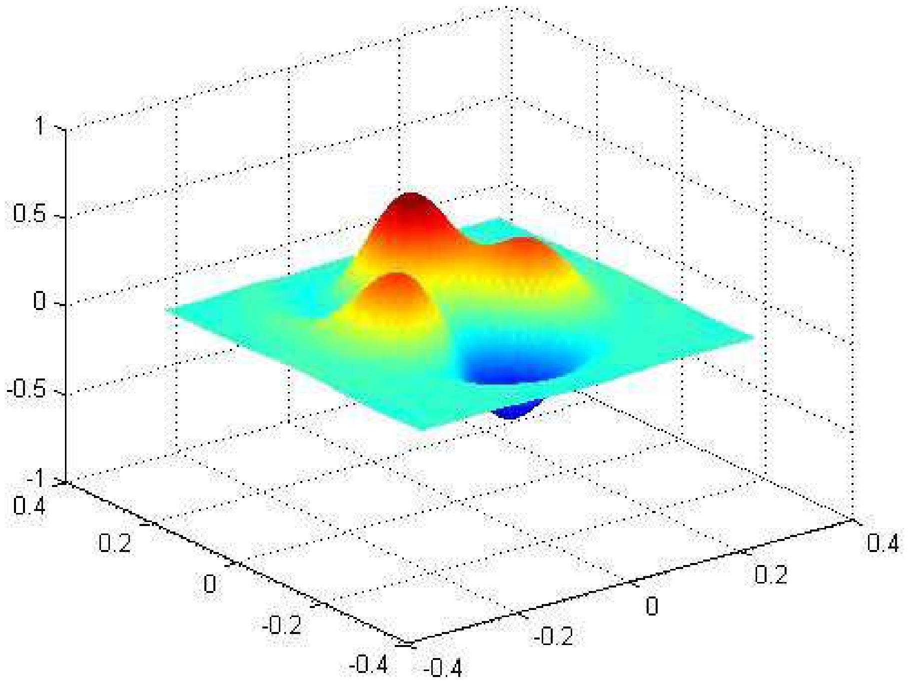

Figure 8: Reconstruction at k=41.

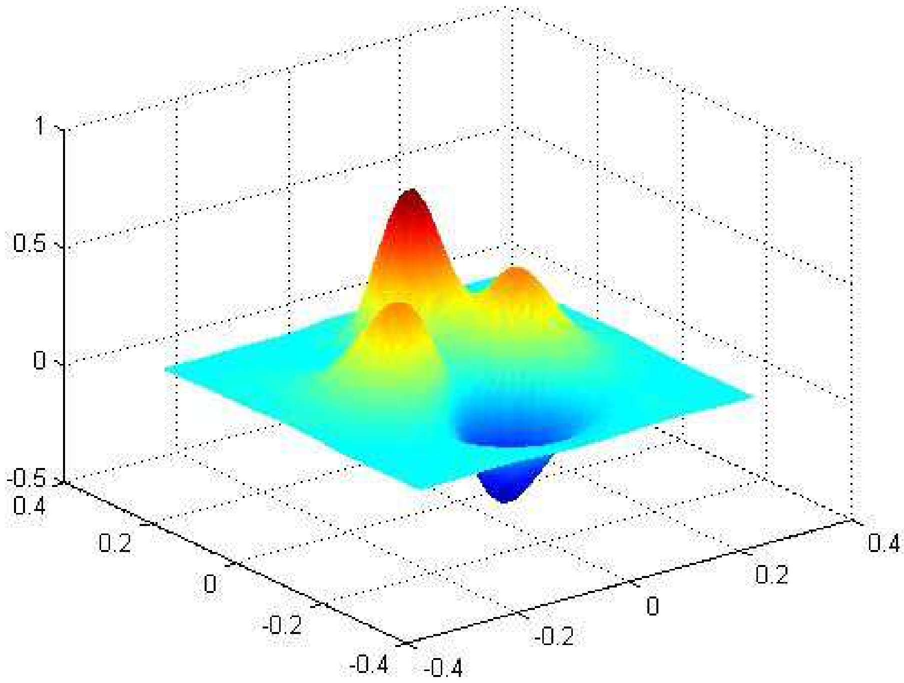

Figure 9: Reconstruction at k=61.

Future works

The presented work can be found in the two papers:

- *Multi-Frequency Inverse Source Problem,* to appear in JDE 2010.
- *Numerical solution of the inverse source problem for the Helmholtz Equation with multiple frequency data,* submitted.

Here are some future works:

i Multi-Frequency Inverse Source Problem in an inhomogeneous medium.

ii Multi-Frequency Inverse Source Problem with phase less data.

Thanks