Compressive Wave Computation

Laurent Demanet[1] Gabriel Peyré[2]

[1]Department of Mathematics
Stanford University

[2]Ceremade
University of Paris Dauphine

Sept 2008 :: UCL

Special thanks to: Felix Herrmann, Lexing Ying

Compressive wave computation

$$\sigma^2(x)\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad u|_0 = u_0, \frac{\partial u}{\partial t}|_0 = u_1, \quad \text{B.C.}$$

Objective: Solve without timestepping.

Strategy: Fourier transform in time: $t \mapsto \omega$

$$v_\omega''(x) + \omega^2\sigma^2(x)v_\omega(x) = 0, \qquad \text{same B.C.},$$

but do not use all ω

Recover $u(x,t)$ from a few v_ω using L^1 minimization.

“Compressive sensing in the Helmholtz domain"

Outline

1. L^2 vs. L^1 minimization
2. Why it works: new PDE estimates
3. How it works: algorithms
4. Numerical examples

In your textbook

$$\sigma^2(x)\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad u|_0 = u_0, \frac{\partial u}{\partial t}|_0 = u_1, \quad \text{B.C.}$$

$$v_\omega''(x) + \omega^2\sigma^2(x)v_\omega(x) = 0, \qquad \text{same B.C.}$$

B.C. are Dirichlet or Neumann.

The v_ω are orthogonal for the inner product

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g}(x)\sigma^2(x)dx$$

In your textbook

Solution of the wave equation:

$$c_\omega(t) = \langle u(\cdot, t), v_\omega(x) \rangle$$

$$u(x, t) = \sum_{\text{all } \omega} c_\omega(t) v_\omega(x)$$

with

$$c_\omega(t) = \langle u_0, v_\omega \rangle \cos \omega t + \langle u_1, v_\omega \rangle \frac{\sin \omega t}{\omega}$$

(bounded domain $\leftrightarrow$ discrete spectrum)

Missing components

$$c_\omega(t) = \langle u_0, v_\omega \rangle \cos \omega t + \langle u_1, v_\omega \rangle \frac{\sin \omega t}{\omega}, \qquad \omega \in \Omega$$

Fix $t > 0$. Least squares:

$$\min \|u\|_{L^2_{\sigma^2}} \quad \text{s.t.} \quad \langle u, v_\omega \rangle = c_\omega(t), \quad \text{for} \quad \omega \in \Omega.$$

Then

$$u^\sharp(x) = \sum_{\omega \in \Omega} c_\omega(t) v_\omega(x).$$

Error:

$$\|u^\sharp - u(\cdot, t)\|^2_{L^2_{\sigma^2}} = \sum_{\omega \notin \Omega} |c_\omega(t)|^2.$$

The compressive viewpoint

Basic strategy: from

$$v_\omega, \qquad c_\omega(t), \qquad \omega \in \Omega,$$

solve for $u(x,t)$ using L^1 minimization:

$$\min \int \sigma(x)|u(x,t)|\,dx \quad \text{s.t.} \quad \langle u, v_\omega \rangle = c_\omega(t), \quad \text{for} \quad \omega \in \Omega.$$

Basic question:

Find conditions on $u(x,t)$ and $\sigma(x)$ to guarantee recovery.

The compressive viewpoint

Bold stroke answer from compressed sensing: (Donoho, Stark 1989, Candès, Romberg, Tao 2004 and 2005, Donoho 2004)

- $u(x, t)$ should be sparse (small support size in x);
- $v_\omega(x)$ should be extended;
- $\omega \in \Omega$ should be drawn nearly uniformly at random;

Then $|\Omega| < N$ can guarantee exact reconstruction w.h.p.

A form of uncertainty principle

Algorithm for solving waves

1. Draw Ω and compute $v_\omega(x)$, for $\omega \in \Omega$;
2. Form $\langle u_0, v_\omega \rangle$ and $\langle u_1, v_\omega \rangle$, for $\omega \in \Omega$;
3. Form

$$c_\omega(t) = \langle u_0, v_\omega \rangle \cos \omega t + \langle u_1, v_\omega \rangle \frac{\sin \omega t}{\omega}, \quad \omega \in \Omega;$$

4. Solve

$$\min_u \int_0^1 \sigma(x) |u(x,t)| \, dx, \quad \text{such that } \langle u, v_\omega \rangle = c_\omega(t), \quad \omega \in \Omega.$$

Quality of recovery

Sketch of Theorem (D-Peyré, 2008)

Let $\mathrm{Var}(\log \sigma) < \pi$. When

$$|\Omega| = K \geqslant C(\sigma) \cdot S \log N \cdot \log^2(S) \log(S \log N),$$

then total error $\leqslant$

- discretization error
- $+C \cdot$ error of truncation to a sparse S-term approximation,
- $+C \cdot$ error of computing egval and egvec,

with very high probability. It suffices to take

$$C(\sigma) = C \cdot \exp(2\,\mathrm{Var}(\log \sigma)) \cdot \frac{\pi + \mathrm{Var}(\log \sigma)}{\pi - \mathrm{Var}(\log \sigma)}.$$

Set up the problem so that $S << N$.

Why it works

The ℓ_1-Helmholtz problem

The three conditions of compressed sensing:

- Is $u(x, t)$ sparse in x?
- Are the v_ω incoherent? (extended)
- How uniform is the randomness of the sampling scheme?

Sparsity of the solution

Theorem (D-Peyré, 2008)

Assume $Var(\log\sigma) < 1$ *and* $t \leqslant 1/\sigma_{\min}$. *Then*

$$\int_0^1 \sigma(x)|u(x,t)|dx \leqslant D(\sigma)\cdot\left(\int_0^1 \sigma(x)|u_0(x)|dx + \int_0^1 \sigma^2(x)\left|\int_0^x u_1(y)dy\right|dx\right),$$

where

$$D(\sigma) = C\sqrt{\frac{\sigma_{\max}}{\sigma_{\min}}}\frac{1}{1 - Var(\log\sigma)}.$$

Sparsity of the solution

Generalization to *n* dimensions, $\sigma \in C^\infty$: curvelets.

Extension of the eigenfunctions

Theorem (D-Peyré, 2008)

Assume $Var(\log\sigma) < \infty$. *Then*

$$\|v_\omega\|_{L^\infty} \leqslant \sqrt{2}\exp(Var(\log\sigma)) \cdot \|v_\omega\|_{L^2_{\sigma^2}}.$$

Then extension $\Rightarrow$ incoherence.

Eigenvalue gaps

Linear algebra methods: find $\lambda = -\omega^2$ closest to a shift chosen at random.

Theorem (D-Peyré, 2008)

Assume $Var(\log \sigma) < \pi$. *Then*

$$|\omega_1 - \omega_2| \geqslant \frac{\pi - Var(\log \sigma)}{\int_0^1 \sigma(x)\, dx}.$$

How it works

Algorithm for solving waves

- How to compute all these eigenvectors?
  - Distribute on a cluster!
  - Power method with shifting and inversion.
  - Preconditioner: e.g. discrete symbol calculus.
- How to solve an ℓ_1 problem?
  - Iterative thresholding
  - Parallelizable too.

Numerical examples

Smooth medium

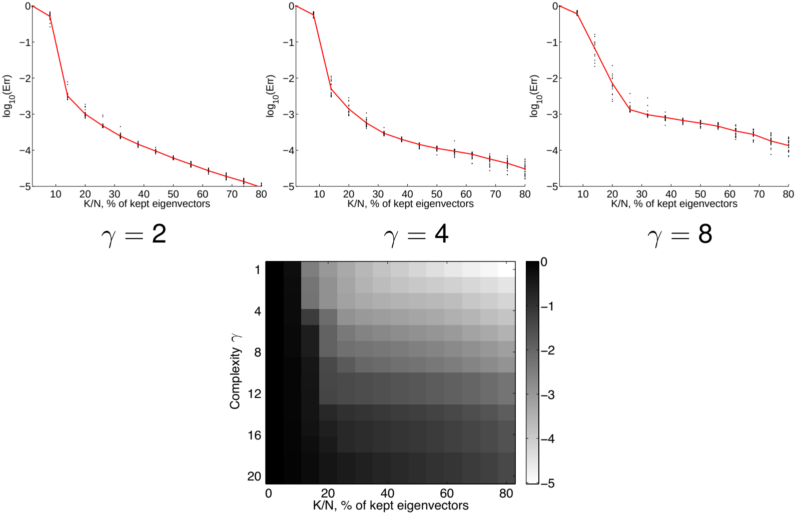
- # oscillations $\gamma \in [1, 20]$, contrast $\sigma_{\max} \in [1, 10]$ proportional to γ:

$$\sigma_\gamma[j] = \frac{\sigma_{\max} + 1}{2} + \frac{\sigma_{\max} - 1}{2} \left(\sin(2\pi\gamma j/N) + 3\right).$$

- Average recovery error: times $t_i = Ti/n_t$, random set $|\Omega| = K$:

$$\mathrm{Err}(\sigma, K/N)^2 = \frac{1}{Nn_t|\Omega_K|\,\|u\|} \sum_{\Omega\in\Omega_K} \sum_{i=0}^{n_t-1} \sum_{j=0}^{N-1} |u[j](t_i) - \tilde{u}[j](t_i)|^2.$$

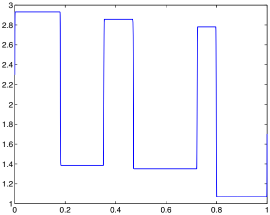
Smooth medium



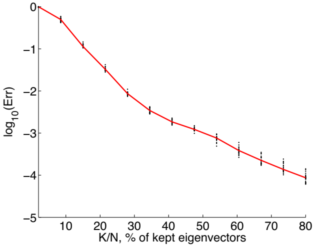
$\gamma = 2$ $\gamma = 4$ $\gamma = 8$

$\log_{10}(\mathrm{Err}(\sigma_\gamma, K/N))$

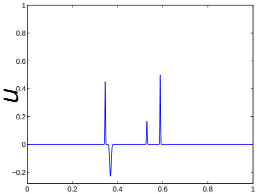
Piecewise Smooth Medium



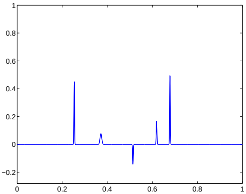
Speed σ^{-1}



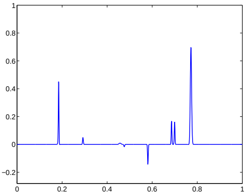
$\log_{10}(\mathrm{Err}(\sigma_\gamma, K/N))$



$t = .2$

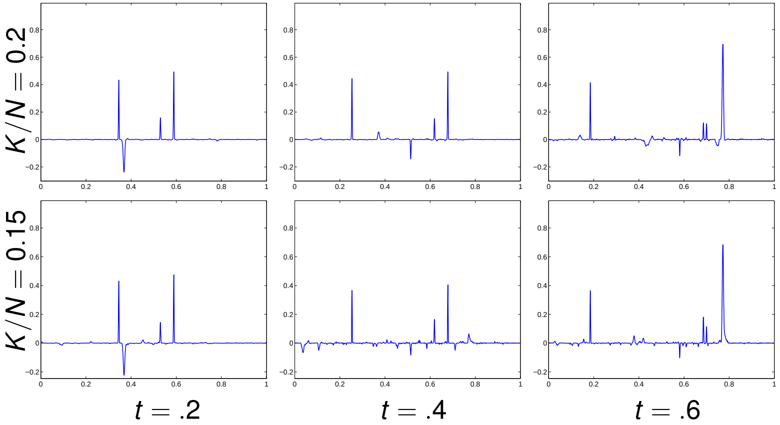


$t = .4$



$t = .6$

Piecewise Smooth Medium

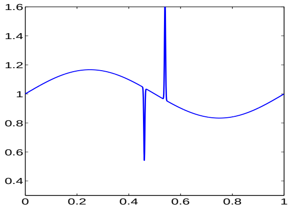


Significance

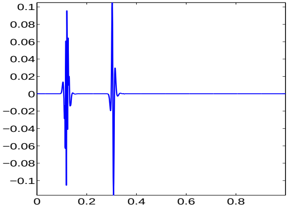
- Perturbated medium: smooth σ_0, reflectors $r(x)$

$$\sigma^2(x) = \sigma_0^2(x) + r(x).$$

- Seismic inversion: recover $r(x)$ from reflected wave d_1 at $t = T$.



$\sigma^2 = \sigma_0^2 + r$

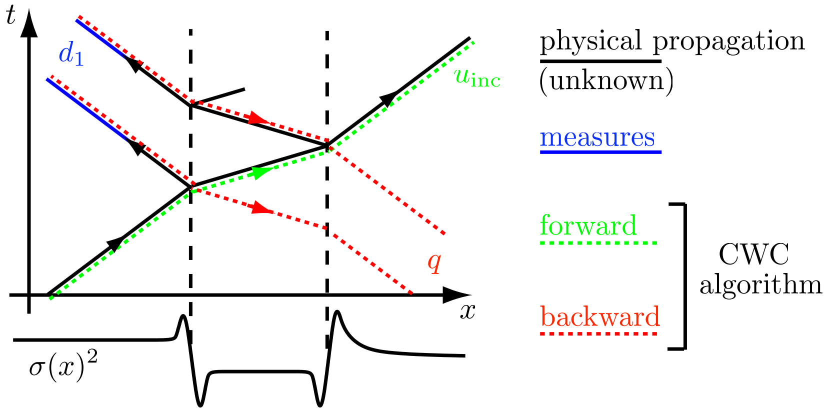


d_1 at time T

Reverse Time Migration

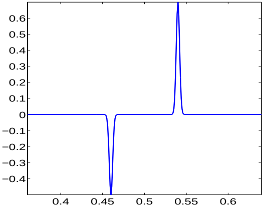
- Reverse-time migration: compute integrals

$$\tilde{r}(x) = -\int_0^T q(x,t)\frac{\partial^2 u_{\text{inc}}}{\partial t^2}(x,t)dt$$

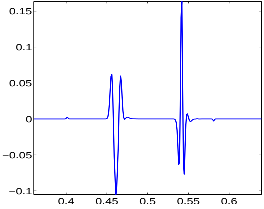


Compressive RTM

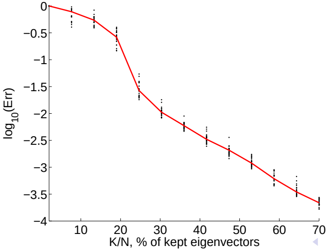
Compressive RTM: compute $\tilde{r}$ without timestepping, i.e, without the associated memory overhead.



$r(x)$ (zoom)



$\tilde{r}(x)$ (zoom)



Conclusion

1D:

- sparsity: remove $\text{Var}(\log \sigma) < 1$.

2D and 3D:

- sparsity in curvelets / wave atoms
- incoherence scale-by-scale

Large computational scale reflection seismology

Deleted slides

Compressed sensing

For φ_k orthogonal in $\mathbb{R}^N$, $f_0 \in \mathbb{R}^N$. Let $c_k = f_0 \cdot \varphi_k + z$.

$$\min \|f\|_1 \qquad s.t. \qquad \|f \cdot \varphi_k - c_k\|_2 \leqslant \varepsilon, \qquad k = 1, \ldots, K$$

Call it (P_1)

- f should be sparse: $\|f_0 - f_{0,S}\|_1 \to 0$ quickly as $S \to \infty$;
- φ_k should be incoherent: $\mu = \sqrt{N} \max_{j,k} |\varphi_k[j]|$ is small;
- φ_k should be drawn uniformly at random from rows of an orthogonal matrix.

Compressed sensing

Theorem (Candès, Romberg, Tao, 2005 + Rudelson, Vershynin, 2007)

Let φ_k, $k = 1, \ldots, K$ be rows drawn uniformly at random from an orthogonal matrix. There exists $C, C' > 0$ such that, for all S and K that obey

$$K \geqslant C \cdot \mu^2 \cdot S \log N \cdot \log^2(S) \log(S \log N),$$

then, with probability $1 - O(N^{-C'})$, the solution $\tilde{f}$ of (P_1)) is unique and obeys

$$\|\tilde{f} - f\|_2 \leqslant C_1\, \varepsilon + C_2 \frac{\|f - f_S\|_{\ell_1}}{\sqrt{S}}.$$

Intrinsically probabilistic.