A Wavelet Tour of Signal Processing

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The Sparse Way

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with contributions from
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AMSTERDAM • BOSTON • HEIDELBERG • LONDON
NEW YORK • OXFORD • PARIS • SAN DIEGO
SAN FRANCISCO • SINGAPORE • SYDNEY • TOKYO
Academic Press is an imprint of Elsevier

Academic Press is an imprint of Elsevier
30 Corporate Drive, Suite 400
Burlington, MA 01803

This book is printed on acid-free paper. ∞

Library of Congress Cataloging-in-Publication Data
Application submitted

ISBN 13: 978-0-12-374370-1

For information on all Academic Press publications,
visit our Website at *www.books.elsevier.com*

Printed in the United States
08 09 10 11 12 10 9 8 7 6 5 4 3 2 1

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À la mémoire de mon père, Alexandre.
Pour ma mère, Francine.

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Preface to the Sparse Edition

I cannot help but find striking resemblances between scientific communities and schools of fish. We interact in conferences and through articles, and we move together while a global trajectory emerges from individual contributions. Some of us like to be at the center of the school, others prefer to wander around, and a few swim in multiple directions in front. To avoid dying by starvation in a progressively narrower and specialized domain, a scientific community needs also to move on. Computational harmonic analysis is still very much alive because it went beyond wavelets. Writing such a book is about decoding the trajectory of the school and gathering the pearls that have been uncovered on the way. Wavelets are no longer the central topic, despite the previous edition's original title. It is just an important tool, as the Fourier transform is. Sparse representation and processing are now at the core.

In the 1980s, many researchers were focused on building time-frequency decompositions, trying to avoid the uncertainty barrier, and hoping to discover the ultimate representation. Along the way came the construction of wavelet orthogonal bases, which opened new perspectives through collaborations with physicists and mathematicians. Designing orthogonal bases with Xlets became a popular sport with compression and noise-reduction applications. Connections with approximations and sparsity also became more apparent. The search for sparsity has taken over, leading to new grounds where orthonormal bases are replaced by redundant dictionaries of waveforms.

During these last seven years, I also encountered the industrial world. With a lot of naiveness, some bandlets, and more mathematics, I cofounded a start-up with Christophe Bernard, Jérome Kalifa, and Erwan Le Pennec. It took us some time to learn that in three months good engineering should produce robust algorithms that operate in real time, as opposed to the three years we were used to having for writing new ideas with promising perspectives. Yet, we survived because mathematics is a major source of industrial innovations for signal processing. Semiconductor technology offers amazing computational power and flexibility. However, ad hoc algorithms often do not scale easily and mathematics accelerates the trial-and-error development process. Sparsity decreases computations, memory, and data communications. Although it brings beauty, mathematical understanding is not a luxury. It is required by increasingly sophisticated information-processing devices.

New Additions

Putting sparsity at the center of the book implied rewriting many parts and adding sections. Chapters 12 and 13 are new. They introduce sparse representations in redundant dictionaries, and inverse problems, super-resolution, and compressive sensing. Here is a small catalog of new elements in this third edition:

- Radon transform and tomography
- Lifting for wavelets on surfaces, bounded domains, and fast computations
- JPEG-2000 image compression
- Block thresholding for denoising
- Geometric representations with adaptive triangulations, curvelets, and bandlets
- Sparse approximations in redundant dictionaries with pursuit algorithms
- Noise reduction with model selection in redundant dictionaries
- Exact recovery of sparse approximation supports in dictionaries
- Multichannel signal representations and processing
- Dictionary learning
- Inverse problems and super-resolution
- Compressive sensing
- Source separation

Teaching

This book is intended as a graduate-level textbook. Its evolution is also the result of teaching courses in electrical engineering and applied mathematics. A new website provides software for reproducible experimentations, exercise solutions, together with teaching material such as slides with figures and MATLAB software for numerical classes of *http://wavelet-tour.com*.

More exercises have been added at the end of each chapter, ordered by level of difficulty. Level^1 exercises are direct applications of the course. Level^2 exercises requires more thinking. Level^3 includes some technical derivation exercises. Level^4 are projects at the interface of research that are possible topics for a final course project or independent study. More exercises and projects can be found in the website.

Sparse Course Programs

The Fourier transform and analog-to-digital conversion through linear sampling approximations provide a common ground for all courses (Chapters 2 and 3). It introduces basic signal representations and reviews important mathematical and algorithmic tools needed afterward. Many trajectories are then possible to explore and teach sparse signal processing. The following list notes several topics that can orient a course's structure with elements that can be covered along the way.

Sparse representations with bases and applications:

- Principles of linear and nonlinear approximations in bases (Chapter 9)
- Lipschitz regularity and wavelet coefficients decay (Chapter 6)
- Wavelet bases (Chapter 7)
- Properties of linear and nonlinear wavelet basis approximations (Chapter 9)
- Image wavelet compression (Chapter 10)
- Linear and nonlinear diagonal denoising (Chapter 11)

Sparse time-frequency representations:

- Time-frequency wavelet and windowed Fourier ridges for audio processing (Chapter 4)
- Local cosine bases (Chapter 8)
- Linear and nonlinear approximations in bases (Chapter 9)
- Audio compression (Chapter 10)
- Audio denoising and block thresholding (Chapter 11)
- Compression and denoising in redundant time-frequency dictionaries with best bases or pursuit algorithms (Chapter 12)

Sparse signal estimation:

- Bayes versus minimax and linear versus nonlinear estimations (Chapter 11)
- Wavelet bases (Chapter 7)
- Linear and nonlinear approximations in bases (Chapter 9)
- Thresholding estimation (Chapter 11)
- Minimax optimality (Chapter 11)
- Model selection for denoising in redundant dictionaries (Chapter 12)
- Compressive sensing (Chapter 13)

Sparse compression and information theory:

- Wavelet orthonormal bases (Chapter 7)
- Linear and nonlinear approximations in bases (Chapter 9)
- Compression and sparse transform codes in bases (Chapter 10)
- Compression in redundant dictionaries (Chapter 12)
- Compressive sensing (Chapter 13)
- Source separation (Chapter 13)

Dictionary representations and inverse problems:

- Frames and Riesz bases (Chapter 5)
- Linear and nonlinear approximations in bases (Chapter 9)
- Ideal redundant dictionary approximations (Chapter 12)
- Pursuit algorithms and dictionary incoherence (Chapter 12)
- Linear and thresholding inverse estimators (Chapter 13)
- Super-resolution and source separation (Chapter 13)
- Compressive sensing (Chapter 13)

Geometric sparse processing:

- Time-frequency spectral lines and ridges (Chapter 4)
- Frames and Riesz bases (Chapter 5)
- Multiscale edge representations with wavelet maxima (Chapter 6)
- Sparse approximation supports in bases (Chapter 9)
- Approximations with geometric regularity, curvelets, and bandlets (Chapters 9 and 12)
- Sparse signal compression and geometric bit budget (Chapters 10 and 12)
- Exact recovery of sparse approximation supports (Chapter 12)
- Super-resolution (Chapter 13)

ACKNOWLEDGMENTS

Some things do not change with new editions, in particular the traces left by the ones who were, and remain, for me important references. As always, I am deeply grateful to Ruzena Bajcsy and Yves Meyer.

I spent the last few years with three brilliant and kind colleagues—Christophe Bernard, Jérome Kalifa, and Erwan Le Pennec—in a pressure cooker called a "start-up." Pressure means stress, despite very good moments. The resulting sauce was a blend of what all of us could provide, which brought new flavors to our personalities. I am thankful to them for the ones I got, some of which I am still discovering.

This new edition is the result of a collaboration with Gabriel Peyré, who made these changes not only possible, but also very interesting to do. I thank him for his remarkable work and help.

Stéphane Mallat

Notations

$\langle f, g \rangle$	Inner product (A.6)
$\|f\|$	Euclidean or Hilbert space norm
$\|f\|_1$	$\mathbf{L^1}$ or $\mathbf{l}^1$ norm
$\|f\|_\infty$	$\mathbf{L}^\infty$ norm
$f[n] = O(g[n])$	Order of: there exists K such that $f[n] \leqslant Kg[n]$
$f[n] = o(g[n])$	Small order of: $\lim_{n \to +\infty} \frac{f[n]}{g[n]} = 0$
$f[n] \sim g[n]$	Equivalent to: $f[n] = O(g[n])$ and $g[n] = O(f[n])$
$A < +\infty$	A is finite
$A \gg B$	A is much bigger than B
z^*	Complex conjugate of $z \in \mathbb{C}$
$\lfloor x \rfloor$	Largest integer $n \leqslant x$
$\lceil x \rceil$	Smallest integer $n \geqslant x$
$(x)_+$	$\max(x, 0)$
$n \bmod N$	Remainder of the integer division of n modulo N

Sets

$\mathbb{N}$	Positive integers including 0
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{C}$	Complex numbers
$\|\Lambda\|$	Number of elements in a set Λ

Signals

$f(t)$	Continuous time signal
$f[n]$	Discrete signal
$\delta(t)$	Dirac distribution (A.30)
$\delta[n]$	Discrete Dirac (3.32)
$\mathbf{1}_{[a,b]}$	Indicator of a function that is 1 in $[a, b]$ and 0 outside

Spaces

$\mathbf{C}_0$	Uniformly continuous functions (7.207)
$\mathbf{C}^p$	p times continuously differentiable functions
$\mathbf{C}^\infty$	Infinitely differentiable functions
$\mathbf{W}^s(\mathbb{R})$	Sobolev^s times differentiable functions (9.8)
$\mathbf{L^2}(\mathbb{R})$	Finite energy functions $\int \|f(t)\|^2 \, dt < +\infty$
$\mathbf{L^p}(\mathbb{R})$	Functions such that $\int \|f(t)\|^p \, dt < +\infty$
$\ell^2(\mathbb{Z})$	Finite energy discrete signals $\sum_{n=-\infty}^{+\infty} \|f[n]\|^2 < +\infty$
$\ell^p(\mathbb{Z})$	Discrete signals such that $\sum_{n=-\infty}^{+\infty} \|f[n]\|^p < +\infty$
$\mathbb{C}^N$	Complex signals of size N
$\mathbf{U} \oplus \mathbf{V}$	Direct sum of two vector spaces

$\mathbf{U} \otimes \mathbf{V}$	Tensor product of two vector spaces (A.19)
NullU	Null space of an operator U
ImU	Image space of an operator U

Operators

Id	Identity
$f'(t)$	Derivative $\frac{df(t)}{dt}$
$f^{(p)}(t)$	Derivative $\frac{d^p f(t)}{dt^p}$ of order p
$\vec{\nabla} f(x,y)$	Gradient vector (6.51)
$f \star g(t)$	Continuous time convolution (2.2)
$f \star g[n]$	Discrete convolution (3.33)
$f \circledast g[n]$	Circular convolution (3.73)

Transforms

$\hat{f}(\omega)$	Fourier transform (2.6), (3.39)
$\hat{f}[k]$	Discrete Fourier transform (3.49)
$Sf(u,s)$	Short-time windowed Fourier transform (4.11)
$P_S f(u,\xi)$	Spectrogram (4.12)
$Wf(u,s)$	Wavelet transform (4.31)
$P_W f(u,\xi)$	Scalogram (4.55)
$P_V f(u,\xi)$	Wigner-Ville distribution (4.120)

Probability

X	Random variable
$E\{X\}$	Expected value
$\mathcal{H}(X)$	Entropy (10.4)
$\mathcal{H}_d(X)$	Differential entropy (10.20)
$\mathrm{Cov}(X_1, X_2)$	Covariance (A.22)
$F[n]$	Random vector
$R_F[k]$	Autocovariance of a stationary process (A.26)

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