

A Wavelet Tour of Signal Processing

A Wavelet Tour of Signal Processing

The Sparse Way

Stéphane Mallat
with contributions from
Gabriel Peyré



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*À la mémoire de mon père, Alexandre.
Pour ma mère, Francine.*

Contents

	Preface to the Sparse Edition	xv
	Notations	xix
CHAPTER 1	Sparse Representations	1
1.1	Computational Harmonic Analysis	1
1.1.1	The Fourier Kingdom	2
1.1.2	Wavelet Bases	2
1.2	Approximation and Processing in Bases	5
1.2.1	Sampling with Linear Approximations	7
1.2.2	Sparse Nonlinear Approximations	8
1.2.3	Compression	11
1.2.4	Denoising	11
1.3	Time-Frequency Dictionaries	14
1.3.1	Heisenberg Uncertainty	15
1.3.2	Windowed Fourier Transform	16
1.3.3	Continuous Wavelet Transform	17
1.3.4	Time-Frequency Orthonormal Bases	19
1.4	Sparsity in Redundant Dictionaries	21
1.4.1	Frame Analysis and Synthesis	21
1.4.2	Ideal Dictionary Approximations	23
1.4.3	Pursuit in Dictionaries	24
1.5	Inverse Problems	26
1.5.1	Diagonal Inverse Estimation	27
1.5.2	Super-resolution and Compressive Sensing	28
1.6	Travel Guide	30
1.6.1	Reproducible Computational Science	30
1.6.2	Book Road Map	30
CHAPTER 2	The Fourier Kingdom	33
2.1	Linear Time-Invariant Filtering	33
2.1.1	Impulse Response	33
2.1.2	Transfer Functions	35
2.2	Fourier Integrals	35
2.2.1	Fourier Transform in $L^1(\mathbb{R})$	35
2.2.2	Fourier Transform in $L^2(\mathbb{R})$	38
2.2.3	Examples	40
2.3	Properties	42
2.3.1	Regularity and Decay	42
2.3.2	Uncertainty Principle	43

	2.3.3	Total Variation	46
	2.4	Two-Dimensional Fourier Transform	51
	2.5	Exercises	55
CHAPTER 3		Discrete Revolution	59
	3.1	Sampling Analog Signals	59
	3.1.1	Shannon-Whittaker Sampling Theorem	59
	3.1.2	Aliasing	61
	3.1.3	General Sampling and Linear Analog Conversions ..	65
	3.2	Discrete Time-Invariant Filters	70
	3.2.1	Impulse Response and Transfer Function	70
	3.2.2	Fourier Series	72
	3.3	Finite Signals	75
	3.3.1	Circular Convolutions	76
	3.3.2	Discrete Fourier Transform	76
	3.3.3	Fast Fourier Transform	78
	3.3.4	Fast Convolutions	79
	3.4	Discrete Image Processing	80
	3.4.1	Two-Dimensional Sampling Theorems	80
	3.4.2	Discrete Image Filtering	82
	3.4.3	Circular Convolutions and Fourier Basis	83
	3.5	Exercises	85
CHAPTER 4		Time Meets Frequency	89
	4.1	Time-Frequency Atoms	89
	4.2	Windowed Fourier Transform	92
	4.2.1	Completeness and Stability	94
	4.2.2	Choice of Window	98
	4.2.3	Discrete Windowed Fourier Transform	101
	4.3	Wavelet Transforms	102
	4.3.1	Real Wavelets	103
	4.3.2	Analytic Wavelets	107
	4.3.3	Discrete Wavelets	112
	4.4	Time-Frequency Geometry of Instantaneous Frequencies ...	115
	4.4.1	Analytic Instantaneous Frequency	115
	4.4.2	Windowed Fourier Ridges	118
	4.4.3	Wavelet Ridges	129
	4.5	Quadratic Time-Frequency Energy	134
	4.5.1	Wigner-Ville Distribution	136
	4.5.2	Interferences and Positivity	140
	4.5.3	Cohen's Class	145
	4.5.4	Discrete Wigner-Ville Computations	149
	4.6	Exercises	151

CHAPTER 5	Frames	155
5.1	Frames and Riesz Bases.....	155
5.1.1	Stable Analysis and Synthesis Operators	155
5.1.2	Dual Frame and Pseudo Inverse	159
5.1.3	Dual-Frame Analysis and Synthesis Computations... ..	161
5.1.4	Frame Projector and Reproducing Kernel	166
5.1.5	Translation-Invariant Frames	168
5.2	Translation-Invariant Dyadic Wavelet Transform	170
5.2.1	Dyadic Wavelet Design	172
5.2.2	Algorithme à Trous	175
5.3	Subsampled Wavelet Frames	178
5.4	Windowed Fourier Frames	181
5.4.1	Tight Frames	183
5.4.2	General Frames	184
5.5	Multiscale Directional Frames for Images	188
5.5.1	Directional Wavelet Frames	189
5.5.2	Curvelet Frames	194
5.6	Exercises	201
CHAPTER 6	Wavelet Zoom	205
6.1	Lipschitz Regularity	205
6.1.1	Lipschitz Definition and Fourier Analysis	205
6.1.2	Wavelet Vanishing Moments	208
6.1.3	Regularity Measurements with Wavelets	211
6.2	Wavelet Transform Modulus Maxima	218
6.2.1	Detection of Singularities	218
6.2.2	Dyadic Maxima Representation	224
6.3	Multiscale Edge Detection	230
6.3.1	Wavelet Maxima for Images	230
6.3.2	Fast Multiscale Edge Computations	239
6.4	Multifractals	242
6.4.1	Fractal Sets and Self-Similar Functions	242
6.4.2	Singularity Spectrum	246
6.4.3	Fractal Noises	254
6.5	Exercises	259
CHAPTER 7	Wavelet Bases	263
7.1	Orthogonal Wavelet Bases.....	263
7.1.1	Multiresolution Approximations.....	264
7.1.2	Scaling Function	267
7.1.3	Conjugate Mirror Filters.....	270
7.1.4	In Which Orthogonal Wavelets Finally Arrive	278
7.2	Classes of Wavelet Bases.....	284
7.2.1	Choosing a Wavelet.....	284

	7.2.2	Shannon, Meyer, Haar, and Battle-Lemarié Wavelets .	289
	7.2.3	Daubechies Compactly Supported Wavelets	292
7.3		Wavelets and Filter Banks	298
	7.3.1	Fast Orthogonal Wavelet Transform	298
	7.3.2	Perfect Reconstruction Filter Banks	302
	7.3.3	Biorthogonal Bases of $\ell^2(\mathbb{Z})$	306
7.4		Biorthogonal Wavelet Bases	308
	7.4.1	Construction of Biorthogonal Wavelet Bases	308
	7.4.2	Biorthogonal Wavelet Design	311
	7.4.3	Compactly Supported Biorthogonal Wavelets	313
7.5		Wavelet Bases on an Interval	317
	7.5.1	Periodic Wavelets	318
	7.5.2	Folded Wavelets	320
	7.5.3	Boundary Wavelets	322
7.6		Multiscale Interpolations	328
	7.6.1	Interpolation and Sampling Theorems	328
	7.6.2	Interpolation Wavelet Basis	333
7.7		Separable Wavelet Bases	338
	7.7.1	Separable Multiresolutions	338
	7.7.2	Two-Dimensional Wavelet Bases	340
	7.7.3	Fast Two-Dimensional Wavelet Transform	346
	7.7.4	Wavelet Bases in Higher Dimensions	348
7.8		Lifting Wavelets	350
	7.8.1	Biorthogonal Bases over Nonstationary Grids	350
	7.8.2	Lifting Scheme	352
	7.8.3	Quincunx Wavelet Bases	359
	7.8.4	Wavelets on Bounded Domains and Surfaces	361
	7.8.5	Faster Wavelet Transform with Lifting	367
7.9		Exercises	370
CHAPTER 8		Wavelet Packet and Local Cosine Bases	377
8.1		Wavelet Packets	377
	8.1.1	Wavelet Packet Tree	377
	8.1.2	Time-Frequency Localization	383
	8.1.3	Particular Wavelet Packet Bases	388
	8.1.4	Wavelet Packet Filter Banks	393
8.2		Image Wavelet Packets	395
	8.2.1	Wavelet Packet Quad-Tree	395
	8.2.2	Separable Filter Banks	399
8.3		Block Transforms	400
	8.3.1	Block Bases	401
	8.3.2	Cosine Bases	403
	8.3.3	Discrete Cosine Bases	406
	8.3.4	Fast Discrete Cosine Transforms	407

8.4	Lapped Orthogonal Transforms	410
8.4.1	Lapped Projectors	410
8.4.2	Lapped Orthogonal Bases	416
8.4.3	Local Cosine Bases	419
8.4.4	Discrete Lapped Transforms	422
8.5	Local Cosine Trees	426
8.5.1	Binary Tree of Cosine Bases	426
8.5.2	Tree of Discrete Bases	429
8.5.3	Image Cosine Quad-Tree	429
8.6	Exercises	432
CHAPTER 9	Approximations in Bases	435
9.1	Linear Approximations	435
9.1.1	Sampling and Approximation Error	435
9.1.2	Linear Fourier Approximations	438
9.1.3	Multiresolution Approximation Errors with Wavelets	442
9.1.4	Karhunen-Loève Approximations	446
9.2	Nonlinear Approximations	450
9.2.1	Nonlinear Approximation Error	451
9.2.2	Wavelet Adaptive Grids	455
9.2.3	Approximations in Besov and Bounded Variation Spaces	459
9.3	Sparse Image Representations	463
9.3.1	Wavelet Image Approximations	464
9.3.2	Geometric Image Models and Adaptive Triangulations	471
9.3.3	Curvelet Approximations	476
9.4	Exercises	478
CHAPTER 10	Compression	481
10.1	Transform Coding	481
10.1.1	Compression State of the Art	482
10.1.2	Compression in Orthonormal Bases	483
10.2	Distortion Rate of Quantization	485
10.2.1	Entropy Coding	485
10.2.2	Scalar Quantization	493
10.3	High Bit Rate Compression	496
10.3.1	Bit Allocation	496
10.3.2	Optimal Basis and Karhunen-Loève	498
10.3.3	Transparent Audio Code	501
10.4	Sparse Signal Compression	506
10.4.1	Distortion Rate and Wavelet Image Coding	506
10.4.2	Embedded Transform Coding	516

10.5	Image-Compression Standards	519
10.5.1	JPEG Block Cosine Coding	519
10.5.2	JPEG-2000 Wavelet Coding	523
10.6	Exercises	531
CHAPTER 11	Denoising	535
11.1	Estimation with Additive Noise	535
11.1.1	Bayes Estimation	536
11.1.2	Minimax Estimation	544
11.2	Diagonal Estimation in a Basis	548
11.2.1	Diagonal Estimation with Oracles	548
11.2.2	Thresholding Estimation	552
11.2.3	Thresholding Improvements	558
11.3	Thresholding Sparse Representations	562
11.3.1	Wavelet Thresholding	563
11.3.2	Wavelet and Curvelet Image Denoising	568
11.3.3	Audio Denoising by Time-Frequency Thresholding ..	571
11.4	Nondiagonal Block Thresholding	575
11.4.1	Block Thresholding in Bases and Frames	575
11.4.2	Wavelet Block Thresholding	581
11.4.3	Time-Frequency Audio Block Thresholding	582
11.5	Denoising Minimax Optimality	585
11.5.1	Linear Diagonal Minimax Estimation	587
11.5.2	Thresholding Optimality over Orthosymmetric Sets	590
11.5.3	Nearly Minimax with Wavelet Estimation	595
11.6	Exercises	606
CHAPTER 12	Sparsity in Redundant Dictionaries	611
12.1	Ideal Sparse Processing in Dictionaries	611
12.1.1	Best M -Term Approximations	612
12.1.2	Compression by Support Coding	614
12.1.3	Denoising by Support Selection in a Dictionary	616
12.2	Dictionaries of Orthonormal Bases	621
12.2.1	Approximation, Compression, and Denoising in a Best Basis	622
12.2.2	Fast Best-Basis Search in Tree Dictionaries	623
12.2.3	Wavelet Packet and Local Cosine Best Bases	626
12.2.4	Bandlets for Geometric Image Regularity	631
12.3	Greedy Matching Pursuits	642
12.3.1	Matching Pursuit	642
12.3.2	Orthogonal Matching Pursuit	648
12.3.3	Gabor Dictionaries	650
12.3.4	Coherent Matching Pursuit Denoising	655

12.4	l^1 Pursuits	659
12.4.1	Basis Pursuit	659
12.4.2	l^1 Lagrangian Pursuit	664
12.4.3	Computations of l^1 Minimizations.....	668
12.4.4	Sparse Synthesis versus Analysis and Total Variation Regularization	673
12.5	Pursuit Recovery	677
12.5.1	Stability and Incoherence	677
12.5.2	Support Recovery with Matching Pursuit	679
12.5.3	Support Recovery with l^1 Pursuits	684
12.6	Multichannel Signals.....	688
12.6.1	Approximation and Denoising by Thresholding in Bases.....	689
12.6.2	Multichannel Pursuits	690
12.7	Learning Dictionaries	693
12.8	Exercises	696
CHAPTER 13	Inverse Problems	699
13.1	Linear Inverse Estimation	700
13.1.1	Quadratic and Tikhonov Regularizations.....	700
13.1.2	Singular Value Decompositions	702
13.2	Thresholding Estimators for Inverse Problems	703
13.2.1	Thresholding in Bases of Almost Singular Vectors ...	703
13.2.2	Thresholding Deconvolutions	709
13.3	Super-resolution	713
13.3.1	Sparse Super-resolution Estimation	713
13.3.2	Sparse Spike Deconvolution	719
13.3.3	Recovery of Missing Data	722
13.4	Compressive Sensing	728
13.4.1	Incoherence with Random Measurements	729
13.4.2	Approximations with Compressive Sensing	735
13.4.3	Compressive Sensing Applications	742
13.5	Blind Source Separation	744
13.5.1	Blind Mixing Matrix Estimation.....	745
13.5.2	Source Separation	751
13.6	Exercises	752
APPENDIX	Mathematical Complements	753
Bibliography		765
Index		795

Preface to the Sparse Edition

I cannot help but find striking resemblances between scientific communities and schools of fish. We interact in conferences and through articles, and we move together while a global trajectory emerges from individual contributions. Some of us like to be at the center of the school, others prefer to wander around, and a few swim in multiple directions in front. To avoid dying by starvation in a progressively narrower and specialized domain, a scientific community needs also to move on. Computational harmonic analysis is still very much alive because it went beyond wavelets. Writing such a book is about decoding the trajectory of the school and gathering the pearls that have been uncovered on the way. Wavelets are no longer the central topic, despite the previous edition's original title. It is just an important tool, as the Fourier transform is. Sparse representation and processing are now at the core.

In the 1980s, many researchers were focused on building time-frequency decompositions, trying to avoid the uncertainty barrier, and hoping to discover the ultimate representation. Along the way came the construction of wavelet orthogonal bases, which opened new perspectives through collaborations with physicists and mathematicians. Designing orthogonal bases with Xlets became a popular sport with compression and noise-reduction applications. Connections with approximations and sparsity also became more apparent. The search for sparsity has taken over, leading to new grounds where orthonormal bases are replaced by redundant dictionaries of waveforms.

During these last seven years, I also encountered the industrial world. With a lot of naiveness, some bandlets, and more mathematics, I cofounded a start-up with Christophe Bernard, Jérôme Kalifa, and Erwan Le Pennec. It took us some time to learn that in three months good engineering should produce robust algorithms that operate in real time, as opposed to the three years we were used to having for writing new ideas with promising perspectives. Yet, we survived because mathematics is a major source of industrial innovations for signal processing. Semiconductor technology offers amazing computational power and flexibility. However, ad hoc algorithms often do not scale easily and mathematics accelerates the trial-and-error development process. Sparsity decreases computations, memory, and data communications. Although it brings beauty, mathematical understanding is not a luxury. It is required by increasingly sophisticated information-processing devices.

New Additions

Putting sparsity at the center of the book implied rewriting many parts and adding sections. Chapters 12 and 13 are new. They introduce sparse representations in redundant dictionaries, and inverse problems, super-resolution, and

compressive sensing. Here is a small catalog of new elements in this third edition:

- Radon transform and tomography
- Lifting for wavelets on surfaces, bounded domains, and fast computations
- JPEG-2000 image compression
- Block thresholding for denoising
- Geometric representations with adaptive triangulations, curvelets, and bandlets
- Sparse approximations in redundant dictionaries with pursuit algorithms
- Noise reduction with model selection in redundant dictionaries
- Exact recovery of sparse approximation supports in dictionaries
- Multichannel signal representations and processing
- Dictionary learning
- Inverse problems and super-resolution
- Compressive sensing
- Source separation

Teaching

This book is intended as a graduate-level textbook. Its evolution is also the result of teaching courses in electrical engineering and applied mathematics. A new website provides software for reproducible experimentations, exercise solutions, together with teaching material such as slides with figures and MATLAB software for numerical classes of <http://wavelet-tour.com>.

More exercises have been added at the end of each chapter, ordered by level of difficulty. Level¹ exercises are direct applications of the course. Level² exercises requires more thinking. Level³ includes some technical derivation exercises. Level⁴ are projects at the interface of research that are possible topics for a final course project or independent study. More exercises and projects can be found in the website.

Sparse Course Programs

The Fourier transform and analog-to-digital conversion through linear sampling approximations provide a common ground for all courses (Chapters 2 and 3). It introduces basic signal representations and reviews important mathematical and algorithmic tools needed afterward. Many trajectories are then possible to explore and teach sparse signal processing. The following list notes several topics that can orient a course's structure with elements that can be covered along the way.

Sparse representations with bases and applications:

- Principles of linear and nonlinear approximations in bases (Chapter 9)
- Lipschitz regularity and wavelet coefficients decay (Chapter 6)
- Wavelet bases (Chapter 7)
- Properties of linear and nonlinear wavelet basis approximations (Chapter 9)
- Image wavelet compression (Chapter 10)
- Linear and nonlinear diagonal denoising (Chapter 11)

Sparse time-frequency representations:

- Time-frequency wavelet and windowed Fourier ridges for audio processing (Chapter 4)
- Local cosine bases (Chapter 8)
- Linear and nonlinear approximations in bases (Chapter 9)
- Audio compression (Chapter 10)
- Audio denoising and block thresholding (Chapter 11)
- Compression and denoising in redundant time-frequency dictionaries with best bases or pursuit algorithms (Chapter 12)

Sparse signal estimation:

- Bayes versus minimax and linear versus nonlinear estimations (Chapter 11)
- Wavelet bases (Chapter 7)
- Linear and nonlinear approximations in bases (Chapter 9)
- Thresholding estimation (Chapter 11)
- Minimax optimality (Chapter 11)
- Model selection for denoising in redundant dictionaries (Chapter 12)
- Compressive sensing (Chapter 13)

Sparse compression and information theory:

- Wavelet orthonormal bases (Chapter 7)
- Linear and nonlinear approximations in bases (Chapter 9)
- Compression and sparse transform codes in bases (Chapter 10)
- Compression in redundant dictionaries (Chapter 12)
- Compressive sensing (Chapter 13)
- Source separation (Chapter 13)

Dictionary representations and inverse problems:

- Frames and Riesz bases (Chapter 5)
- Linear and nonlinear approximations in bases (Chapter 9)
- Ideal redundant dictionary approximations (Chapter 12)
- Pursuit algorithms and dictionary incoherence (Chapter 12)
- Linear and thresholding inverse estimators (Chapter 13)
- Super-resolution and source separation (Chapter 13)
- Compressive sensing (Chapter 13)

Geometric sparse processing:

- Time-frequency spectral lines and ridges (Chapter 4)
- Frames and Riesz bases (Chapter 5)
- Multiscale edge representations with wavelet maxima (Chapter 6)
- Sparse approximation supports in bases (Chapter 9)
- Approximations with geometric regularity, curvelets, and bandlets (Chapters 9 and 12)
- Sparse signal compression and geometric bit budget (Chapters 10 and 12)
- Exact recovery of sparse approximation supports (Chapter 12)
- Super-resolution (Chapter 13)

ACKNOWLEDGMENTS

Some things do not change with new editions, in particular the traces left by the ones who were, and remain, for me important references. As always, I am deeply grateful to Ruzena Bajcsy and Yves Meyer.

I spent the last few years with three brilliant and kind colleagues—Christophe Bernard, Jérôme Kalifa, and Erwan Le Pennec—in a pressure cooker called a “start-up.” Pressure means stress, despite very good moments. The resulting sauce was a blend of what all of us could provide, which brought new flavors to our personalities. I am thankful to them for the ones I got, some of which I am still discovering.

This new edition is the result of a collaboration with Gabriel Peyré, who made these changes not only possible, but also very interesting to do. I thank him for his remarkable work and help.

Stéphane Mallat

Notations

$\langle f, g \rangle$	Inner product (A.6)
$\ f\ $	Euclidean or Hilbert space norm
$\ f\ _1$	\mathbf{L}^1 or \mathbf{l}^1 norm
$\ f\ _\infty$	\mathbf{L}^∞ norm
$f[n] = O(g[n])$	Order of: there exists K such that $f[n] \leq Kg[n]$
$f[n] = o(g[n])$	Small order of: $\lim_{n \rightarrow +\infty} \frac{f[n]}{g[n]} = 0$
$f[n] \sim g[n]$	Equivalent to: $f[n] = O(g[n])$ and $g[n] = O(f[n])$
$A < +\infty$	A is finite
$A \gg B$	A is much bigger than B
z^*	Complex conjugate of $z \in \mathbb{C}$
$\lfloor x \rfloor$	Largest integer $n \leq x$
$\lceil x \rceil$	Smallest integer $n \geq x$
$(x)_+$	$\max(x, 0)$
$n \bmod N$	Remainder of the integer division of n modulo N

Sets

\mathbb{N}	Positive integers including 0
\mathbb{Z}	Integers
\mathbb{R}	Real numbers
\mathbb{R}^+	Positive real numbers
\mathbb{C}	Complex numbers
$ \Lambda $	Number of elements in a set Λ

Signals

$f(t)$	Continuous time signal
$f[n]$	Discrete signal
$\delta(t)$	Dirac distribution (A.30)
$\delta[n]$	Discrete Dirac (3.32)
$\mathbf{1}_{[a,b]}$	Indicator of a function that is 1 in $[a, b]$ and 0 outside

Spaces

\mathbf{C}_0	Uniformly continuous functions (7.207)
\mathbf{C}^p	p times continuously differentiable functions
\mathbf{C}^∞	Infinitely differentiable functions
$\mathbf{W}^s(\mathbb{R})$	Sobolev ^s times differentiable functions (9.8)
$\mathbf{L}^2(\mathbb{R})$	Finite energy functions $\int f(t) ^2 dt < +\infty$
$\mathbf{L}^p(\mathbb{R})$	Functions such that $\int f(t) ^p dt < +\infty$
$\ell^2(\mathbb{Z})$	Finite energy discrete signals $\sum_{n=-\infty}^{+\infty} f[n] ^2 < +\infty$
$\ell^p(\mathbb{Z})$	Discrete signals such that $\sum_{n=-\infty}^{+\infty} f[n] ^p < +\infty$
\mathbb{C}^N	Complex signals of size N
$\mathbf{U} \oplus \mathbf{V}$	Direct sum of two vector spaces

$\mathbf{U} \otimes \mathbf{V}$	Tensor product of two vector spaces (A.19)
Null \mathbf{U}	Null space of an operator U
Im \mathbf{U}	Image space of an operator U

Operators

Id	Identity
$f'(t)$	Derivative $\frac{df(t)}{dt}$
$f^{(p)}(t)$	Derivative $\frac{d^p f(t)}{dt^p}$ of order p
$\vec{\nabla}f(x, y)$	Gradient vector (6.51)
$f \star g(t)$	Continuous time convolution (2.2)
$f \star g[n]$	Discrete convolution (3.33)
$f \otimes g[n]$	Circular convolution (3.73)

Transforms

$\hat{f}(\omega)$	Fourier transform (2.6), (3.39)
$\hat{f}[k]$	Discrete Fourier transform (3.49)
$Sf(u, s)$	Short-time windowed Fourier transform (4.11)
$P_Sf(u, \xi)$	Spectrogram (4.12)
$Wf(u, s)$	Wavelet transform (4.31)
$P_Wf(u, \xi)$	Scalogram (4.55)
$P_Vf(u, \xi)$	Wigner-Ville distribution (4.120)

Probability

X	Random variable
$E\{X\}$	Expected value
$\mathcal{H}(X)$	Entropy (10.4)
$\mathcal{H}_d(X)$	Differential entropy (10.20)
$\text{Cov}(X_1, X_2)$	Covariance (A.22)
$F[n]$	Random vector
$R_F[k]$	Autocovariance of a stationary process (A.26)

A Wavelet Tour of Signal Processing

